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Some Continuous Field Quantizations, Equivalent to the C*-Weyl Quantization

By

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Abstract

Starting from a (possibly infinite dimensional) pre-symplectic space (E, σ), we study a class of modified Weyl quantizations. For each value of the real Planck parameter $\hbar$ we have a C*-Weyl algebra $\mathcal{W}(E, \hbar\sigma)$, which altogether constitute a continuous field of C*-algebras, as discussed in previous works. For $\hbar = 0$ we construct a Fréchet–Poisson algebra, densely contained in $\mathcal{W}(E, 0)$, as the classical observables to be quantized. The quantized Weyl elements are decorated by so-called quantization factors, indicating the kind of normal ordering in specific cases. Under some assumptions on the quantization factors, the quantization map may be extended to the Fréchet–Poisson algebra. It is demonstrated to constitute a strict and continuous deformation quantization, equivalent to the Weyl quantization, in the sense of Rieffel and Landsman. Realizing the C*-algebraic quantization maps in regular and faithful Hilbert space representations leads to quantizations of the unbounded field expressions.

§1. Introduction

Initiated especially by the seminal paper [1] there has been in the last years an extensive study of various forms of the so-called deformation quantization (cf., e.g., [2], [3], [4], [5], [6], [7], [8] and references therein), which associates a quantum mechanical algebraic structure with a rather arbitrary, finite dimensional Poisson manifold. Perhaps the most elaborated mathematical realization of Dirac's abstract q-number quantization goes in terms of representation independent C*-algebras. In connection with the deformation concept it has been

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called *strict deformation quantization* by Rieffel and Landsman (cf., e.g., [9], [10], [11], [12], [13]). Already for finite dimensional Poisson manifolds mathematical quantization theory has clarified important relationships between the classical observables (and states) and their quantum mechanical counterparts. For systems with infinitely many degrees of freedom, usually called 'field systems', the procedure of a mathematically rigorous quantization is, of course, much more complicated.

In Section 2 we recapitulate the general notions of strict and continuous deformation quantizations and of equivalent strict quantizations, supplementing some structural insights, onto which our subsequent investigations are based.

In Section 3 we present the conceptual frame for our intended class of quantizations, starting with a general, but preferentially infinite dimensional, pre–symplectic space (E, σ) and making use of the uniquely associated C*-Weyl algebras $\mathcal{W}(E, \hbar\sigma)$, as introduced in [14]. The Planck parameter $\hbar$ varies in a subset I of $\mathbb{R}$ and accumulates at zero. For each $\hbar$ the algebra $\mathcal{W}(E, \hbar\sigma)$ may be viewed as the twisted group C*-algebra of the vector group E with respect to the multiplier $(f, g) \mapsto \exp\left\{-\frac{i}{2}\hbar\sigma(f, g)\right\}$ on E. The generating Weyl elements $W^\hbar(f)$, $f \in E$, satisfy the canonical commutations relations (CCR) in Weyl form, to which we shall refer as the 'Weyl relations',

$$W^\hbar(f)W^\hbar(g) = \exp\{-\tfrac{i}{2}\hbar\sigma(f, g)\}W^\hbar(f + g)\,, \quad W^\hbar(f)^* = W^\hbar(-f)\,.$$

The case $\hbar = 0$ signifies the classical theory (without twisting). According to Dirac's ingenious suggestion [15], [16], the start out for quantization is not the commutative but the Poisson product. We equip first the linear hull $\Delta(E, 0)$, of classical Weyl elements $W^0(f)$, $f \in E$, with an algebraic Poisson bracket. In order to enrich the set of classical observables to be quantized we complete $\Delta(E, 0)$ by means of a Fréchet-topology to the Poisson algebra $\mathcal{P}^\infty_\varsigma$. For defining the phase space manifold in terms of a dual E'_τ of E, a locally convex topology τ is introduced on E. A function realization on E'_τ of the classical Weyl elements allows for the introduction of the classical (smeared) fields $\Phi^0(f)$, and their powers, by differentiation. The fundamental Poisson brackets of the (infinite dimensional) phase space are expressed either by the Poisson brackets of the Weyl elements or by those of the smeared fields.

A continuous field of the C*-Weyl algebras is introduced for discussing continuity properties in dependence of the varying parameter $\hbar \in I$.

In Section 4 a generalization of the C*-algebraic Weyl quantization of [14] is put forward by associating the classical Weyl elements $W^0(f)$ with the quantum mechanical ones $W^\hbar(f)$ times a 'quantization factor'. There are various reasons for doing so. Since we have in mind the application of the quantization theory to

QED (in non-covariant gauges for a Hamiltonian formalism) we were compelled by the different kinds of normally ordered correlation functions in quantum optics. In fact, our class of quantizations constitutes an extreme generalization of the one-parameter family of quantum n-point functions, apparently first discussed in [17], [18], and then reproduced in text books on this subject (cf., e.g., [19]). It is also connected with the well known method in general quantum field theory to factor out the state-dependent 2-point functions from the n-point functions for revealing the true interaction effects.

From the more mathematical point of view it is tempting to replace the 2-cocycle for the Weyl quantization by equivalent ones. It speaks for the inherent consistency of the involved notions, that equivalent 2-cocycles give rise to equivalent strict deformation quantizations in the sense of [13]. The more difficult part is, to demonstrate the characteristic mathematical properties for the modified Weyl quantizations. Accommodating the line of reasoning of [14] to the altered quantization maps, we prove, in fact, that we again have strict and continuous deformation quantizations, provided we use the concept of a continuous field of C*-algebras in the unrestricted sense of Dixmier to cope with the unbounded quantization factors. Also the Poisson algebra and the range I of the Planck parameter have to be carefully adjusted, if the quantization is to display the desired features.

In Section 5 we deal with the quantum mechanical observables in Hilbert space representations. Since for infinitely many degrees of freedom there exists an abundance of essentially different representations, it is very satisfactory, that the C*-algebraic results, especially the norm-continuity of the *-algebraic operations in $\hbar$, offer devices for formulating the 'correspondence asymptotics' to the classical theory by using arbitrary regular faithful Hilbert space representations of the C*-Weyl algebras.

Much more subtle is what we call 'direct field quantization', where the classical field expressions are directly mapped onto unbounded operator expressions in a Hilbert space. In this work, we only indicate how the represented quantizations of Weyl elements may lead to direct field quantizations, where the higher products of field operators incorporate various ordering conditions, leaving the physical and mathematical details to another work [20].

§2. Strict and Continuous Quantizations

Let us first introduce some basic notational conventions. If not specified otherwise every (bi-) linear map is understood to be (bi-) linear over the complex field $\mathbb{C}$. By the linear hull $\mathrm{LH}\{M\}$ we mean all (finite) complex linear combinations of the elements of the set M. We consider exclusively *-algebras over the complex field $\mathbb{C}$, equipped with an associative, but possibly non-commutative product (in general denoted without a multiplication symbol) and with an involutive, antilinear, and anti-multiplicative *-operation. A linear functional ω on a *-algebra $\mathcal{A}$ with identity $\mathbb{1}$, expressed in terms of the duality bracket $\langle\omega; A\rangle$, $A \in \mathcal{A}$, is called a state, if it satisfies the *positivity* condition $\langle\omega; A^*A\rangle \geq 0$ for all $A \in \mathcal{A}$ and the *normalization* $\langle\omega; \mathbb{1}\rangle = 1$. A representation $(\Pi, \mathcal{H}_\Pi)$ of the *-algebra $\mathcal{A}$ is a *-homomorphism Π from $\mathcal{A}$ into the C*-algebra $\mathcal{L}(\mathcal{H}_\Pi)$ of all *bounded* operators of a complex Hilbert space $\mathcal{H}_\Pi$. $(\Pi, \mathcal{H}_\Pi)$ is called non-degenerate, if $\Pi(\mathcal{A})\mathcal{H}_\Pi$ is dense in $\mathcal{H}_\Pi$, or equivalently, if $\Pi(\mathbb{1}) = \mathbb{1}_\Pi$. A C*-norm $\|.\|$ on a *-algebra $\mathcal{A}$ is an algebra norm which satisfies the C*-property $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$.

For a set P the *-algebraic operations of functions $A : \mathsf{P} \to \mathbb{C}$ and $B : \mathsf{P} \to \mathbb{C}$ are defined pointwise in the usual way

(2.1)
$$(A + zB)[F] := A[F] + zB[F] \,, \quad (AB)[F] := A[F]B[F] \,, \quad A^*[F] := \overline{A[F]} \,,$$

for $z \in \mathbb{C}$ and for all $F \in \mathsf{P}$. That are especially the algebraic operations of the commutative *-algebra $\mathrm{C}^\infty(\mathsf{P})$, consisting of the infinitely differentiable functions $A : \mathsf{P} \to \mathbb{C}$ on a differentiable manifold P. Whenever we consider the set P as a (possibly infinite dimensional) phase space we write the functional dependence as $\mathsf{P} \ni F \mapsto A[F]$, using the square bracket. As the natural C*-norm $\|.\|_0$ on *-algebras of bounded continuous functions A on P, the sup-norm

$$\|A\|_0 := \sup\{|A[F]| \mid F \in \mathsf{P}\} \tag{2.2}$$

is chosen.

A Poisson algebra $(\mathcal{P}, \{.,.\})$ is here defined purely algebraically, without reference to a function representation. It consists of a commutative *-algebra $\mathcal{P}$ endowed with a (bilinear) Poisson bracket $\{.,.\} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$, which is anticommutative, real, and fulfills the Jacobi identity, in formulas

$$\{A, B\} = -\{B, A\} \,, \quad \{A, B\}^* = \{A^*, B^*\} \,, \quad \forall A, B \in \mathcal{P} \,,$$
$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0 \,, \quad \forall A, B, C \in \mathcal{P} \,.$$

The Poisson bracket is supposed to satisfy the Leibniz rule with respect to the commutative algebraic product

$$\{A, BC\} = \{A, B\}C + B\{A, C\} \,, \quad \forall A, B, C \in \mathcal{P} \,.$$

Note, as mentioned in our above notational conventions, a Poisson algebra $\mathcal{P}$ is always meant as a *-algebra over the complex field: A Poisson algebra $\mathcal{P}_\mathbb{R}$ over the real field leads by complexification $\mathcal{P} = \mathcal{P}_{\mathbb{R}} + i\mathcal{P}_{\mathbb{R}}$ to a complex Poisson algebra, and conversely, the selfadjoint part $\mathcal{P}_{\mathbb{R}} := \{A \in \mathcal{P} \mid A^* = A\}$ of a complex Poisson algebra $\mathcal{P}$ constitutes a real Poisson algebra.

In order to work within a C*-algebraic frame it is assumed that $\mathcal{P}$ is $\|.\|_0$-dense in a commutative C*-algebra $\mathcal{A}^0$, where $\|.\|_0$ denotes the norm on $\mathcal{A}^0$. In general the Poisson bracket $\{.,.\}$ is not $\|.\|_0$-continuous, and hence cannot be continued to all of the C*-algebra $\mathcal{A}^0$, as especially may be seen from the example in Eq. (2.3) below.

In classical Hamiltonian mechanics Poisson algebras $\mathcal{P}$ usually arise as *-algebras of differentiable, $\mathbb{C}$-valued functions on a Poisson manifold P, where P is the phase space of the classical (field) system. Each Poisson bracket $\{.,.\}$ on a differentiable finite dimensional manifold P has the form

$$\{A, B\} = \Sigma(dA, dB)\,, \quad \forall A, B \in \mathrm{C}^\infty(\mathsf{P})\,, \tag{2.3}$$

with a Poisson tensor Σ, a smooth antisymmetric bivector tensor field on the complexified cotangent bundle ${}^{\mathbb{C}}T^*\mathsf{P}$ [21, III.8.6]. The Poisson algebra $\mathcal{P}$ is then a sub-*-algebra of $\mathrm{C}^\infty(\mathsf{P})$. As an example one may take for $\mathcal{P}$ the infinitely differentiable functions with compact support $\mathrm{C}_c^\infty(\mathsf{P})$. If P is a flat space, the continuous almost periodic functions all derivatives of which are also almost periodic, constitute another useful Poisson algebra (cf. Subsection 3.3). As C*-norm $\|.\|_0$ we have the sup-norm from Eq. (2.2), so that for our examples one obtains as C*-algebra $\mathcal{A}^0$ the continuous functions vanishing at infinity, resp. the continuous almost periodic functions on P.

A quantum (field) system at the value $\hbar \neq 0$ is described by a non-commutative C*-algebra $\mathcal{A}^\hbar$, for which the $\hbar$-scaled commutator is defined by

$$[A, B]_\hbar := \tfrac{i}{\hbar}(AB - BA)\,, \quad \forall A, B \in \mathcal{A}^\hbar\,. \tag{2.4}$$

In order to perform the classical correspondence limit $\hbar \to 0$, we introduce for the $\hbar$-values a subset I of the real line $\mathbb{R}$, so that $0 \in I$ is an accumulation point of $I_0 := I\backslash\{0\}$.

For each $\hbar \in I_0$ a quantization map $Q_\hbar : \mathcal{P} \to \mathcal{A}^\hbar$ associates a quantum observable with each classical one from $\mathcal{P}$. $Q_\hbar$ should be linear and should map selfadjoint classical observables onto selfadjoint quantum observables. Thus one demands the *-preservation $Q_\hbar(A^*) = Q_\hbar(A)^*$ for all $A \in \mathcal{P}$. Since the quantum algebra $\mathcal{A}^\hbar$ is non-commutative, a quantization map cannot respect products, and so cannot be a *-algebraic homomorphism. Thus, the image $Q_\hbar(\mathcal{P})$ is a *-invariant subspace of $\mathcal{A}^\hbar$, but in general not a sub-*-algebra of $\mathcal{A}^\hbar$.

It has been Dirac [22], who had formalized Bohr's classical correspondence limit $I_0 \ni \hbar \to 0$ as an asymptotic equality of the $\hbar$-scaled commutator with the Poisson bracket. The various concise mathematical formulations of this limit developed, however, much later. We follow here the C*-algebraic version of Rieffel and Landsman.

Definition 2.1 (Strict Deformation Quantization). Let $I \subseteq \mathbb{R}$ be as above. A strict quantization $(\mathcal{A}^\hbar, Q_\hbar)_{\hbar \in I}$ of the Poisson algebra $(\mathcal{P}, \{.,.\})$ consists for each value $\hbar \in I$ of a linear, *-preserving map

$$Q_\hbar : \mathcal{P} \to \mathcal{A}^\hbar ,$$

where $\mathcal{A}^\hbar$ is a C*-algebra with norm $\|.\|_\hbar$, such that Q_0 is the identical embedding of $\mathcal{P}$ into $\mathcal{A}^0$, and such that for all $A, B \in \mathcal{P}$ the following conditions are satisfied:

(a) [**Dirac's Condition**] The $\hbar$-scaled commutator (2.4) approaches the Poisson bracket as $I_0 \ni \hbar \to 0$, that is, $\lim_{\hbar \to 0} \|[Q_\hbar(A), Q_\hbar(B)]_\hbar - Q_\hbar(\{A, B\})\|_\hbar = 0$.

(b) [**von Neumann's Condition**] In the limit $\hbar \to 0$ one has the asymptotic product homomorphy $\lim_{\hbar \to 0} \|Q_\hbar(A)Q_\hbar(B) - Q_\hbar(AB)\|_\hbar = 0$.

(c) [**Rieffel's Condition**] $I \ni \hbar \mapsto \|Q_\hbar(A)\|_\hbar$ is continuous.

The strict quantization $(\mathcal{A}^\hbar, Q_\hbar)_{\hbar \in I}$ of the Poisson algebra $(\mathcal{P}, \{.,.\})$ is called a strict deformation quantization, if for each $\hbar \in I_0$ one has in addition:

(d) [**Deformation Condition**] The map $Q_\hbar : \mathcal{P} \to \mathcal{A}^\hbar$ is injective, and its image $Q_\hbar(\mathcal{P})$ is closed with respect to the product of $\mathcal{A}^\hbar$, i.e., $Q_\hbar(A)Q_\hbar(B) \in Q_\hbar(\mathcal{P})$ for all $A, B \in \mathcal{P}$ (the latter is equivalent for $Q_\hbar(\mathcal{P})$ being a sub-*-algebra of $\mathcal{A}^\hbar$).

Dirac's condition is part of any quantization prescription in one form or the other. The C*-algebraic framework conforms to the tradition in quantum field theory. Von Neumann's condition has no counterpart in theoretical physics, and von Neumann has rather emphasized its invalidity in the sense of a strict homomorphism. It is independent from Dirac's condition in virtue of a different scaling. Rieffel's condition is a technically non-trivial smoothness demand. A richness property for the image of the quantization map is always obtainable by the restriction of $\mathcal{A}^\hbar$ to the smallest sub-C*-algebra containing $Q_\hbar(\mathcal{P})$.

In the case of a strict deformation quantization one gets the connection to the usual deformation quantization by equipping the space of classical observables $\mathcal{P}$ with the deformed, non-commutative product $\cdot_\hbar$ according to

$$A \cdot_\hbar B := Q_\hbar^{-1}(Q_\hbar(A)Q_\hbar(B)), \quad \forall A, B \in \mathcal{P}.$$

This renders $\mathcal{P}$ into a *-algebra with product $\cdot_\hbar$, which is *-algebraically isomorphic to the image $Q_\hbar(\mathcal{P})$, a possibility which has inspired the mathematical research work on quantization.

Even stronger continuity conditions, which may be of interest for the classical limit in quantum field theory, are expressible by means of continuous fields of C*-algebras in the sense of J. Dixmier [23, Chapter 10].

For our above subset $I \subseteq \mathbb{R}$ let $\prod_{\hbar \in I} \mathcal{A}^\hbar$ be the cartesian product of the family of C*-algebras $\mathcal{A}^\hbar$, $\hbar \in I$, which may be considered as a bundle over the base manifold I. The elements K of $\prod_{\hbar \in I} \mathcal{A}^\hbar$ are then considered as sections $I \ni \hbar \mapsto K(\hbar) \in \mathcal{A}^\hbar$, which we also write as $[\hbar \mapsto K(\hbar)] \in \prod_{\hbar \in I} \mathcal{A}^\hbar$. If the *-algebraic operations (scalar multiplication, addition, product, *-operation) are taken pointwise, then $\prod_{\hbar \in I} \mathcal{A}^\hbar$ becomes a *-algebra. For each $\hbar \in I$ the point evaluation is denoted by $\alpha_\hbar$, that is, $\alpha_\hbar(K) := K(\hbar)$.

Definition 2.2 (Continuous Field of C*-Algebras)**.** A continuous field of C*-algebras $(\{\mathcal{A}^\hbar\}_{\hbar \in I}, \mathcal{K})$ consists of a sub-*-algebra $\mathcal{K}$ of $\prod_{\hbar \in I} \mathcal{A}^\hbar$ such that the following conditions are valid:

(a) $I \ni \hbar \mapsto \|K(\hbar)\|_\hbar$ is continuous for all $K \in \mathcal{K}$.

(b) For each $\hbar \in I$ the set $\{K(\hbar) \mid K \in \mathcal{K}\}$ coincides with $\mathcal{A}^\hbar$.

(c) $\mathcal{K}$ is locally complete, that is, $K \in \prod_{\hbar \in I} \mathcal{A}^\hbar$ is an element of $\mathcal{K}$, if for each $\hbar_0 \in I$ and each $\varepsilon > 0$ there exists an $H \in \mathcal{K}$ and a neighborhood U_0 of $\hbar_0$ so that $\|K(\hbar) - H(\hbar)\|_\hbar < \varepsilon \; \forall \hbar \in U_0$.

The elements K of $\mathcal{K}$ are called *continuous sections.*

Let us refer a subsequently used result from [23]:

Lemma 2.3. *If* $K \in \mathcal{K}$ *and* $u : I \to \mathbb{C}$ *is continuous, then* $[\hbar \mapsto u(\hbar)K(\hbar)] \in \mathcal{K}$.

For a given continuous field of C*-algebras $\mathcal{K}$, the bounded continuous sections $K \in \mathcal{K}$ constitute the C*-algebra $\mathcal{K}_\mathrm{b}$ with the C*-norm

$$\|K\|_{\sup} := \sup_{\hbar \in I} \|K(\hbar)\|_\hbar \,. \tag{2.5}$$

Dixmier and also Landsman restrict to the sub-C*-algebra of continuous sections $I \ni \hbar \mapsto \|K(\hbar)\|_\hbar$ vanishing at infinity.

We now give a notion of a continuous quantization, which is slightly more general than that in [13], but is adapted to the Weyl quantization and especially to the equivalent quantizations treated in the present investigation.

Definition 2.4 (Continuous Quantization)**.** Let be given a Poisson algebra $(\mathcal{P}, \{.,.\})$, a continuous field of C*-algebras $(\{\mathcal{A}^\hbar\}_{\hbar\in I}, \mathcal{K})$ with $I \subseteq \mathbb{R}$ as above, and a linear, *-preserving map

$$Q : \mathcal{P} \longrightarrow \mathcal{K}\,.$$

Then the tripel $(\{\mathcal{A}^\hbar\}_{\hbar\in I}, \mathcal{K}; Q)$ is called a continuous quantization of $(\mathcal{P}, \{.,.\})$, if the following conditions are valid:

(a) $\mathcal{P} \subseteq \mathcal{A}^0$, and $\alpha_0(Q(A)) = A$ for all $A \in \mathcal{P}$.

(b) Dirac's condition is fulfilled for $Q_\hbar := \alpha_\hbar \circ Q$, $\hbar \in I$.

In a continuous quantization $(\{\mathcal{A}^\hbar\}_{\hbar\in I}, \mathcal{K}; Q)$, the first entry sets the C*-algebraic frame, the second entry selects the special continuous sections, especially the behavior of $\hbar \mapsto \|K(\hbar)\|_\hbar$, and the third specifies the chosen connections between the classical observables and the continuous sections. Thus, the above mapping $Q : \mathcal{P} \to \mathcal{K}$ is a global quantization map, which has strong continuity properties, but does not require unnecessary boundedness conditions (or even vanishing norm) for the sections in the unphysical limit $\hbar \to \pm\infty$, provided $I = \mathbb{R}$.

If $(\{\mathcal{A}^\hbar\}_{\hbar\in I}, \mathcal{K}; Q)$ is a continuous quantization, then $(\mathcal{A}^\hbar, Q_\hbar)_{\hbar\in I}$ is a strict quantization for $Q_\hbar = \alpha_\hbar \circ Q$. (To prove this it remains only to check the validity of von Neumann's condition: For $K := Q(A)Q(B) - Q(AB) \in \mathcal{K}$ it is $K(\hbar) = Q_\hbar(A)Q_\hbar(B) - Q_\hbar(AB)$, especially $\hbar \mapsto \|K(\hbar)\|_\hbar$ is continuous by Definition 2.2(a).) Note, that a continuous quantization does in general not induce a strict deformation quantization.

The converse reasoning, starting with a family of strict quantizations, is covered by the following immediate result.

Proposition 2.5. *Let $(\mathcal{A}^\hbar, Q_\hbar)_{\hbar\in I}$ be a strict quantization of the Poisson algebra $(\mathcal{P}, \{.,.\})$ so that the *-algebraic span of $Q_\hbar(\mathcal{P})$ is dense in $\mathcal{A}^\hbar$ for each $\hbar \in I$ (richness condition). Then the following assertions are equivalent*:

(i) $I \ni \hbar \mapsto \|P(Q_\hbar(A_1)\cdots Q_\hbar(A_m))\|_\hbar$ *is continuous for all $A_k \in \mathcal{P}$, all $m \in \mathbb{N}$, and all polynomials P on $\mathbb{R}^m$.*

(ii) *There exists a continuous quantization* $(\{\mathcal{A}^\hbar\}_{\hbar\in I}, \mathcal{K}; Q)$ *of* $(\mathcal{P}, \{.,.\})$ *satisfying* $Q_\hbar = \alpha_\hbar \circ Q$ *for every* $\hbar \in I$.

Since only the $\hbar \to 0$-properties are specified, a strict quantization of a Poisson algebra is highly non-unique. Rieffel's continuity property allows for an elegant notion of equivalent strict quantizations (cf. [13]):

Definition 2.6 (Equivalent Quantizations)**.** Two strict quantizations $(\mathcal{A}^\hbar, Q_\hbar)_{\hbar\in I}$ and $(\mathcal{A}'^{\,\hbar}, Q'_\hbar)_{\hbar\in I}$ of the same Poisson algebra $\mathcal{P}$ are called equivalent, if $\mathcal{A}^\hbar = \mathcal{A}'^{\,\hbar}$ for all $\hbar \in I$, and if

$$I \ni \hbar \mapsto \|Q_\hbar(A) - Q'_\hbar(A)\|_\hbar$$

is continuous for every $A \in \mathcal{P}$.

Since $Q_0(A) = A = Q'_0(A)$ in virtue of Definition 2.1, the norm difference $\|Q_\hbar(A) - Q'_\hbar(A)\|_\hbar$ for two equivalent quantizations vanishes for $\hbar \to 0$.

Observation 2.7. *If two strict quantizations* $Q_\hbar : \mathcal{P} \to \mathcal{A}^\hbar$ *resp.* $Q'_\hbar : \mathcal{P} \to \mathcal{A}^\hbar$ *arise from continuous quantizations* $Q : \mathcal{P} \to \mathcal{K}$ *resp.* $Q' : \mathcal{P} \to \mathcal{K}$ *with the same continuous field of C*-algebras* $(\{\mathcal{A}^\hbar\}_{\hbar\in I}, \mathcal{K})$, *then* $(\mathcal{A}^\hbar, Q_\hbar)_{\hbar\in I}$ *and* $(\mathcal{A}^\hbar, Q'_\hbar)_{\hbar\in I}$ *are automatically equivalent.*

§3. Bundle of Weyl Algebras

In the present Section we start from an arbitrary but fixed *pre-symplectic space* (E, σ), that is a real vector space E equipped with a non-trivial pre-symplectic form $\sigma : E \times E \to \mathbb{R}$, which is by definition $\mathbb{R}$-bilinear and anti-symmetric. The pre-symplectic form σ is called non-degenerate, if $\sigma(f, g) = 0$ $\forall g \in E$ implies $f = 0$, in which case it is called a symplectic form.

§3.1. The C*-Weyl algebra

We may regard E as a vector group equipped with the discrete topology, over which we define for each $\hbar \in \mathbb{R}$ the multiplier

$$E \times E \ni (f, g) \longmapsto \exp\{-\tfrac{i}{2}\hbar\sigma(f, g)\}\,. \tag{3.1}$$

For constructing the twisted group Banach- resp. C*-algebra we start from an abstract *-algebra given as the linear hull

$$\Delta(E, \hbar\sigma) := \mathrm{LH}\{W^\hbar(f) \mid f \in E\}$$

of *linearly independent* elements $W^\hbar(f)$, $f \in E$, called Weyl elements, which may be realized in various ways. Equipped with the twisted product (arising from the multiplier (3.1)) and the *-operation according to the Weyl relations

$$\begin{aligned} W^\hbar(f)W^\hbar(g) &= \exp\{-\tfrac{i}{2}\hbar\sigma(f,g)\}W^\hbar(f+g)\,, \\ W^\hbar(f)^* &= W^\hbar(-f)\,, \quad \forall f,g \in E\,, \end{aligned} \tag{3.2}$$

the linear hull $\Delta(E,\hbar\sigma)$ becomes a *-algebra (since every polynomial of the Weyl elements reduces to a linear combination and thus to an element of $\Delta(E,\hbar\sigma)$). The *-algebra $\Delta(E,\hbar\sigma)$ has the identity $\mathbb{1}^\hbar := W^\hbar(0)$, and the Weyl elements $W^\hbar(f)$ are algebraic unitaries. Note that an arbitrary element $A \in \Delta(E,\sigma)$ decomposes uniquely into a linear combination $A = \sum\limits_{k=1}^{n} z_k W(f_k)$ with *different* f_k's from E and coefficients $z_k \in \mathbb{C}$.

The completion $\overline{\Delta(E,\hbar\sigma)}^1$ of the *-algebra $\Delta(E,\hbar\sigma)$ with respect to the norm

$$\Big\| \sum_{k=1}^{n} z_k W^\hbar(f_k) \Big\|_1 := \sum_{k=1}^{n} |z_k|$$

($n \in \mathbb{N}$, $z_k \in \mathbb{C}$, different f_k's from E) is the twisted group Banach-*-algebra of the discrete vector group E. The *Weyl algebra* $\mathcal{W}(E,\hbar\sigma)$ arises as the enveloping C*-algebra of the Banach-*-algebra $\overline{\Delta(E,\hbar\sigma)}^1$ [23], and thus constitutes the twisted group C*-algebra of E with respect to the multiplier (3.1). The C*-norm on $\mathcal{W}(E,\hbar\sigma)$ is denoted by $\|.\|_\hbar$; it varies with the values $\hbar \in \mathbb{R}$ in contrast to the Banach norm $\|.\|_1$. Furthermore,

$$\|A\|_\hbar \leq \|A\|_1\,, \quad \forall A \in \overline{\Delta(E,\hbar\sigma)}^1\,, \tag{3.3}$$

and $\overline{\Delta(E,\hbar\sigma)}^1$ is the proper, but $\|.\|_\hbar$-dense sub-*-algebra of $\mathcal{W}(E,\hbar\sigma)$ consisting of those $A \in \mathcal{W}(E,\hbar\sigma)$ which possess the unique decomposition $A = \sum_k z_k W(f_k)$ with *different* f_k's from E, $k \in \mathbb{N}$, and coefficients $z_k \in \mathbb{C}$, satisfying $\|A\|_1 = \sum_k |z_k| < \infty$. Note, we briefly write $\sum_k \ldots$ for the possibly infinite series $\sum\limits_{k=1}^{\infty} \ldots$.

The Weyl algebra can be characterized uniquely as follows, where it is not necessary to demand the above linear independence of the Weyl elements [24]:

Theorem 3.1. *$\mathcal{W}(E,\hbar\sigma)$ is the unique C*-algebra, which is generated by* non-zero *elements $W^\hbar(f)$, $f \in E$, satisfying* (3.2), *and for which every projective group representation $\pi_\hbar$ of E with multiplier* (3.1), *arises from one of its representations $\Pi_\hbar$, such that $\pi_\hbar(f) = \Pi_\hbar(W^\hbar(f))$ for all $f \in E$.*

It holds, moreover, that $\mathcal{W}(E, \hbar\sigma)$ is simple, if and only if σ is non-degenerate and $\hbar \neq 0$.

For twisted group algebras we refer e.g. to [25], [26], [27], [28]; for further results concerning the Weyl algebra, see also [29], [30], [31], [32], [33], [34], [35].

§3.2. Continuous field of C*-Weyl algebras

Having so far treated the Weyl algebras $\mathcal{W}(E, \hbar\sigma)$ for each parameter $\hbar \in \mathbb{R}$ separately, we now piece them together to arrive at their cartesian product bundle $\prod_{\hbar\in\mathbb{R}} \mathcal{W}(E, \hbar\sigma)$, consisting of sections $K = [\hbar \mapsto K(\hbar)]$ with $K(\hbar) \in \mathcal{W}(E, \hbar\sigma)$ $\forall \hbar \in \mathbb{R}$. We follow [14] for the construction of a continuous field of C*-Weyl algebras, where here I is set equal to $\mathbb{R}$.

The Weyl relations (3.2) ensure that the linear hull

$$\Delta_{\mathrm{WF}}(E, \sigma) := \mathrm{LH}\{\, [\hbar \mapsto \exp\{-i\hbar s\} W^{\hbar}(f)] \mid (s, f) \in \mathbb{R} \times E\} \tag{3.4}$$

is a sub-*-algebra of the bundle *-algebra $\prod_{\hbar\in\mathbb{R}} \mathcal{W}(E, \hbar\sigma)$ ('WF' for 'Weyl algebra field'). The generating elements $[\hbar \mapsto \exp\{-i\hbar s\} W^{\hbar}(f)]$, where (s, f) varies in the cartesian product $\mathbb{R} \times E$, are linearly independent.

The Weyl relations imply that $W^{\hbar}(f)^* W^{\hbar}(g)^* W^{\hbar}(f) W^{\hbar}(g) = \exp\{-i\hbar\sigma (f, g)\}\mathbb{1}^{\hbar}$, and thus the *-algebra $\Delta_{\mathrm{WF}}(E, \sigma)$ is *-algebraically generated by the sections $[\hbar \mapsto W^{\hbar}(f)]$, $f \in E$, since we assumed σ non–trivial. Here '*-algebraically generated' means that $\Delta_{\mathrm{WF}}(E, \sigma)$ is the smallest sub-*-algebra of $\prod_{\hbar\in\mathbb{R}} \mathcal{W}(E, \hbar\sigma)$ which contains all of the $[\hbar \mapsto W^{\hbar}(f)]$, $f \in E$. That $\{K(\hbar) \mid K \in \Delta_{\mathrm{WF}}(E, \sigma)\} = \Delta(E, \hbar\sigma)$ is $\|.\|_{\hbar}$-dense in $\mathcal{W}(E, \hbar\sigma)$ is already known, and one may show that the sections from $\Delta_{\mathrm{WF}}(E, \sigma)$ are continuous. Now $\Delta_{\mathrm{WF}}(E, \sigma)$ can be locally completed to a continuous field $\mathcal{K}$ in the sense of Definition 2.2(c): $\mathcal{K}$ consists of those sections $K \in \prod_{\hbar\in\mathbb{R}} \mathcal{W}(E, \hbar\sigma)$, for which there exists for each $\hbar_0 \in \mathbb{R}$ and each $\varepsilon > 0$ an $H \in \Delta_{\mathrm{WF}}(E, \sigma)$ and a neighborhood U_0 of $\hbar_0$ with $\|K(\hbar) - H(\hbar)\|_{\hbar} < \varepsilon$ $\forall \hbar \in U_0$.

Theorem 3.2. *There exists a unique continuous field of C*-algebras* $(\{\mathcal{W}(E, \hbar\sigma)\}_{\hbar\in\mathbb{R}}, \mathcal{K})$ *such that* $[\hbar \mapsto W^{\hbar}(f)] \in \mathcal{K}$ *for all* $f \in E$.

Obviously, $[\hbar \mapsto \exp\{-i\hbar s\} W^{\hbar}(f)] \in \mathcal{K}_{\mathrm{b}}$ for each tuple $(s, f) \in \mathbb{R} \times E$ for the C*-algebra $\mathcal{K}_{\mathrm{b}}$ of bounded continuous sections of our continuous field $\mathcal{K}$ of C*-Weyl algebras. Consequently, $\Delta_{\mathrm{WF}}(E, \sigma)$ is a sub-*-algebra of $\mathcal{K}_{\mathrm{b}}$, the norm-closure of which is denoted by $C^*_{\mathrm{WF}}(E, \sigma)$ (the C*-norm $\|.\|_{\sup}$ of $\mathcal{K}_{\mathrm{b}}$ is given in Eq. (2.5)). Clearly, $C^*_{\mathrm{WF}}(E, \sigma)$ is the smallest sub-C*-algebra of $\mathcal{K}_{\mathrm{b}}$ containing all the sections $[\hbar \mapsto W^{\hbar}(f)]$, $f \in E$.

Lemma 3.3. *If $K \in C^*_{WF}(E,\sigma)$ and $u : \mathbb{R} \to \mathbb{C}$ is continuous and almost periodic, then* $[\hbar \mapsto u(\hbar)K(\hbar)] \in C^*_{WF}(E,\sigma)$.

Proof. We have that $\big\|[\hbar \mapsto (\sum_k z_k \exp\{-i\hbar s_k\})\mathbb{1}^\hbar]\big\|_{\sup} = \sup_\hbar |\sum_k z_k \exp\{-i\hbar s_k\}|$ is just the sup-norm on the C*-algebra of almost periodic functions on $\mathbb{R}$, for which the functions $\hbar \mapsto \exp\{-i\hbar s\}$ are total [36, Section 101]. So $[\hbar \mapsto u(\hbar)\mathbb{1}^\hbar] \in C^*_{\mathrm{WF}}(E,\sigma)$ for every continuous almost periodic function $u : \mathbb{R} \to \mathbb{C}$. Now take the product with $K \in C^*_{\mathrm{WF}}(E,\sigma)$. □

Because of the linear independence of the $[\hbar \mapsto \exp\{-i\hbar s\}W^\hbar(f)]$, $(s,f) \in \mathbb{R} \times E$, we may introduce into our *-algebra $\Delta_{\mathrm{WF}}(E,\sigma)$ the vector space norm $\|.\|_1$ by

$$\Big\|\sum_{k=1}^n z_k[\hbar \mapsto \exp\{-i\hbar s_k\}W^\hbar(f_k)]\Big\|_1 := \sum_{k=1}^n |z_k|$$

for *different* tuples $(s_k, f_k) \in \mathbb{R}\times E$ and arbitrary $z_k \in \mathbb{C}$ and $n \in \mathbb{N}$. Obviously,

$$\|K\|_{\sup} \le \|K\|_1 \,, \quad \forall K \in \Delta_{\mathrm{WF}}(E,\sigma)\,.$$

As for the Weyl algebra in Subsection 3.1 we go over to the $\|.\|_1$-completion of $\Delta_{\mathrm{WF}}(E,\sigma)$, denoted by $\overline{\Delta_{\mathrm{WF}}(E,\sigma)}^1$. The $\|.\|_1$-continuous extension of the *-algebraic operations from $\Delta_{\mathrm{WF}}(E,\sigma)$ ensures $\overline{\Delta_{\mathrm{WF}}(E,\sigma)}^1$ to be a Banach-*-algebra. In [14] it is shown that $\overline{\Delta_{\mathrm{WF}}(E,\sigma)}^1$ is a $\|.\|_{\sup}$-dense sub-*-algebra of $C^*_{\mathrm{WF}}(E,\sigma)$, which consists of those sections $K \in C^*_{\mathrm{WF}}(E,\sigma)$ which have the unique decomposition

$$K = [\hbar \mapsto K(\hbar)] = \textstyle\sum_k z_k[\hbar \mapsto \exp\{-i\hbar s_k\}W^\hbar(f_k)] \tag{3.5}$$

with coefficients $z_k \in \mathbb{C}$ satisfying $\|K\|_1 = \sum_k |z_k| < \infty$ for *different* tuples $(s_k, f_k) \in \mathbb{R}\times E$, $k \in \mathbb{N}$. Note that by Subsection 3.1 $K(\hbar) = \sum_k z_k \exp\{-i\hbar s_k\} W^\hbar(f_k) \in \overline{\Delta(E,\hbar\sigma)}^1$. We summarize the above relations in terms of the following inclusions

$$\Delta_{\mathrm{WF}}(E,\sigma) \subseteq \overline{\Delta_{\mathrm{WF}}(E,\sigma)}^1 \subseteq C^*_{\mathrm{WF}}(E,\sigma) \subseteq \mathcal{K}_{\mathrm{b}} \subseteq \mathcal{K}\,, \tag{3.6}$$

which have the meaning of being sub-*-algebras. The identity of each *-algebra is given by $[\hbar \mapsto \mathbb{1}^\hbar]$, and for every $(s,f) \in \mathbb{R} \times E$ the continuous section $[\hbar \mapsto \exp\{-i\hbar s\}W^\hbar(f)]$ is a unitary.

§3.3. Classical poisson algebras

The bundle of Weyl algebras $\{\mathcal{W}(E,\hbar\sigma)\}_{\hbar\in\mathbb{R}}$ has for $\hbar = 0$ the fiber $\mathcal{W}(E,0)$ corresponding to commuting Weyl elements $W^0(f)$, $f \in E$. Especially $\Delta(E,0)$, $\overline{\Delta(E,0)}^1$, as well as $\mathcal{W}(E,0)$ are commutative *-algebras.

Because of the linear independence of the Weyl elements, the bilinear extension of

$$\{W^0(f), W^0(g)\} := \sigma(f,g)W^0(f+g)\,, \quad \forall f,g \in E\,, \tag{3.7}$$

leads to a well defined Poisson bracket $\{.,.\}$ on the *-algebra $\Delta(E,0)$.

In order to construct larger Poisson algebras we follow [14] and suppose from now on the existence of a semi-norm ς on E such that

$$|\sigma(f,g)| \le c\,\varsigma(f)\,\varsigma(g)\,, \quad \forall f,g \in E\,, \tag{3.8}$$

for some constant $c > 0$. For non-degenerate σ one easily concludes that ς has to be a norm on E. The definition

$$\mathcal{P}_\varsigma^\infty := \{A = \textstyle\sum_k z_k W^0(f_k) \in \overline{\Delta(E,0)}^1 \mid \sum_k \varsigma(f_k)^m\,|z_k| < \infty \text{ for all } m \in \mathbb{N}\}$$

leads to a Fréchet-*-algebra (as sub-*-algebra of $\overline{\Delta(E,0)}^1$) with the locally convex Hausdorff topology arising from the increasing system of norms (different f_k's)

$$\left\|\textstyle\sum_k z_k W^0(f_k)\right\|_\varsigma^n := \sum_{m=0}^{n} \textstyle\sum_k \varsigma(f_k)^m\,|z_k|\,, \quad n \in \mathbb{N}\,. \tag{3.9}$$

By construction, $\Delta(E,0)$ is a dense sub-*-algebra of $\mathcal{P}_\varsigma^\infty$ with respect to its Fréchet-topology. Especially, for every $n \in \mathbb{N}$ there is a constant $c_n \ge 1$ with

$$\|AB\|_\varsigma^n \le c_n\,\|A\|_\varsigma^n\,\|B\|_\varsigma^n\,, \quad \forall A,B \in \mathcal{P}_\varsigma^\infty\,. \tag{3.10}$$

By use of the estimation (3.8) one may show that the Poisson bracket $\{.,.\}$ defined in (3.7) is jointly continuous with respect to the Fréchet-topology, and thus extends continuously from $\Delta(E,0)$ to a jointly Fréchet-continuous Poisson bracket on $\mathcal{P}_\varsigma^\infty$, making $(\mathcal{P}_\varsigma^\infty, \{.,.\})$ to a Poisson algebra.

The Poisson bracket $\{.,.\}$ may be realized in terms of a bivector field, applied to differentials. For this we introduce an arbitrary locally convex Hausdorff vector space topology τ on E. On the topological dual space E'_τ of E we consider the $\sigma(E'_\tau, E)$-topology, and so the bidual is given by $(E'_\tau)' = E$. According to [24] the commutative C*-Weyl algebra $\mathcal{W}(E,0)$ is *-isomorphic to the C*-algebra of the almost periodic, $\sigma(E'_\tau, E)$-continuous functions on E'_τ, and we may regard each element $A \in \mathcal{W}(E, 0)$ as an almost periodic function $A : E'_\tau \to \mathbb{C}$. The Weyl elements $W^0(f)$ are realized in terms of the periodic functions

$$W^0(f) : E'_\tau \to \mathbb{C}\,, \quad F \mapsto \exp\{iF(f)\} \equiv W^0(f)[F]\,. \tag{3.11}$$

The topological dual E'_τ may be considered as the *phase space manifold* of our classical field theory. One may introduce a differentiable structure onto E'_τ, by means of which the usual Poisson bracket has the form

$$\begin{aligned} \{A, B\}[F] := \; & -\sigma(d_F A_1, d_F B_1) - i\sigma(d_F A_1, d_F B_2) \\ & -i\sigma(d_F A_2, d_F B_1) + \sigma(d_F A_2, d_F B_2) \end{aligned} \tag{3.12}$$

with constant bivector field (cf. Eq. (2.3)), where A and B are differentiable, $\mathbb{C}$-valued functions on the manifold E'_τ, having real resp. imaginary parts A_1, B_1 resp. A_2, B_2. For the periodic functions $W^0(f)$ from Eq. (3.11) one obtains the total differentials $d_F W^0(f) = iW^0(f)[F]\, f$, which leads back to the Poisson bracket relations Eq. (3.7).

This illustrates, that the above Poissonian structure is independent of the chosen locally convex topology τ on E (leading to the phase space manifold E'_τ). Hence our Poisson and C*-algebras of the classical field theory do not depend on the phase space but arise functorially from the pre-symplectic test function space (E, σ). Modified and larger Poisson algebras are elaborated in [37].

§4. Equivalent Strict Deformation Quantizations

Again we are given in this Section an arbitrary pre-symplectic space (E, σ) with non-trivial σ. For convenience we use here the quantization concepts of Section 2 for $I = \mathbb{R}$; the quantization results for a smaller $I \subset \mathbb{R}$ are then immediate, see Corollary 4.7 below.

The various quantizations, we are considering here, are indexed by a certain class of functions $w : \mathbb{R} \times E \to \mathbb{C}$. The physical motivation for these factors $w(\hbar, f)$ is mentioned in the Introduction and further elucidated in the forthcoming publication [20].

Definition 4.1 (Quantization Factor). A mapping $w : \mathbb{R} \times E \to \mathbb{C}$, $(\hbar, f) \mapsto w(\hbar, f)$ is called a quantization factor, if

(a) $w(\hbar, f) \neq 0$ and $\overline{w(\hbar, f)} = w(\hbar, -f)$ for all $f \in E$ and all $\hbar \in \mathbb{R}$.

(b) $w(0, f) = 1 = w(\hbar, 0)$ for each $f \in E$ and every $\hbar \in \mathbb{R}$.

(c) $\mathbb{R} \ni \hbar \mapsto w(\hbar, f)$ is continuous for every $f \in E$.

The set of all quantization factors is denoted by QF.

A quantization factor w is called $\hbar$-locally bounded, if for each $\hbar_0 \in \mathbb{R}$ there exists a neighborhood U_0 so that the restricted map $U_0 \times E \ni (\hbar, f) \mapsto w(\hbar, f)$ is bounded. The set of all $\hbar$-locally bounded quantization factors is denoted by $\mathrm{QF_b}$.

Of course, a $w \in \mathrm{QF_b}$ may be globally unbounded as $\hbar \to \pm\infty$.

Let us first consider our continuous field of C*-Weyl algebras $(\{\mathcal{W}(E, \hbar\sigma)\}_{\hbar\in\mathbb{R}}, \mathcal{K})$ from Theorem 3.2, with associated C*-algebra $\mathcal{K}_\mathrm{b}$ of the bounded continuous sections, and with the smallest sup-C*-algebra $C^*_\mathrm{WF}(E, \sigma)$ containing the fields $[\hbar \mapsto W^\hbar(f)]$, $f \in E$. For each $w \in \mathrm{QF}$ we introduce the sections

$$K^w(f) := [\hbar \mapsto w(\hbar, f)W^\hbar(f)] \in \textstyle\prod_{\hbar\in\mathbb{R}} \mathcal{W}(E, \hbar\sigma)\,.$$

Proposition 4.2. *Let $w \in \mathrm{QF}$. Then the section $K^w(f)$ is an element of $\mathcal{K}$ for every $f \in E$. Furthermore, $(\{\mathcal{W}(E, \hbar\sigma)\}_{\hbar\in\mathbb{R}}, \mathcal{K})$ is the unique continuous field of C*-algebras such that $K^w(f) \in \mathcal{K}$ for all $f \in E$.*

For each $f \in E$ it holds: If $\mathbb{R} \ni \hbar \mapsto w(\hbar, f)$ is bounded, then $K^w(f) \in \mathcal{K}_b$, and if $\mathbb{R} \ni \hbar \mapsto w(\hbar, f)$ is almost periodic, then $K^w(f) \in C^*_{WF}(E, \sigma)$.

Proof. $K^w(f) \in \mathcal{K}$ follows immediately by Lemma 2.3 and Definition 4.1(c), cf. also Lemma 3.3. Since $w(\hbar, f) \neq 0$ for all $\hbar \in \mathbb{R}$, the inverse function $\hbar \mapsto w(\hbar, f)^{-1}$ exists, being continuous, too. Thus $K^w(f) \in \mathcal{K}$, if and only if $K^1(f) \in \mathcal{K}$ ($K^1(f)$ means with constant $w \equiv 1$). The uniqueness statement follows now from that in Theorem 3.2. □

For each quantization factor w we define a global quantization map $Q^w : \Delta(E, 0) \to \mathcal{K}$ by the linear extension of

$$Q^w(W^0(f)) := K^w(f) = [\hbar \mapsto w(\hbar, f)W^\hbar(f)]\,, \tag{4.1}$$

which is *-preserving by the relation $\overline{w(\hbar, f)} = w(\hbar, -f)$ in Definition 4.1(a). Using the point evaluation $\alpha_\hbar$ from Section 2 for our continuous field of C*-Weyl algebras, we arrive for each $\hbar \in \mathbb{R}$ at the quantization map

$$Q^w_\hbar := \alpha_\hbar \circ Q^w : \Delta(E, 0) \to \mathcal{W}(E, \hbar\sigma)\,, \quad Q^w_\hbar(W^0(f)) = w(\hbar, f)W^\hbar(f)\,, \tag{4.2}$$

which has to be linearly extended. Provided a $\hbar$-locally bounded quantization factor w, one may extend the quantization maps Q^w resp. $Q^w_\hbar$ from $\Delta(E, 0)$ to the Banach-*-algebra $\overline{\Delta(E, 0)}^1$.

Proposition 4.3. *Suppose $w \in \mathrm{QF}_b$. Then it follows for not necessarily different f_k's from E, $k \in \mathbb{N}$, and for coefficients $z_k \in \mathbb{C}$, satisfying $\sum_k |z_k| < \infty$, that*

$$\textstyle\sum_k z_k K^w(f_k) = [\hbar \mapsto \sum_k z_k w(\hbar, f_k) W^\hbar(f_k)] \in \mathcal{K}\,.$$

Consequently, the above quantization maps Q^w and $Q^w_\hbar$ extend $\|.\|_1$-continuously from $\Delta(E,0)$ to $\overline{\Delta(E,0)}^1$. More precisely, for summable coefficients $z_k \in \mathbb{C}$ we have

$$\begin{aligned} &(4.3) \qquad Q^w(\textstyle\sum_k z_k W^0(f_k)) = \sum_k z_k K^w(f_k) \in \mathcal{K}\,, \\ &Q^w_\hbar(\textstyle\sum_k z_k W^0(f_k)) = \sum_k z_k w(\hbar, f_k) W^\hbar(f_k) \in \overline{\Delta(E,\hbar\sigma)}^1\,. \end{aligned}$$

Moreover, if w is (globally) bounded, i.e. $|w(\hbar, f)| \leq d$, $\forall \hbar \in \mathbb{R}$, $\forall f \in E$, for some constant $d > 0$, then $Q^w(A) \in \mathcal{K}_b$ for all $A \in \overline{\Delta(E,0)}^1$ (cf. the inclusions (3.6)).

Proof. w being $\hbar$-locally bounded, there exists for each $\hbar_0 \in \mathbb{R}$ a neigborhood U_0 of $\hbar_0$ and a constant $a_0 > 0$ with $|w(\hbar, f)| \leq a_0$ for all $f \in E$ and all $\hbar \in U_0$. Thus for each $\hbar \in \mathbb{R}$ it follows that $\sum_k z_k w(\hbar, f_k) W^\hbar(f_k) \in \overline{\Delta(E,\hbar\sigma)}^1$ by Subsection 3.1. Consequently, we obtain the well defined section $K := \sum_k z_k K^w(f_k)$ in the cartesian product bundle $\prod_{\hbar\in\mathbb{R}} \mathcal{W}(E,\hbar\sigma)$. But for such a local neighborhood U_0 we have

$$\|K(\hbar) - K_n(\hbar)\|_\hbar \leq \|K(\hbar) - K_n(\hbar)\|_1 \leq a_0 \sum_{k=n+1}^{\infty} |z_k|\,, \quad \forall \hbar \in U_0\,,$$

where $K_n := \sum_{k=1}^{n} z_k K^w(f_k) \in \mathcal{K}$, $n \in \mathbb{N}$. That is, K is approximated locally uniformly in $\hbar$ from $\mathcal{K}$, which yields $K \in \mathcal{K}$ by the local completeness of $\mathcal{K}$ in Definition 2.2(c). Now it is obvious that Q^w is continuous with respect to the Banach norm $\|.\|_1$ on $\Delta(E,0)$ and the locally uniform convergence in $\mathcal{K}$. The rest is immediate. $\square$

Note that for the $\|.\|_1$-continuous extension of the quantization map $Q^w_\hbar$ from (4.3), it suffices that $E \ni f \mapsto w(\hbar, f)$ is bounded only for the given value $\hbar \neq 0$.

In virtue of the above Proposition the quantization maps Q^w and $Q^w_\hbar$ are defined on the enlarged Poisson algebra $(\mathcal{P}^\infty_\varsigma, \{.,.\})$ from Subsection 3.3 for $\hbar$-locally bounded quantization factors w, only. So let us treat in the subsequent investigations the two cases, which we describe with the use of a variable symbol Γ, running through the different sets of quantization factors, as follows:

(1) For the Poisson algebra $\mathcal{P} := \Delta(E,0)$ we take the whole set $\Gamma := \mathrm{QF}$ as range for the allowed quantization factors w.

(2) For the enlarged Poisson algebra $\mathcal{P} := \mathcal{P}_\varsigma^\infty$ we restrict the range of allowed quantization factors w to the $\hbar$-locally bounded ones, i.e. to $\Gamma := \mathrm{QF}_\mathrm{b}$.

Case (2) assumes the existence of a semi–norm ς on E satisfying the estimation (3.8). We formulate now our first main result.

Theorem 4.4 (Continuous and Strict Deformation Quantizations)**.** *Let the quantization maps Q^w and $Q^w_\hbar$ be as above. Then for both cases* (1) *and* (2) *the following assertions are valid for each $w \in \Gamma$:*

(a) $(\{\mathcal{W}(E,\hbar\sigma)\}_{\hbar\in\mathbb{R}}, \mathcal{K}; Q^w)$ *constitutes a continuous quantization of* $(\mathcal{P},\{.,.\})$.

(b) $(\mathcal{W}(E,\hbar\sigma), Q^w_\hbar)_{\hbar\in\mathbb{R}}$ *constitutes a strict deformation quantization of* $(\mathcal{P},\{.,.\})$.

Proof. Part (a) for case (1). We only have to establish Dirac's condition from Definition 2.1(a) for $Q^w_\hbar = \alpha_\hbar \circ Q^w$, $\hbar \in \mathbb{R}$. With the Poisson bracket expressions (3.7), the Weyl relations (3.2), and $\left\|W^\hbar(f+g)\right\|_1 = 1$ we obtain for arbitrary $f,g \in E$ that

$$\begin{aligned}
&\left\|[Q^w_\hbar(W^0(f)), Q^w_\hbar(W^0(g))]_\hbar - Q^w_\hbar(\{W^0(f), W^0(g)\})\right\|_1 \\
&\quad = \left\|w(\hbar,f)w(\hbar,g)\tfrac{i}{\hbar}\big(W^\hbar(f)W^\hbar(g) - W^\hbar(g)W^\hbar(f)\big)\right. \\
&\qquad \left. - w(\hbar,f+g)\sigma(f,g)W^\hbar(f+g)\right\|_1 \\
&\quad = \left|w(\hbar,f)w(\hbar,g)\tfrac{i}{\hbar}\left(\exp\{-\tfrac{i}{2}\hbar\sigma(f,g)\} - \exp\{\tfrac{i}{2}\hbar\sigma(f,g)\}\right)\right. \\
&\qquad \left. - w(\hbar,f+g)\sigma(f,g)\right| \\
&\quad = \left| w(\hbar,f)w(\hbar,g)\left(i\frac{\exp\{-\frac{i}{2}\hbar\sigma(f,g)\}-1}{\hbar} - i\frac{\exp\{\frac{i}{2}\hbar\sigma(f,g)\}-1}{\hbar}\right)\right. \\
&\qquad \left. - w(\hbar,f+g)\sigma(f,g)\right| \\
&\quad \stackrel{\hbar\to 0}{\longrightarrow} 0\,,
\end{aligned}$$

by the differential limits $\lim\limits_{\hbar\to 0}\frac{\exp\{\pm\frac{i}{2}\hbar\sigma(f,g)\}-1}{\hbar} = \frac{d\exp\{\pm\frac{i}{2}\hbar\sigma(f,g)\}}{d\hbar}|_{\hbar=0} = \pm\frac{i}{2}\sigma(f,g)$, and since $\hbar \mapsto w(\hbar,h)$ is continuous with $w(0,h) = 1$ for each $h \in E$ by the Definition 4.1 of a quantization factor. With the triangle inequality one immediately checks that Dirac's condition is valid for all $A, B \in \Delta(E,0)$

with respect to the Banach norm $\|.\|_1$, and by (3.3) also for the C*-norms $\|.\|_\hbar$, $\hbar \in \mathbb{R}$.

Part (b) for case (1). Since for fixed $\hbar \in \mathbb{R}$ the factors $w(\hbar, f)$ represent only constants we have $Q^w_\hbar(\Delta(E,0)) = \Delta(E, \hbar\sigma)$, which is indeed a sub-*-algebra of $\mathcal{W}(E, \hbar\sigma)$. The Weyl elements $W^\hbar(f)$, $f \in E$, being linearly independent for each $\hbar \in \mathbb{R}$, also for $\hbar = 0$, proves $Q^w_\hbar$ to be injective.

Part (a) for case (2). Since w is supposed to be $\hbar$-locally bounded, there exists for $0 = \hbar_0 \in \mathbb{R}$ a neigborhood U_0 and a constant $b_0 > 0$ with $|w(\hbar, f)| \leq b_0$ for all $f \in E$ and all $\hbar \in U_0$. The mean value theorem of differential calculus for one real variable implies

$$\left| \frac{\exp\{\pm\frac{i}{2}\hbar\sigma(f,g)\} - 1}{\hbar} \right| \leq \frac{1}{2}\, |\sigma(f,g)| \leq \frac{c}{2} \varsigma(f)\, \varsigma(g)\,, \quad \forall\, 0 \neq \hbar \in \mathbb{R}\,, \quad \forall f, g \in E\,,$$

leading to the inequality $\big\| [Q^w_\hbar(W^0(f)), Q^w_\hbar(W^0(g))]_\hbar - Q^w_\hbar(\{W^0(f), W^0(g)\}) \big\|_1 \leq 2 b_0 c\, \varsigma(f) \varsigma(g)$ for all $0 \neq \hbar \in U_0$ and all $f, g \in E$ by (3.8). Since for arbitrary $A = \sum_k u_k W^0(f_k)$ and $B = \sum_l v_l W^0(g_l)$ from $\mathcal{P}^\infty_\varsigma$ (with coefficients $u_k, v_l \in \mathbb{C}$ and $f_k, g_l \in E$) the majorant is summable, we may subsequently exchange the limit $\hbar \to 0$ with $\sum_{k,l} \ldots$ by Lebesgue's dominated convergence theorem, and so we get Dirac's condition,

$$\begin{aligned} &\|[Q^w_\hbar(A), Q^w_\hbar(B)]_\hbar - Q^w_\hbar(\{A, B\})\|_1 \\ &\leq \sum_{k,l=1}^{\infty} |u_k|\, |v_l|\, \big\| [Q^w_\hbar(W^0(f)), Q^w_\hbar(W^0(g))]_\hbar \\ &\quad - Q^w_\hbar(\{W^0(f), W^0(g)\}) \big\|_1 \stackrel{\hbar \to 0}{\longrightarrow} 0\,. \end{aligned}$$

Part (b) for case (2). The injectivity of $Q^w_\hbar$ follows as above. It remains to prove that $Q^w_\hbar(A) Q^w_\hbar(B) \in Q^w_\hbar(\mathcal{P}^\infty_\varsigma)$ for all $A, B \in \mathcal{P}^\infty_\varsigma$. Since $Q^w_\hbar(A) Q^w_\hbar(B) = Q^w_\hbar(C)$ with $C \in \overline{\Delta(E,0)}^1$ defined by $C := \sum_{k,l} w(\hbar, f_k) w(\hbar, g_l) \exp\{-\frac{i}{2}\hbar\sigma (f_k, g_l)\} u_k v_l W^0(f_k + g_l)$ for our above $A = \sum_k u_k W^0(f_k)$ and $B = \sum_l v_l W^0(g_l)$, we only have to show that $C \in \mathcal{P}^\infty_\varsigma$. Up to the terms $w(\hbar, f_k) w(\hbar, g_l) \exp\{-\frac{i}{2}\hbar\sigma(f_k, g_l)\}$, this is just the commutative product of A and B from Subsection 3.3. Since these terms are uniformly bounded in the test functions f_k and g_l ($|w(\hbar, f)| \leq a_0$ for all $f \in E$ with some constant $a_0 > 0$ by the $\hbar$-local boundedness of w), we conclude from the structure of the norms (3.9) that estimation (3.10) remains valid, i.e., $\|C\|^n_\varsigma \leq a_0^2 c_n \|A\|^n_\varsigma \|B\|^n_\varsigma$ for all $n \in \mathbb{N}$, and find that in fact $C \in \mathcal{P}^\infty_\varsigma$. □

Clearly, when we restrict, for given $\hbar$-locally bounded factor w, the quantization maps Q^w and $Q^w_\hbar$ to an arbitrary sub-Poisson algebra $\widetilde{\mathcal{P}}$ with $\Delta(E,0) \subset$

$\tilde{\mathcal{P}} \subset \mathcal{P}_\varsigma^\infty$ (proper inclusions), then we arrive again at a continuous quantization Q^w, which provides, however, only a strict quantization, but in general not a strict deformation quantization $(Q_\hbar^w)_{\hbar\in\mathbb{R}}$ of $(\tilde{\mathcal{P}}, \{.,.\})$ by the following reasoning: On $\tilde{\mathcal{P}}$ the $Q_\hbar^w$ act injectively, too, but $Q_\hbar^w(A)Q_\hbar^w(B)$ possibly may not be an element of $Q_\hbar^w(\tilde{\mathcal{P}})$ for some $A, B \in \tilde{\mathcal{P}}\backslash\Delta(E,0)$, which in view of Definition 2.1(d) prevents $(Q_\hbar^w)_{\hbar\in\mathbb{R}}$ from being a strict deformation quantization of $\tilde{\mathcal{P}}$.

Remark 4.5 (Weyl Quantization). The results on the Weyl quantization of [14] follow as a special case from the foregoing Theorem by selecting the trivial quantization factor $w \equiv 1$. Especially, we have $Q^1(A) \in \overline{\Delta_{\mathrm{WF}}(E,\sigma)}^1$ for all $A \in \overline{\Delta(E,0)}^1$.

Since by Proposition 4.2 we have for every quantization factor w the same continuous field of C*-Weyl algebras $\mathcal{K}$, we immediately conclude from Observation 2.7:

Theorem 4.6 (Equivalent Quantizations)**.** *Let all be as in Theorem* 4.4*. For each of the two cases* (1) *and* (2) *it holds*: *The strict deformation quantizations* $(\mathcal{W}(E,\hbar\sigma), Q_\hbar^w)_{\hbar\in\mathbb{R}}$ *of* $(\mathcal{P}, \{.,.\})$*, where w varies in the pertinent* Γ*, are mutually equivalent in the sense of Definition* 2.6*. Especially, each of them is equivalent to the Weyl quantization.*

The above conclusions may be restricted to some smaller subsets I than $\mathbb{R}$:

Corollary 4.7. *Suppose* $w \in \mathrm{QF}$ *to be only* $\hbar$*–locally bounded for* $\hbar$ *from a subset* $I \subset \mathbb{R}$ *as in Section* 2*. Then Theorem* 4.4(b) *remains valid for the enlarged Poisson algebra* $\mathcal{P} = \mathcal{P}_\varsigma^\infty$ *when restricting from* $\mathbb{R}$ *to* I*. That is, beside* $(\mathcal{W}(E,\hbar\sigma), Q_\hbar^w)_{\hbar\in\mathbb{R}}$ *being a strict deformation quantization of* $(\Delta(E,0), \{.,.\})$*, it follows that* $(\mathcal{W}(E,\hbar\sigma), Q_\hbar^w)_{\hbar\in I}$ *is a strict deformation quantization of* $(\mathcal{P}_\varsigma^\infty, \{.,.\})$*.*

The equivalence Theorem 4.6 *is valid for* $\mathcal{P} = \mathcal{P}_\varsigma^\infty$*, too, when restricting to* I*, and when* Γ *consists of those* $w \in \mathrm{QF}$ *which are* $\hbar$*–locally bounded within* I*.*

Theorem 4.4(a) *also may be formulated for* I *instead of* $\mathbb{R}$; *we omit details.*

A quantization factor $w \in \mathrm{QF}$ may be transformed by means of Bogoliubov *-automorphisms $\alpha_T^\hbar$ on $\mathcal{W}(E,\hbar\sigma)$, which by construction satisfy

$$\alpha_T^\hbar(W^\hbar(f)) = W^\hbar(Tf)\,, \quad \forall f \in E\,,$$

for each $\hbar \in \mathbb{R}$, respectively. Here T is an element of the group $\mathrm{symp}(E,\sigma)$ of all symplectic transformations on the pre-symplectic space (E,σ). (Recall that $T \in \mathrm{symp}(E,\sigma)$ is an $\mathbb{R}$-linear bijection on E with $\sigma(f,g) = \sigma(Tf,Tg)$; [24]). With Eq. (4.2) it is immediately checked that for each $T \in \mathrm{symp}(E,\sigma)$ and every $\hbar \in \mathbb{R}$ one gets

$$(\alpha_T^\hbar)^{-1} \circ Q_\hbar^w \circ \alpha_T^0 = Q_\hbar^{w^T},$$

where the quantization factor $w^T \in \mathrm{QF}$ is given by

$$w^T(\hbar, f) := w(\hbar, Tf)\,, \quad \forall f \in E\,, \quad \forall \hbar \in \mathbb{R}\,.$$

The strict deformation quantization $(\mathcal{W}(E,\hbar\sigma), Q_\hbar^w)_{\hbar\in\mathbb{R}}$ of $(\Delta(E,0),\{.,.\})$ is in this manner transformed into the strict deformation quantization $(\mathcal{W}(E,\hbar\sigma), Q_\hbar^{w^T})_{\hbar\in\mathbb{R}}$ of the Poisson algebra $(\Delta(E,0),\{.,.\})$. Similar deformations of the quantization factor arise from gauge transformations of the second kind $\alpha_{F^\hbar}^\hbar (W^\hbar(f)) := \exp\{iF^\hbar(f)\}\,W^\hbar(f)$ with $F^\hbar \in E'_\tau$ a real smeared field, and by certain completely positive mappings (e.g., [34]), which depend continuously on $\hbar$. These constructions may be extended to the enlarged Poisson algebra $\mathcal{P}_\varsigma^\infty$ only under certain boundedness resp. continuity conditions on w and T resp. $F^\hbar$. (The gauge transformations do not preserve the field-compatibility of w given in Definition 5.2 below).

Let us sketch a connection to the equivalence concept of 2-cocycles in the theory of central group extensions (e.g., [25]). For our given pre-symplectic space (E,σ) one easily checks that for each $w \in \mathrm{QF}$ the map

$$\Sigma_\hbar^w : E \times E \to \mathbb{C}'\,, \quad (f,g) \mapsto \frac{w(\hbar,f)w(\hbar,g)}{w(\hbar,f+g)} \exp\{-\tfrac{i}{2}\hbar\sigma(f,g)\}$$

is a 2-cocycle over the multiplicative complex group $\mathbb{C}' := \mathbb{C}\backslash\{0\}$. This covers Eq. (3.1) for the trivial factor $w \equiv 1$, corresponding to the Weyl quantization, cf., Remark 4.5. One may introduce something like a generalized Heisenberg group $H(\Sigma_\hbar^w)$ by extending the additive group E centrally by $\mathbb{C}'$ via $\Sigma_\hbar^w$, which is performed by equipping the cartesian product $E \times \mathbb{C}'$ with the group multiplication

$$(f,u) \bullet_w (g,v) := (f+g, u\,v\,\Sigma_\hbar^w(f,g))\,.$$

(If σ and w are smooth in a τ-topology, then $H(\Sigma_\hbar^w)$ may be made to a possibly infinite dimensional Lie group, as is, e.g., carried through in [38] for nuclear symplectic spaces.)

Observation 4.8 (Equivalent Cocycles and Isomorphic Heisenberg Groups). *The 2-cocycles $\Sigma_\hbar^w$, where w varies in* QF, *are mutually equivalent. Thus the group extensions $H(\Sigma_w^\hbar)$, $w \in$* QF, *are isomorphic.*

For every quantization factor $w \in \mathrm{QF}$ *the corresponding quantized Weyl elements*

$$Q_w^\hbar(W^0(f)) = w(\hbar, f)W^\hbar(f) =: W_w^\hbar(f)$$

satisfy the modified Weyl relations

$$W_w^\hbar(f)W_w^\hbar(g) = \Sigma_\hbar^w(f,g)W_w^\hbar(f+g)\,, \quad W_w^\hbar(f)^* = W_w^\hbar(-f)\,.$$

Observation 4.8 illustrates, that a Heisenberg group is not a characterization of a quantization proper, being the same for all quantizations in our investigation, but characterizes, in a certain sense, the algebraic frame for quantizations.

§5. Field Quantizations

The preceding discussions on strict quantization dealt with the abstract C*-Weyl algebras. Many investigations in the literature, which concern deformation or even strict quantization methods, use special Hilbert space representations. In order to find the connection to usual quantum field theory one applies Stone's theorem to the Weyl elements, unitarily represented in a Hilbert space. That is, one differentiates for $\hbar \neq 0$ in a regular representation $\Pi_\hbar$ of $\mathcal{W}(E, \hbar\sigma)$ the unitaries $\Pi_\hbar(W^\hbar(tf))$ to the parameter $t \in \mathbb{R}$:

$$\Phi_\Pi^\hbar(f) \equiv \Phi_{\Pi_\hbar}(f) := -i\,\frac{d}{dt}\Pi_\hbar(W^\hbar(tf))\bigg|_{t=0}\,, \quad f \in E\,, \tag{5.1}$$

(regularity means that for each $f \in E$ the map $\mathbb{R} \ni t \mapsto \Pi_\hbar(W^\hbar(tf))$ is strongly continuous, cf. [35]). The field operators $\Phi_\Pi^\hbar(f)$, $f \in E$, and thus the physical contents of the theory, may differ essentially from each other in the various representations of $\mathcal{W}(E, \hbar\sigma)$. Perturbation theory is in a certain sense connected with series of increasing field powers and thus should, in a mathematical realization, depend on the special representation.

Our C*-algebraic quantization theory of Section 4 provides a strategy for a representation dependent quantization, which then is used to quantize also field expressions. Here we follow this procedure of field quantizations. The forthcoming publication [20] is devoted to the converse process, which is more in the spirit of quantum field theory. There quantizations are formulated directly in terms of fields, which under certain conditions may give rise to the previous C*-algebraic quantizations using Weyl elements.

Let us select for the following a fixed family $\Pi \equiv (\Pi_\hbar)_\hbar$ of regular, non-degenerate, and faithful representations $\Pi_\hbar$ of $\mathcal{W}(E, \hbar\sigma)$, $0 \neq \hbar \in \mathbb{R}$. Recall that the algebras $\mathcal{W}(E,\hbar\sigma)$ are simple for non–degenerate σ, and so all representations are faithful. Observe the norm preservation of any faithful representation. For dealing with the fields it may be advantageous to choose the elements of a representation family Π compatible with each other by setting $\Pi_\hbar := \Pi_{\hbar\equiv 1} \circ \beta_\hbar$ with suitable *-isomorphisms $\beta_\hbar$ from $\mathcal{W}(E,\hbar\sigma)$ onto $\mathcal{W}(E,\sigma)$ for the values $\hbar \neq 0$. The next result gives a scheme for constructing such $\beta_\hbar$, which is further expounded in [37].

Lemma 5.1. *For $\hbar \neq 0$ let $T_\hbar : E \to E$ be an $\mathbb{R}$-linear bijection such that $\sigma(T_\hbar f, T_\hbar g) = \hbar\sigma(f,g)$ for all $f,g \in E$. Then there exists a unique *-isomorphism $\beta_\hbar$ from $\mathcal{W}(E,\hbar\sigma)$ onto $\mathcal{W}(E,\sigma)$ such that $\beta_\hbar(W^\hbar(f)) = W^1(T_\hbar f)$ for all $f \in E$.*

To formulate quantizations of field observables we have need for a special subclass of quantization factors.

Definition 5.2 (Field-Compatible Quantization Factor)**.** The quantization factor $w \in \mathrm{QF}$ is called to be field-compatible, if for each $\hbar \in \mathbb{R}$ and all $f,g \in E$ the mapping $\mathbb{R} \ni t \mapsto w(\hbar, tf+g)$ is infinitely differentiable, and if $\frac{d}{dt} w(\hbar, tf)|_{t=0} = 0$. The set of field-compatible quantization factors is denoted by $\mathrm{QF}_{\mathrm{fc}}$, and that of $\hbar$-locally bounded, field-compatible quantization factors by $\mathrm{QF}_{\mathrm{b,fc}}$.

Beside the family Π, let us choose a fixed field-compatible quantization factor $w \in \mathrm{QF}_{\mathrm{fc}}$. Then the Π-dependent quantization mappings $Q^w_{\Pi,\hbar} : \mathcal{P} \to \Pi_\hbar(\mathcal{W}(E,\hbar\sigma))$ are defined for each $\hbar \neq 0$ similarly to Eq. 4.2, namely by the linear and possibly $\|.\|_1$-continuous extension of

$$Q^w_{\Pi,\hbar}(W^0(f)) := w(\hbar,f)\Pi_\hbar(W^\hbar(f)), \quad \forall f \in E. \tag{5.2}$$

If $w \in \mathrm{QF}_{\mathrm{fc}}$ is not $\hbar$-locally bounded, then only $\mathcal{P} = \Delta(E,0)$ is possible, but for $w \in \mathrm{QF}_{\mathrm{b,fc}}$ also $\mathcal{P} = \mathcal{P}^\infty_\varsigma$ is allowed, in accordance with the cases (1) and (2) from Section 4.

Being formulated in terms of the representation independent norm topology the previous $\hbar$-asymptotic results on the $Q^w_\hbar$ take over to the $Q^w_{\Pi,\hbar}$. In a completely analogous manner as before one may introduce a continuous family $(\{\Pi_\hbar(\mathcal{W}(E,\hbar\sigma))\}_{\hbar\in I}, \mathcal{K}_\Pi)$ of represented C*-Weyl algebras and a global quantization map $Q^w_\Pi : \mathcal{P} \to \mathcal{K}_\Pi$.

Proposition 5.3. *Let $\Pi \equiv (\Pi_\hbar)_\hbar$ be an arbitrary family of regular, non-degenerate, and faithful representations $\Pi_\hbar$ of $\mathcal{W}(E,\hbar\sigma)$, $0 \neq \hbar \in \mathbb{R}$. Then the following assertions are valid:*

(a) $(\{\Pi_\hbar(\mathcal{W}(E,\hbar\sigma))\}_{\hbar\in I}, \mathcal{K}_\Pi; Q^w_\Pi)$ *constitutes a continuous quantization of* $(\mathcal{P}, \{.,.\})$.

(b) $(\Pi_\hbar(\mathcal{W}(E,\hbar\sigma)), Q^w_{\Pi,\hbar})_{\hbar\in\mathbb{R}}$ *constitutes a strict quantization of* $(\mathcal{P}, \{.,.\})$.

For discussing field expressions let us employ in the classical case $\hbar = 0$ the realization of $\mathcal{W}(E,0)$ by almost periodic functions on some phase space manifold E'_τ, known from Subsection 3.3. Here the field observable functions $\Phi^0(f) : E'_\tau \to \mathbb{R}$, $F \mapsto F(f)$ are obtained in a similar way as the quantum fields in Eq. (5.1), namely by pointwise differentiation

$$\Phi^0(f)[F] := -i\,\frac{d}{dt} W^0(tf)[F]\bigg|_{t=0} = F(f)\,, \quad \forall F \in E'_\tau\,,$$

of the periodic phase space functions $W^0(f)$ from (3.11).

As a guiding idea we approximate quantized higher field powers by bounded elements of the represented Weyl algebra analogously as for the classical field expressions, where for the latter one has

$$\Phi^0(f_1)\cdots\Phi^0(f_n) = (-i)^n\,\frac{\partial^n}{\partial t_1\cdots\partial t_n} W^0(\sum_{k=1}^n t_k f_k)\bigg|_{t_1=\cdots=t_n=0}.$$

This suggests for each value $\hbar \neq 0$ the prolongation of the quantization mapping $Q^w_{\Pi,\hbar}$ from (5.2) to unbounded quantum field polynomials by the linear extension of

(5.3)

$$\begin{aligned} Q^w_{\Pi,\hbar}(\Phi^0(f_1)\cdots\Phi^0(f_n)) &:= (-i)^n\,\frac{\partial^n}{\partial t_1\cdots\partial t_n} Q^w_{\Pi,\hbar}\big(W^0(\sum_{k=1}^n t_k f_k)\big)\bigg|_{t_1=\cdots=t_n=0} \\ &= (-i)^n\,\frac{\partial^n}{\partial t_1\cdots\partial t_n} w(\hbar, \sum_{k=1}^n t_k f_k)\Pi_\hbar\big(W^\hbar(\sum_{k=1}^n t_k f_k)\big)\bigg|_{t_1=\cdots=t_n=0}, \end{aligned}$$

where $n \in \mathbb{N}$ resp. the $f_k \in E$ are arbitrary, and the t_k are real differentiation parameters. Since $\frac{d}{dt} w(\hbar, tf)|_{t=0} = 0$ by Definition 5.2, one realizes for $n = 1$ that it holds

$$Q^w_{\Pi,\hbar}(\Phi^0(f)) = \Phi^\hbar_\Pi(f)\,, \quad \forall f \in E\,,$$

the so-called 'field compatibility' of w.

Differentiating the Weyl relations one arrives at the $\hbar$-scaled commutators (defined in Eq. (2.4)) for the quantum fields,

$$[\Phi^\hbar_\Pi(f), \Phi^\hbar_\Pi(g)]_\hbar = -\sigma(f,g)\mathbb{1}^\hbar_\Pi\,, \quad \forall f, g \in E\,,$$

the original canonical commutation relations (CCR). With (3.12) it is immediate to check that the classical field observables realize the Poisson bracket relations

$$\{\Phi^0(f), \Phi^0(g)\} = -\sigma(f,g)\mathbb{1}^0 \,, \quad \forall f, g \in E \,,$$

where $\mathbb{1}^0[F] \equiv 1$ is the unit function on E'_τ. These two formulas are formally in accordance with the Dirac condition in the stationary sense.

The higher derivatives in Eq. (5.3) lead to modified products of the quantized fields, such as may arise by various kinds of normal ordering (e.g., [20]).

Observe that for each field polynomial only a finite set of test functions is involved. It is known that for a finite dimensional subspace M of E there always exists a common subspace of entire analytic vectors for the selfadjoint field operators $\Phi_\Pi^\hbar(f)$ with $f \in M$, which is dense in the representation Hilbert space of $\Pi_\hbar$.

Since the Weyl relations and the quantizations factors do not depend on the representation family Π the preceding definitions of the quantized field monomials lead — up to the domains of definition — to isomorphic commutators, for all $\Pi_\hbar$, $\hbar \neq 0$. In this sense there is inherent, also for field expressions, a common algebraic structure. Fields $\Phi_\Pi^\hbar(f)$, with f in the null space of σ, commute with all other field expressions and thus constitute superselection observables. They are affiliated with the center of each representation von Neumann algebra, but in general, these centers contain also non-trivial limits of bounded functions of the fields.

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References

[1] Bayen, F., Flato, M., Fronsdal, C., Lichnerovicz, A. and Sternheimer, D., Deformation theory and quantization, *J. Operator Theory*, **3** (1980), 237-269.

[2] DeWilde, M. and Lecompte, P. B. A., Existence of star-products and of formal deformations of a Poisson Lie algebra of arbitrary symplectic manifolds, *Lett. Math. Phys.*, **7** (1983), 487-496.

[3] ———, Formal deformations of a Poisson Lie algebra of a symplectic manifold and star products. existence, equivalence, derivations, In M. Hazewinkel and M. Gerstenhaber eds., *Deformation Theory of Algebras and Structures and Applications*, Kluwer, Dordrecht, 1988.

[4] Omori, H., Maeda, Y. and Yoshioka, A., Existence of a closed star product, *Lett. Math. Phys.*, **26** (1992), 285-294.

[5] Weinstein, A., Deformation quantization, *Séminaire Bourbaki 46ème annee*, 789, (1994).

[6] Fedosov, B. V., A simple geometric construction of deformation quantization, *J. Differential Geom.*, **40** (1994), 213-238.

[7] Fedosov, B., *Deformation Quantization and Index Theory*, Academie-Verlag, Berlin, 1996.

[8] Bordemann, M., Römer, H. and Waldmann, S., KMS-states and star product quantization, *Rep. Math. Phys.*, **44** (1999).

[9] Rieffel, M. A., Deformation quantization for actions of $\mathbb{R}^d$, *Mem. Amer. Math. Soc.*, **106** (1993).

[10] ———, Quantization and C*-algebras, In R. S. Doran, ed., *C*-Algebras*: 1943–1993, pages 67-97. Contemp. Math. **167**, Providence, RI, Amer. Math. Soc., 1994.

[11] ———, Questions on quantization, *Berkeley*, quant-ph/9712009, 1998.

[12] Landsman, N. P., Strict quantization of coadjoint orbits, *Amsterdam*, math-ph/9807027, 1998.

[13] ———, *Mathematical Topics Between Classical and Quantum Mechanics*, Springer, New York, 1998.

[14] Binz, E., Honegger, R. and Rieckers, A., Field-theoretic Weyl quantization as a strict and continuous deformation quantization, *Ann. Henri Poincaré*, **5** (2004), 327-346.

[15] Dirac, P. A. M., Quantum mechanics, *Proc. Roy. Soc. A*, **110** (1926), 561-579.

[16] ———, On quantum algebra, *Proc. Cambr. Phil. Soc.*, **23** (1926), 412-418.

[17] Titulaer, U. M. and Glauber, R. J., Correlation functions and coherent fields, *Phys. Rev.*, **140** (1965), B676-B682.

[18] Titulaer, U. M. and Glauber, R. J., Density operators for coherent fields, *Phys. Rev.*, **145** (1966), 1041-1050.

[19] Vogel, W., Welsch, D.-G. and Wallentowitz, S., *Quantum Optics, An Introduction.* WILEY-VCH, Berlin, 2001.

[20] Honegger, R. and Rieckers, A., Direct field quantizations and strict deformation quantizations, *Preprint Mannheim, Tübingen*, 2003.

[21] Libermann, P. and Marle, C.-M., *Symplectic Geometry and Analytical Mechanics*, D. Reidel Publ. Co, Dordrecht, 1987.

[22] Dirac, P. A. M., Quantum mechanics and a preliminary investigation of the hydrogen atom, *Proc. Roy. Soc. A*, **110** (1926), 561-579.

[23] Dixmier, J., *C*-Algebras*, North-Holland, Amsterdam, 1977.

[24] Binz, E., Honegger, R. and Rieckers, A., Construction and uniqueness of the C*-Weyl algebra over a general pre-symplectic form, *J. Math. Phys.*, **45** (2004), 2885-2907.

[25] Edwards, C. M. and Lewis, J. T., Twisted group algebras I, II. *Comm. Math. Phys.*, **13** (1969), 119-141.

[26] Pedersen, G. K., *C*-Algebras and their Automorphism Groups*, Academic Press, London, 1979.

[27] Packer, J. A. and Raeburn, I., Twisted crossed products of C*-algebras, *Math. Proc. Cambridge Philos. Soc.*, **106** (1989), 293-311.

[28] Grundling, H., A group algebra for inductive limit groups. continuity problems of the canonical commutation relations, *Acta Appl. Math.*, **46** (1997), 107-145.

[29] Araki, H. and Woods, E. J., Representations of the Canonical Commutation Relations describing a nonrelativistic infinite free Bose gas, *J. Math. Phys.*, **5** (1963), 637-662.

[30] Miracle-Sole, S., Traitment de la convolution gauche pour les systèmes infinis, *Ann. Inst. Henri Poincaré*, **VI** (1967), 59-71.

[31] Manuceau, J., C*-algèbre de relations de commutation, *Ann. Inst. Henri Poincaré*, **VIII** (1968), 139-161.

[32] Slawny, J., On factor representations and the C*-algebra of the canonical commutation relations, *Comm. Math. Phys.*, **24** (1971), 151-170.

[33] Manuceau, J., Sirugue, M., Testard, D. and Verbeure, A., The smallest C*-algebra for canonical commutation relations, *Comm. Math. Phys.*, **32** (1973), 231-243.

[34] Evans, D. E. and Lewis, J. T., *Dilations of Irreversible Evolutions in Algebraic Quantum Theory*, volume 24 of *Communications of the Dublin Institute of Advanced Studies Series A* (*Theoretical Physics*), Dublin Institute for Advanced Studies, Dublin, 1977.

[35] Bratteli, O. and Robinson, D. W., *Operator Algebras and Quantum Statistical Mechanics*, **II**. Springer-Verlag, New York, 1981.

[36] Riesz, F. and Sz.-Nagy, B., *Vorlesungen über Funktionalanalysis*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1982.

[37] Binz, E., Honegger, R. and Rieckers, A., Field-theoretic Weyl quantization of large Poisson algebras, *Preprint Mannheim, Tübingen*, 2003.

[38] Hida, T., *Brownian Motion*, Springer-Verlag, New York, Berlin, 1980.