

From Exact-WKB towards Singular Quantum Perturbation Theory[†]

By

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Abstract

We use exact WKB analysis to derive some concrete formulae in singular quantum perturbation theory, for Schrödinger eigenvalue problems on the real line with polynomial potentials of the form $(q^M + gq^N)$, where $N > M > 0$ even, and $g > 0$. Mainly, we establish the $g \rightarrow 0$ limiting forms of global spectral functions such as the zeta-regularized determinants and some spectral zeta functions.

§1. Introduction

The purpose of this work is to set up a path to obtain precise statements of a quantum perturbative nature with the help of exact WKB analysis. The RIMS has always played a major and pioneering role in the inception and growth of exact asymptotic analysis, and earlier, in the development of some of its fundamental tools (such as hyperfunctions and holomorphic microlocal analysis). This influence is testified by the Proceedings volume of a recent Kyoto conference, which offers a very complete view of the subject [5]. It is therefore a great honor and a proper tribute to the RIMS to write here about a subject which grew thanks to the crucial participation and encouragements of RIMS researchers.

A prototype problem in quantum perturbation theory is the quartic anharmonic oscillator,

$$(1.1) \quad \left(-\frac{d^2}{dq^2} + q^2 + gq^4 - E \right) \Psi(q) = 0, \quad q \in \mathbb{R}, \quad g \geq 0.$$

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This problem has a purely discrete eigenvalue spectrum $\{E_k(g)\}$ for all $g \geq 0$. A typical task in (Rayleigh–Schrödinger) perturbation theory is to compute individual eigenvalues $E_k(g)$ (or their eigenfunctions) as formal power series of the coupling constant g [1]. This is of important practical use when the unperturbed ($g = 0$) problem is exactly solvable, here a harmonic oscillator; a major drawback is however that the coupling term has the higher degree, hence the formalism is singular. Thus, the perturbation series diverges for any $g \neq 0$; it only gives an asymptotic expansion for $g \rightarrow 0$, which is moreover non-uniform in the quantum number k .

As our theoretical discussion can readily include all binomials potentials, we immediately turn to the more general Schrödinger equation

$$(1.2) \quad \left(-\frac{d^2}{dq^2} + U_g(q) - E\right)\Psi(q) = 0,$$

$$U_g(q) \stackrel{\text{def}}{=} q^M + gq^N, \quad q \in \mathbb{R}, \quad N > M \geq 2 \text{ even}, \quad g \geq 0;$$

we keep $E_k(g)$ as a generic notation for the eigenvalues of this problem ((N, M) -dependences now being implied).

(Exact WKB formalisms accommodate non-even potentials as well [2, 8]; for instance, eq. (1.2) could be considered with odd N or M but on the half-line $[0, +\infty)$ [8]; however, this extension is not essential here while it does complicate the classification when $M = 1$, so we omit it in the present work.)

A very basic fact (*Symanzik scaling* property) is that a simple coordinate dilation, $q \mapsto g^{-1/(N+2)}q$, establishes a unitary equivalence between the two Schrödinger operators

$$(1.3) \quad v^{2/(M+2)}(-d^2/dq^2 + U_g(q)) \quad \text{and} \quad \widehat{H}_v \stackrel{\text{def}}{=} -d^2/dq^2 + V_v(q),$$

$$(1.4) \quad \text{where} \quad v \equiv g^{-(M+2)/(N+2)} \quad \text{and} \quad V_v(q) \stackrel{\text{def}}{=} q^N + vq^M.$$

Thus, eq. (1.2) is equivalent to

$$(1.5) \quad \left(-\frac{d^2}{dq^2} + q^N + vq^M + \lambda\right)\Psi(q) = 0, \quad v \equiv g^{-(M+2)/(N+2)}, \quad \lambda \equiv -v^{2/(M+2)}E.$$

In this transformed Schrödinger equation, the interaction term is now vq^M and has the lower degree, so that v can act as a regular deformation parameter; the former perturbative regime $g \rightarrow 0$ translates as the asymptotic $v \rightarrow +\infty$ regime. However, at no finite v is the problem (1.5) solvable in any traditional sense, and this has severely limited practical uses of this reparametrization. On

the other hand, this deformation can be fully studied by exact WKB analysis, which now handles general (1D) polynomial potentials. One earlier detailed study of this sort is based on resurgence theory [2]. Another such path from exact-WKB to perturbation theory lies in proving the Zinn-Justin conjectures about multi-instantons [3, 9]. Here we will develop still another line of calculations started in [8] (Secs. 3–4), using an exact WKB framework built upon Sibuya’s formalism [6]: i.e., we will fully compute how the *spectral determinants* themselves (and related spectral functions) *asymptotically depend on the coupling parameter* $v \rightarrow +\infty$ (or $g \rightarrow 0$). Spectral functions being symmetric functions of all eigenvalues $E_k(g)$ together, the non-uniformity in k of perturbative approximations must show up somehow, and the $g \rightarrow 0$ behavior of such objects may not be obviously traceable to existing (fixed- k) perturbative results.

To give a concrete example, we ask: how do the spectral zeta functions $Z_g(s) = \sum_{k=0}^{\infty} E_k(g)^{-s}$ precisely behave for $g \rightarrow 0$? Specially at $s = 1$ when $M = 2$: then, that series converges for any $g > 0$ but term by term it becomes the divergent (odd) harmonic series $\sum_k (2k + 1)^{-1}$ at $g = 0$. The latter series admits a fundamental regularization by means of a sharp summation cutoff K :

$$(1.6) \quad \sum_{k=0}^{K-1} \frac{1}{2k + 1} \sim \frac{1}{2}(\log K + \gamma + 2 \log 2), \quad K \rightarrow +\infty,$$

which is (in just a slight variant form) the basic definition of Euler’s constant γ . Now, the eigenvalues themselves obey $E_k(g) \sim 2k+1$ for $g \rightarrow 0$ (k fixed) by perturbation theory, but $E_k(g) \propto [g(2k+1)^N]^{2/(N+2)}$ for $k \rightarrow +\infty$ (g fixed) by the asymptotic Bohr–Sommerfeld condition. Hence the series $Z_g(1) = \sum_k E_k(g)^{-1}$ naturally provides another regularization to the odd harmonic series, now by *gradually* forcing it into a convergent regime; the crossover zone roughly lies around $k \propto K_g \stackrel{\text{def}}{=} g^{-2/(N-2)}$ [4]. By substituting $K = K_g$ into eq. (1.6), we intuitively conjecture $Z_g(1) \sim -\frac{1}{N-2} \log g + C_N$, which will prove correct for the logarithmic slope; such an approach is however very crude, and it cannot guess the additive constant C_N . By contrast, exact WKB analysis will yield *more precise, rigorous asymptotic results* for such zeta-values, such as the final formula (5.13) for $Z_g(1)$.

The outline of the paper is as follows. §2 recalls essential prerequisites and definitions for the exact WKB approach to be used here. §3 presents the asymptotic problem and its conceptual resolution by exact WKB theory, namely eq. (3.10). §4 performs the key computational steps: a class of specific *improper action integrals* $\int_0^{+\infty} \Pi(q) dq$ are explicitly evaluated (where $\Pi(q)$ are

essentially classical momentum functions, and the integrals are primitively very divergent). Finally, §5 processes all intermediate calculations into concrete formulae, mainly eqs. (5.1), (5.5).

§2. Some Notions from Exact WKB Theory

We recall the essential facts and notations to be used later concerning the exact WKB treatment of a Schrödinger operator $\widehat{H} \stackrel{\text{def}}{=} -d^2/dq^2 + V(q)$ on $L^2(\mathbb{R})$ with a polynomial potential $V(q) = +q^N + [\text{lower-degree terms}]$, here taken *real and even*. (Details and justifications have to be omitted: cf. [8] Sec. 1, and references therein). Such an operator is self-adjoint, has a compact resolvent and commutes with the parity operator ($q \mapsto -q$). A frequently needed quantity (which we call the *order* of the problem) is

$$(2.1) \quad \mu(N) \stackrel{\text{def}}{=} \frac{1}{2} + \frac{1}{N}.$$

As standard notations, we will also use $\psi(z) \equiv [\Gamma' / \Gamma](z)$, and $\gamma = \text{Euler's constant}$.

§2.1. Improper action integral, and residual polynomial

Important quantities enter at the classical dynamical level around the (complexified) momentum function,

$$(2.2) \quad \Pi_\lambda(q) \stackrel{\text{def}}{=} (V(q) + \lambda)^{1/2},$$

where the constant $(-\lambda)$ stands for the classical energy and, say, $\lambda > -\inf V$ (initially). *Improper action integrals* taken over *semi-infinite paths* prove most useful: $\int_q^{+\infty} \Pi_\lambda(q') dq'$ (primitively divergent) is very naturally defined as the analytical continuation to $s = 0$, *when finite*, of

$$(2.3) \quad I_q(s, \lambda) \stackrel{\text{def}}{=} \int_q^{+\infty} (V(q') + \lambda)^{-s+1/2} dq' \quad (\text{convergent for } \text{Re}(s) > \mu(N)).$$

Now, any s -plane singularities of $I_q(s, \lambda)$ arise in $\{\text{Re}(s) \leq \mu(N)\}$ from the large- q' behavior of the integrand. Specifically, the $q' \rightarrow +\infty$ expansion

$$(2.4) \quad (V(q') + \lambda)^{-s+1/2} \sim \sum_{\rho} \beta_{\rho}(s) (q')^{\rho-Ns} \quad (\rho = N/2, N/2 - 1, \dots)$$

(where $\beta_\rho(s)$ are polynomials in s , explicitly computable one by one), implies the singular decomposition

$$(2.5) \quad I_q(s, \lambda) \sim - \sum_{\rho} \beta_\rho(s) \frac{q^{\rho+1-Ns}}{\rho+1-Ns};$$

hence at $s = 0$, $I_q(s, \lambda)$ has at most a simple pole, generated by the $\rho = -1$ term (if any):

$$(2.6) \quad \text{Res}_{s=0} I_q(s, \lambda) = \beta_{-1}(0)/N,$$

a value actually independent of λ (save when $N = 2$) and of q . Remark: for non-monic $V(q) = vq^N + \dots$, $\beta_\rho(s)$ will also carry a non-polynomial but trivial factor $\propto v^{-s}$ (as in eq. (3.5) below).

A central distinction sets in at this point: if the “residual” polynomial $\beta_{-1}(s) \equiv 0$, the Schrödinger problem $(\widehat{H} + \lambda)\Psi = 0$ will behave more simply (“normal” type, **N**); otherwise, “anomaly” terms will occur (type **A**). Wholly generic polynomials $(V(q) + \lambda)$ are of type **A**; still, in a sense, “a majority” of them have $\beta_{-1}(s) \equiv 0$: among the even ones, already all those having a *degree N multiple of 4* (and, among the non-even ones, all those of *odd degree*).

Thus for the **N** type, $\int_q^{+\infty} \Pi_\lambda(q') dq'$ is readily defined as the analytical continuation of $I_q(s, \lambda)$ to the (regular) point $s = 0$. As for a more general specification including type **A**, the best choice is not the bare finite part of $I_q(s, \lambda)$ at the pole $s = 0$ (denoted $\text{FP}_{s=0} I_q(s, \lambda)$), but rather ([8], eq. (32))

$$(2.7) \quad \int_q^{+\infty} \Pi_\lambda(q') dq' \stackrel{\text{def}}{=} \text{FP}_{s=0} I_q(s, \lambda) + 2(1 - \log 2) \beta_{-1}(0)/N$$

in order to preserve the basic identities (2.15) below. Additivity is also maintained:

$$(2.8) \quad \int_q^{+\infty} \Pi_\lambda(q') dq' = \int_q^{q''} \Pi_\lambda(q') dq' + \int_{q''}^{+\infty} \Pi_\lambda(q') dq' \quad \text{for all } q, q'' \text{ finite}$$

(because a finite integral $\int_q^{q''} (V(q') + \lambda)^{-s+1/2} dq'$ is entire in s).

Remarks. (i) $\int_q^{+\infty} \Pi_\lambda(q') dq'$ is an “Agmon distance from q to $+\infty$ ”, suitably renormalized; (ii) this procedure is a classical counterpart to zeta-regularization at the quantum level; (iii) like the extra term in eq. (2.7), all anomaly terms *here* will just be proportional to “the residue” $\beta_{-1}(0)$ (residue of $\Pi_\lambda(q)$ at ∞), but more general anomaly terms occur elsewhere [8].

§2.2. Spectral functions

A (1D) operator \widehat{H} as above has a purely discrete real spectrum $\{\lambda_0 < \lambda_1 < \lambda_2 < \dots\}$, $(\lambda_k \uparrow +\infty)$, where even (resp. odd) k correspond to eigenfunctions of even (resp. odd) parity.

Generalized (*à la* Hurwitz) *spectral zeta functions* can be defined for each parity:

$$(2.9) \quad Z^\pm(s, \lambda) \stackrel{\text{def}}{=} \sum_{\substack{k \text{ even} \\ \text{odd}}} (\lambda_k + \lambda)^{-s} \quad \text{for } \text{Re } s > \mu(N)$$

and, say, $\lambda > -\lambda_0$; yet some results will separate more neatly upon a *skew* versus a *full* zeta function, respectively defined as

$$(2.10) \quad Z^P \equiv Z^+ - Z^-, \quad Z \equiv Z^+ + Z^-.$$

Spectral determinants $D^\pm(\lambda) \equiv \det^\pm(\widehat{H} + \lambda)$ are defined next by zeta-regularization:

$$(2.11) \quad D^\pm(\lambda) \stackrel{\text{def}}{=} \exp[-\partial_s Z^\pm(s, \lambda)]_{s=0} \quad (\text{and } D^P \equiv D^+ / D^-, \quad D \equiv D^+ D^-),$$

where $s = 0$ is reached by analytical continuation from $\{\text{Re } s > \mu(N)\}$. These functions also admit more explicit characterizations:

- **their Weierstrass infinite products** (written for $\mu(N) < 2$, as is the case here):

$$(2.12) \quad D^\pm(\lambda) \equiv D^\pm(0) e^{\text{FP}_{s=1} Z^\pm(s,0) \lambda} \prod_{\substack{k \text{ even} \\ \text{odd}}} (1 + \lambda/\lambda_k) e^{-\lambda/\lambda_k},$$

$$(2.13) \quad \equiv D^\pm(0) \prod_{\substack{k \text{ even} \\ \text{odd}}} (1 + \lambda/\lambda_k) \quad \text{when } \mu(N) < 1, \text{ i.e., } N > 2$$

and likewise for D, D^P ; this shows that the determinants continue to *entire* functions (of order $\mu(N)$) in the variable λ (except for D^P , *meromorphic*);

- **the basic identities of the exact-WKB method:** let $\Psi_\lambda(q)$ be the *canonical recessive* solution of the differential equation, specified through its $q \rightarrow +\infty$ *asymptotic form*

$$(2.14) \quad \Psi_\lambda(q) \sim \Pi_\lambda(q)^{-1/2} e^{\int_q^{+\infty} \Pi_\lambda(q') dq'},$$

where $\int_q^{+\infty} \Pi_\lambda(q') dq'$ is *fixed according to* eq. (2.7); then, under that precise normalization,

$$(2.15) \quad D^-(\lambda) \equiv \Psi_\lambda(0), \quad D^+(\lambda) \equiv -\Psi'_\lambda(0),$$

(also valid for a rescaled potential, i.e., $V(q) = uq^N + \dots$, with $u > 0$).

Remark. the solutions (2.14) are proportional to Sibuya's subdominant solutions [6], but the two normalizations fully coincide only when the type is **N**.

Finally, we will need the transformation rules for spectral functions under a *global spectral dilation* ($\lambda_k \mapsto r\lambda_k$, $r = \text{cst.} > 0$). Obviously,

$$(2.16) \quad Z^\pm(s, \lambda) \mapsto r^{-s} Z^\pm(s, \lambda/r) \quad \text{for } \text{Re } s > \mu(N)$$

(and likewise for Z , Z^P); hence upon continuation to $s = 0$, and applying eq. (2.11),

$$(2.17) \quad D(\lambda) \mapsto r^{Z(0, \lambda/r)} D(\lambda/r), \quad D^P(\lambda) \mapsto r^{Z^P(0, \lambda/r)} D^P(\lambda/r)$$

where, moreover, ([8], eqs. (27), (37))

$$(2.18) \quad Z(0, \lambda) \equiv -2\beta_{-1}(0)/N, \quad Z^P(0, \lambda) \equiv 1/2.$$

§3. The Asymptotic $v \rightarrow +\infty$ Problem

We now return to the Schrödinger operator $\widehat{H}(v) = -d^2/dq^2 + q^N + vq^M$, as in eq. (1.5) ($N > M \geq 2$ both even, $v > 0$). We will find the asymptotic behaviors of its spectral determinants in the regime of singular perturbation theory for eq. (1.1):

$$(3.1) \quad D^\pm(\lambda, v) \equiv \det^\pm(\widehat{H}(v) + \lambda) \quad (\lambda > 0), \quad v \equiv g^{-(M+2)/(N+2)} \rightarrow +\infty.$$

By the lowest-order $g \rightarrow 0$ perturbation theory, the *individual* eigenvalues $\lambda_k(v)$ of $\widehat{H}(v)$ ("coupled problem") become asymptotic to those of $\widehat{H}_0(v) = -d^2/dq^2 + vq^M$ ("uncoupled problem"). We then expect $\det^\pm(\widehat{H}(v) + \lambda)$ to somehow behave like $\det^\pm(\widehat{H}_0(v) + \lambda)$ as $v \rightarrow +\infty$, but the latter regime is singular and moreover non-uniform in k ; hence the actual behavior of the determinants is an open problem. In [8] (Secs. 3–4), we tackled it for a few binomial potentials and exclusively at $\lambda = 0$; here we treat it in full generality.

§3.1. Detailed anomaly types

As argued in §2.1, it is essential to distinguish between normal (**N**) and anomalous (**A**) cases, but now this has to be done for each problem $\widehat{H}(v)$ and $\widehat{H}_0(v)$ independently.

- **the coupled problem** ($\Pi_\lambda(q)^2 = q^N + vq^M + \lambda$): the residual polynomial is the coefficient of q^{-1-Ns} in the generalized binomial expansion for $q^{N(1/2-s)}(1 + vq^{M-N} + \lambda q^{-N})^{1/2-s}$. When $\Pi_\lambda(q)^2$ has degree $\neq 2$, like here, the residual polynomial $\beta_{-1}(s)$ cannot depend on λ ; concretely, it evaluates as follows:

$$(3.2) \quad \beta_{-1}(s) \equiv 0 \text{ unless } \frac{N+2}{2(N-M)} = j \in \mathbb{N}^* \text{ ("anomaly condition } \mathbf{A}_j \text{ of level } j\text{")};$$

thus, anomalies require *exponent pairs* (N, M) obeying *special congruence relations*:

$$(3.3) \quad (\text{level } j :) \quad N = 2jm - 2, M = N - m \text{ for } m \in \mathbb{N}^* \\ \text{(with } m \text{ even for even potentials),}$$

and then

$$(3.4) \quad \beta_{-1}(s) \equiv (-1)^j \frac{\Gamma(s+j-1/2)}{\Gamma(s-1/2)j!} v^j \quad \left[\beta_{-1}(0) = (-1)^{j-1} \frac{(2j-2)!}{2^{2j-1}(j-1)!j!} v^j \right].$$

- **the uncoupled problem** ($\Pi_{0,\lambda}(q)^2 = vq^M + \lambda$): the same calculation now simply yields

$$(3.5) \quad \beta_{-1}(s) \equiv v^{-1/2-s} \lambda (1/2 - s) \text{ if } M = 2 \quad [\mathbf{A}_1 \text{ for } \lambda \neq 0], \\ \text{otherwise } \beta_{-1}(s) \equiv 0 \quad [\mathbf{N}].$$

The *harmonic oscillator* ($\Pi(q)^2 = vq^2 + \lambda$) thus gives the prime example of anomaly, and the only case (among all potentials) where the residue depends on the spectral parameter; all other binomials $\{vq^M + \lambda\}$ ($M \neq 2$) are of type **N**.

The type can jump either way in the $v \rightarrow +\infty$ limit, giving birth to four distinct variants (and the “basic” example of eq. (1.1) is not the simplest!):

- N** \rightarrow **N**: e.g., $V(q) = q^8 + vq^4$;
- A_j** \rightarrow **N**: e.g., $V(q) = q^6 + vq^4$, of level $j = 2$;
- N** \rightarrow **A₁**: e.g., $V(q) = q^4 + vq^2$ (the “basic” example) when $\lambda \neq 0$;
- A_j** \rightarrow **A₁**: only one case, $V(q) = q^6 + vq^2$ when $\lambda \neq 0$, for which $j = 1$.

§3.2. The main estimate

We can relate the coupled and uncoupled spectral determinants very easily through a key result of exact WKB theory, the basic identities (2.15). These are to be written for both (coupled and uncoupled) problems independently:

$$(3.6) \quad \begin{aligned} \det^-(\widehat{H}(v) + \lambda) &\equiv \Psi_\lambda(0, v), & \det^+(\widehat{H}(v) + \lambda) &\equiv -\Psi'_\lambda(0, v), \\ \det^-(\widehat{H}_0(v) + \lambda) &\equiv \Psi_{0,\lambda}(0, v), & \det^+(\widehat{H}_0(v) + \lambda) &\equiv -\Psi'_{0,\lambda}(0, v), \end{aligned}$$

where $\Psi_\lambda(q, v)$, resp. $\Psi_{0,\lambda}(q, v)$ are the canonical recessive solutions of $(\widehat{H}(v) + \lambda)\Psi = 0$, resp. $(\widehat{H}_0(v) + \lambda)\Psi_0 = 0$. So, the problem boils down to relating $\Psi_\lambda(q, v)$ and $\Psi_{0,\lambda}(q, v)$ near $q = 0$ as $v \rightarrow +\infty$.

Now, as soon as $|q|^{N-M} \ll v$, the term q^N becomes a negligible perturbation of vq^M within the Schrödinger equation, hence the recessive solution $\Psi_\lambda(q, v)$ has to become *asymptotically proportional to* $\Psi_{0,\lambda}(q, v)$ in that regime. Indeed, the alternative normalization of recessive solutions *based at* $q = 0$,

$$(3.7) \quad \begin{aligned} \Psi_\lambda(q, v) &\sim \Pi_\lambda(q, v)^{-1/2} e^{-\int_0^q \Pi_\lambda(q', v) dq'}, \\ \Psi_{0,\lambda}(q, v) &\sim \Pi_{0,\lambda}(q, v)^{-1/2} e^{-\int_0^q \Pi_{0,\lambda}(q', v) dq'} \end{aligned}$$

(valid for $\Pi_\lambda(q) \rightarrow +\infty$ in the $\{q > 0\}$ domain) immediately entails

$$(3.8) \quad \Psi_\lambda(q, v) \sim \Psi_{0,\lambda}(q, v) \quad \text{for } v \rightarrow +\infty, \quad |q|^{N-M} \ll v,$$

simply by invoking the asymptotic equivalence $\Pi_\lambda(q', v) \sim \Pi_{0,\lambda}(q', v)$ all over the *bounded* interval $[0, q]$ when $v \rightarrow +\infty$ at fixed q .

The only remaining issue is then to relate the two normalizations, our canonical one of eq. (2.14) “based at $q = +\infty$ ”, and the latter one of eq. (3.7) based at $q = 0$. Thanks to eq. (2.8), the answer is simply

$$(3.9) \quad \Psi_\lambda(q, v) \equiv e^{\int_0^{+\infty} \Pi_\lambda(q', v) dq'} \Psi_\lambda(q, v) \quad (\text{and likewise for } \Psi_{0,\lambda} \text{ with } \Pi_{0,\lambda}).$$

Finally, putting together eqs. (3.6)–(3.9), we end up with the comparison formula

$$(3.10) \quad \det^\pm(\widehat{H}(v) + \lambda) \sim e^{S(\lambda, v)} e^{-S_0(\lambda, v)} \det^\pm(\widehat{H}_0(v) + \lambda) \quad (v \rightarrow +\infty)$$

(stated in most general terms), where

$$(3.11) \quad S(\lambda, v) = \int_0^{+\infty} \Pi_\lambda(q, v) dq, \quad \text{resp.} \quad S_0(\lambda, v) = \int_0^{+\infty} \Pi_{0,\lambda}(q, v) dq,$$

are coupled, resp. uncoupled, improper action integrals. Specifically here,

$$(3.12) \quad S(\lambda, v) = \int_0^{+\infty} (q^N + vq^M + \lambda)^{1/2} dq, \text{ resp. } S_0(\lambda, v) = \int_0^{+\infty} (vq^M + \lambda)^{1/2} dq.$$

The problem has thus been reduced to the separate asymptotic ($v \rightarrow +\infty$) evaluations of *the two (improper) action integrals* in eq. (3.12).

§4. Explicit Formulae for Improper Action Integrals

This Section constitutes a kind of technical digression; it derives a number of formulae for improper action integrals $\int_0^{+\infty} \Pi(q) dq$ which might be of interest for their own sake. Basically, those integrals will be given *exactly* for any binomial $\Pi(q)^2$, by eqs. (4.4), (4.8); whereas the integral for a trinomial $\Pi(q)^2 = q^N + vq^M + \lambda$ will be given *asymptotically* for $v \rightarrow +\infty$, by eq. (4.16).

§4.1. Binomial $\Pi(q)^2$: exact evaluation

We compute the improper action integral $\int_0^{+\infty} \Pi(q) dq$ *exactly* for a *binomial* $\Pi(q)^2 = uq^N + vq^M$, in the rather general setting $N > M \geq 0$, $u, v > 0$, resulting in the extensive formulae (4.4) and (4.8) (where N and M might even be non-integers).

At the core, by eq. (2.3), $\int_0^{+\infty} \Pi(q) dq = \lim_{s \rightarrow 0} I_0(s)$ where

$$(4.1) \quad I_0(s) \stackrel{\text{def}}{=} \int_0^{+\infty} (uq^N + vq^M)^{1/2-s} dq \quad (\text{Re } s > \frac{1}{2} + \frac{1}{N}),$$

as long as the limit (understood as the analytical continuation to $s = 0$) is finite. Now the right-hand side reduces to a Eulerian integral, of the form

$$(4.2) \quad \int_0^{+\infty} (ax + b)^{1/2-s} x^{\alpha-1} dx \equiv a^{-\alpha} b^{1/2+\alpha-s} \frac{\Gamma(\alpha) \Gamma(s - \alpha - 1/2)}{\Gamma(s - 1/2)}$$

(under the change of variable $q^{N-M} = u^{-1}vx$; specifically here, $\alpha = [M(1 - 2s) + 2]/[2(N - M)]$ and

$$(4.3) \quad I_0(s) \equiv \frac{\Gamma(\frac{M(1-2s)+2}{2(N-M)}) \Gamma(-\frac{N(1-2s)+2}{2(N-M)})}{(N - M) \Gamma(s - 1/2)} u^{-\frac{M(1-2s)+2}{2(N-M)}} v^{\frac{N(1-2s)+2}{2(N-M)}}.$$

Consequently, at $s = 0$,

$$(4.4) \quad \int_0^{+\infty} (uq^N + vq^M)^{1/2} dq = \frac{\Gamma(\frac{M+2}{2(N-M)}) \Gamma(-\frac{N+2}{2(N-M)})}{(N - M) \Gamma(-1/2)} u^{-\frac{M+2}{2(N-M)}} v^{\frac{N+2}{2(N-M)}}$$

in the normal case, i.e., when the right-hand side is finite, meaning here $\frac{N+2}{2(N-M)} \notin \mathbb{N}$.

Concrete examples of this \mathbf{N} type are:

$$(4.5) \quad \int_0^{+\infty} (q^4 + vq^2)^{1/2} dq = -v^{3/2}/3$$

$$(4.6) \quad \int_0^{+\infty} (uq^N + \lambda)^{1/2} dq = -\frac{\Gamma(1 + \frac{1}{N})\Gamma(-\frac{1}{2} - \frac{1}{N})}{2\sqrt{\pi}} u^{-\frac{1}{N}} \lambda^{\frac{1}{2} + \frac{1}{N}} \quad (N \neq 2).$$

Now, the right-hand side of eq. (4.4) turns infinite whenever $(2j - 1)N = 2(jM + 1)$ for some $j \in \mathbb{N}^*$ ($j = 0$ cannot occur): this is precisely the anomaly condition \mathbf{A}_j of level j . All these exceptional cases are readily (albeit tediously) handled by applying eq. (2.7) to $I_0(s)$. (The formula (4.11) below, resp. also (4.5), were already implicitly derived in [8] (as eqs. (115), resp. (84)) by a variant route.) First, the residue is

$$(4.7) \quad \beta_{-1}(0) = (-1)^{j-1} \frac{(2j-2)!}{2^{2j-1}(j-1)! j!} u^{1/2-j} v^j;$$

then, the finite part at $s = 0$ of eq. (4.3) gets extracted as

$$\beta_{-1}(0) \left\{ \frac{2j}{N+2} \left[\psi(j+1) - \log v + \frac{M}{N} \left(-\psi(j-1/2) + \log u \right) \right] - \frac{1}{N} \psi(-1/2) \right\};$$

so that finally, when $M = [(j - 1/2)N - 1]/j$ (condition \mathbf{A}_j), eq. (2.7) yields

$$(4.8) \quad \int_0^{+\infty} (uq^N + vq^M)^{1/2} dq = \frac{2j \beta_{-1}(0)}{N+2} \times \left[-\log v + \sum_{m=1}^j \frac{1}{m} + \frac{2M}{N} \left(\log 2 + \frac{1}{2} \log u - \sum_{m=1}^{j-1} \frac{1}{2m-1} \right) \right].$$

The cases with $j = 1$ are of special interest. First comes the harmonic oscillator ($N = 2$) at a general energy value $(-\lambda)$, for which

$$(4.9) \quad \int_0^{+\infty} (vq^2 + \lambda)^{1/2} dq = \frac{1}{4} v^{-1/2} \lambda (1 - \log \lambda).$$

The higher binomials of type \mathbf{A}_1 occur in *supersymmetric quantum mechanics* (at zero energy) [7], [8] Sec. 4:

$$(4.10) \quad \Pi(q)^2 = uq^N + vq^M \quad \text{with } N = 2M + 2 \quad (M > 0),$$

in which case eq. (4.8) distinctly simplifies to

$$(4.11) \quad \int_0^{+\infty} (uq^N + vq^{N/2-1})^{1/2} dq = \frac{u^{-1/2}v}{N+2} \times \left[-\log v + 1 + \frac{N-2}{N} (\log 2 + \frac{1}{2} \log u) \right] \quad (j = 1).$$

§4.2. Trinomial $\Pi(q)^2$: asymptotic $v \rightarrow \infty$ evaluation

We now consider a trinomial $\Pi(q)^2$ of the form $q^N + vq^M + \lambda$, with even $N > M > 0$, and a systematically *constant* third term: the spectral parameter itself, $\lambda (> 0)$ ($-\lambda =$ the total energy). One of the coefficients can always be scaled out to unity, and we have done this for the highest power initially.

In the fully trinomial case, we can no longer compute the action integral $\int_0^{+\infty} \Pi(q) dq$ exactly. In view of eq. (3.10), however, we mainly need its large- v behavior, specially for $v \rightarrow +\infty$ in order to recover singular perturbation theory according to eq. (1.3) (but as in [8], we expect the results to remain valid over suitable sectors in the complex v -plane).

According to the zeta-regularization idea, we must start from the large- v behavior of $I(s; \lambda, v) \stackrel{\text{def}}{=} \int_0^{+\infty} (q^N + vq^M + \lambda)^{1/2-s} dq$; this problem is rather delicate, so any brute-force expansion scheme is dubious. Instead, we apply the following general idea: if the function $I(v)$ under study is an inverse Mellin transform,

$$(4.12) \quad I(v) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \tilde{I}(\sigma) v^\sigma d\sigma,$$

then the singularities of $\tilde{I}(\sigma)$ in the half-plane $\{\text{Re } \sigma < c\}$ encode the large- v behavior of $I(v)$. In particular, (by the residue calculus) any polar part of the form $A(\sigma - \sigma_0)^{-2} + B(\sigma - \sigma_0)^{-1}$ in $\tilde{I}(\sigma)$ expresses an asymptotic contribution $v^{\sigma_0}(A \log v + B)$ to $I(v)$. This perfectly works for $I(s; \lambda, v)$, because its direct Mellin transform $\tilde{I}(s; \lambda, \sigma) \stackrel{\text{def}}{=} \int_0^{+\infty} I(s; \lambda, v) v^{-\sigma-1} dv$ is *exactly computable* (by the same formula (4.2) as for the binomial case but now used *twice in succession*), and *meromorphic in σ* : formally,

$$(4.13) \quad \begin{aligned} & \int_0^{+\infty} dv v^{-\sigma-1} (q^N + vq^M + \lambda)^{1/2-s} \\ &= \frac{\Gamma(-\sigma) \Gamma(s + \sigma - 1/2)}{\Gamma(s - 1/2)} q^{M\sigma} (q^N + \lambda)^{1/2-s-\sigma} \\ &\implies \tilde{I}(s; \lambda, \sigma) = \frac{\Gamma(-\sigma) \Gamma(\frac{M\sigma+1}{N}) \Gamma(s + \sigma - \frac{1}{2} - \frac{M\sigma+1}{N})}{N \Gamma(s - 1/2)} \lambda^{-s - \frac{N-M}{N}\sigma + \frac{1}{2} + \frac{1}{N}} \end{aligned}$$

(using the change of variable $q^N = \lambda r$ for the q -integration). In addition, the Mellin integral has to be genuinely defined somewhere: here, all integrations converge in some strip $\sigma' < \text{Re } \sigma < 0$ provided $\text{Re } s > \mu(N)$, and the inverse transformation (4.12) applies with $c = -0$. Consequently, the poles $\sigma(s)$ relevant to the current asymptotic problem are those which lie in $\{\text{Re } \sigma < 0\}$

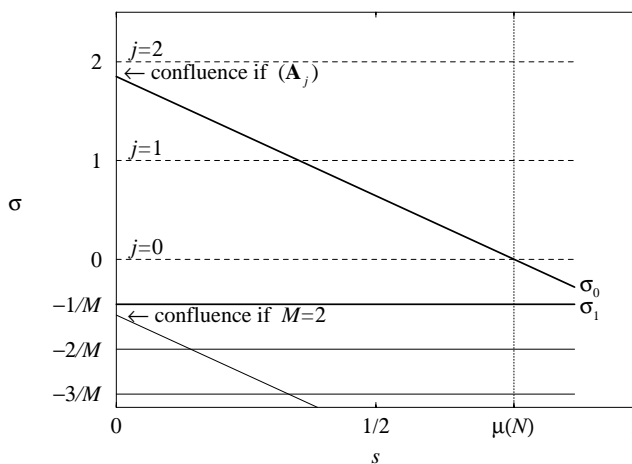


Figure 1. Schematic depiction (using some non-integer N, M) of the poles $\sigma(s)$ of the Mellin transform $\tilde{I}(s; \lambda, \sigma)$ in eq. (4.13). The two main contributing poles (in the $v \rightarrow +\infty$ limit) are drawn with bold lines; non-contributing poles are drawn with dashed lines.

when $\text{Re } s > \mu(N)$, and their contributions are then to be analytically continued to $s = 0$. Overall, the poles in eq. (4.13) form three arithmetic progressions, one for each Gamma factor in numerator; they are real for real s (Fig. 1). At $s = 0$, any pole $\sigma(s)$ will contribute an asymptotic term of degree $d_v = \sigma(0)$ in v (on general grounds) and of degree $d_\lambda = \frac{1}{2} + \frac{1}{N} - \frac{N-M}{N}\sigma(0)$ in λ (by examination of eq. (4.13)). At the end, we plan to keep the terms of degree $d_g \leq 0$ in the perturbative coupling constant g (discarding $o(1)$ terms when $g \rightarrow 0$); now the Symanzik scaling (eq. (1.5) at fixed E) entails $d_g \equiv -\frac{M+2}{N+2}(d_v + \frac{2}{M+2}d_\lambda) = -(M\sigma(0)+1)/N$; altogether, $d_g \leq 0$ then amounts to keeping only the poles for which $\sigma(0) \geq -1/M$.

When $M \geq 2$ (as here), only two poles $\sigma(s)$ satisfy both criteria, (in real form) $\sigma(s) < 0$ for $s > \mu(N)$ and $\sigma(0) \geq -1/M$: they are, in decreasing order at $s = 0$,

$$(4.14) \quad \sigma_0(s) \equiv \frac{N}{N-M} \left(\frac{1}{2} + \frac{1}{N} - s \right) \quad (\text{leading}) \quad \text{and} \quad \sigma_1(s) \equiv -\frac{1}{M} \quad (\text{subleading}).$$

They are generically simple, with two exceptions at $s = 0$: $\sigma_0(0) = \frac{N+2}{2(N-M)}$ becomes confluent with the (fixed) pole $\sigma = +j$ when the *coupled* problem is of anomalous type \mathbf{A}_j ; and independently, σ_1 becomes confluent with the (next

mobile) pole $\frac{N}{N-M}(\frac{1}{2} + \frac{1}{N} - s - 1)$ when the *uncoupled* problem is anomalous, i.e., $M = 2$. The latter confluence will induce a usual double-pole contribution; the former confluence is worse, making the inverse-Mellin representation *singular* as the integration path gets *pinched* between the two poles.

We now specifically evaluate the two dominant polar contributions at $s = 0$, from σ_0 and σ_1 .

- **the leading pole** $\sigma_0(0) = \frac{N+2}{2(N-M)}$: if the coupled problem is of type **N** this pole remains simple, and its asymptotic contribution $[\text{Res}_{\sigma_0(0)} \tilde{I}(0; \lambda, \sigma)] v^{\sigma_0(0)}$ turns out (by inspection) to be just $\int_0^{+\infty} (q^N + vq^M)^{1/2} dq$ (as given by eq. (4.4) at $u = 1$). Furthermore, $\partial_\lambda I(s; \lambda, \sigma) \propto I(s + 1; \lambda, \sigma)$, an operation which precisely annihilates this leading pole part in all cases, so the latter has to be a constant in λ ; then, its computation at $\lambda = 0$ precisely yields $\int_0^{+\infty} (q^N + vq^M)^{1/2} dq$, *now including the confluent cases* (**A_j**).

- **the subleading pole** $\sigma_1 = -1/M$: if $M > 2$ this pole remains simple, and its asymptotic contribution $[\text{Res}_{\sigma_1} \tilde{I}(0; \lambda, \sigma)] v^{\sigma_1}$ coincides with $\int_0^{+\infty} (vq^M + \lambda)^{1/2} dq$ as given by eq. (4.6). Under confluence ($M = 2$), the contribution becomes that of the double pole of eq. (4.13) at $\sigma = -1/2$, i.e., $\int_0^{+\infty} (vq^2 + \lambda)^{1/2} dq + \frac{N}{N-2} A_1(\lambda, v)$ where the action integral is given by eq. (4.9), and

$$(4.15) \quad A_1(\lambda, v) = \frac{1}{4} v^{-1/2} \lambda (\log v + 2 \log 2).$$

All in all, the asymptotic $v \rightarrow +\infty$ formula for the trinomial action integral is then

$$(4.16) \quad \int_0^{+\infty} (q^N + vq^M + \lambda)^{1/2} dq \sim \int_0^{+\infty} (q^N + vq^M)^{1/2} dq + \int_0^{+\infty} (vq^M + \lambda)^{1/2} dq + \delta_{M,2} \frac{N}{N-2} A_1(\lambda, v),$$

where $\int_0^{+\infty} (q^N + vq^M)^{1/2} dq$ is specified through eq. (4.4) if the coupled problem is of type **N**, or else eq. (4.8) if the coupled problem is of type **A_j** (i.e., if $\frac{N+2}{2(N-M)} = j \in \mathbb{N}^*$); whereas $\int_0^{+\infty} (vq^M + \lambda)^{1/2} dq$ is given by eq. (4.6) if $M > 2$, or eq. (4.9) if $M = 2$ (and $\delta_{M,2}$ is a Kronecker delta symbol).

§5. Application to Spectral Functions

The theoretical results of § 3–4 translate into concrete asymptotic formulae for spectral functions in the $v \rightarrow +\infty$ regime.

§5.1. The spectral determinants

Returning to the end output of §3, eq. (3.10), if we substitute the explicit formulae of §4 therein, then $\int_0^{+\infty} (vq^M + \lambda)^{1/2} dq$ cancels out, and a slightly simpler $v \rightarrow +\infty$ formula results:

$$(5.1) \quad \det^\pm(-d^2/dq^2 + q^N + vq^M + \lambda) \sim e^{\int_0^{+\infty} (q^N + vq^M)^{1/2} dq + \delta_{M,2} \frac{N}{N-2} A_1(\lambda, v)} \det^\pm(-d^2/dq^2 + vq^M + \lambda),$$

where $\int_0^{+\infty} (q^N + vq^M)^{1/2} dq$ is given through eq. (4.4) if the coupled problem is of type **N**, or (4.8) if it is of type **A_j**, and $A_1(\lambda, v)$ by eq. (4.15).

Being homogeneous, the uncoupled potentials obey the scaling eq. (1.3) in a simpler form: $(-d^2/dq^2 + vq^M)$ is unitarily equivalent to $v^{2/(M+2)}(-d^2/dq^2 + q^M)$. Then the scaling laws (2.17) apply with $r = v^{2/(M+2)}$; but these laws are more awkward for the \det^\pm than for the full and skew determinants \det and \det^P , so we now switch to the latter combinations and explicitly obtain

$$(5.2) \quad \det(-d^2/dq^2 + vq^M + \lambda) \equiv \det(-d^2/dq^2 + q^M + v^{-2/(M+2)}\lambda) \quad (M \neq 2)$$

$$(5.3) \quad \det(-d^2/dq^2 + vq^2 + \lambda) \equiv v^{-v^{-1/2}\lambda/4} 2^{-v^{-1/2}\lambda/2} \sqrt{2\pi} / \Gamma(\frac{1}{2}(1 + v^{-1/2}\lambda))$$

$$(5.4) \quad \det^P(-d^2/dq^2 + vq^M + \lambda) \equiv v^{1/(M+2)} \det^P(-d^2/dq^2 + q^M + v^{-2/(M+2)}\lambda).$$

Remark. as the harmonic-oscillator determinants are known in closed form ([8], eqs. (155)), we immediately wrote eq. (5.3) in a fully explicit form (needed later).

Thus, eqs. (5.1) plus (5.2)–(5.4) can supply the $v \rightarrow +\infty$ behaviors at fixed λ of the coupled determinants in terms of the corresponding uncoupled determinants at $\lambda = 0$ (which are computable numbers, cf. [8], eq. (136)).

However, our main concern here is the singular perturbation limit: $v \equiv g^{-(M+2)/(N+2)} \rightarrow +\infty$ with $v^{-2/(M+2)}\lambda \stackrel{\text{def}}{=} (-E)$ fixed, according to eq. (1.5). The corresponding results, deduced from eqs. (5.1)–(5.4) after rescaling both sides, are

$$(5.5) \quad \det(-d^2/dq^2 + q^M + gq^N - E) / \det(-d^2/dq^2 + q^M - E) \sim \begin{cases} e^{2 \int_0^{+\infty} (q^N + vq^M)^{1/2} dq} e^{-\delta_{M,2} \frac{1}{N-2} (\frac{N+2}{4} \log v + N \log 2) E} & \text{for type } \mathbf{N} \\ v^{\frac{4\beta_{-1}(0)}{N(M+2)}} e^{2 \int_0^{+\infty} (q^N + vq^M)^{1/2} dq} e^{-\delta_{M,2} \frac{1}{N-2} (\frac{N+2}{4} \log v + N \log 2) E} & \text{for type } \mathbf{A}_j \end{cases}$$

($\beta_{-1}(0)$ (given by eq. (4.7)), and *type*, both refer to the coupled problem). By contrast, the skew determinants always behave straightforwardly:

$$(5.6) \quad \det^P(-d^2/dq^2 + q^M + gq^N - E) \sim \det^P(-d^2/dq^2 + q^M - E).$$

The main non-trivial result here is the asymptotic ratio in eq. (5.5). Its essential singularity for $g \rightarrow 0$ should relate to the non-uniformity of this limit with respect to the quantum number k . The dependence of its logarithm upon E is *purely affine* (with $\{E = 0\}$ intercepts already determined in [8] for some cases). The basic example (1.1), being of type \mathbf{N} , thus gives

$$(5.7) \quad \det(-d^2/dq^2 + q^2 + gq^4 - E) \sim e^{-2/3g} e^{(\log g/2 - 2 \log 2)E} \det(-d^2/dq^2 + q^2 - E).$$

(Note the “instanton-like” structure of the first prefactor, computed by eq. (4.5).)

§5.2. The spectral zeta functions

Over the spectrum $\{E_k(g)\}$ of the rescaled operator $(-d^2/dq^2 + q^M + gq^N)$, we can consider the full and skew spectral zeta functions

$$(5.8) \quad Z_g(s; E) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (E_k(g) - E)^{-s}, \quad Z_g^P(s; E) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} (-1)^k (E_k(g) - E)^{-s}$$

for, say, integer $s \in \mathbb{N}^*$, in which case they converge for $g > 0$ and relate to the spectral determinants in a simpler way than for general s ,

$$(5.9) \quad Z_g(s; E) \equiv -\frac{1}{(s-1)!} \frac{\partial^s}{\partial E^s} \log \det(-d^2/dq^2 + q^M + gq^N - E),$$

(obtained from eq. (2.13) upon rescaling; and likewise for (Z^P, \det^P)).

Assuming all previous estimates are stable under E -differentiations (as is usually the case in WKB theory), the preceding formulae imply the *regular behaviors* (see Fig. 2, left)

$$(5.10) \quad Z_g(s; E) \sim Z_0(s; E), \quad Z_g^P(s; E) \sim Z_0^P(s; E) \quad (g \rightarrow 0),$$

except for $Z_g(1; E)$ (the resolvent trace) when $M = 2$, which gives the *singular case* ($Z_0(1; E)$ infinite, while $Z_0^P(1; E)$ stays finite). Those patterns were conjectured in [8] (Sec. 3), but not the precise divergent behavior of $Z_g(1; E)$, which required the E -linear terms now appearing in exponent in eqs. (5.5). For $s = 1$, eq. (5.9) also needs to be regularized at $g = 0$, as

$$(5.11) \quad -(d/dE) \log \det(-d^2/dq^2 + q^2 - E) \equiv -\frac{1}{2}[\psi(\frac{1}{2}(1 - E)) + \log 2]$$

(using the known closed form (5.3) of the harmonic-oscillator determinant).

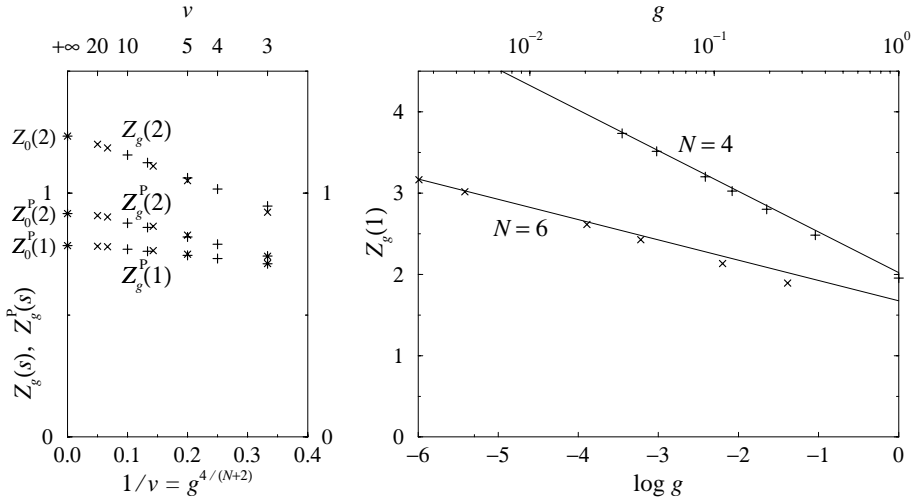


Figure 2. Illustration of the two $g \rightarrow 0$ behaviors of spectral zeta functions at $E = 0$, $Z_g(s) \stackrel{\text{def}}{=} \sum_k E_k(g)^{-s}$ and $Z_g^P(s) \stackrel{\text{def}}{=} \sum_k (-1)^k E_k(g)^{-s}$, as computed from numerical spectra (by means of eqs. (5.8)) for a sample of potentials $V(q) = q^2 + gq^N$ with $N = 4$ (+) or 6 (×). Left: *regular* behaviors verifying eq. (5.10), plotted against $1/v$ (visually a convenient variable); remark: $Z_0^P(1) = \pi/4$, $Z_0(2) = \pi^2/8$, and $Z_0^P(2) = \text{Catalan's constant}$ (≈ 0.9159656). Right: *singular* behavior of $Z_g(1)$, plotted against $\log g$; the straight lines show the theoretical asymptotic predictions (eq. (5.13)).

Then the logarithmic differentiation of eqs. (5.5) for $M = 2$ yields the $g \rightarrow 0$ behavior of $Z_g(1; E)$ for all potentials $q^2 + gq^N$ (irrespective of type), in the following (singular) form:

$$(5.12) \quad Z_g(1; E) \sim \frac{1}{N-2}(-\log g + N \log 2) - \frac{1}{2} [\psi(\frac{1}{2}(1-E)) + \log 2].$$

For instance, at $E = 0$ this gives (see Fig. 2, right)

$$(5.13) \quad \sum_{k=0}^{\infty} E_k(g)^{-1} \sim -\frac{1}{N-2} \log g + \frac{1}{2} \left(\gamma + \frac{3N-2}{N-2} \log 2 \right) \quad (g \rightarrow 0)$$

$$\sim -\frac{1}{2} \log g + \frac{1}{2}(\gamma + 5 \log 2) \quad \text{for } N = 4$$

$$\sim -\frac{1}{4} \log g + \frac{1}{2} \gamma + 2 \log 2 \quad \text{for } N = 6, \dots$$

(to be compared with the sharp cutoff regularization of eq. (1.6)).

§5.3. Concluding remarks

We have completed here one “exercise in exact quantization” begun in [8]: we gave the $g \rightarrow 0^+$ behavior of the spectral determinants $\det^\pm(-d^2/dq^2 + q^M + gq^N - E)$, now for general parameter values. While it may appear wasteful to use a wholly exact approach for perturbative calculations, exact WKB analysis actually proved quite efficient for the task; inversely, such problems help to strengthen the practical sides of that field, which still need further development.

We are also confident that the above approach can be extended further, both to complex parameter asymptotics and towards higher orders in powers of g .

References

- [1] Bender, C. M. and Wu, T. T., Anharmonic oscillator, *Phys. Rev.*, **184** (1969), 1231-1260.
- [2] Delabaere, É. and Pham, F., Unfolding the quartic oscillator, *Ann. Phys.*, **261** (1997), 180-218 (and refs. therein).
- [3] Delabaere, É., Dillinger, H. and Pham, F., Exact semiclassical expansions for one-dimensional quantum oscillators, *J. Math. Phys.*, **38** (1997), 6126-6184.
- [4] Hioe, F.T., MacMillen, D. and Montroll, E. W., Quantum theory of anharmonic oscillators. II. Energy levels of oscillators with $x^{2\alpha}$ anharmonicity, *J. Math. Phys.*, **17** (1976), 1320-1337.
- [5] C. Howls, T. Kawai and Y. Takei eds., *Toward the exact WKB analysis of differential equations, linear or non-linear*, (Proceedings), RIMS, Kyoto, Japan, 1998, Kyoto University Press, 2000.
- [6] Sibuya, Y., *Global Theory of a Second Order Linear Ordinary Differential Operator with a Polynomial Coefficient*, North-Holland, 1975.
- [7] Suzuki, J., Functional relations in Stokes multipliers – Fun with $x^6 + \alpha x^2$ potential –, *J. Stat. Phys.*, **102** (2001), 1029-1047.
- [8] Voros, A., Exercises in exact quantization, *J. Phys. A*, **33** (2000), 7423-7450. [In this work, we mistakenly used “quasi-exactly solvable” for “supersymmetric” potentials throughout.]
- [9] Zinn-Justin, J., *Quantum field theory and critical phenomena*, (Chap. 43), Oxford University Press, 4th edition, 2002 (and refs. therein).