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Algebraic Local Cohomology Classes Attached to Quasi-Homogeneous Hypersurface Isolated Singularities

By

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Abstract

The purpose of this paper is to study hypersurface isolated singularities by using partial differential operators based on $\mathcal{D}$-modules theory. Algebraic local cohomology classes supported at a singular point that constitute the dual space of the Milnor algebra are considered. It is shown that an isolated singularity is quasi-homogeneous if and only if an algebraic local cohomology class generating the dual space can be characterized as a solution of a holonomic system of first order partial differential equations.

§1. Introduction

In this paper, we consider isolated hypersurface singularities and give in particular characterization of quasi-homogeneity of these singularities from the viewpoint of the theory of $\mathcal{D}$-modules.

Let us recall the following theorem concerning to the quasi-homogeneous singularities due to K. Saito;

Theorem (K. Saito [8]). Let $f = f(z)$ be a holomorphic function in a neighbourhood of the origin in $\mathbb{C}^n$ defining an isolated singularity at the origin O. The following conditions are equivalent;

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1. There is a holomorphic coordinate transformation φ such that $\varphi(f)$ is a weighted-homogeneous polynomial.

2. There exist holomorphic functions $a_j(z) \in \mathcal{O}_{X,O}$, $j = 1, \ldots, n$ such that

$$f(z) = a_1(z)\frac{\partial f(z)}{\partial z_1} + \cdots + a_n(z)\frac{\partial f(z)}{\partial z_n}.$$

In 1996, Y.-J. Xu and S. S.-T.Yau ([12]) gave a characterization of quasihomogeneity of a hypersurface singularity in terms of its moduli algebra (i.e., Tjurina algebra).

Apart from the hypersurface case, characterization of quasihomogeneity have been studied by G.-M. Greuel ([2]), G.-M. Greuel, B. Martin and G. Pfister ([3]), J. Wahl ([11]) for isolated complete intersection singularities. They showed that, for several cases, the quasihomogeneity can be characterized by the equality of Milnor number and Tjurina number. More recently, H. Vosegaard ([10]) extended this characterization to any isolated complete intersection singularities.

In this paper, we derive a new characterization of quasihomogeneity of hypersurface isolated singularities by considering $\mathcal{D}$-module properties of algebraic local cohomology classes. The main objects of examination are an algebraic local cohomology class, denoted by σ, which generates the dual space of Milnor algebra, and the associated holonomic system of first order differential equations.

In §2, we introduce the ideal $\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma)$ generated by annihilating differential operators for a generator σ of order at most one and give a description of the solution space in the algebraic local cohomologies $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ of the holonomic system $\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma)$ (Theorem 2.1). In §3, we give an equivalent condition, in terms of the holonomic system, for isolated singularities to be quasihomogeneous (Theorem 3.1 and Proposition 3.1). In §4, we give examples.

The approach adopted in this paper can be applied to a study of non quasihomogeneous isolated singularities. Some applications to unimodal singularities will be treated elsewhere ([6]).

§2. The First Order Differential Operators Acting on the Dual Space

Let X be a neighbourhood of the origin O of $\mathbb{C}^n$ and $\mathcal{O}_X$ the sheaf of germs of holomorphic functions in X. Let $f = f(z_1, \ldots, z_n) \in \mathcal{O}_{X,O}$ be a germ of a holomorphic function defining an isolated singularity at the origin O. Let $\mathcal{J}_f$ be the ideal in $\mathcal{O}_{X,O}$ generated by partial derivatives $f_{z_j} = \dfrac{\partial f}{\partial z_j}$ $(j = 1, \ldots, n)$ of f:

$$\mathcal{J}_f = (f_{z_1}, \ldots, f_{z_n}).$$

Let Σ_f denote the space consisting of algebraic local cohomology classes annihilated by the Jacobi ideal $\mathcal{J}_f$:

$$\Sigma_f = \{\eta \in \mathcal{H}^n_{[O]}(\mathcal{O}_X) \mid g\eta = 0, \ {}^\forall g \in \mathcal{J}_f\}.$$

Σ_f can be identified with $\mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}_f, \mathcal{O}_X)$. We can also identify the Milnor algebra $\mathcal{O}_X/\mathcal{J}_f$ with $\Omega^n_X/\mathcal{J}_f\Omega^n_X$ where Ω^n_X is the sheaf of holomorphic differential n-forms. Then, by the non-degeneracy of the Grothendieck local duality

$$\Omega^n_X/\mathcal{J}_f\Omega^n_X \times \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}_f, \mathcal{O}_X) \to \mathbb{C}_0,$$

Σ_f can be considered as the dual space of the Milnor algebra $\mathcal{O}_X/\mathcal{J}_f$ by treating them as finite dimensional vector spaces.

The dual space Σ_f can be generated by a single algebraic local cohomology class, denoted by σ, over $\mathcal{O}_{X,O}$:

$$\Sigma_f = \mathcal{O}_{X,O}\sigma.$$

Let us consider first order differential operators that annihilate σ in the sheaf $\mathcal{D}_{X,O}$ of linear partial differential operators. We have the following fundamental property;

Lemma 2.1. *Let σ be an algebraic local cohomology class which generates Σ_f over $\mathcal{O}_{X,O}$. Annihilating differential operators of order one for the cohomology class σ act on the space Σ_f.*

Proof. Let $P = \sum_{j=1}^n a_j(z)\dfrac{\partial}{\partial z_j} + a_0(z)$ be an annihilator of σ where $a_j(z) = a_j(z_1, \ldots, z_n) \in \mathcal{O}_{X,O}$ $(j = 0, 1, \ldots, n)$. Put $v_P = \sum_{j=1}^n a_j(z)\dfrac{\partial}{\partial z_j}$. Since any class η in Σ_f can be written as $\eta = h(z)\sigma$ with some holomorphic function $h(z) = h(z_1, \ldots, z_n) \in \mathcal{O}_{X,O}$, we have

$$\begin{aligned} P\eta &= P(h(z)\sigma) \\ &= (PQ - QP)\sigma + h(z)P\sigma \\ &= (v_P h(z))\sigma \in \Sigma_f \end{aligned}$$

where Q is the multiplication operator in $\mathcal{D}_{X,O}$ defined by $Q = h(z)$. □

Let $\mathcal{L}_f$ be the set of linear partial differential operators of order at most 1 which annihilate σ:

$$\mathcal{L}_f = \left\{ P = \sum_{j=1}^{n} a_j(z)\frac{\partial}{\partial z_j} + a_0(z) \mid P\sigma = 0,\, a_j(z) \in \mathcal{O}_{X,O},\, j = 0, 1, \ldots, n \right\}.$$

It is obvious from the proof of Lemma 2.1 that the condition whether a given first order differential operator P acts on Σ_f or not depends only on the first order part v_P of P. We denote by Θ_f the set of differential operators of the form $\sum_{j=1}^{n} a_j(z)\partial/\partial z_j$ with $a_j(z) \in \mathcal{O}_{X,O}$, $j = 1, \ldots, n$ acting on Σ_f. Then, an operator v is in Θ_f if and only if v satisfies the condition $vg(z) \in \mathcal{J}_f$ for all $g(z) = g(z_1, \ldots, z_n) \in \mathcal{J}_f$, i.e.,

$$\Theta_f = \left\{ v = \sum_{j=1}^{n} a_j(z)\frac{\partial}{\partial z_j} \mid vg(z) \in \mathcal{J}_f, {}^{\forall} g(z) \in \mathcal{J}_f, \right.$$
$$\left. a_j(z) \in \mathcal{O}_{X,O},\, j = 1, \ldots, n \right\}.$$

Lemma 2.2. *The mapping, from $\mathcal{L}_f$ to Θ_f, which associates the first order part $v_P \in \Theta_f$ to a first order differential operator $P \in \mathcal{L}_f$ is surjective.*

Proof. For any $v \in \Theta_f$, there exists a holomorphic function $h(z) \in \mathcal{O}_{X,O}$ such that $v\sigma = h(z)\sigma$. Thus the operator $P = v - h(z)$ is in $\mathcal{L}_f$. □

Let $P \in \mathcal{L}_f$ be an annihilator of σ of the form $P = \sum_{j=1}^{n} a_j(z)\frac{\partial}{\partial z_j} + a_0(z)$. If an algebraic local cohomology class $\eta = h(z)\sigma \in \Sigma_f$ is a solution of the homogeneous differential equation $P\eta = 0$, we have

$$v_P h(z) = \sum_{j=1}^{n} a_j(z)\frac{\partial h(z)}{\partial z_j} \in \mathcal{J}_f$$

where $v_P \in \Theta_f$ is the first order part of the operator P. It is obvious that, in order to represent $\eta \in \Sigma_f$ in the form $\eta = h(z)\sigma$, it suffices to take the modulo class in $\mathcal{O}_{X,O}/\mathcal{J}_f$ of the holomorphic function $h(z) \in \mathcal{O}_{X,O}$. Furthermore any element v in Θ_f induces a linear operator acting on $\mathcal{O}_{X,O}/\mathcal{J}_f$ which is also denoted by v:

$$v : \mathcal{O}_{X,O}/\mathcal{J}_f \to \mathcal{O}_{X,O}/\mathcal{J}_f.$$

Now we make the following definition;

Definition. A solution space $\mathcal{H}_f$ is the set of solutions in $\mathcal{O}_{X,O}/\mathcal{J}_f$ of differential equations $vh(z) = 0$ for all $v \in \Theta_f$:

$$\mathcal{H}_f = \{h(z) \in \mathcal{O}_{X,O}/\mathcal{J}_f \mid vh(z) = 0, {}^\forall v \in \Theta_f\}.$$

Then, by Lemma 2.2, we have the following result;

Lemma 2.3.

$$\mathcal{H}_f = \{h(z) \in \mathcal{O}_{X,O}/\mathcal{J}_f \mid P(h(z)\sigma) = 0, {}^\forall P \in \mathcal{L}_f\}.$$

From the above definition, $\mathcal{H}_f$ does not depend on the choice of a generator σ.

Let $\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma)$ be a left ideal in $\mathcal{D}_{X,O}$ defined to be $\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma) = \mathcal{D}_{X,O}\mathcal{L}_f$. By the above Lemma 2.3, we have the following result;

Theorem 2.1. *Let $f \in \mathcal{O}_{X,O}$ define an isolated singularity at the origin. Let σ be a generator of Σ_f over $\mathcal{O}_{X,O}$. Then*

$$Hom_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)) = \{h(z)\sigma \mid h(z) \in \mathcal{H}_f\}.$$

Proof. Since $\mathcal{D}_{X,O}\mathcal{J}_f \subset \mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma)$, we have

$$\begin{aligned} &Hom_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)) \\ &\quad\subset Hom_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\mathcal{D}_{X,O}\mathcal{J}_f, \mathcal{H}^n_{[O]}(\mathcal{O}_X)). \end{aligned}$$

Since $Hom_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\mathcal{D}_{X,O}\mathcal{J}_f, \mathcal{H}^n_{[O]}(\mathcal{O}_X)) = \Sigma_f$, the above inclusion relation implies that any solution of the holonomic system $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma)$ can be represented in the form $h(z)\sigma$ with some $h(z) \in \mathcal{O}_{X,O}/\mathcal{J}_f$. Thus the theorem follows from Lemma 2.3. □

§3. The Quasi-Homogeneous Singularities

Let $f \in \mathcal{O}_{X,O}$ be a function which defines an isolated singularity at the origin and $\mathcal{J}_f$ the Jacobi ideal of f. Let σ be a generator of Σ_f over $\mathcal{O}_{X,O}$.

Proposition 3.1. *Assume that a function f is quasi-homogeneous. Then the set $\mathcal{H}_f$ is an one-dimensional vector space* $\mathrm{Span}_{\mathbb{C}}\{1\}$.

Proof. Let $\mathbf{w} = (w_1, \dots, w_n)$ be the quasi-weight of the quasihomogeneous function f with $w_1, \dots, w_n \in \mathbb{N}^+$. By a suitable holomorphic coordinate transformation, f is transformed into a weighted-homogeneous function of the same type $\mathbf{w}$. Since the assertion does not depend on the choice of coordinates, we may assume that f is a weighted-homogeneous function. Denote by σ_f the algebraic local cohomology class $\left[\dfrac{1}{f_{z_1}\dots f_{z_n}}\right] \in \mathcal{H}^n_{[O]}(\mathcal{O}_X)$ corresponding to the Grothendieck symbol $\left[\begin{array}{c} 1 \\ f_{z_1}\dots f_{z_n}\end{array}\right] \in \mathcal{E}xt^n_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}_f, \mathcal{O}_X)$. Then $\Sigma_f = \mathcal{O}_{X,O}\sigma_f$ holds. The Euler operator $v = \sum_{j=1}^n w_j z_j \partial/\partial z_j$ is in Θ_f. Let E be the set of all exponents of basis monomials of $\mathcal{O}_{X,O}/\mathcal{J}_f$. A function $h(z)$ in $\mathcal{H}_f$ can be written in the form

$$h(z) = b_{\mathbf{0}} + \sum_{\mathbf{k}\in E\setminus\{\mathbf{0}\}} b_{\mathbf{k}} z^{\mathbf{k}}$$

with $b_{\mathbf{0}}, b_{\mathbf{k}} \in \mathbb{C}$. We have

$$\begin{aligned} vh(z) &= \sum_{\mathbf{k}\in E\setminus\{\mathbf{0}\}} b_{\mathbf{k}}(w_1k_1 + \cdots + w_nk_n)z^{\mathbf{k}} \\ &= 0. \end{aligned}$$

Thus, $b_{\mathbf{k}}(w_1k_1 + \cdots + w_nk_n) = 0$ hold for all $\mathbf{k} \in E \setminus \{\mathbf{0}\}$. Since $w_j > 0$ $(j = 1, \dots, n)$, we have $b_{\mathbf{k}} = 0$ for all $\mathbf{k} \in E \setminus \{\mathbf{0}\}$. This implies $h(z) = b_{\mathbf{0}}$. □

Let $\mathcal{A}nn_{\mathcal{D}_{X,O}}(\sigma)$ be a left ideal in $\mathcal{D}_{X,O}$ consisting of all annihilators of the algebraic local cohomology class σ.

Theorem 3.1. *Let $f \in \mathcal{O}_{X,O}$ define a hypersurface isolated singularity at the origin. The following three conditions are equivalent*;

(i) $(f, \mathcal{J}_f) = \mathcal{J}_f$.

(ii) $\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma) = \mathcal{A}nn_{\mathcal{D}_{X,O}}(\sigma)$.

(iii) $Hom_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)) = \mathrm{Span}_{\mathbb{C}}\{\sigma\}$.

Proof. The equivalence of the condition (ii) and (iii) is obvious from the simplicity of the holonomic system $\mathcal{D}_{X,O}/\mathcal{A}nn_{\mathcal{D}_{X,O}}(\sigma)$. The implication (i)⇒(ii) follows immediately from Theorem 2.1 and Proposition 3.1. We only have to prove the implication (iii)⇒(i).

(iii)⇒(i): Assuming $f \notin \mathcal{J}_f$, we have $f\sigma \neq 0$. Let us denote by $F \in \mathcal{D}_{X,O}$ the multiplication operator defined by $F = f \in \mathcal{O}_{X,O} \subset \mathcal{D}_{X,O}$. For an annihilator $P = \sum_{j=1}^{n} a_j(z)\dfrac{\partial}{\partial z_j} + a_0(z) \in \mathcal{L}_f$ of σ, we have

$$\begin{aligned} P(f\sigma) &= PF\sigma \\ &= (PF - FP)\sigma + FP\sigma \\ &= (v_P f)\sigma. \end{aligned}$$

Since $v_P f = \sum_{j=1}^{n} a_j(z)\dfrac{\partial f}{\partial z_j}$ being in $\mathcal{J}_f$, $P(f\sigma) = 0$ holds. As σ and $f\sigma$ are linearly independent algebraic local cohomology classes in Σ_f, we have

$$\dim Hom_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)) \geq 2.$$

□

§4. Examples

In this section, we give two examples: one is about a quasi-homogeneous case and the other is about a non quasi-homogeneous case.

Let f_0 be a function defined by a polynomial $x^3 + y^7$, which is weighted homogeneous of the weighted-degree 21 with the weight $(7, 3)$.

Example 1. Let f_1 be a function defined by a polynomial $f_0 + xy^4 = x^3 + y^7 + xy^4$. The weighted-degree 19 of the monomial xy^4 is smaller than that of the function f_0. The standard basis of the Jacobi ideal $\mathcal{J}_{f_1}$ of the function f_1 with respect to the lexicographical ordering is

$$\{y^7, 7y^6 + 4xy^3, y^4 + 3x^2\}.$$

The monomial basis of $\mathcal{O}_{X,O}/\mathcal{J}_{f_1}$ is given by $\{xy^2, xy, x, y^6, y^5, y^4, y^3, y^2, y, 1\}$. The dual space Σ_{f_1} is spanned by the following 10 algebraic local cohomology classes;

$$\left[\frac{1}{x^2y^3}\right], \left[\frac{1}{x^2y^2}\right], \left[\frac{1}{x^2y}\right], \left[\frac{1}{xy^7} - \frac{1}{3}\frac{1}{x^3y^3} - \frac{7}{4}\frac{1}{x^2y^4}\right], \left[\frac{1}{xy^6} - \frac{1}{3}\frac{1}{x^3y^2}\right],$$
$$\left[\frac{1}{xy^5} - \frac{1}{3}\frac{1}{x^3y}\right], \left[\frac{1}{xy^4}\right], \left[\frac{1}{xy^3}\right], \left[\frac{1}{xy^2}\right], \left[\frac{1}{xy}\right]$$

where $[\cdot]$ is a standard Čech covering representation of algebraic local cohomolog classes. The space Θ_{f_1} is generated by first order differential operators

$$4x\frac{\partial}{\partial y} + (4y^3 - 35xy^2)\frac{\partial}{\partial x}, \quad 16y\frac{\partial}{\partial y} + (-28y^3 + 147xy^2 + 32x)\frac{\partial}{\partial x}$$

and operators in $\{y^7\frac{\partial}{\partial x}, y^7\frac{\partial}{\partial y}, (7y^6+4xy^3)\frac{\partial}{\partial x}, (7y^6+4xy^3)\frac{\partial}{\partial y}, (y^4+3x^2)\frac{\partial}{\partial x}, (y^4+3x^2)\frac{\partial}{\partial y}\}$.

Solving the simultaneous differential equations $vh(z) = 0$ for above generators v of Θ_{f_1}, we find $\mathcal{H}_{f_1} = \mathrm{Span}_{\mathbb{C}}\{1\}$. Thus the function f_1 is quasi-homogeneous. For instance, we can obtain a representation

$$\Big[-\frac{3176523}{16384}\frac{1}{xy} + \frac{49}{64}\frac{1}{x^3y} + \frac{1029}{256}\frac{1}{x^2y^2} + \frac{21609}{1024}\frac{1}{xy^3} - \frac{1}{12}\frac{1}{x^3y^3} - \frac{7}{16}\frac{1}{x^2y^4} - \frac{147}{64}\frac{1}{xy^5} + \frac{1}{4}\frac{1}{xy^7}\Big]$$

of the cohomology class $\sigma_{f_1} = \Big[\frac{1}{f_{1x}f_{1y}}\Big]$ by solving first order partial differential equations $P\sigma_{f_1} = 0$, $\forall P \in \mathcal{A}nn^{(1)}_{\mathcal{D}_{X,O}}(\sigma_{f_1})$ where $f_{1x} = \frac{\partial f_1}{\partial x}$ and $f_{1y} = \frac{\partial f_1}{\partial y}$. Note that (see [8]), the function f_1 satisfies $Df_1 = f_1$, where D is a differential operator defined by $D = \dfrac{1}{48+441y^2}\{(16x + 147xy^2 - 8y^3)\frac{\partial}{\partial x} + (8y + 6x + 63y^3)\frac{\partial}{\partial y}\}$.

Example 2. Let f_2 be a function defined by a polynomial $f_0 + xy^5 = x^3 + y^7 + xy^5$. The weighted-degree 22 of the monomial xy^5 is greater than that of the function f_0. The standard basis of the Jacobi ideal $\mathcal{J}_{f_2}$ of the function f_2 with respect to the lexicographic ordering is

$$\{y^8, 7y^6 + 5xy^4, y^5 + 3x^2\}.$$

The monomial basis of $\mathcal{O}_{X,O}/\mathcal{J}_{f_2}$ is given by

$$\{xy^3, xy^2, xy, x, y^7, y^6, y^5, y^4, y^3, y^2, y, 1\}.$$

The following 12 algebraic local cohomology classes constitute a basis of the dual space Σ_{f_2};

$$\Big[\frac{1}{x^2y^4}\Big], \Big[\frac{1}{x^2y^3}\Big], \Big[\frac{1}{x^2y^2}\Big], \Big[\frac{1}{x^2y}\Big], \Big[\frac{1}{xy^8} - \frac{7}{5}\frac{1}{x^2y^6} - \frac{1}{3}\frac{1}{x^3y^3} + \frac{7}{15}\frac{1}{x^4y}\Big],$$
$$\Big[\frac{1}{xy^7} - \frac{7}{5}\frac{1}{x^2y^5} + \frac{1}{3}\frac{1}{x^3y^2}\Big], \Big[\frac{1}{xy^6} - \frac{1}{3}\frac{1}{x^3y}\Big], \Big[\frac{1}{xy^5}\Big], \Big[\frac{1}{xy^4}\Big], \Big[\frac{1}{xy^3}\Big], \Big[\frac{1}{xy^2}\Big], \Big[\frac{1}{xy}\Big].$$

Any operator in Θ_{f_2} is given as a linear combination of first order differential operators $xy^3\frac{\partial}{\partial x}$, $y^7\frac{\partial}{\partial x}$, $y^6\frac{\partial}{\partial x}$, $(5y^5 - 21xy^2)\frac{\partial}{\partial x}$, $xy^3\frac{\partial}{\partial y}$, $xy^2\frac{\partial}{\partial y}$, $2xy\frac{\partial}{\partial y} - 7xy^2\frac{\partial}{\partial x}$, $30x\frac{\partial}{\partial y} + (35y^4 - 252xy)\frac{\partial}{\partial x}$, $y^7\frac{\partial}{\partial y}$, $y^6\frac{\partial}{\partial y}$, $y^5\frac{\partial}{\partial y}$, $y^4\frac{\partial}{\partial y}$, $2y^3\frac{\partial}{\partial y} + 5xy^2\frac{\partial}{\partial x}$, $42y^2\frac{\partial}{\partial y} + (5y^4 + 84xy)\frac{\partial}{\partial x}$ and operators belonging to the set $\mathcal{J}_{f_2}\frac{\partial}{\partial x} + \mathcal{J}_{f_2}\frac{\partial}{\partial y}$ of first order differential operators with coefficients in the ideal $\mathcal{J}_{f_2}$. Consequently, Θ_{f_2} is generated over $\mathcal{O}_{X,O}$ by first order differential operators $v_1 = 30x\frac{\partial}{\partial y} + (35y^4 - 252xy)\frac{\partial}{\partial x}$, $v_2 = 42y^2\frac{\partial}{\partial y} + (5y^4 + 84xy)\frac{\partial}{\partial x}$ and operators in $\{y^8\frac{\partial}{\partial x}, y^8\frac{\partial}{\partial y}, (7y^6 + 5xy^4)\frac{\partial}{\partial x}, (7y^6 + 5xy^4)\frac{\partial}{\partial y}, (y^5 + 3x^2)\frac{\partial}{\partial x}, (y^5 + 3x^2)\frac{\partial}{\partial y}\}$.

Solving the simultaneous differential equations $v_i h(z) = 0$, $i = 1, 2$, we find $\mathcal{H}_{f_2} = \mathrm{Span}_{\mathbb{C}}\{1, y^7\}$. Thus the function f_2 is not quasihomogeneous and the local cohomology class σ which generates Σ_{f_2} can not be characterized uniquely as a solution of first order holonomic system of partial differential equations. In fact, the function f_2 is known as a normal form of an exceptional family of E_{12}-type unimodal singularities.

Actually, in order to obtain the following representation of cohomology class $\sigma_{f_2} = \left[\frac{1}{f_{2x} f_{2y}}\right]$ by solving a system of linear partial differential operators, one needs to employ a system of second order differential equations ([6]);

$$\left[\frac{1}{f_{2x} f_{2y}}\right] = \Big[-\frac{30517578125}{218041257467152161}\frac{1}{xy} + \frac{9765625}{1441471195647}\frac{1}{x^2y} - \frac{3125}{9529569}\frac{1}{x^3y} + \frac{1}{63}\frac{1}{x^4y}$$
$$+ \frac{1220703125}{1483273860320763}\frac{1}{xy^2} - \frac{390625}{9805926501}\frac{1}{x^2y^2} + \frac{125}{64827}\frac{1}{x^3y^2} - \frac{48828125}{10090298369529}\frac{1}{xy^3}$$
$$+ \frac{15625}{66706983}\frac{1}{x^2y^3} - \frac{5}{441}\frac{1}{x^3y^3} + \frac{1953125}{68641485507}\frac{1}{xy^4} - \frac{625}{453789}\frac{1}{x^2y^4} - \frac{78125}{466948881}\frac{1}{xy^5}$$
$$+ \frac{25}{3087}\frac{1}{x^2y^5} + \frac{3125}{3176523}\frac{1}{xy^6} - \frac{1}{21}\frac{1}{x^2y^6} - \frac{125}{21609}\frac{1}{xy^7} + \frac{5}{147}\frac{1}{xy^8}\Big].$$

It should be mentioned that, in [9], T. Torrelli recently gave, by a completely different manner from this paper, the same characterization for complete intersection isolated singularities to be quasi-homogeneous in his study of Berenstein polynomials.

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