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A Survey on a Class of Exact Solutions of the Navier-Stokes Equations and a Model for Turbulence†

By

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Abstract

In this paper we review some classes of exact solutions of the Navier-Stokes equations under a time-independent external straining flow, centering around the celebrated Burgers vortex. The objectives are (i) to clarify the relationship between them and (ii) to examine them as models of turbulence. Particularly we study the Lundgren spiral model for turbulence in the presence of azimuthal vorticity (that is, with axial velocity). The implication on linear stability of the Burgers vortex is briefly discussed.

§1. Introduction

At present there is no known method of integrating the Navier-Stokes equations in a systematic fashion. In this sense one should not be surprised at the paucity of exact solutions known to date. There are a number of attempts to collect and tabulate the exact solutions, which are scattered rather sporadically over a range of the literature [B63, W89, W91, O97]. Some of them are discovered independently by different researchers and some of them belong to the folklore. The purpose of this article is two-fold. First, we review a class of exact solutions centering around the celebrated exact solution of the Burgers vortex

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[B48]. Second, we consider its application a model for small-scale motion in turbulence.

There are a number of attempts to represent turbulent flow field as an assembly of exact solutions of Navier-Stokes or Euler equations. Fourier spectrum of energy $E(k)$ may be used as a rough but convenient measure of each model; see [S97] for a nice survey of such researches. One such attempt uses an inviscid solution of Hill's vortex [SL43]. This first model suffers from a defect that it cannot sustain energy transfer, in that any odd-order moments vanish in this model. See also [AP01]. Another attempt uses vortex tubes and layers [T51, S68]. It was shown that a vortex tube alone deterministically yields $E(k) \propto k^{-1}$ and a vortex layer yields $E(k) \propto k^{-2}$.

We will be interested in the model which utilizes the Burgers vortex but has helical components in addition to it. More specifically, we consider the Lundgren's spiral model for turbulent small scale motion [L82]. This consists of axial vorticity, helically wound up around the Burgers vortex tube, see [M93] for more general helical models in flow fields. It was found that the energy spectrum of the spiral model associated with axial vorticity yields a form of $k^{-5/3}$, something between the values predicted by tube and layer models, and more importantly consistent with Kolmogorov's similarity theory of turbulence [L82]. This model has been discussed subsequently in [L93, PS93, PBS94].

We ask what happens to the model if there is azimuthal vorticity, which were put to zero in the Lundgren's model. To estimate the spectrum coming from the azimuthal components we will first prepare a suitable class of exact solutions of the Navier-Stokes equations.

There is a phenomenological theory [G93] regarding the Lundgren's spiral model which explains $k^{-5/3}$ for axial vorticity and predicts (if straightforwardly extended) $k^{-7/3}$ for azimuthal vorticity. In this sense the azimuthal contribution would not mask the axial one, thereby suggesting robustness of the $k^{-5/3}$ form of the latter. We will examine validity of this phenomenology by asymptotic solutions.

The rest of this paper is organized as follows. In Section 2, we list the Burgers vortex and a similar, but different class of solutions investigated by Takaoka in [T91] as a model for vortex reconnection. In Section 3, a slightly more general class is considered, which is obtained by direct superposition of Burgers' and Takaoka's. Alternatively, this class will be understood in an unified manner by a simple construction for quasi two-dimensional Navier-Stokes flows. In Section 4, a brief summary of Lundgren's model for turbulence is given. As an application of the construction, we derive the energy spectrum

$E^{\perp}(k)$ associated with axial velocity in the spirit of the original paper [L82] in Section 5,[1] where we consider the Lundgren's model using the general superposed class of solutions. Finally, Section 6 will be devoted to summary and discussion.

§2. Some Classes of Stretched Solutions

§2.1. Burgers vortex

The celebrated Burgers vortex tube [B48] is an exact solution under an external time-independent strain field. With the external field the solution dose not belong in L_2, but the physical meaning of it is that it mimics the effect of other vortices residing at far distances. The Burgers vortex solution was independently found by Rott [R58]. There, some of its properties such as pressure distribution were studied in detail.

Originally, the Burgers solution was given as a stationary solution of the Navier-Stokes equations. In [R58] a non-stationary form is described. More generally it can be realized as a large-time asymptotic state of non-stationary solutions [K84, M86]. Consider a flow under a constant strain rate $\alpha(> 0)$, whose velocity and vorticity are respectively given by

$$\boldsymbol{u} = (-\alpha x + u_1(x, y, t), -\alpha y + u_2(x, y, t), 2\alpha z) \tag{2.1}$$

and

$$\boldsymbol{\omega} = (0, 0, \omega_3),\ \omega_3 = \partial_x u_2 - \partial_y u_1. \tag{2.2}$$

The vorticity equation reads

$$\frac{\partial \omega_3}{\partial t} + u_1 \frac{\partial \omega_3}{\partial x} + u_2 \frac{\partial \omega_3}{\partial y} = \alpha x \frac{\partial \omega_3}{\partial x} + \alpha y \frac{\partial \omega_3}{\partial y} - \gamma \omega_3 + \nu \triangle_2 \omega_3, \tag{2.3}$$

where $\triangle_2 \equiv \partial_x^2 + \partial_y^2$. If the flow is axisymmetric, we have

$$u_1 \frac{\partial \omega_3}{\partial x} + u_2 \frac{\partial \omega_3}{\partial y} = \frac{\partial(\omega_3, \psi)}{\partial(x, y)} = 0,$$

where ψ is the stream function. Introducing the transformations $\Omega_3 = \omega_3 e^{\gamma t}$ and $X = xe^{\alpha t}$, $Y = ye^{\alpha t}$, $T = (e^{2\alpha t} - 1)/(2\alpha)$, the vorticity equations reduces to a linear diffusion equation

$$\frac{\partial \Omega_3}{\partial T} = \nu \left(\frac{\partial^2 \Omega_3}{\partial X^2} + \frac{\partial^2 \Omega_3}{\partial Y^2} \right). \tag{2.4}$$

[1] It should be emphasized that the same result has been obtained by [PL01] by a more sophisticated treatment following [PS93, PBS94].

This can be solved by using a heat-kernel as

$$\Omega_3(X,Y,T) = \frac{1}{4\pi\nu T}\iint \omega_3(X',Y',0) \times \exp\left(-\frac{(X-X')^2+(Y-Y')^2}{4\nu T}\right) dX'dY'. \tag{2.5}$$

In the original variables we find

$$\omega_3(r,t) = \frac{\alpha}{2\pi\nu}\frac{e^{2\alpha t}}{e^{2\alpha t}-1}\iint \omega_3(X',Y',0) \times \exp\left(-\frac{\alpha}{2\nu}\frac{(e^{\alpha t}x-X')^2+(e^{\alpha t}y-Y')^2}{e^{2\alpha t}-1}\right) dX'dY', \tag{2.6}$$

so we find by dominated convergence that

$$\omega_3(r,t) \to \frac{\alpha\Gamma}{2\pi\nu}\exp\left(-\frac{\alpha r^2}{2\nu}\right) \text{ as } t\to\infty.$$

This is the Burgers vortex tube, obtained as a balance between stretching by the external strain field and viscous diffusion.

§2.2. A class of solutions studied by Takaoka

This class is complementary to the Burgers vortex, in that vorticity has two non-zero components and velocity one non-zero component. Axisymmetric case can be obtained by applying Lundgren's transformation to a class of quasi 2D solutions (see below) and is only marginally novel. In [T91] this class has been used as a model for vortex reconnection, see also, [T90] for the two-dimensional case.

The velocity and vorticity are

$$\boldsymbol{u} = (-\alpha x, -\beta y, -\gamma z + u_3(x,y,t)) \tag{2.7}$$

and

$$\boldsymbol{\omega} = (\partial_y u_3, -\partial_x u_3, 0), \tag{2.8}$$

where $\alpha,\beta > 0,\ \alpha+\beta+\gamma = 0$.

The vorticity equations read

$$\frac{\partial\omega_1}{\partial t} = \alpha x\frac{\partial\omega_1}{\partial x} + \beta y\frac{\partial\omega_1}{\partial y} - \alpha\omega_1 + \nu\triangle_2\omega_1, \tag{2.9}$$

$$\frac{\partial\omega_2}{\partial t} = \alpha x\frac{\partial\omega_2}{\partial x} + \beta y\frac{\partial\omega_2}{\partial y} - \beta\omega_2 + \nu\triangle_2\omega_2, \tag{2.10}$$

where use has been made of an identity

$$\boldsymbol{\omega}\cdot\nabla u_3 = \omega_1\partial_x u_3 + \omega_2\partial_y u_3 = 0.$$

By transforming dependent and independent variables as[2]

$$\Omega_1 = \omega_1 e^{\alpha t},\ \Omega_2 = \omega_2 e^{\beta t},$$
$$U_3 = u_3 e^{-\gamma t},$$
$$X = xe^{\alpha t},\ Y = ye^{\beta t},\ T = t$$

we find

$$\frac{\partial\Omega_1}{\partial T} = \nu\left(e^{2\alpha T}\frac{\partial^2}{\partial X^2} + e^{2\beta T}\frac{\partial^2}{\partial Y^2}\right)\Omega_1. \tag{2.11}$$

Noting that

$$\Omega_1 = \frac{\partial U_3}{\partial Y}$$

we verify that the axial velocity obeys the same equation

$$\frac{\partial U_3}{\partial T} = \nu\left(e^{2\alpha T}\frac{\partial^2}{\partial X^2} + e^{2\beta T}\frac{\partial^2}{\partial Y^2}\right)U_3. \tag{2.12}$$

This can be solved as

$$U_3(X,Y,T) = \frac{\sqrt{\alpha\beta}}{2\pi\nu\sqrt{(e^{2\alpha T}-1)(e^{2\beta T}-1)}} \tag{2.13}$$

$$\times\iint u_3(X',Y',0)\exp\left(-\frac{\alpha(X-X')^2}{2\nu(e^{2\alpha T}-1)} - \frac{\beta(Y-Y')^2}{2\nu(e^{2\beta T}-1)}\right)dX'dY',$$

that is,

$$u_3(x,y,t) = \frac{e^{\gamma t}\sqrt{\alpha\beta}}{2\pi\nu\sqrt{(e^{2\alpha t}-1)(e^{2\beta t}-1)}} \tag{2.14}$$

$$\times\iint u_3(X',Y',0)\exp\left(-\frac{\alpha(xe^{\alpha t}-X')^2}{2\nu(e^{2\alpha t}-1)} - \frac{\beta(ye^{\beta t}-Y')^2}{2\nu(e^{2\beta t}-1)}\right)dX'dY'.$$

Unlike the Burgers vortex, it does not converge to a nontrivial stationary solution in the limit $t\to\infty$. Rather it decays to zero as

$$u_3(x,y,t) \approx \frac{\sqrt{\alpha\beta}}{2\pi\nu}\left(\iint u_3(x',y',0)dx'dy'\right)\exp\left(-\frac{\alpha x^2+\beta y^2}{2\nu}\right)\exp(2\gamma t),$$

where $\gamma = -(\alpha+\beta) < 0$.

[2] In [T91] a slightly different scaling was used for U_3.

§3. Superposed Class of Solutions

Generally speaking, because the Navier-Stokes equations are nonlinear, a superposition of two of their exact solutions does not necessarily yield another solution. Nevertheless,

Proposition 3.1. *A superposition of Burgers' and Takaoka's class of solutions does yield another class of solution.*

Proof. Let us consider a class of solutions of the form

$$\boldsymbol{u} = (-\alpha x + u_1(x,y,t), -\beta y + u_2(x,y,t), -\gamma z + u_3(x,y,t)) \tag{3.1}$$

and

$$\boldsymbol{\omega} = (\partial_y u_3, -\partial_x u_3, \partial_x u_2 - \partial_y u_1). \tag{3.2}$$

Recall that

$$u_1 = \partial_y \psi,\ u_2 = -\partial_x \psi,$$

$$\omega_1 = \partial_y u_3,\ \omega_2 = -\partial_x u_3,\ \omega_3 = \partial_x u_2 - \partial_y u_1.$$

The vorticity equations read

$$\begin{aligned}\frac{\partial \omega_1}{\partial t} &= -\alpha\omega_1 + \left(\alpha x \frac{\partial \omega_1}{\partial x} + \beta y \frac{\partial \omega_1}{\partial y}\right) - \left(u_1 \frac{\partial \omega_1}{\partial x} + u_2 \frac{\partial \omega_1}{\partial y}\right) \\ &\quad + \left(\omega_1 \frac{\partial u_1}{\partial x} + \omega_2 \frac{\partial u_1}{\partial y}\right) + \nu \triangle_2 \omega_1,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\frac{\partial \omega_2}{\partial t} &= -\beta\omega_2 + \left(\alpha x \frac{\partial \omega_2}{\partial x} + \beta y \frac{\partial \omega_2}{\partial y}\right) - \left(u_1 \frac{\partial \omega_2}{\partial x} + u_2 \frac{\partial \omega_2}{\partial y}\right) \\ &\quad + \left(\omega_1 \frac{\partial u_2}{\partial x} + \omega_2 \frac{\partial u_2}{\partial y}\right) + \nu \triangle_2 \omega_2,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\frac{\partial \omega_3}{\partial t} &= -\gamma\omega_3 + \left(\alpha x \frac{\partial \omega_3}{\partial x} + \beta y \frac{\partial \omega_3}{\partial y}\right) - \left(u_1 \frac{\partial \omega_3}{\partial x} + u_2 \frac{\partial \omega_3}{\partial y}\right) \\ &\quad + \left(\omega_1 \frac{\partial u_3}{\partial x} + \omega_2 \frac{\partial u_3}{\partial y}\right) + \nu \triangle_2 \omega_3.\end{aligned} \tag{3.5}$$

We have, as in the Takaoka vortex,

$$\boldsymbol{\omega} \cdot \nabla u_3 = \omega_1 \partial_x u_3 + \omega_2 \partial_y u_3 = 0,$$

so the equation for the axial vorticity decouples from those for the azimuthal.

If we assume axisymmetry[3] (so, necessarily $\alpha = \beta$ is required), we have as in the case of the Burgers vortex

$$u_1 \frac{\partial \omega_3}{\partial x} + u_2 \frac{\partial \omega_3}{\partial y} \equiv \frac{\partial(\omega_3, \psi)}{\partial(x, y)} = 0.$$

The same equation results as the one for the Burgers vortex for axial vorticity

$$\frac{\partial \omega_3}{\partial t} = -\gamma\omega_3 + \alpha \left(x \frac{\partial \omega_3}{\partial x} + y \frac{\partial \omega_3}{\partial y} \right) + \nu \triangle_2 \omega_3.$$

It should be noted that $u_3,\ \omega_1,\ \omega_2$ are functions of x, y, t but $u_1,\ u_2,\ \omega_3$ are functions of r, t only.

We can make use of the transformation of variables (writing for the general case $\alpha \neq \beta$ to show how the time-dependent coefficients appear in the equations)

$$X = xe^{\alpha t},\ Y = ye^{\beta t},$$

$$U_1 = e^{-\alpha t} u_1,\ U_2 = e^{-\beta t} u_2,\ U_3 = e^{-\gamma t} u_3,$$

$$\Omega_1 = e^{\alpha t} \omega_1,\ \Omega_2 = e^{\beta t} \omega_2,\ \Omega_3 = e^{\gamma t} \omega_3.$$

Defining

$$\Omega_1 = \frac{\partial U_3}{\partial Y},\ \Omega_2 = -\frac{\partial U_3}{\partial X},\ \Omega_3 = \frac{\partial U_2}{\partial X} - \frac{\partial U_1}{\partial Y},$$

we may write the first two equations as

$$\text{(3.6)}\quad \frac{\partial \Omega_1}{\partial t} = -\left(e^{2\alpha t} U_1 \frac{\partial \Omega_1}{\partial X} + e^{2\beta t} U_2 \frac{\partial \Omega_1}{\partial Y} \right) + e^{2\alpha t} \left(\Omega_1 \frac{\partial U_1}{\partial X} + \Omega_2 \frac{\partial U_1}{\partial Y} \right) + \nu \left(e^{2\alpha t} \frac{\partial^2 \Omega_1}{\partial X^2} + e^{2\beta t} \frac{\partial^2 \Omega_1}{\partial Y^2} \right),$$

$$\text{(3.7)}\quad \frac{\partial \Omega_2}{\partial t} = -\left(e^{2\alpha t} U_1 \frac{\partial \Omega_2}{\partial X} + e^{2\beta t} U_2 \frac{\partial \Omega_2}{\partial Y} \right) + e^{2\beta t} \left(\Omega_1 \frac{\partial U_2}{\partial X} + \Omega_2 \frac{\partial U_2}{\partial Y} \right) + \nu \left(e^{2\alpha t} \frac{\partial^2 \Omega_2}{\partial X^2} + e^{2\beta t} \frac{\partial^2 \Omega_2}{\partial Y^2} \right).$$

For the symmetric case $\alpha = \beta$, by

$$T = \frac{\exp(2\alpha t) - 1}{2\alpha}$$

[3]Without axisymmetry the equation for ω_3 is decoupled, but is not linearized.

the above equations are simplified as

(3.8)
$$\frac{\partial \Omega_1}{\partial T} = -\left(U_1 \frac{\partial \Omega_1}{\partial X} + U_2 \frac{\partial \Omega_1}{\partial Y}\right) + \left(\Omega_1 \frac{\partial U_1}{\partial X} + \Omega_2 \frac{\partial U_1}{\partial Y}\right) + \nu \left(\frac{\partial^2 \Omega_1}{\partial X^2} + \frac{\partial^2 \Omega_1}{\partial Y^2}\right),$$

(3.9)
$$\frac{\partial \Omega_2}{\partial T} = -\left(U_1 \frac{\partial \Omega_2}{\partial X} + U_2 \frac{\partial \Omega_2}{\partial Y}\right) + \left(\Omega_1 \frac{\partial U_2}{\partial X} + \Omega_2 \frac{\partial U_2}{\partial Y}\right) + \nu \left(\frac{\partial^2 \Omega_2}{\partial X^2} + \frac{\partial^2 \Omega_2}{\partial Y^2}\right).$$

Or, using
$$\Psi = \psi,\ U_1 = \frac{\partial \Psi}{\partial Y},\ U_2 = -\frac{\partial \Psi}{\partial X},$$
we can write alternatively
$$\frac{\partial \Omega_1}{\partial T} = \frac{\partial(\Psi, \Omega_1)}{\partial(X, Y)} + \frac{\partial(U_1, U_3)}{\partial(X, Y)} + \nu \left(\frac{\partial^2 \Omega_1}{\partial X^2} + \frac{\partial^2 \Omega_1}{\partial Y^2}\right),$$
$$\frac{\partial \Omega_2}{\partial T} = \frac{\partial(\Psi, \Omega_2)}{\partial(X, Y)} + \frac{\partial(U_2, U_3)}{\partial(X, Y)} + \nu \left(\frac{\partial^2 \Omega_2}{\partial X^2} + \frac{\partial^2 \Omega_2}{\partial Y^2}\right).$$
together with
$$\frac{\partial \Omega_3}{\partial T} = \nu \left(\frac{\partial^2 \Omega_3}{\partial X^2} + \frac{\partial^2 \Omega_3}{\partial Y^2}\right).$$
The axial vorticity Ω_3 is obtained by solving a diffusion equation. Then $U_1,\ U_2$(or Ψ) is known explicitly and we may solve the linear equations for Ω_1 and Ω_2. □

§3.1. An alternative interpretation of the superposed solutions

A method is known for constructing a class of quasi two-dimensional ($\partial/\partial z = 0$) Navier-Stokes flows, see, e.g. [M85]. It has axisymmetric azimuthal velocity and axial velocity.

Lemma 3.1. *Consider a flow in cylindrical coordinates* (r, θ, z),
$$u_r = 0, u_\theta = u_\theta(r, t), u_z = u_z(r, \theta, t).$$
The Navier-Stokes equations can be written as

$$\frac{\partial u_\theta}{\partial t} = \nu \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right), \tag{3.10}$$

$$\frac{\partial u_z}{\partial t} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} = \nu \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} \right), \tag{3.11}$$

and

$$\frac{u_\theta^2}{r} = \frac{\partial p}{\partial r}. \tag{3.12}$$

Proof. This is a direct consequence of the axisymmetric Navier-Stokes equations. □

Remark. The components of vorticity are

$$\omega_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta}, \omega_\theta = -\frac{\partial u_z}{\partial r}, \omega_z = \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r}.$$

Remark. When $\nu = 0$, we may solve them easily as

$$u_r = 0, u_\theta = u_\theta(r), u_z = u_z\left(r, \theta - t\frac{u_\theta(r)}{r}\right).$$

As time elapses, u_z has an increasingly oscillating structure. It is well known that as $t \to \infty$ such a solution does not converge to anything in the normal sense, but it displays weak convergence. In this case as $t \to \infty$, u_z converges in a weak sense to the θ-average of its initial value, see e.g. [Y00, BG98]. If $\nu > 0$, u_z eventually decays to zero by viscous diffusion.

Proposition 3.2. *For the axisymmetric case $\alpha = \beta$, the superposed class of solutions mentioned above actually belongs to this category.*

Proof. To see this we write the equations for velocity

$$\frac{\partial u_1}{\partial t} = -\alpha u_1 + \left(\alpha x \frac{\partial u_1}{\partial x} + \beta y \frac{\partial u_1}{\partial y}\right) - \left(u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y}\right) - \frac{\partial p_2}{\partial x} + \nu \triangle_2 u_1, \tag{3.13}$$

$$\frac{\partial u_2}{\partial t} = -\beta u_2 + \left(\alpha x \frac{\partial u_2}{\partial x} + \beta y \frac{\partial u_2}{\partial y}\right) - \left(u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y}\right) - \frac{\partial p_2}{\partial y} + \nu \triangle_2 u_2, \tag{3.14}$$

$$\frac{\partial u_3}{\partial t} = -\gamma u_3 + \left(\alpha x \frac{\partial u_3}{\partial x} + \beta y \frac{\partial u_3}{\partial y}\right) - \left(u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y}\right) + \nu \triangle_2 u_3, \tag{3.15}$$

where we have introduced p_2 defined by

$$p(x, y, z, t) = -\frac{1}{2}\left(\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2\right) + p_2(x, y, t).$$

Under the set of transformations, the last equation (3.15) becomes

$$\frac{\partial U_3}{\partial T} + U_1 \frac{\partial U_3}{\partial X} + U_2 \frac{\partial U_3}{\partial Y} = \nu \left(\frac{\partial^2 U_3}{\partial X^2} + \frac{\partial^2 U_3}{\partial Y^2} \right),$$

or, in cylindrical coordinates

$$\frac{\partial U_3}{\partial T} + \frac{U_\Theta}{R} \frac{\partial U_3}{\partial \Theta} = \nu \left(\frac{\partial^2 U_3}{\partial R^2} + \frac{1}{R} \frac{\partial U_3}{\partial R} + \frac{1}{R^2} \frac{\partial^2 U_3}{\partial \Theta^2} \right),$$

which is the same as (3.11). □

Note that U_3 behaves like a passive scalar because of the decoupling of axial and planar components of velocity. We also note that

$$\Omega_R = \frac{1}{R} \frac{\partial U_3}{\partial \Theta}, \ \Omega_\Theta = -\frac{\partial U_3}{\partial R}.$$

§4. Lundgren's Spiral Model for Turbulence

In a seminal paper Lundgren considered a spiral model for small-scale turbulent motion on the basis of the Burgers vortex. Lundgren's asymptotic solution of the 2D Navier-Stokes equations takes the following form

$$\omega(r, \theta, t) = \sum_{n=-\infty}^{\infty} \omega_n(r, t) \exp(in\theta), \tag{4.1}$$

$$\omega_n = e^{2\alpha t} f_n^{||}(R) \exp(-in\ \sigma\ (R)T - \nu n^2 \sigma'(R) T^3/3), (n \neq 0) \tag{4.2}$$

$$\omega_0(r, t) = e^{2\alpha t} \left(f_0^{||}(R, T) + g^{||}(R, T) \right), \tag{4.3}$$

$$R(r, t) = r e^{\alpha t}, T = (e^{2\alpha t} - 1)/(2\alpha), \tag{4.4}$$

where

$$\frac{1}{R} \frac{d}{dR} \left(R^2 \sigma(R) \right) = f_0^{||}(R) + g^{||}(R). \tag{4.5}$$

The expression (4.5) relates vorticity and circulation in cylindrical coordinates

$$\omega = \frac{1}{r} \frac{\partial}{\partial r} (r u_\theta(r)).$$

Here $f_0^{\|}(R,T)$ describes θ-averaged vorticity of the spiral component and $g^{\|}(R,T)$ the background vorticity.

The following comment may help clarify the meaning of the approximate solution. If, in the totally inviscid case, we take $f_n^{\|}(\rho) = f^{\|}(\rho)$ (independent of n) we have

$$\omega(r,\theta,t) = 2\pi e^{2\alpha t} f^{\|}(R)\delta(\theta - \sigma(R)T) + e^{2\alpha t} g^{\|}(R),$$

where δ is the Dirac delta function. Therefore the underlying mechanism in this case is rolling-up of a nearly circular vortex layer due to Kelvin-Helmholtz instability by differential rotation σ. For general $f_n^{\|}$, such a simple interpretation is not available.

Using this asymptotic solution, Lundgren computed the averaged energy spectrum $\overline{E(k)}$ as

$$\overline{E(k)} = \overline{E_0(k)} + \overline{E_{\|}(k)},$$

where $\overline{E_0(k)} \propto k^{-1}$ is the contribution from the vortex tube,[4] whose unaveraged form is given by

$$E_0(k) = \frac{N\Gamma^2}{4\pi k} \exp\left(-\frac{\nu k^2}{\alpha}\right). \tag{4.6}$$

The spectrum $\overline{E_{\|}(k)}$ due to the spiral contributions is

$$\overline{E_{\|}(k)} = Ak^{-5/3} \exp\left(-\frac{\nu k^2}{3\alpha}\right), \tag{4.7}$$

$$A = \frac{4\pi}{3} N(2\alpha)^{1/3} \sum_{n=1}^{\infty} n^{-4/3} \int_0^{\infty} \frac{|f_n^{\|}(\rho)|^2 \rho d\rho}{(-\sigma'(\rho))^{4/3}} \tag{4.8}$$

Under the circumstances where the prefactor of the core spectrum is smaller than the one of the spiral, we find $E(k) \propto k^{-5/3}$. We observe that this result is independent from the choice of initial conditions, because $f_n^{\|}(\rho)$ is included in A.

This model consists of three parts, each of them are of importance in their own right in applied analysis of the Navier-Stokes equations. The first two are required for the approximate solutions and the final for estimation of the spectrum.

[4]Care should be taken in averaging $E_0(k)$, as a cut-off is needed for time integration [PS93, PBS94].

1. **Lundgren's transform**

We have already made use of it a couple of times in this paper. Given a 3D flow with external strain fields, we may reduce it to 2D flow without them. Conversely, this boosts a 2D flow without external strain fields to a 3D flow with external strain fields. For an axially symmetric case it goes likes this.

Suppose that there is a 3D flow with strain fields which obeys

$$\frac{\partial\omega}{\partial t} - \alpha r\frac{\partial\omega}{\partial r} + \frac{1}{r}\frac{\partial(\omega,\psi)}{\partial(r,\theta)} = 2\alpha\omega + \nu\triangle\omega, \tag{4.9}$$

where

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta},\ u_\theta = -\frac{\partial\psi}{\partial r}. \tag{4.10}$$

We introduce transforms for independent variables

$$R = e^{\alpha t}r,\ \Theta = \theta,\ T = \frac{e^{2\alpha t}-1}{2\alpha}$$

and those for dependent variables

$$\Omega(R,\Theta,T) = e^{-2\alpha t}\omega(r,\theta,t),$$

$$\Psi(R,\Theta,T) = \psi(r,\theta,t).$$

Then we can reduce the dynamical equations to

$$\frac{\partial\Omega}{\partial T} + \frac{1}{R}\frac{\partial(\Omega,\Psi)}{\partial(R,\Theta)} = \nu\left(\frac{\partial^2\Omega}{\partial R^2} + \frac{1}{R}\frac{\partial\Omega}{\partial R} + \frac{1}{R^2}\frac{\partial^2\Omega}{\partial\Theta^2}\right), \tag{4.11}$$

which is a 2D flow without strain fields. Note that

$$U_R = \frac{1}{R}\frac{\partial\Psi}{\partial\Theta},\ U_\Theta = -\frac{\partial\Psi}{\partial R}. \tag{4.12}$$

2. **Asymptotic axisymmetrization of advecting velocity**

This means that at large times ($t\to\infty$), dynamics of vorticity is well approximated by

$$\frac{\partial \Omega}{\partial T} + \frac{U_0(R)}{R}\frac{\partial \Omega}{\partial \Theta} = 0, \tag{4.13}$$

where U_0 is the axisymmetric azimuthal velocity induced by vorticity Ω_0. For simplicity, viscosity has been discarded.

It means that as the vorticity winds up with excitation on small scales, its non-axisymmetric perturbations become rapidly varying in radial direction, which makes vorticity-stream function coupling cancel at leading order [BL94, BG98]. Thus, as $T \to \infty$ the vorticity behaves as if it *were* a passive scaler advected by a flow induced by Ω_0. This large time asymptotic analysis has been recast using a kind of WKB analysis [BL94] under a linear approximation. While a rigorous mathematical justification for the fully-nonlinear case seems to be lacking, validity of the asymptotic theory has been obtained by numerical simulations, see e.g. [L93, BG98]. See also related works [LB95, BG99, MBG02, HBG03].

3. **Statistical average of the energy spectrum**

It is assumed that there are a large number of vortices in the box. In that box each vortex has the identical structure but is in different stages of time evolution. These vortices are assumed to be supplied at some constant rate N, the rate of creation of vortex length per unit time per unit volume $[L^{-2}T^{-1}]$ to take into account the increasing length of the structures by vortex stretching. This may be viewed as a kind of ergodic hypothesis.

The Lundgren's model assumes that azimuthal vorticity is absent. If it is present, it might influence the shape of the spectrum $\overline{E(k)}$.

A phenomenological theory based on the cascade argument (essentially the so-called β-model for turbulence) is given by Gilbert in [G93]. According to it, the spectrum due to axial vorticity leads to

$$\overline{E_{\parallel}(k)} \propto k^{-5/3},$$

consistent with Lundgren's model. If we apply the same phenomenology to azimuthal component of vorticity, it is readily verified that

$$\overline{E_\perp(k)} \propto k^{-7/3},$$

whose slope is steeper than $-5/3$.

It may be in order to briefly describe the derivation here. Consider a blob of vortex in an axisymmetric field $\boldsymbol{u} = (0, u_\theta(r), 0)$. The three components of vorticity evolve as $\omega_\theta \approx \omega_0 \sigma t,\ \omega_r \approx \omega_0,\ \omega_z \approx \omega_0$, where $\sigma \approx r\partial_r(u_\theta/r)$ is the differential rotation. If we turn on the straining filed $\boldsymbol{u} = (-\alpha r, 0, 2\alpha z)$, the vorticity is stretched as

$$\omega_r(r,\theta,z,t) = e^{-\alpha t}\Omega_R(R,\Theta,Z,T),$$

$$\omega_\theta(r,\theta,z,t) = e^{-\alpha t}\Omega_\Theta(R,\Theta,Z,T)$$

and

$$\omega_z(r,\theta,z,t) = e^{2\alpha t}\Omega_R(Z,\Theta,Z,T),$$

where

$$R = e^{\alpha t}r,\ \Theta = \theta,\ Z = e^{-2\alpha t}z,\ T = \frac{e^{2\alpha t}-1}{2\alpha}.$$

Thus, axial and azimuthal components of vorticity are transformed as

$$\omega_z \approx \omega_0 \to \omega_0 e^{2\alpha t}$$

and

$$\omega_\theta \approx \omega_0\sigma t \to \omega_0 e^{-\alpha t}\sigma\frac{e^{2\alpha t}-1}{2\alpha} \approx \frac{\omega_0\sigma}{\alpha}e^{\alpha t}.$$

On the other hand, the height, length and thickness of the blob undergo respectively transformations as

$$l_0 \to l_0 e^{2\alpha t},\ l_0\sigma t \to \frac{l_0\sigma}{\alpha}e^{\alpha t},\ \frac{l_0}{\sigma t} \to \frac{l_0\alpha}{\sigma}e^{-3\alpha t}(\equiv l).$$

Now, considering the eddy turn-over time $O(1/\alpha)$ and reading the smallest scale by $l(=1/k)$, the enstrophy spectra $Q_\parallel(k)$ and $Q_\perp(k)$ associated respectively with axial and azimuthal vorticity are estimated as

$$kQ_\parallel(k) \approx \frac{l_0^3}{\alpha}(\omega_0 e^{2\alpha t})^2 \propto l^{-4/3} \approx k^{4/3}$$

and

$$kQ_\perp(k) \approx \frac{l_0^3}{\alpha}\left(\frac{\omega_0\sigma}{\alpha}e^{\alpha t}\right)^2 \propto l^{-2/3} \approx k^{2/3}.$$

These are Gilbert's phenomenological predictions.

Thus, as far as the phenomenology is concerned, the azimuthal component is expected to have little effect on the spectrum because it could not spoil $k^{-5/3}$ for large k. Because we have an appropriate class of superposed solutions we may address the prediction analytically by means of asymptotic solutions.

§5. Properties of the Superposed Solution

Now we apply Lundgren's methods to the superposed class of solutions. The rest is basically reiteration of his argument for the derivation of the energy spectrum due to axial velocity (or, in other words, azimuthal vorticity).

§5.1. Asymptotic solution

Proposition 5.1. *Consider a flow of the following form*

$$u_r = -\alpha r,\ u_\theta = u_\theta(r,t),\ u_z(\equiv u_3) = 2\alpha z + u_z(r,\theta,t),\ (\alpha > 0), \tag{5.1}$$

$$\omega_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta},\ \omega_\theta = -\frac{\partial u_z}{\partial r},\ \omega_z = \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r}. \tag{5.2}$$

At large time $t \to \infty$, an asymptotic solution is given by

$$u_{3,n} \approx S^{-1} h_n(S^{1/2} r) \exp\left(-in\sigma(S^{1/2}r)P - \nu\frac{n^2}{3}\sigma'(S^{1/2}r)^2 P^3\right),$$

$$\omega_{\theta,n} \approx f_n^{\perp}(S^{1/2}r) S^{1/2} r \sigma'(S^{1/2}r) P \exp\left(-in\sigma(S^{1/2}r)P - \nu\frac{n^2}{3}\sigma'(S^{1/2}r)^2 P^3\right),$$

and

$$\omega_{r,n} \approx S^{-1/2} f_n^{\perp}(S^{1/2}r) \exp\left(-in\sigma(S^{1/2}r)P - \nu\frac{n^2}{3}\sigma'(S^{1/2}r)^2 P^3\right)$$

for Fourier expansions such as

$$u_3(r,\theta,t) = \sum_{n=-\infty}^{\infty} u_{3,n}(r,t)\exp(in\theta).$$

Here we have introduced

$$\sigma(r) = \frac{u_\theta(r,0)}{r},\ S = \exp(2\alpha t),\ P = \frac{\exp(2\alpha t) - 1}{2\alpha},$$

and $f_n^{\perp}(r) = inh_n(r)/r$ with $h_n(r)$ is an arbitrary function of r. An asymptotic solution of axial vorticity $\omega = \omega_z$ is still given by (4.1)*.*

Proof. We start with the equation for the axial velocity

$$\frac{\partial U_3}{\partial T} + \frac{U_\Theta}{R}\frac{\partial U_3}{\partial \Theta} = \nu\left(\frac{\partial^2 U_3}{\partial R^2} + \frac{1}{R}\frac{\partial U_3}{\partial R} + \frac{1}{R^2}\frac{\partial^2 U_3}{\partial \Theta^2}\right), \tag{5.3}$$

which has the same form as the equation for axial vorticity. We define its Fourier coefficient by

$$U_3 = \sum_{n=-\infty}^{\infty} U_{3,n}(R,T)e^{in\Theta}$$

and

$$\sigma(R,T) = \frac{U_\Theta(R,T)}{R}.$$

The axial component of velocity U_3 is a passive scalar. As in [L82] we may treat $\sigma(R,T)$ as a constant; $\sigma(R,T) = \sigma(R,0)$, see also [BG98, BG99]. The validity of such an approximation is different from the (more difficult, nonlinear) vorticity problem, where asymptotic axisymmetrization is the key property. Here for a passive scalar problem, the large time asymptotic analysis holds because dynamics is linear by definition [RW83].

To absorb rapidly varying exponentials, we introduce $h_n(R,T) = U_{3,n}\exp(in\sigma T)$ then it satisfies

$$\begin{aligned}\frac{\partial h_n}{\partial T} = \nu\Bigg[&\frac{\partial^2 h_n}{\partial R^2} + \left(\frac{1}{R} - 2in\frac{d\sigma}{dR}T\right)\frac{\partial h_n}{\partial R}\\ &+\left(-in\frac{d^2\sigma}{dR^2}T - n^2\left(\frac{d\sigma}{dR}\right)^2 T^2 - \frac{in}{R}\frac{d\sigma}{dR}T - \frac{n^2}{R^2}\right)h_n\Bigg].\end{aligned}$$

Retaining the dominant T^2-term at large times in the viscous term we find

$$\frac{\partial h_n}{\partial T} \approx -\nu n^2\left(\frac{d\sigma}{dR}\right)^2 T^2 h_n.$$

This can be solved to give

$$U_{3,n} \approx h_n(R)\exp\left(-in\sigma(R)T - \nu\frac{n^2}{3}\left(\frac{d\sigma}{dR}\right)^2 T^3\right), \tag{5.4}$$

with an arbitrary function $h_n(R)$ of R. So we find

$$U_3 \approx \sum_{n=-\infty}^{\infty} h_n(R)\exp\left(-in\sigma(R)T - \nu\frac{n^2}{3}\left(\frac{d\sigma}{dR}\right)^2 T^3 + in\Theta\right). \tag{5.5}$$

We may derive the two components of vorticity;

(5.6)

$$\Omega_R = \frac{1}{R}\frac{\partial U_3}{\partial \Theta} \approx \sum_{n=-\infty}^{\infty} f_n^{\perp}(R)\exp\left(-in\sigma(R)T - \nu\frac{n^2}{3}\left(\frac{d\sigma}{dR}\right)^2 T^3 + in\Theta\right)$$

and

(5.7)

$$\Omega_\Theta = -\frac{\partial U_3}{\partial R} \approx \sum_{n=-\infty}^{\infty}\left[g_n^{\perp}(R) + Rf_n^{\perp}(R)\frac{d\sigma}{dR}T + \frac{in\nu}{3}Rf_n^{\perp}(R)\frac{d}{dR}\left(\frac{d\sigma}{dR}\right)^2 T^3\right] \times\exp\left(-in\sigma(R)T - \nu\frac{n^2}{3}\left(\frac{d\sigma}{dR}\right)^2 T^3 + in\Theta\right),$$

where

$$f_n^{\perp}(R) \equiv \frac{inh_n(R)}{R}$$

and

$$g_n^{\perp}(R) = -h_n'(R).$$

For the Fourier coefficients, we find

$$\Omega_{R,n} \approx f_n^{\perp}(R)\exp\left(-in\sigma(R)T - \nu\frac{n^2}{3}\left(\frac{d\sigma}{dR}\right)^2 T^3\right), \tag{5.8}$$

$$\Omega_{\Theta,n} \approx \left(g_n^{\perp}(R) + f_n^{\perp}(R)R\frac{d\sigma}{dR}T + \frac{3i\nu}{n}Rf_n^{\perp}(R)\frac{d}{dR}\left(\frac{d\sigma}{dR}\right)^2 T^3\right) \times\exp\left(-in\sigma(R)T - \nu\frac{n^2}{3}\left(\frac{d\sigma}{dR}\right)^2 T^3\right), \tag{5.9}$$

where

$$\Omega_R = \sum_{n=-\infty}^{\infty}\Omega_{R,n}(R,T)e^{in\Theta}, \Omega_\Theta = \sum_{n=-\infty}^{\infty}\Omega_{\Theta,n}(R,T)e^{in\Theta}.$$

Finally, after performing the Lundgren's transforms for independent and dependent variables, we find in terms of the original coordinates

$$\omega_{r,n} \approx S^{-1/2}f_n^{\perp}(S^{1/2}r)\exp\left(-in\sigma(S^{1/2}r)P - \nu\frac{n^2}{3}\sigma'(S^{1/2}r)^2P^3\right) \tag{5.10}$$

and

$$\omega_{\theta,n} \approx \Big[S^{-1/2} g_n^{\perp}(S^{1/2}r) + f_n^{\perp}(S^{1/2}r) S^{1/2} r \sigma'(S^{1/2}r) P \tag{5.11}$$
$$+S\frac{3i\nu}{n}\rho\left(\sigma'(\rho)^2\right)' f_n^{\perp}(\rho) P^3\Big]$$
$$\times \exp\left(-in\sigma(S^{1/2}r)P - \nu\frac{n^2}{3}\sigma'(S^{1/2}r)^2 P^3\right),$$

where

$$S = \exp(2\alpha t),\ P = \frac{\exp(2\alpha t) - 1}{2\alpha}.$$

As expected, $\omega_{r,n}$ is negligible compared with $\omega_{\theta,n}$. We also note that the third term (with explicit ν-dependence) in the brackets in $\omega_{\theta,n}$ is small when the Reynolds number is large.

□

Remark. As $T \to \infty$ we have $U_3 \to 0$. Therefore addition of axial velocity does not give rise to another stationary solution different from the Burgers vortex.

§5.2. Estimation of the energy spectrum

We will derive the energy spectrum associated with axial velocity following the method in [L82].

Theorem 5.1. *Consider the superposed class of solutions of the proposition* 5.1. *The energy spectrum due to planar components of vorticity is given by*

$$\overline{E_{\perp}(k)} = Bk^{-7/3} \exp\left(-\frac{\nu k^2}{3\alpha}\right), \tag{5.12}$$

$$B = \frac{4\pi}{3}\frac{N}{(2\alpha)^{7/3}}\sum_{n=1}^{\infty} n^{-2/3} \int_0^{\infty} |f_n^{\perp}(\rho)|^2 (-\sigma'(\rho))^{4/3} \rho^3 d\rho, \tag{5.13}$$

where $f_n^{\perp}(\rho)$ represents an amplitude of Fourier coefficient of planar vorticity.

Proof. We will closely follow the footsteps of Lundgren to evaluate the spectral from azimuthal component. We write

$$E(k,t) = E_0(k,t) + E_{\parallel}(k,t) + E_{\perp}(k,t), \tag{5.14}$$

where $E_0(k,t)$ is the spectrum of the associated with the Burgers vortex, $E_\parallel(k,t)$ is that of the spiral of the axial vorticity, and $E_\perp(k,t)$ is that of the spiral of the planar vorticity. It is given by

$$E_\perp(k,t) = 2\pi k \sum_{n=1}^{\infty} |I_n(k,t)|^2, \tag{5.15}$$

where

$$I_n(k,t) = \int_0^\infty J_n(kr) u_{3,n}(r,t) r dr \tag{5.16}$$

is the Hankel transform. Using an integral representation of Bessel functions

$$J_n(kr) = \frac{1}{2}\left(\frac{2}{\pi kr}\right)^{1/2} ((-i)^{n+1/2} e^{ikr} + i^{n+1/2} e^{-ikr}),$$

we find

$$I_n(k,t) = \int_0^\infty \frac{1}{2}\left(\frac{2}{\pi kr}\right)^{1/2} ((-i)^{n+1/2} e^{ikr} + i^{n+1/2} e^{-ikr}) \omega_{\theta,n}(r,t) r dr.$$

Inserting the approximate solution obtained above

$$I_n(k,t) \approx \int_0^\infty W(r_n,t)\left((-i)^{n+1/2}\exp\left(ikr - in\sigma(S^{1/2}r)P\right)\right.$$
$$\left. + i^{n+1/2}\exp\left(-ikr - in\sigma(S^{1/2}r)P\right)\right) dr,$$

where

$$W_n(r,t) \equiv \frac{1}{2}\left(\frac{2}{\pi kr}\right)^{1/2} S^{-1} h_n(S^{1/2}r) \exp\left(-\nu \frac{n^2}{3}\sigma'(S^{1/2}r)^2 P^3\right).$$

Assuming

$$\sigma'(S^{1/2}r) < 0,\ \sigma''(S^{1/2}r) > 0,$$

we can evaluate the above integral using the method of stationary phase[5] as

$$I_n(k,t) \approx i^{n-1/2} e^{i\pi/4}\left(\frac{2\pi}{n\sigma''(S^{1/2}r_n)PS}\right)^{1/2}$$
$$\times W_n(r_n,t)\exp\left(-ikr_n - in\sigma(S^{1/2}r_n)P\right),$$

[5] Note that $\int_{-\infty}^{\infty} e^{iax^2} dx = \sqrt{\frac{\pi}{a}} e^{i\pi/4},\ a > 0.$

where r_n is determined by the stationarity condition

$$k + n\sigma'(S^{1/2}r_n)PS^{1/2} = 0.$$

The azimuthal spectrum becomes

$$E_\perp(k,t) = 2\pi k \sum_{n=1}^{\infty} \frac{2\pi}{n\sigma''(S^{1/2}r_n)PS} (W_n(r_n,t))^2$$
$$= 2\sum_{n=1}^{\infty} \frac{\pi}{n\sigma''(S^{1/2}r_n)PS} r_n \left(S^{-1}h_n(S^{1/2}r_n)\right)^2 \exp\left(-2\nu\frac{n^2}{3}\sigma'(S^{1/2}r)^2P^3\right).$$

Inserting

$$n = -\frac{k}{S^{1/2}P\sigma'(S^{1/2}r_n)},$$

we find

$$E_\perp(k,t) = \frac{2\pi}{k}\sum_{n=1}^{\infty} \frac{-\sigma'(S^{1/2}r_n)}{\sigma''(S^{1/2}r_n)} S^{-1/2} r_n \left(S^{-1}h_n(S^{1/2}r_n)\right)^2$$
$$\times \exp\left(-2\nu\frac{n^2}{3}\sigma'(S^{1/2}r)^2P^3\right).$$

This is a deterministic form of the azimuthal spectrum.

It is assumed that there are a large number of vortices in a box of size L^3, then the averaged energy spectrum can be written as

$$\overline{E_\perp(k,t)} = \sum_j \frac{l_j}{L^3} E_\perp(k,t),$$

where l_j denotes a segment along which each vortex. We replace the above ensemble average with a long-time average

$$\sum_j = N_c \int_0^\infty dt$$

together with $l_j = S(t)l_0$. The averaged azimuthal spectrum is

$$\overline{E_\perp(k,t)} = \frac{2\pi}{k}\frac{N_c l_0}{L^3}\int_0^\infty \sum_{n=1}^{\infty} \frac{-\sigma'(S^{1/2}r_n)}{\sigma''(S^{1/2}r_n)} S^{-1/2} r_n \left(S^{-1}h_n(S^{1/2}r_n)\right)^2$$
$$\times \exp\left(-2\nu\frac{n^2}{3}\sigma'(S^{1/2}r)^2P^3\right) S dt.$$

Changing variables to $\rho = S^{1/2}r_n$ and noting $Sdt = dP$ and

$$-\frac{\sigma'(\rho)}{\sigma''(\rho)}dP = \frac{2}{3}P d\rho,$$

we find

$$\overline{E_\perp(k)} = \frac{4\pi}{3}\frac{N}{2\alpha}k^{-1}\exp\left(-\frac{\nu k^2}{3\alpha}\right)\sum_{n=1}^{\infty}\int_0^\infty \rho|h_n(\rho)|^2 PS^{-3}\rho d\rho,$$

with

$$P = \left(-\frac{k}{n\sigma'(\rho)(2\alpha)^{1/2}}\right)^{2/3},$$

where we have defined $N = N_c l_0/L^3$. Because $P/S \approx 1/(2\alpha)$, we obtain

$$\overline{E_\perp(k)} = Bk^{-7/3}\exp\left(-\frac{\nu k^2}{3\alpha}\right), \tag{5.17}$$

$$B = \frac{4\pi}{3}\frac{N}{(2\alpha)^{7/3}}\sum_{n=1}^{\infty} n^{4/3}\int_0^\infty |h_n(\rho)|^2(-\sigma'(\rho))^{4/3}\rho d\rho. \tag{5.18}$$

In terms of an amplitude $f_n^\perp(\rho)$ of planar vorticity components we may write

$$B = \frac{4\pi}{3}\frac{N}{(2\alpha)^{7/3}}\sum_{n=1}^{\infty} n^{-2/3}\int_0^\infty |f_n^\perp(\rho)|^2(-\sigma'(\rho))^{4/3}\rho^3 d\rho. \tag{5.19}$$

□

§6. Summary and Discussion

If we extend Gilbert's phenomenological argument to planar components of vorticity in a straightforward fashion, it predicts

$$E_\perp(k) \propto k^{-7/3}.$$

The above asymptotic solutions is consistent with it, in that they give the same exponent. It is of interest to note that there is a subtle difference between $\overline{E_\perp(k)}$ and the Lundgren's spectrum

(6.1)

$$\overline{E_\parallel(k)} = Ak^{-5/3}\exp\left(-\frac{\nu k^2}{3\alpha}\right),\ A = \frac{4\pi}{3}N(2\alpha)^{1/3}\sum_{n=1}^{\infty} n^{-4/3}\int_0^\infty \frac{|f_n^\parallel(\rho)|^2\rho d\rho}{(-\sigma'(\rho))^{4/3}}.$$

While we can take $f_n^\parallel(\rho) = f^\parallel(\rho)$ in $E_\parallel(k)$, we cannot take $f_n^\perp(\rho) = f^\perp(\rho)$ in $E_\perp(k)$, because the summation w.r.t. n would be divergent. This suggests that the spatial structure of planar vorticity cannot be a simple roll-up mechanism of a thin vortex layer. With this reservation in mind, we can say that the Lundgren's $-5/3$ is robust as it cannot be masked by the contribution from azimuthal vorticity.

Some comments of related problems are in order. The present study is related with of works of [PA84, M85, N, C89, S90, KKTY97] which investigate motion of diffusive vortex in a simple shear. The difference is that in the present case the axial velocity $u_z(r, \theta)$ decays in x-y plane, rather than a simple shear *e.g.* $u_z \propto x$ which become indefinitely large at $r = \infty$.

The Burgers vortex has been generalized in a number of different ways. In [S59] Sullivan obtained an exact solution which represents two-cell vortices, which may be of interest as another basis for a modeling turbulence. However, it is a three-dimensional, nonlinear solution and techniques of reduction to linear diffusion equations does not work there. More work would be necessary to determine the small-scale motion associated with it.

In [GFD99, OG00], the Burgers vortex was generalized in a different way, which has an indefinitely growing strain field in one direction. The blow-up problem of fluid equations has been investigated in a class of solutions.

Finally, in spite of some works [RS84, PP95, C98], the linear stability property of the Burgers vortex has not yet been fully investigated. The main difficulty stems from the very existence of the imposed strain fields which behave wildly at far distances. In the superposed class of solutions, the added component of axial velocity (planar vorticity) eventually decays in time. It does not define a new stationary solution other than the Burgers vortex. This is consistent with a conjecture that the Burgers vortex is stable with infinitesimal perturbations. There is another kind of stationary solution known as the Burgers vortex layer. For its stability, and in particular its transition into virtex tubes, see e.g. [BK96, B97a, B97b] and references cited therein.

Note: After submitting this manuscript, the author was informed of an overlooked work [PL01], where the same result as Theorem 5.1 was derived. He would like to thank Professors Pullin and Lundgren for kindly pointing this out.

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