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The Bost–Connes phase transition and unitary representations

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Abstract. We construct a family of unitary representations of the $ax + b$ group $\mathbb{Q} \rtimes \mathbb{Q}_{+}^{\times}$. We show that this family of representations exhibits a "phase transition" analogous to that observed by Bost and Connes [2] and then explain how these representations are related to the equilibrium states of Bost and Connes.

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1. Introduction

In their paper [2], Bost and Connes constructed a C*-dynamical system from number-theoretic data, and showed that this system possesses a unique equilibrium state at high temperatures but many distinct equilibria at low temperatures. Since the publication of [2], a number of other systems have been constructed, exhibiting a similar "phase transition": see the recent book of Connes and Marcolli [4] for a discussion of some of the developments and references to others.

The purpose of this note is to describe a manifestation of the Bost–Connes phase transition in the context of unitary representations. We construct a natural family of unitary representations of the rational $ax + b$ group, parametrised by the half-plane $\operatorname{Re} z \geq 0$. We show that this family of representations exhibits a phase transition, in the following sense:

- The representations corresponding to $\operatorname{Re} z = 0$ are one-dimensional and mutually inequivalent.
- The representations corresponding to $0 < \operatorname{Re} z \leq 1$ are mutually inequivalent, irreducible and infinite-dimensional.
- The representations corresponding to $\operatorname{Re} z > 1$ are all mutually equivalent and reducible, with irreducible constituents parametrised by the Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}, \mathbb{Q})$.

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The real part of our parameter z thus corresponds to the inverse temperature β of Bost and Connes. The imaginary part $\operatorname{Im} z$ is related to the time evolution on the Bost–Connes algebra, as will be explained later in the paper.

The $ax + b$ group that we consider is isomorphic to the semidirect product, $\mathbb{Q} \rtimes \mathbb{Q}_+^\times$, for the action of the positive rationals on $\mathbb{Q}$ by multiplication. Recall the following standard procedure for producing irreducible representations of such a group: one takes a character $\varphi \in \widehat{\mathbb{Q}}$ and a character χ of the stabiliser $(\mathbb{Q}_+^\times)_\varphi$, and induces the representation $\varphi \rtimes \chi$ from $\mathbb{Q} \rtimes (\mathbb{Q}_+^\times)_\varphi$ to $\mathbb{Q} \rtimes \mathbb{Q}_+^\times$ [9], §14. There are two extreme cases of this construction: if φ is the trivial character, the representations obtained in this way are one-dimensional; at the other extreme, where the stabiliser of φ is trivial, we get $\operatorname{Ind}_{\mathbb{Q}}^{\mathbb{Q} \rtimes \mathbb{Q}_+^\times} \varphi$. The representations constructed in this paper are, for $\operatorname{Re} z = 0$, of the former type, while the irreducible constituents for $\operatorname{Re} z > 1$ are of the latter type. The semidirect product $\mathbb{Q} \rtimes \mathbb{Q}_+^\times$ is not regular (in the sense of [9]), so it also possesses irreducible representations not accessible through the above construction; the representations we construct for $0 < \operatorname{Re} z \leq 1$ are of this more exotic type.

After establishing some notation, we define in Section 2 the representations in question, and give a precise statement of our main result. Our construction is related, in the case $z = 1$, to papers of Blackadar [1] and Matthews [10]. Section 3 contains the proof of our result, part of which relies on an argument due to Neshveyev [11]. In Section 4 we show how our construction connects, via results of Laca [7], [8], to the equilibrium states of Bost and Connes.

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Notation. Our notation agrees, for the most part, with that of [2]. We refer to [5], [13] and [3] for more details about adeles and ideles. $\mathcal{A}$ and $\mathcal{A}^\times$ denote, respectively, the locally compact ring of *finite* adeles of $\mathbb{Q}$, and the locally compact group of *finite* ideles. Recall the definition of $\mathcal{A}$ and $\mathcal{A}^\times$ as restricted direct products over the primes,

$$\mathcal{A} = \prod_p{}' (\mathbb{Q}_p, \mathbb{Z}_p), \quad \mathcal{A}^\times = \prod_p{}' (\mathbb{Q}_p^\times, \mathbb{Z}_p^\times).$$

We normalise additive Haar measure on each $\mathbb{Q}_p$ so that $\mathbb{Z}_p$ has measure one; the product of these measures is a Haar measure on the additive group $\mathcal{A}$, giving measure one to the compact open subring $\mathcal{R} = \prod_p \mathbb{Z}_p$. Similarly, we normalise Haar measure on $\mathbb{Q}_p^\times$ so that $\mathbb{Z}_p^\times$ has measure one; the product measure on $\mathcal{A}^\times$ is a Haar measure, giving measure one to the compact open subgroup $W = \prod_p \mathbb{Z}_p^\times$.

$\mathbb{Q}$ embeds diagonally into $\mathcal{A}$ as a dense subring, and $\mathbb{Q}_+^\times$ embeds diagonally into $\mathcal{A}^\times$ as a discrete and co-compact subgroup. There is an "absolute value" $|\cdot| \colon \mathcal{A}^\times \to \mathbb{Q}_+^\times$, which may be defined as the Radon–Nikodym derivative

$$|a| = \frac{\mathrm{d}(ax)}{\mathrm{d}(x)},$$

where $\mathrm{d}x$ is the additive Haar measure on $\mathcal{A}$. Alternatively, $|\cdot|$ may be defined as the product of the p-adic absolute values. One has $|a| = a^{-1}$ for all $a \in \mathbb{Q}_+^\times$, and W is the kernel of the absolute value. In this way, we can decompose $\mathcal{A}^\times$ as the direct product $\mathbb{Q}_+^\times \times W$. Note that $W \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}, \mathbb{Q})$ (by the Kronecker–Weber theorem, [3], VII, §5.7).

The Schwartz–Bruhat space $\mathcal{S}(\mathcal{A})$ can be defined as the space of locally constant, compactly supported, complex-valued functions on $\mathcal{A}$. It is a $*$-algebra, under convolution (with respect to additive Haar measure) and the involution $f^*(x) = \overline{f(-x)}$. $\mathcal{S}(\mathcal{A})$ is a direct limit of finite-dimensional spaces, so it is nuclear in the direct-limit topology, and every linear functional is continuous.

The additive group $\mathcal{A}$ is (non-canonically) self-dual. We fix an identification $\mathcal{A} \cong \widehat{\mathcal{A}}$ as follows. For each prime p, let ψ_p be a character on $\mathbb{Q}_p$ that is trivial on $\mathbb{Z}_p$ but non-trivial on $p^{-1}\mathbb{Z}_p$. Then $\psi := \bigotimes_p \psi_p$ is a character of $\mathcal{A}$, and we associate to each $a \in \mathcal{A}$ the character $\psi_a \colon x \mapsto \psi(ax)$. The Fourier transform arising from this identification, $\hat{f}(x) = \int_{\mathcal{A}} f(y)\psi(xy)\,\mathrm{d}y$, is a linear automorphism of $\mathcal{S}(\mathcal{A})$, and our character ψ was chosen so that this automorphism fixes the characteristic function $f_{\mathcal{R}}$ of $\mathcal{R}$.

2. The phase transition

We will construct representations of the "$ax + b$ group"

$$P_{\mathbb{Q}}^+ = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Q}_+^\times,\ b \in \mathbb{Q} \right\}.$$

This group embeds into the adelic $ax + b$ group,

$$P_{\mathcal{A}} = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a \in \mathcal{A}^\times,\ b \in \mathcal{A} \right\}.$$

$P_{\mathcal{A}}$ has a natural action on $\mathcal{A}$, by affine transformations, giving rise to a linear representation on $\mathcal{S}(\mathcal{A})$,

$$\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} f(x) = f(ax + b).$$

The group $P_{\mathcal{A}}$ is not unimodular: its modular function is $\left(\begin{smallmatrix} 1 & b \\ 0 & a \end{smallmatrix}\right) \mapsto |a|$. Twisting the above representation by the complex powers of this function, we obtain a family of representations α_z parametrised by $z \in \mathbb{C}$. In order to make our parametrisation agree with that of [2], we define

$$\alpha_z \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} f(x) = |a|^{1-\frac{z}{2}} f(ax + b).$$

This is a standard construction in representation theory: whenever a group G acts on a space X admitting a quasi-invariant measure, one obtains a family of (not necessarily unitary) representations of G on an appropriate space of functions on X, by twisting the usual representation by the complex powers of the Radon–Nikodym derivative. See, for example, the construction of the principal and complementary series for SL_2 in [5], Chapter 2, §3.

We first describe which of the representations α_z are unitarisable, in the sense that $\mathcal{S}(\mathcal{A})$ admits a non-trivial (but possibly degenerate), positive, α_z-invariant hermitian form.

Lemma 2.1. *If* $\operatorname{Re} z < 0$*,* α_z *is not unitarisable.*

If $\operatorname{Re} z \geq 0$*, then* α_z *is unitarisable, and any two unitarisations of* α_z *are equivalent. If* $\operatorname{Re} z = 0$*, the corresponding unitary representation is one-dimensional. If* $\operatorname{Re} z > 0$*, the representation is infinite-dimensional.*

The proof is given in the next section.

We will focus on the unitary case, $\operatorname{Re} z \geq 0$, and for the rest of this section α_z denotes the *unitary* representation of $P_{\mathcal{A}}$ given by Lemma 2.1. We consider the unitary representations

$$\operatorname{Res} \alpha_z = \operatorname{Res}_{P_{\mathbb{Q}}^+}^{P_{\mathcal{A}}} \alpha_z$$

of $P_{\mathbb{Q}}^+$, obtained by restricting α_z to this subgroup.

Theorem 2.2. *If* $\operatorname{Re} z = 0$*, then* $\operatorname{Res} \alpha_z$ *is equivalent to the one-dimensional representation* $\left(\begin{smallmatrix} 1 & b \\ 0 & a \end{smallmatrix}\right) \mapsto a^{i \operatorname{Im} z/2}$.

If $0 < \operatorname{Re} z \leq 1$*, then* $\operatorname{Res} \alpha_z$ *is irreducible and infinite-dimensional, and is not induced by any character of* $\mathbb{Q}$ *(as described in Section 1). If* $z \neq z'$ *in this strip, then* $\operatorname{Res} \alpha_z$ *and* $\operatorname{Res} \alpha_{z'}$ *are inequivalent.*

If $\operatorname{Re} z > 1$*, then there is a direct-integral decomposition*

$$\operatorname{Res} \alpha_z \cong \int_W^{\oplus} \operatorname{Ind}_{\mathbb{Q}}^{P_{\mathbb{Q}}^+} \psi_w \, \mathrm{d}w,$$

and the representations $\operatorname{Ind} \psi_w$ *are mutually inequivalent and irreducible. In particular,* $\operatorname{Res} \alpha_z$ *and* $\operatorname{Res} \alpha_{z'}$ *are unitarily equivalent whenever* $\operatorname{Re} z$*,* $\operatorname{Re} z' > 1$.

Remarks 2.3. (1) In the course of proving Theorem 2.2, we will see that the representations α_z themselves undergo a "phase transition" of a similar (albeit less drastic) kind: α_z is irreducible whenever $\operatorname{Re} z \geq 0$; for $\operatorname{Re} z = 0$, α_z is one-dimensional; for $0 < \operatorname{Re} z \leq 1$, α_z is irreducible, and is not induced by any character of $\mathcal{A}$; for $\operatorname{Re} z > 1$, $\alpha_z \cong \operatorname{Ind}_{\mathcal{A}}^{P_{\mathcal{A}}} \psi$.

(2) $P_{\mathbb{Q}}^+$ is not closed in $P_{\mathcal{A}}$; its closure is the subgroup

$$\mathcal{A} \rtimes \mathbb{Q}_+^\times = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Q}_+^\times,\ b \in \mathcal{A} \right\},$$

and the decomposition $\mathcal{A}^\times \cong \mathbb{Q}_+^\times \times W$ gives

$$P_{\mathcal{A}} \cong (\mathcal{A} \rtimes \mathbb{Q}_+^\times) \rtimes W.$$

From a representation-theoretic point of view, it might appear more natural to consider the restriction of α_z to the *closed* subgroup $\mathcal{A} \rtimes \mathbb{Q}_+^\times$. The statement of Theorem 2.2 would remain the same – the representations α_z are continuous, so questions about their reducibility and equivalence have the same answers for $\mathcal{A} \rtimes \mathbb{Q}_+^\times$ as for its dense subgroup $P_{\mathbb{Q}}^+$.

3. Proofs

Proof of Lemma 2.1. Fix z. Every translation-invariant hermitian form on $\mathcal{S}(\mathcal{A})$ has the form

$$\langle f_1, f_2 \rangle = D(f_1 * f_2^*)$$

for some tempered distribution $D \in \mathcal{S}(\mathcal{A})^*$ (this follows from the kernel theorem, as in [6], II, §3.5). If such a form is to be invariant also under the group $\alpha_z \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{A}^\times \end{pmatrix}$, then D must satisfy the homogeneity equation

$$D(f(ax)) = |a|^{\operatorname{Re} z - 1} D(f(x)) \quad \text{for all } a \in \mathcal{A}^\times,\ f \in \mathcal{S}(\mathcal{A}). \tag{3.1}$$

Weil showed in [15] that the space of distributions satisfying this equation is one-dimensional. Moreover, the assertions about positivity and degeneracy follow from the explicit construction of these distributions described in [15]. In particular, for $\operatorname{Re} z = 0$ the distribution in question is additive Haar measure, or in other words the Fourier transform of the Dirac distribution at 0, and so the resulting inner product is degenerate on a subspace of codimension 1. □

The assertion of Theorem 2.2 in the case $\operatorname{Re} z = 0$ is now easily dealt with: the map $\mathcal{S}(\mathcal{A}) \to \mathbb{C}$ given by integration with respect to Haar measure establishes a unitary equivalence between $\operatorname{Res} \alpha_z$ and the given one-dimensional representation.

For each z with $\operatorname{Re} z > 0$, let us fix a choice of α_z-invariant inner product on $\mathcal{S}(\mathcal{A})$ by requiring that the characteristic function $f_{\mathcal{R}}$ of $\mathcal{R}$ be a unit vector. From now on, α_z will denote the corresponding unitary representation.

We turn to the proof of Theorem 2.2. Rather than work with the representations α_z directly, it will be convenient to consider a family of unitarily equivalent representations, which we now define.

For each $\beta > 0$, let μ_β be the regular Borel measure on $\mathcal{A}$ characterised by the properties

(1) $\mathrm{d}\mu_\beta(ax) = |a|^\beta \mathrm{d}\mu_\beta(x)$ for every $a \in \mathcal{A}^\times$,

(2) $\mu_\beta(\mathcal{R}) = 1$.

(Note that each μ_β is the Fourier transform of a distribution satisfying (3.1) with $\beta = \operatorname{Re} z$. The existence and uniqueness of such a measure thus follows from [15].)

For each $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, define a unitary representation $\hat{\alpha}_z$ of $P_{\mathcal{A}}$ on $L^2(\mathcal{A}, \mu_{\operatorname{Re} z})$ by

$$\hat{\alpha}_z \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} f(x) = |a|^{-z/2} \overline{\psi}(a^{-1}xb) f(a^{-1}x).$$

A routine calculation shows that, under our choice of normalisations, the Fourier transform $\mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{A})$ implements a unitary equivalence between α_z and $\hat{\alpha}_z$.

Theorem 2.2 will be proved as a consequence of the following lemma, which is a generalisation of [11], Proposition, with a similar proof (cf. also [2], §7).

Lemma 3.2. *Suppose that* $0 < \beta \leq 1$. *For each* $w \in \mathbb{C}$ *with* $\operatorname{Re} w \geq 0$, *let* $H_{\beta,w}$ *denote the subspace of* $L^2(\mathcal{R}, \mu_\beta)$ *consisting of functions* f *with* $f(ax) = a^{-w} f(x)$ *for all positive integers* a. *Then*

$$\dim H_{\beta,w} = \begin{cases} 1 & \text{if } w = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The case $w = 0$ was shown by Neshveyev in [11]. The same proof applies, with mostly cosmetic changes, to the case $w \neq 0$. Instead of reproducing Neshveyev's argument here, let us point out only that part which is responsible for the difference between the two cases.

Following Neshveyev, one finds that $H_{\beta,w} = 0$ if and only if

$$\lim_{t \to 1+} \frac{L(\chi, t + \overline{w})}{\zeta(t)} = 0$$

for all Dirichlet characters χ, where L is the corresponding Dirichlet L-function and ζ is Riemann's function. Elementary properties of Dirichlet series (as explained in [12], VI, §3, for instance) imply that if $w \neq 0$, the numerator in the limit remains bounded, while the denominator diverges. So the limit is equal to zero for all χ when $w \neq 0$.

(Note that the same conclusion is not valid when $w = 0$ and χ is a principal character: in this case both the numerator and the denominator have a simple pole at $t = 1$.) □

Proof of Theorem 2.2. We already took care of the case of $\operatorname{Re} z = 0$ above.

Suppose $0 < \operatorname{Re} z \leq 1$. Any bounded operator on $L^2(\mathcal{A}, \mu_{\operatorname{Re} z})$ commuting with $\hat{\alpha}_z \left(\begin{smallmatrix} 1 & \mathbb{Q} \\ 0 & 1 \end{smallmatrix}\right)$ is given by pointwise multiplication by some function $f \in L^\infty(\mathcal{A}, \mu_{\operatorname{Re} z})$ (because $\mathbb{Q}$ is dense in $\mathcal{A}$). If such an operator is to commute with $\hat{\alpha}_z \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathbb{Q}_+^\times \end{smallmatrix}\right)$, we must have $f(ax) = f(x)$ for all $a \in \mathbb{Q}_+^\times$. Any such function is determined by its restriction to $\mathcal{R}$, and Lemma 3.2 then implies that f is a constant, and so $\operatorname{Res}\alpha_z$ is irreducible.

Similarly, if $t \in \mathbb{R}$, then any intertwining operator from $\operatorname{Res}\alpha_z$ to $\operatorname{Res}\alpha_{z+it}$ is given by an element of $H_{\operatorname{Re} z,-it/2}$, so Lemma 3.2 implies that these two representations are inequivalent unless $t = 0$.

Lemma 3.2 also implies that for $0 < \beta < \beta' \leq 1$, the measures μ_β and $\mu_{\beta'}$ are inequivalent. Indeed, if they were equivalent then we would have

$$\left.\frac{\mathrm{d}\mu_{\beta'}}{\mathrm{d}\mu_\beta}\right|_{\mathcal{R}} \in H_{\beta,\beta'-\beta} = 0.$$

If z, z' have $0 < \operatorname{Re} z < \operatorname{Re} z' \leq 1$, it follows that $\operatorname{Res}\alpha_z$ and $\operatorname{Res}\alpha_{z'}$ are inequivalent (they are even inequivalent as representations of $\mathbb{Q}$). To finish with the case $0 < \operatorname{Re} z \leq 1$, we note that the representation $\operatorname{Res}\alpha_z$ cannot be obtained by the "standard procedure" described in Section 1 because its restriction to $\mathbb{Q}$ corresponds to a properly ergodic measure class on $\widehat{\mathbb{Q}}$.

Now suppose that $\operatorname{Re} z > 1$. In this case the measure $\mu_{\operatorname{Re} z}$ is concentrated on the subset $\mathcal{A}^\times \subset \mathcal{A}$ [11], and the operator

$$I_z \colon L^2(\mathcal{A}, \mu_{\operatorname{Re} z}) \to L^2(\mathcal{A}^\times), \quad I_z f(x) = |x|^{z/2} f(x),$$

is a positive multiple of a unitary. I_z intertwines $\hat{\alpha}_z$ with the unitary representation $\operatorname{Ind}_{\mathcal{A}}^{P_{\mathcal{A}}} \psi$. Transitivity of induction, coupled with the decomposition

$$P_{\mathcal{A}} \cong \overline{P_{\mathbb{Q}}^+} \rtimes W,$$

now implies that

$$\operatorname{Res}\alpha_z \cong \int_W^\oplus \operatorname{Ind}_{\mathbb{Q}}^{P_{\mathbb{Q}}^+} \psi_w \, \mathrm{d}w.$$

Each of the representations appearing in this direct integral is irreducible, because no element of $\mathbb{Q}_+^\times$ fixes any ψ_w. They are all mutually inequivalent, because the characters ψ_w and $\psi_{w'}$ lie in distinct $\mathbb{Q}_+^\times$-orbits whenever $w \neq w'$. □

4. KMS states on the Bost–Connes C*-algebra

We now briefly explain how the representations appearing in Theorem 2.2 are related, via work of Laca [7], [8], to the results of [2]. The arguments become most transparent when considered from the adelic point of view, and we begin by recalling this perspective on the Bost–Connes algebra and its representations.

Bost and Connes consider the Hecke C*-algebra $C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$ associated to the almost normal subgroup $P_{\mathbb{Z}}^+ = \left(\begin{smallmatrix}1 & \mathbb{Z}\\ 0 & 1\end{smallmatrix}\right) \subset P_{\mathbb{Q}}^+$. A general result of Tzanev [14], which was observed in this particular case by Laca [8], implies that the Bost–Connes

C*-algebra is isomorphic to the full corner of the group C*-algebra $C^*(\mathcal{A} \rtimes \mathbb{Q}_+^\times)$ determined by the projection $e_1 \otimes f_{\mathcal{R}} \in C_c(\mathbb{Q}_+^\times \times \mathcal{A})$ (here e_1 denotes the characteristic function of $1 \in \mathbb{Q}_+^\times$, and $f_{\mathcal{R}}$ the characteristic function of $\mathcal{R} \subset \mathcal{A}$).

Suppose that π is a unitary representation of $P_{\mathbb{Q}}^+$ on a Hilbert space $\mathcal{H}$. If this representation extends to a unitary representation of $\mathcal{A} \rtimes \mathbb{Q}_+^\times$, then it can be integrated to a $*$-representation of $C^*(\mathcal{A} \rtimes \mathbb{Q}_+^\times)$, and then compressed by $\pi(e_1 \otimes f_{\mathcal{R}})$ to obtain a $*$-representation of the Hecke algebra $C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$ on the space $\pi(e_1 \otimes f_{\mathcal{R}})\mathcal{H} = \mathcal{H}^{\mathcal{R}}$ of $\mathcal{R}$-fixed vectors. We use the same symbol to denote both the original unitary representation of $P_{\mathbb{Q}}^+$, and the resulting representation of $C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$.

Let ρ denote the right-regular representation of $\mathbb{Q}_+^\times$, which we view as a representation of $P_{\mathbb{Q}}^+$ via the quotient map $P_{\mathbb{Q}}^+ \to \mathbb{Q}_+^\times$. For each $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, consider the unitary representation $\rho \otimes \operatorname{Res} \alpha_z$ of $P_{\mathbb{Q}}^+$. These representations all extend to $\mathcal{A} \rtimes \mathbb{Q}_+^\times$ and so induce $*$-representations of the Bost–Connes C*-algebra. The function $e_1 \otimes f_{\mathcal{R}}$ may be viewed as a vector f_z in the representation $\rho \otimes \operatorname{Res} \alpha_z$, and our choices of normalisation ensure that f_z is a unit vector, fixed by the subgroup $\mathcal{R} \subset \mathcal{A} \rtimes \mathbb{Q}_+^\times$.

Proposition 4.1. *For each z with $\operatorname{Re} z > 0$, let φ_z denote the unique W-invariant KMS$_{\operatorname{Re} z}$ state on $C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$. The GNS representation of φ_z is equivalent to the representation $\rho \otimes \operatorname{Res} \alpha_z$, with f_z corresponding to the distinguished cyclic vector. In particular,*

$$\varphi_z(a) = \langle \rho \otimes \operatorname{Res} \alpha_z(a) f_z, f_z \rangle.$$

Proof. Let $\hat{f}_z$ denote the vector $e_1 \otimes f_{\mathcal{R}}$ in $\ell^2(\mathbb{Q}_+^\times) \otimes L^2(\mathcal{A}, \mu_{\operatorname{Re} z})$; the notation is justified by the fact that the equivalence $\alpha_z \to \hat{\alpha}_z$ fixes $f_{\mathcal{R}}$.

Let π_z denote the unitary representation of $\mathcal{A}$ associated with the measure $\mu_{\operatorname{Re} z}$ on $\mathcal{A} = \widehat{\mathcal{A}}$ (in other words, π_z is the restriction from $P_{\mathcal{A}}$ to $\mathcal{A}$ of the representation $\hat{\alpha}_z$). Results of Laca ([7], Theorem 34, [8], Proposition 3.1) imply that the W-invariant KMS$_{\operatorname{Re} z}$ state on $C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$ is the vector state in the representation

$$\operatorname{Ind}_{\mathcal{A}}^{\mathcal{A} \rtimes \mathbb{Q}_+^\times} \pi_z \colon C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+) \to \mathcal{B}(\ell^2(\mathbb{Q}_+^\times) \otimes L^2(\mathcal{A}, \mu_{\operatorname{Re} z}))^{\mathcal{R}}$$

corresponding to the $\mathcal{R}$-fixed cyclic unit vector $\hat{f}_z$.

We therefore seek a unitary operator U on $\ell^2(\mathbb{Q}_+^\times) \otimes L^2(\mathcal{A}, \mu_{\operatorname{Re} z})$ that intertwines the representations $\operatorname{Ind}_{\mathbb{Q}}^{P_{\mathbb{Q}}^+} \pi_z$ and $\rho \otimes \operatorname{Res} \hat{\alpha}_z$, and fixes the vector $\hat{f}_z$. The formula

$$Uf(q, x) = q^{-z/2} f(q, qx),$$

for $q \in \mathbb{Q}_+^\times$, $x \in \mathcal{A}$ and $f \in C_c(\mathbb{Q}_+^\times \times \mathcal{A})$, defines such an operator. □

We conclude with some remarks on the interpretation of our parameters $\operatorname{Re} z$ and $\operatorname{Im} z$ in the thermodynamical setting of Bost and Connes. Proposition 4.1 shows how our parameter $\operatorname{Re} z$ may be identified with the inverse temperature β of [2]. The parameter $\operatorname{Im} z$ corresponds to the time evolution on the Bost–Connes algebra, in the following sense. Denoting by $\{\sigma_t \mid t \in \mathbb{R}\}$ the one-parameter automorphism group of $C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Q}}^+)$ defined in [2], and viewing each $\operatorname{Res} \alpha_z$ as a representation of this C*-algebra, we have

$$\operatorname{Res} \alpha_z \circ \sigma_t = \operatorname{Res} \alpha_{z+2it}.$$

The assertions in Theorem 2.2 regarding (in)equivalence of the representations $\operatorname{Res} \alpha_z$ and $\operatorname{Res} \alpha_{z+it}$ thus admit the following "dynamical" formulation:

Proposition 4.2. *If* $\operatorname{Re} z > 1$, *then the representation* $\operatorname{Res} \alpha_z$ *can be incorporated into a covariant representation* $(\operatorname{Res} \alpha_z, U)$ *of the* C*-*dynamical system* $(C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+), \sigma)$. *If* $0 < \operatorname{Re} z \leq 1$, *then* $\operatorname{Res} \alpha_z$ *does not admit such a covariant extension.*

(By contrast, the representation $\rho \otimes \operatorname{Res} \alpha_z$ can *always* be incorporated into a covariant representation since it is the GNS representation for the σ-invariant state φ_z.)

References

[1] B. E. Blackadar, The regular representation of restricted direct product groups. *J. Functional Analysis* **25** (1977), 267–274. Zbl 0364.22004 MR 0439979

[2] J.-B. Bost and A. Connes, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory. *Selecta Math.* (*N.S.*) **1** (1995), 411–457. Zbl 0842.46040 MR 1366621

[3] J. W. S. Cassels and A. Fröhlich (Eds.), *Algebraic number theory*. Academic Press, London 1967. Zbl 0153.07403 MR 0215665

[4] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives*. Amer. Math. Soc. Colloq. Publ. 55, Amer. Math. Soc., Providence, RI, 2008. Zbl 1209.58007 MR 2371808

[5] I. M. Gel'fand, M. I. Graev, and I. I. Pyatetskii-Shapiro, *Representation theory and automorphic functions*. Academic Press, Boston 1990. Zbl 0718.11022 MR 1071179

[6] I. M. Gel'fand and N. Ya. Vilenkin, *Applications of harmonic analysis*. Academic Press, New York 1964. Zbl 0136.11201 MR 0173945

[7] M. Laca, Semigroups of *-endomorphisms, Dirichlet series, and phase transitions. *J. Funct. Anal.* **152** (1998), 330–378. Zbl 0957.46039 MR 1608003

[8] M. Laca, From endomorphisms to automorphisms and back: dilations and full corners. *J. London Math. Soc.* (2) **61** (2000), 893–904. Zbl 0973.46066 MR 1766113

[9] G. W. Mackey, Induced representations of locally compact groups I. *Ann. of Math.* (2) **55** (1952), 101–139. Zbl 0046.11601 MR 0044536

[10] C. R. Matthews, Spectral analysis of the action of ideles on adeles. *J. London Math. Soc.* (2) **32** (1985), 392–398. Zbl 0614.12009 MR 825913

[11] S. Neshveyev, Ergodicity of the action of the positive rationals on the group of finite adeles and the Bost-Connes phase transition theorem. *Proc. Amer. Math. Soc.* **130** (2002), 2999–3003. Zbl 1031.46077 MR 1908923

[12] J.-P. Serre, *A course in arithmetic*. Graduate Texts in Math. 7, Springer-Verlag, New York 1973. Zbl 0256.12001 MR 0344216

[13] J. Tate, Fourier analysis in number fields and Hecke's zeta function. In *Algebraic number theory*. Academic Press, London 1967, 305-–347. Zbl 0153.07403 MR 0217026

[14] K. Tzanev, Hecke C^*-algebras and amenability. *J. Operator Theory* **50** (2003), 169–178. Zbl 1036.46054 MR 2015025

[15] A. Weil, Fonction zêta et distributions. In *Séminaire Bourbaki*, Vol. 9, Exp. No. 312, Soc. Math. France, Paris 1995, 523-–531. Zbl 0226.12008 MR 1610983 http://archive.numdam.org/article/SB_1964-1966__9__523_0.pdf

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