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Smoothness of Solutions for Schrödinger Equations with Unbounded Potentials

By

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Dedicated to Professor Kunihiko Kajitani on his sixty-second birthday

Abstract

We consider a Schrödinger equation with linearly bounded magnetic potentials and a quadratically bounded electric potential when the coefficients of the principal part do not necessarily converge to constants near infinity. Assuming that there exists a suitable function $f(x)$ near infinity which is convex with respect to the Hamilton vector field generated by the (scalar) principal symbol, we show a microlocal smoothing effect, which says that the regularity of the solution increases for all time $t \in (0, T]$ at every point that is not trapped backward by the geodesic flow if the initial data decays in an incoming region in the phase space. Here T depends on the potentials; we can choose $T = \infty$ if the magnetic potentials are sublinear and the electric potential is subquadratic. Our method regards the growing potentials as perturbations; so it is applicable to matrix potentials as well.

§1. Introduction

Let $H(t)$ be a time dependent Schrödinger operator acting on $\mathbf{C}^n$-valued functions:

$$H(t) = \sum_{j,k=1}^{d} (D_j - a_j(t,x)) g^{jk}(x)(D_k - a_k(t,x)) + V(t,x), \quad (t,x) \in \mathbf{R} \times \mathbf{R}^d.$$

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Here $D_j = -i\partial_j = -i\partial/\partial x_j$; $M_n(\mathbf{C})$ is the space of all $n \times n$ complex matrices; $g^{jk} = g^{kj} \in C^\infty(\mathbf{R}^d, \mathbf{R})$, and $(g^{jk}(x))$ is positive definite for each x; $\partial_x^\alpha a_j$, $\partial_x^\alpha V \in C(\mathbf{R}_t \times \mathbf{R}_x^d, M_n(\mathbf{C}))$ for all $\alpha \in \mathbf{N}_0^d$, and $a_j(t,x)$, $V(t,x)$ are Hermitian matrices for each (t,x).

Under suitable conditions, the Cauchy problem for the Schrödinger equation

$$(\partial_t + iH(\cdot))u = 0 \text{ in } \mathcal{D}'(\mathbf{R} \times \mathbf{R}^d, \mathbf{C}^n), \quad u(t_0) = u_0,$$

is well-posed in the scale of spaces associated with the oscillator $H_{\mathrm{osc}} = 1 - \Delta + |x|^2$. Let $S(t,t_0)$ $(t, t_0 \in \mathbf{R})$ denote the propagator, or the solution operator. This paper is concerned with the smoothing effect of $S(t,t_0)$ and the smoothness of its distribution kernel $K(t,t_0,x,y)$ under general conditions on the coefficients, when

(a) $c_1 I_d \le (g^{jk}(x)) \le c_2 I_d$ on $\mathbf{R}^d$ for some $c_1, c_2 > 0$,

(b) $(g^{jk}(x))$ does not necessarily converge to a constant matrix, and

(c) $|a_j(t,x)| = O(|x|)$ and $|V(t,x)| = O(|x|^2)$ as $|x| \to \infty$ uniformly on every compact time interval.

Remark. If $\mathbf{R}^d$ has a positive density $v(x)dx$, $v \in C^\infty(\mathbf{R}^d)$, it is natural to consider the Schrödinger operator of the following form

$$\tilde{H}(t) = v(x)^{-1} \sum_{j,k=1}^{d} (D_j - a_j(t,x)) v(x) g^{jk}(x) (D_k - a_k(t,x)) + \tilde{V}(t,x),$$

where $\tilde{V}$ is a Hermitian potential like V. Then $v(x)^{1/2}\, \tilde{H}(t)\, v(x)^{-1/2} = H(t)$ with $V(t,x) = \tilde{V}(t,x) + (\frac{1}{2}\Delta_{g,v} \log v(x) - \frac{1}{4} g_x(d\log v, d\log v)) I_n$. Here for $f \in C^\infty(\mathbf{R}^d)$ we set $\Delta_{g,v} f(x) = v(x)^{-1} \sum_{j,k=1}^d \partial_j \left(v(x) g^{jk}(x) \partial_k f(x) \right)$ and $g_x(df, df) = \sum_{j,k=1}^d g^{jk}(x)(\partial_j f(x))(\partial_k f(x))$.

What are our difficulties? When $(g^{jk}(x)) = (\delta^{jk})$, the previous works have regarded the potentials of the maximal order in (c) as part of the principal part and used the Hamilton flow of this "principal symbol" to construct important operators such as the fundamental solution, a parametrix, and a conjugate operator; this construction has called for deriving detailed estimates of the Hamilton flow, which has required stronger conditions on the derivatives of the potentials. When $(g^{jk}(x))$ does not converge to a constant matrix as $|x| \to \infty$, the nontrapped bicharacteristic curve of the principal symbol $h_0(x,\xi) = \sum_{j,k=1}^d g^{jk}(x)\xi_j\xi_k$ has no asymptotic velocity in general, because the short-range condition, $|\nabla_x g^{jk}(x)| = O(|x|^{-1-\varepsilon})$ as $|x| \to \infty$ for some $\varepsilon > 0$, fails; so it seems hopeless to derive detailed estimates, or precise asymptotic behavior, of the Hamilton flow of the "principal symbol" when the maximally growing potentials are present. When $n \geq 2$, the "principal symbol" is no more scalar, and hence the Hamilton flow cannot be defined. These are typical difficulties.

Our remedy is simple: we should regard the potentials of order (c) as *perturbations* and use only qualitative properties of the Hamilton flow of the principal symbol. To control the asymptotic behavior of the Hamilton flow, we assume that there exists a suitable strictly convex function $f_{cv} \in C^\infty(\mathbf{R}^d)$ near infinity with respect to the Hamilton vector field $H_{h_0} = \sum_{j=1}^d \left(\frac{\partial h_0}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial h_0}{\partial x_j}\frac{\partial}{\partial \xi_j}\right)$. Then we can regard the potentials of order (c) as perturbations, not for all $t \in \mathbf{R}$ in general, but for all $t \in [0, T]$. Here $T > 0$ is the largest number satisfying

$$T \cdot \lim_{R\to\infty} \sup_{t\in[0,T],|x|\geq R} \sum_{j=1}^d |\nabla_x a_j(t,x)| + T^2 \cdot \lim_{R\to\infty} \sup_{t\in[0,T],|x|\geq R} \frac{|\nabla_x V(t,x)|}{|x|} \leq c$$

for a constant $c = c(d, h_0, f_{cv}) > 0$ independent of the potentials. On this interval, we use a kind of positive commutator method by constructing a conjugate operator as a time dependent scalar pseudodifferential operator whose symbol is an explicit function of h_0, $r = \sqrt{f_{cv}}$, and their Poisson bracket $\{h_0, r\} := H_{h_0} r$. Thus we need no detailed estimates of the Hamilton flow of either the principal symbol or the "principal symbol" (the latter should have been scalar, because the Hamilton flow of a matrix-valued function makes no sense in general). As a by-product, we can largely relax the conditions on the derivatives of the coefficients and handle the matrix potentials as well.

Why can we regard the potentials as perturbations? We shall heuristically explain this when $n = 1$ and $(g^{jk}(x)) = I_d$ outside a compact set (then we can choose $r(x) = \langle x \rangle := \sqrt{1 + |x|^2}$). Let $h(t)$ be the Weyl symbol of $H(t)$ and Φ_{ts} the (2-parameter) Hamilton flow of $h(t)$. Let $K(t)$ be an invertible, time dependent pseudodifferential operator with Weyl symbol $k(t, x, \xi) = e^{\lambda(t,x,\xi)}$ for a nonnegative symbol λ $(0 \leq t \leq T)$. Under suitable conditions, we have

$$K(t)(\partial_t + iH(t))K(t)^{-1} = \partial_t + iH(t) + P(t) + Q(t),$$

where the Weyl symbol of $P(t)$ is $-(\partial_t \lambda(t) + H_{h(t)}\lambda(t))$, and $Q(t)^* + Q(t)$ is bounded. Setting $u(t) = S(t, 0)u_0$, we can show the estimate

$$\|K(t)u(t)\|^2 + \int_0^t (P(\tau)K(\tau)u(\tau), K(\tau)u(\tau))d\tau \leq C\|K(0)u_0\|^2, \quad t \in [0, T],$$

for a constant $C > 0$ independent of u_0 and $t \in [0, T]$. If $-(\partial_t \lambda(t) + H_h \lambda(t))$ is bounded from below, then we can obtain an effective microlocal estimate of

$u(\cdot)$ in the set $A_T = \{(t,x,\xi) \in [0,T] \times T^*\mathbf{R}^d \setminus \{0\}; \lambda(t,x,\xi) > 0\}$. Therefore we require A_T to be backward invariant under the (2-parameter) Hamilton flow of $h(t)$: $\Phi_{ts}(x,\xi) \in A_T$ if $(s,x,\xi) \in A_T$ and $0 \le t \le s \le T$. Sometimes we can replace $h(t)$ by another "principal symbol" in requiring the last condition. This is the case where the potentials are bounded with additional conditions on the derivatives. Then we can choose $A_T = [0,T] \times S$, where

$$S = \left\{ (x,\xi) \in T^*\mathbf{R}^d \setminus 0; \langle x \rangle > R',\ \frac{x \cdot \xi}{\langle x \rangle |\xi|} < -\delta' \right\}\ \ (R' \gg 1,\ 0 < \delta' \ll 1)$$

is backward invariant under the Hamilton flow of $|\xi|^2$. However, when the potentials are unbounded as in (c), we cannot control the order of $\langle x \rangle$ on $[0,T] \times S$. So we require that $\langle x \rangle \le CT|\xi|$ on A_T for a constant $C > 0$ independent of T. In fact, we can choose

$$A_T = \left\{ (t,x,\xi) \in [0,T] \times T^*\mathbf{R}^d \setminus 0; R' < \langle x \rangle < 5(2T-t)|\xi|,\ \frac{x \cdot \xi}{\langle x \rangle |\xi|} < -\delta' \right\}.$$

Then this set is backward invariant under the (2-parameter) Hamilton flow of $h(t)$ if $c(d, h_0, f_{cv})$ is sufficiently small. On this set, we can compare the order of the potentials with that of the principal part, because $\langle x \rangle \le 10T|\xi|$ holds there. Therefore we can regard the potentials as perturbations when $c(d, h_0, f_{cv})$ is sufficiently small.

Let us write the operator $H(t)$ in the following form:

$$H(t) = \left(\sum_{j,k=1}^{d} D_j g^{jk}(x) D_k \right) I_n - \sum_{j=1}^{d} (a^j(t,x) D_j + D_j a^j(t,x)) + b(t,x);$$

$$a^j(t,x) = \sum_{k=1}^{d} g^{jk}(x) a_k(t,x), \quad b(t,x) = V(t,x) + \sum_{j,k=1}^{d} a_j(t,x) g^{jk}(x) a_k(t,x).$$

Then the Weyl symbol $h(t)$ of $H(t)$ is

$$\begin{aligned} h(t,x,\xi) &= h_0(x,\xi) I_n + h_1(t,x,\xi) + h_2(t,x,\xi); \\ h_0(x,\xi) &= \sum_{j,k=1}^{d} g^{jk}(x) \xi_j \xi_k, \quad h_1(t,x,\xi) = -2 \sum_{j=1}^{d} a^j(t,x) \xi_j, \\ h_2(t,x,\xi) &= h_2(t,x) = b(t,x) + \frac{1}{4} \sum_{j,k=1}^{d} \partial_j \partial_k g^{jk}(x)\, I_n. \end{aligned}$$

We recall related results when the operator is scalar ($n = 1$).

(i) Assume $g^{jk}(x) = \delta^{jk}$ and that with some $\varepsilon > 0$

$$|\partial_x^\alpha a_j(t,x)| + |\partial_x^\alpha(\partial_t a_j(t,x) + \partial_j V(t,x))| \le C_\alpha, \quad t \in \mathbf{R}, x \in \mathbf{R}^d,$$
$$|\partial_x^\alpha(\partial_k a_j(t,x) - \partial_j a_k(t,x))| \le C'_\alpha(1+|x|)^{-1-\varepsilon}, \quad t \in \mathbf{R}, x \in \mathbf{R}^d,$$

for all $\alpha \in \mathbf{N}_0^d$ with $|\alpha| \ge 1$. Then $K(t,s,x,y)$ is C^∞ in x, y when $0 < |t-s| \le T$ for some $T > 0$ (see [6] when $a_j = 0$ and [24, 25] in the general case). Remark that V can be eliminated by the change of the unknown function: $u(t,x) \mapsto v(t,x) = u(t,x)\exp(i\int_0^t V(\tau,x)d\tau)$.

(ii) Assume $g^{jk}(x) = \delta^{jk}$, $a_j = 0$, and

$$\lim_{R\to\infty} \sup_{t\in\mathbf{R}, |x|\le R} |\partial_x^\alpha V(t,x)| = 0 \qquad \text{if } |\alpha| = 2;$$
$$|\partial_x^\alpha V(t,x)| \le C_\alpha, \quad t \in \mathbf{R}, x \in \mathbf{R}^d, \text{ if } |\alpha| \ge 3.$$

Then $K(t,s,x,y)$ is C^∞ in x, y when $t \ne s$ ([26]). See also [13].

(iii) Assume $d = 1$, $g^{11}(x) = 1$, $a_1 = 0$, and $V(t,x) = V(x) \ge C(1+|x|)^{2+\varepsilon}$ near infinity for some $\varepsilon > 0$ as well as other technical conditions. Then $K(t,s,x,y)$ is nowhere C^1 ([26]).

(iv) Assume $g^{jk}(x) = \delta^{jk}$, $a_j = 0$, $V(t,x) = |x|^2 + W(t,x)$ with $W(t,x) = o(|x|^2)$ as $|x| \to \infty$. Then $K(t,0,x,y)$ is C^∞ in x for every $y \in \mathbf{R}^d$ and nonresonant $t \notin (\pi/2)\mathbf{Z}$ under general conditions on W, and shows various phenomena such as recurrence and dispersion of singularities for resonant $t \in (\pi/2)\mathbf{Z}$ depending on the growth order of $W(x)$ ([14, 17, 21, 27, 28]).

(v) Assume for some $\varepsilon > 0$ and $\delta > 0$

$$|\partial_x^\alpha(g^{jk}(x) - \delta^{jk})| \le C_\alpha(1+|x|)^{-1-\varepsilon-|\alpha|}, \quad x \in \mathbf{R}^d,$$

and $a^j(t,x) = a^j(x) = O(|x|^{1-\delta})$, $b(t,x) = b(x) = O(|x|^{1-\delta})$ as $|x| \to \infty$ as well as similar conditions on the derivatives. Then the H^s microlocal regularity of a solution for the Cauchy problem increases for all $t > 0$ at a point in $T^*\mathbf{R}^d \setminus 0$ if the point is not trapped backward by the Hamilton flow of h_0 and if the initial data decays along the backward bicharacteristics through that point ([1]).

See [3, 5] for the absence of smoothing effects due to the trapping of the Hamilton flow of the principal symbol. See also [3, 4, 5, 9, 10, 11, 12, 15, 16, 18, 19, 20, 22, 23] for related results in other frameworks.

Our goal is to handle the mixed case of (i) ,(ii), and (v) under relaxed conditions, which allow (a), (b), and (c). The case (iv) will be discussed elsewhere.

We explain the plan of this paper. Section 2 states the main results: the well-posedness of the Cauchy problem for the Schrödinger equation (Subsection 2.1) and the smoothing effect of the associated propagator (Subsection

2.2). Section 3 recalls the Weyl calculus of pseudodifferential operators and proves related lemmas. Section 4 proves two well-posedness theorems of the Cauchy problem: one for the Schrödinger equation in Section 1 and the other for a more general Schrödinger equation appearing in Section 7. Section 5 shows how the Schrödinger operator is transformed when conjugated by an invertible pseudodifferential operator. Section 6 proves first, a smoothing effect of the Schrödinger propagator, local in time and global in an incoming region in $T^*\mathbf{R}^d \setminus 0$, by using Section 5; second, a smoothing effect at every point of $T^*\mathbf{R}^d \setminus 0$ that is not trapped backward by the Hamilton flow of the principal symbol by using the result from Appendix A. Section 7 proves all assertions in Section 2 except for Theorem 2.8. Section 8 discusses the smoothing effect of order half, or the so-called local smoothing effect, from which Theorem 2.8 follows. Appendix A shows an energy estimate along the Hamilton flow of the principal symbol for a general dispersive equation.

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Notation. $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. $C^k(U, V)$ is the set of all C^k maps from U to V ($k \in \mathbf{N}_0 \cup \{\infty\}$), and $C(U, V) = C^0(U, V)$; V is omitted if $V = \mathbf{C}$. For locally convex spaces E and F, $L(E, F)$ is the set of all continuous linear operators from E to F, and $L(E) = L(E, E)$; $L(\mathbf{C}^n)$ is identified with $M_n(\mathbf{C})$. The symbol $(\cdot, \cdot)$ denotes the inner product of $L^2(\mathbf{R}^d)$ or $L^2(\mathbf{R}^d, \mathbf{C}^n)$ by abuse of notation, and $\|\cdot\|$ the norm. For $v \in \mathbf{R}^n$, $\langle v \rangle = (1 + |v|^2)^{1/2}$. For a subset A of $T^*\mathbf{R}^d$, set $\mathrm{cone}(A) = \{(x, t\xi); (x, \xi) \in A,\ t > 0\}$.

§2. Main Results

§2.1. Well-posedness of the Cauchy problem

Throughout Section 2, we assume that the following conditions (H1)–(H4) hold for some $0 < \delta < 1$.

(H1) $c_1 I_d \leq (g^{jk}(x)) \leq c_2 I_d$ on $\mathbf{R}^d$ for some $c_1, c_2 > 0$.

(H2) For every $\alpha \in \mathbf{N}_0^d$ with $|\alpha| \geq 1$, there is $C_\alpha(g) > 0$ such that

$$|\partial_x^\alpha g^{jk}(x)| \leq C_\alpha(g) \langle x \rangle^{-1+\delta(|\alpha|-1)}, \quad x \in \mathbf{R}^d, \quad j, k = 1, \ldots, d.$$

(H3) For every compact set $I \subset \mathbf{R}$ and $\alpha \in \mathbf{N}_0^d$, there is $C_\alpha(a, I) > 0$ such that

$$\begin{aligned} |a^j(t, x)| &\leq C_0(a, I)\langle x \rangle, && t \in I, x \in \mathbf{R}^d,\ j = 1, \ldots, d; \\ |\partial_x^\alpha a^j(t, x)| &\leq C_\alpha(a, I)\langle x \rangle^{\delta(|\alpha|-1)}, && t \in I, x \in \mathbf{R}^d,\ j = 1, \ldots, d, \ \text{ if } |\alpha| \geq 1. \end{aligned}$$

(H4) For every compact set $I \subset \mathbf{R}$ and $\alpha \in \mathbf{N}_0^d$, there is $C_\alpha(b, I) > 0$ such that

$$\begin{aligned} |b(t,x)| &\le C_0(b,I)\langle x\rangle^2, && t \in I, x \in \mathbf{R}^d; \\ |\partial_x^\alpha b(t,x)| &\le C_\alpha(b,I)\langle x\rangle^{1+\delta(|\alpha|-1)}, && t \in I, x \in \mathbf{R}^d, \quad \text{if } |\alpha| \ge 1. \end{aligned}$$

The condition (H1) implies that the Hamilton vector field H_{h_0} is complete on $T^*\mathbf{R}^d$, because h_0 is constant on each integral curve. Let $\Phi_t = \exp(tH_{h_0})$ ($t \in \mathbf{R}$) be the Hamilton flow of H_{h_0}; in other words, $\Phi_t(y,\eta) = (x(t,y,\eta), \xi(t,y,\eta))$ is the solution of the system of ordinary differential equations

$$\begin{aligned} \dot{x}_j(t) &= \partial_{\xi_j} h_0(x(t),\xi(t)), && x_j(0) = y_j, \\ \dot{\xi}_j(t) &= -\partial_{x_j} h_0(x(t),\xi(t)), && \xi_j(0) = \eta_j \qquad (1 \le j \le d). \end{aligned}$$

Next we define Sobolev spaces $\mathcal{B}^s(\mathbf{R}^d)$ ($s \in \mathbf{R}$) (cf. [7]). Let $H_{\rm osc}$ be the self-adjoint extension of the operator $1 - \Delta + |x|^2$ with domain $C_0^\infty(\mathbf{R}^d)$. Then for every $s \in \mathbf{R}$, $H_{\rm osc}^{s/2}$ is continuous on $\mathcal{S}(\mathbf{R}^d)$ and extends to a continuous linear operator on $\mathcal{S}'(\mathbf{R}^d)$ (with the weak* topology), denoted also by $H_{\rm osc}^{s/2}$. We set

$$\mathcal{B}^s(\mathbf{R}^d) = \{u \in \mathcal{S}'(\mathbf{R}^d); H_{\rm osc}^{s/2} u \in L^2(\mathbf{R}^d)\}.$$

These spaces are characterized as follows:

$$\begin{aligned} &\mathcal{B}^s(\mathbf{R}^d) = \{u \in L^2(\mathbf{R}^d); \langle x\rangle^s u \in L^2(\mathbf{R}^d),\ \langle D\rangle^s u \in L^2(\mathbf{R}^d)\} \quad (s \ge 0); \\ &\mathcal{B}^s(\mathbf{R}^d) = \mathcal{B}^{-s}(\mathbf{R}^d)' \quad (s \le 0). \end{aligned}$$

The vector-valued Sobolev spaces $\mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$ are similarly defined.

After preparing the Weyl calculus in Section 3, we shall prove in Lemma 4.1 that for every $s \in \mathbf{R}$ there is $L(s) \gg 1$ such that the operator E_s with Weyl symbol

$$e_s(x,\xi) = (h_0(x,\xi) + |x|^2 + L(s)^2)^{s/2}$$

is a homeomorphism from $\mathcal{B}^{r+s}(\mathbf{R}^d)$ to $\mathcal{B}^r(\mathbf{R}^d)$ for all $r \in \mathbf{R}$. We use $\|E_s \cdot\|$ as a norm of $\mathcal{B}^s(\mathbf{R}^d)$ (or $\mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$), where $\|\cdot\| = \|\cdot\|_{L^2(\mathbf{R}^d)}$ (or $\|\cdot\|_{L^2(\mathbf{R}^d,\mathbf{C}^n)}$).

Now we state our two theorems on the well-posedness of the Cauchy problem.

Theorem 2.1. *Let $s \in \mathbf{R}$, $I = [t_1, t_2]$ ($t_1 < t_2$), and $t_0 \in I$. For every $u_0 \in \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$ and $f \in L^1(I, \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$, there exists $u \in C(I, \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$ satisfying*

$$(\partial_t + iH(\cdot))u = f \ \text{ in } \mathcal{D}'((t_1,t_2) \times \mathbf{R}^d, \mathbf{C}^n), \quad u(t_0) = u_0, \tag{2.1}$$

which is unique in $C(I, \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n))$. *Moreover, the solution* u *satisfies the following estimate*

$$(2.2)\quad e^{-\gamma|t-t_0|}\| E_s u(t) \| \leq \| E_s u(t_0) \| + \left| \int_{t_0}^{t} e^{-\gamma|\tau - t_0|}\| E_s f(\tau) \| \, d\tau \right|, \quad t \in I.$$

Here $\gamma \geq 0$ *depends on* $s \in \mathbf{R}$ *and on the constants* c_1, c_2, $C_\alpha(g)$, $C_\alpha(a, I)$, *and* $C_\alpha(b, I)$ *in* (H1)–(H4), *but not on* f, u_0, *or* u. *In particular,* $\gamma = 0$ *if* $s = 0$.

Theorem 2.2. *Let* $S(t, t_0) \in L(\mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n))$ $(t, t_0 \in \mathbf{R})$ *be the operator mapping* $u_0 \in \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n)$ *to* $u(t) \in \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n)$, *where* $u \in C(\mathbf{R}, \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n))$ *is the solution of the Cauchy problem*

$$(2.3)\qquad (\partial_t + iH(\cdot))u = 0 \text{ in } \mathcal{D}'(\mathbf{R} \times \mathbf{R}^d, \mathbf{C}^n), \quad u(t_0) = u_0.$$

(1) $S(t, t) = 1$ *and* $S(t, s)S(s, r) = S(t, r)$ *on* $\mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n)$ $\quad (t, s, r \in \mathbf{R})$.

(2) *For every compact interval* I, $\{ S(t, t_0)|_{\mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)}; t, t_0 \in I\}$ *is bounded in* $L(\mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$.

(3) $\mathbf{R} \times \mathbf{R} \times \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n) \ni (t, t_0, u_0) \mapsto S(t, t_0)u_0 \in \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$ *is continuous.*

(4) $S(t, t_0)|_{L^2(\mathbf{R}^d, \mathbf{C}^n)} \in L(L^2(\mathbf{R}^d, \mathbf{C}^n))$ *is unitary.*

(5) *If* $H = H(t)$ *is time independent, then* $H|_{C_0^\infty(\mathbf{R}^d, \mathbf{C}^n)}$ *is essentially self-adjoint. If* H *denotes also its self-adjoint extension, then* $e^{-i(t-t_0)H}u_0 = S(t, t_0)u_0$ *for every* $t, t_0 \in \mathbf{R}$ *and* $u_0 \in L^2(\mathbf{R}^d, \mathbf{C}^n)$.

§2.2. Smoothing effects

The asymptotic behavior of Φ_t plays an important role in the smoothing effect of the propagator $S(t, s)$. We introduce several subsets of $T^*\mathbf{R}^d \setminus 0$ consisting of the points which are trapped forward or backward by Φ_t:

$$\begin{aligned} T_+ &= \{X \in T^*\mathbf{R}^d \setminus 0; \lim_{t \to \infty} |\Phi_t(X)| \neq \infty\}, \\ T_- &= \{X \in T^*\mathbf{R}^d \setminus 0; \lim_{t \to -\infty} |\Phi_t(X)| \neq \infty\}; \\ T_{cpt,+} &= \{X \in T^*\mathbf{R}^d \setminus 0; \{\Phi_t(X); t \geq 0\} \text{ is relatively compact}\}, \\ T_{cpt,-} &= \{X \in T^*\mathbf{R}^d \setminus 0; \{\Phi_t(X); t \leq 0\} \text{ is relatively compact}\}. \end{aligned}$$

Put $T_{cpt} = T_{cpt,+} \cap T_{cpt,-}$. To control the asymptotic behavior of Φ_t, we assume the following condition (H5) in addition to (H1)–(H4) stated at the beginning of this section.

(H5) (convexity near infinity). There exists $f_{cv} \in C^\infty(\mathbf{R}^d)$, $\lim_{|x|\to\infty} f_{cv}(x) = \infty$, $f_{cv} \geq 1$, such that for every $\alpha \in \mathbf{N}_0^d$ with $|\alpha| \geq 2$, $\partial^\alpha f_{cv} \in L^\infty(\mathbf{R}^d)$ and that for some $\sigma > 0$, $R > 0$

$$H_{h_0}^2 f_{cv} \geq 2\sigma^2 h_0 \quad \text{on } \{(x,\xi) \in T^*\mathbf{R}^d; r(x) := \sqrt{f_{cv}(x)} \geq R\}.$$

Remark. The function f_{cv} in (H5) satisfies $f_{cv}(x)^{-1} = O(|x|^{-2})$ as $|x| \to \infty$. In fact, take $M > 0$ such that $\{x \in \mathbf{R}^d; |x| \geq M\} \subset \{x \in \mathbf{R}^d; f_{cv}(x) \geq R^2\}$. For $x \in \mathbf{R}^d$, $|x| > M$, take $T > 0$ and $(y,\eta) \in T^*\mathbf{R}^d$ such that $|y| = M$, $h_0(y,\eta) = 1$, $|x(t,y,\eta)| > M$ $(0 < t < T)$, and $x(T,y,\eta) = x$, where $\Phi_t(y,\eta) = (x(t,y,\eta), \xi(t,y,\eta))$. This is possible because Φ_t is a complete geodesic flow. Then $T \geq c|x-y|$ for some $c > 0$ independent of T, x, y by (H1), and

$$\begin{aligned} f_{cv}(x) &= f_{cv}(y) + (H_{h_0} f_{cv})(y,\eta)T + \int_0^1 (1-\theta)(H_{h_0}^2 f_{cv})(\Phi_{\theta T}(y,\eta))d\theta\, T^2 \\ &\geq f_{cv}(y) + (H_{h_0} f_{cv})(y,\eta)T + \sigma^2 T^2 \end{aligned}$$

by (H5). Therefore $\liminf_{|x|\to\infty} f_{cv}(x)/|x|^2 \geq c^2\sigma^2$. □

Remark. If $|\nabla_x g^{jk}(x)| = o(|x|^{-1})$ as $|x| \to \infty$, then (H5) holds with $f_{cv}(x) = 1 + |x|^2$. □

Remark. Let $a \in C^\infty([1,\infty))$ such that $C^{-1} \leq a \leq C$ with $C > 0$ and $\partial^k a(r) = O(r^{-1})$ for all $k \in \mathbf{N}$. Assume $\limsup_{r\to\infty} a'(r)r/a(r) < 1$. If $(g^{jk}(x)) = a(|x|)^2 I$ near infinity, then (H5) is satisfied with $f_{cv}(x) = (\int_1^{|x|} a(r)^{-1}dr)^2$ near infinity. In fact, using the coordinates $t = \int_1^r a(s)^{-1}ds$ $(r = |x|)$ and $\omega = x/|x| \in S^{d-1}$, we have $f_{cv} = t^2$ and $h_0 = \tau^2 + \alpha(t)^2 p$, where τ is the dual variable of t, $-p$ is the principal symbol of the Laplacian on S^{d-1}, and $\alpha(t) = a(r)/r$. Hence $H_{h_0}^2 t^2 = 8\tau^2 + 8\alpha(t)^2 pt/r \cdot (a(r) - ra'(r)) \geq ch_0$ near infinity for some $c > 0$.

For example, when $a(r) = 1 + c\sin(\varepsilon \log r)$ with $c \in \mathbf{R}$ and $\varepsilon > 0$ satisfying $c^2(1+\varepsilon^2) < 1$, then $(H5)$ holds. □

The requirement that $\partial^\alpha f_{cv} \in L^\infty(\mathbf{R}^d)$ for all $|\alpha| \geq 3$ is not essential in (H5), as the following lemma shows.

Lemma 2.3. *Let $f \in C^2(\mathbf{R}^d)$, $f \geq 1$, $\lim_{|x|\to\infty} f(x) = \infty$, such that for every $\alpha \in \mathbf{N}_0^d$ with $|\alpha| = 2$,*

$$\sup_{x\in\mathbf{R}^d} |\partial^\alpha f(x)| < \infty, \qquad \lim_{|h|\to+0} \sup_{x\in\mathbf{R}^d} |\partial^\alpha f(x+h) - \partial^\alpha f(x)| = 0,$$

and that for some $\tilde{\sigma} > 0$, $\tilde{R} > 0$,

$$H_{h_0}^2 f \geq 2\tilde{\sigma}^2 h_0 \quad \text{on } \{(x,\xi) \in T^*\mathbf{R}^d; f(x) \geq \tilde{R}^2\}.$$

Then for every $0 < \sigma < \tilde{\sigma}$ *and* $R > \tilde{R}$, *there exists* $f_{cv} \in C^\infty(\mathbf{R}^d)$ *such that* (H5) *holds with these* σ, R, *and* f_{cv}.

The condition (H5) ensures the existence of a positively (or negatively) invariant set $S_+(R', \sigma')$ (or $S_-(R', \sigma')$) defined below, which asymptotically includes every positive (or negative) orbit that is not relatively compact. The role of this set becomes clearer in Section 6. Let $S^*\mathbf{R}^d = \{X \in T^*\mathbf{R}^d; h_0(X) = 1\}$. Remark that $h_0 \circ \Phi_t = h_0$.

Proposition 2.4 [5, Theorem 3.2]**.** *For* $R' \geq R, 0 < \sigma' < \sigma$, *set*

$$S_+(R', \sigma') = \{X = (x,\xi) \in S^*\mathbf{R}^d; r(x) > R',\ H_{h_0} r(X) > \sigma'\},$$
$$S_-(R', \sigma') = \{X = (x,\xi) \in S^*\mathbf{R}^d; r(x) > R',\ -H_{h_0} r(X) > \sigma'\},$$

where R *and* σ *are the constants in* (H5)*.*

$(1)_+$ $\Phi_t S_+(R', \sigma') \subset S_+(R', \sigma')$ *if* $t \geq 0$.

$(1)_-$ $\Phi_t S_-(R', \sigma') \subset S_-(R', \sigma')$ *if* $t \leq 0$.

$(2)_+$ *For every* $X_0 \in S^*\mathbf{R}^d \setminus T_{cpt,+}$, *there exists* $T > 0$ *such that* $\Phi_t(X_0) \in S_+(R', \sigma')$ *if* $t \geq T$. *In particular,* $T_+ = T_{cpt,+}$.

$(2)_-$ *For every* $X_0 \in S^*\mathbf{R}^d \setminus T_{cpt,-}$, *there exists* $T > 0$ *such that* $\Phi_t(X_0) \in S_-(R', \sigma')$ *if* $t \leq -T$. *In particular,* $T_- = T_{cpt,-}$.

(3) $T_{cpt} \cap S^*\mathbf{R}^d$ *is a compact subset of* $\{(x,\xi) \in T^*\mathbf{R}^d \setminus 0; r(x) < R\}$.

To state our main results, we need some notation. For a bounded interval $I \subset \mathbf{R}$, set

$$\mu_1(I, L) = \sum_{j=1}^d \sup_{t \in I, |x| \geq L} |\nabla_x a^j(t,x)|, \qquad \mu_1(I) = \lim_{L \to \infty} \mu_1(I, L);$$
$$\mu_2(I, L) = \sup_{t \in I, |x| \geq L} \frac{|\nabla_x b(t,x)|}{|x|}, \qquad \mu_2(I) = \lim_{L \to \infty} \mu_2(I, L).$$

Remark. Set $\mu_2'(I, L) = \sup_{t \in I, |x| \geq L} \frac{|\nabla_x h_2(t,x)|}{|x|}$. Then $\lim_{L \to \infty} \mu_2'(I, L) = \mu_2(I)$.

Remark. Set $\mu_1'(I, L) = \sum_{j=1}^d \sup_{t \in I, |x| \geq L} \frac{|a^j(t,x)|}{|x|}$ and $\mu_1'(I) = \lim_{L \to \infty} \mu_1'(I, L)$. Then $\mu_1'(I) \leq \mu_1(I)$, because the equation $a^j(t,x) = a^j(t, \varepsilon x) + \int_\varepsilon^1 \nabla_x a^j(t, \theta x) \cdot x \, d\theta$ gives that $\mu_1'(I, L) \leq \varepsilon \mu_1'(I, \varepsilon L) + (1 - \varepsilon)\mu_1(I, \varepsilon L)$ for every $0 < \varepsilon < 1$ and $L \geq 1$.

Theorem 2.5. *There exists $c(d,h_0,r)>0$ such that for every bounded interval $I=[t_1,t_2]$ $(t_1<t_2)$ satisfying $\mu_1(I)|I|+\mu_2(I)|I|^2\le c(d,h_0,r)$, the assertion below holds: If $a\in S^0_{1,0}=S(1,|dx|^2+\langle\xi\rangle^{-2}|d\xi|^2)$ satisfies that*

$$\operatorname{supp} a\cap T_-=\emptyset\quad (\textit{resp.}\ \operatorname{supp} a\cap T_+=\emptyset)$$

and that $\pi(\operatorname{supp} a)$ is relatively compact, then the mappings

$$\begin{aligned}
&\langle x\rangle^{-\rho}\mathcal{B}^s(\mathbf{R}^d,\mathbf{C}^n)\ni u_0\mapsto |t-t_1|^\rho a^w S(t,t_1)u_0\in C(I_t,\mathcal{B}^{s+\rho}(\mathbf{R}^d,\mathbf{C}^n)),\\
&\langle x\rangle^{-\rho}\mathcal{B}^s(\mathbf{R}^d,\mathbf{C}^n)\ni u_0\\
&\quad\mapsto |t-t_1|^\rho a^w S(t,t_1)u_0\in L^2(I_t,\mathcal{B}^{s+\rho+1/2}(\mathbf{R}^d,\mathbf{C}^n))\\
\Big(&\textit{resp.}\ \langle x\rangle^{-\rho}\mathcal{B}^s(\mathbf{R}^d,\mathbf{C}^n)\ni u_0\mapsto |t-t_2|^\rho a^w S(t,t_2)u_0\in C(I_t,\mathcal{B}^{s+\rho}(\mathbf{R}^d,\mathbf{C}^n)),\\
&\langle x\rangle^{-\rho}\mathcal{B}^s(\mathbf{R}^d,\mathbf{C}^n)\ni u_0\\
&\quad\mapsto |t-t_2|^\rho a^w S(t,t_2)u_0\in L^2(I_t,\mathcal{B}^{s+\rho+1/2}(\mathbf{R}^d,\mathbf{C}^n))\Big)
\end{aligned}$$

are continuous for all $s\in\mathbf{R}$ and $\rho\in[0,\infty)$. Here $\pi:T^*\mathbf{R}^d\ni(x,\xi)\mapsto x\in\mathbf{R}^d$.

Remark. Theorem 2.5 is a corollary of more general theorems (see Theorems 6.2 and 6.5). It suffices to assume that the initial data decays in an incoming region $S_-(R',\sigma')$ (resp. in an outgoing region $S_+(R',\sigma')$) in a sense.

Corollary 2.6. *Let $c(d,h_0,r)>0$ be the constant in Theorem 2.5. Then for every bounded interval $I=[t_1,t_2]$ $(t_1<t_2)$ satisfying $\mu_1(I)|I|+\mu_2(I)|I|^2\le c(d,h_0,r)$, the assertion below holds: For every $u_0\in\mathcal{E}'(\mathbf{R}^d,\mathbf{C}^n)$*

$$\begin{aligned}
WF(S(t,t_0)u_0)&\subset T_-,\quad t_1\le t_0<t\le t_2;\\
WF(S(t,t_0)u_0)&\subset T_+,\quad t_1\le t<t_0\le t_2.
\end{aligned}$$

Corollary 2.7. *Let $c(d,h_0,r)>0$ be the constant in Theorem 2.5. Then for every bounded interval $I=[t_1,t_2]$ $(t_1<t_2)$ satisfying $\mu_1(I)|I|+\mu_2(I)|I|^2\le c(d,h_0,r)$,*

$$\begin{aligned}
WF(K(t,t_0))&\subset(T_-\times T_-)\cup(0\times T_-)\cup(T_-\times 0),\quad t_1\le t_0<t\le t_2;\\
WF(K(t,t_0))&\subset(T_+\times T_+)\cup(0\times T_+)\cup(T_+\times 0),\quad t_1\le t<t_0\le t_2.
\end{aligned}$$

Here 0 is the zero section of $T^*\mathbf{R}^d$.

Theorem 2.8 (smoothing effect of order half). *Let $s\in\mathbf{R}$ and $0<\nu\ll 1$. Let $I=[t_1,t_2]$ $(t_1<t_2)$ and $t_0\in I$.*

(1) *If $T_{cpt} = \emptyset$, then there exists $C > 0$ such that the following estimates hold*:

$$\left| \int_{t_0}^{t} \| \langle x \rangle^{-(1+\nu)/2} E_{s+1/2} u(\tau) \|^2 d\tau \right| \le C \| E_s u(t_0) \|^2 + C \left(\int_{t_0}^{t} \| E_s f(\tau) \| d\tau \right)^2,$$

$$\| E_s u(t) \|^2 + \left| \int_{t_0}^{t} \| \langle x \rangle^{-(1+\nu)/2} E_{s+1/2} u(\tau) \|^2 d\tau \right|$$
$$\le C \| E_s u(t_0) \|^2 + C \left| \int_{t_0}^{t} \| \langle x \rangle^{(1+\nu)/2} E_{s-1/2} f(\tau) \|^2 d\tau \right|$$

for all $t \in I$ and $u \in C^1(I, \mathcal{S}(\mathbf{R}^d, \mathbf{C}^n))$ with $f(t) = (\partial_t + iH(t))u(t)$.

(2) *For every $a \in S(1, |dx|^2 + |d\xi|^2/\langle X \rangle^2)$ satisfying $\overline{\mathrm{cone}(\mathrm{supp}\, a)} \cap T_{cpt} = \emptyset$, there exists $C > 0$ such that the following estimate holds*:

$$\left| \int_{t_0}^{t} \| \langle x \rangle^{-(1+\nu)/2} E_{s+1/2} a^w u(\tau) \|^2 d\tau \right| \le C \| E_s u(t_0) \|^2 + C \left(\int_{t_0}^{t} \| E_s f(\tau) \| d\tau \right)^2$$

for all $t \in I$ and $u \in C^1(I, \mathcal{S}(\mathbf{R}^d, \mathbf{C}^n))$ with $f(t) = (\partial_t + iH(t))u(t)$.

Remark. In contrast to Theorem 2.5, Theorem 2.8 holds for every compact interval I with no distinction between the forward, and backward, propagators (especially, observe the condition $\overline{\mathrm{cone}(\mathrm{supp}\, a)} \cap T_{cpt} = \emptyset$ in (2)). See Section 8 for the comparison among various nontrapping conditions.

Remark. The smoothing effect of order half fails at almost every point in T_{cpt}. See [3, 5] for details in a little different framework.

§3. Weyl Calculus

In this section, we recall the Weyl calculus due to Hörmander (see [8, Chapters 18.4-6] for details) and prove related lemmas.

For a Riemannian metric g on $V = \mathbf{R}^N$ and a positive function $m \in C(\mathbf{R}^N)$, the symbol space $S(m, g)$ is the set of all $a \in C^\infty(\mathbf{R}^N)$ such that for every $k \in \mathbf{N}_0$

$$\|a\|_{k,S(m,g)} = \sum_{j=0}^{k} \sup \left\{ \frac{|\, \partial_{v_1} \cdots \partial_{v_j} a(x) \,|}{m(x) \prod_{i=1}^{j} g_x(v_i)^{1/2}};\ x \in \mathbf{R}^N, 0 \ne v_i \in \mathbf{R}^N \right\} < \infty,$$

where $\partial_v f(x) = (d/dt)|_{t=0} f(x + tv)$ and $g_x(v) = g_x(v, v)$. It is a Fréchet space with seminorms $(\, || \cdot ||_{k,S(m,g)}\,)_{k=0,1,\dots}$. A sequence $(a_n)_{n=1,2,\dots}$ in $S(m, g)$ is said to converge to a weakly in $S(m, g)$, or simply $a_n \to a$ weakly in

$S(m,g)$, if (a_n) is bounded in $S(m,g)$ and converges to a in $C^\infty(\mathbf{R}^N)$ (or equivalently, in $\mathcal{D}'(\mathbf{R}^N)$). Let $S(m,g;M_n(\mathbf{C}))$ denote the $M_n(\mathbf{C})$-valued symbol space $S(m,g)\otimes M_n(\mathbf{C}) = \{(a_{jk})_{1\le j,k\le n}; a_{jk}\in S(m,g)\}$; the seminorms $\|a\|_{k,S(m,g;M_n(\mathbf{C}))}$ are defined similarly to $\|a\|_{k,S(m,g)}$ except that $|a(x)| = \|a(x)\|_{L(\mathbf{C}^n)}$ in the former definition.

From now on, we consider the case where $V=\mathbf{R}^{2d}\cong\mathbf{R}^d\times(\mathbf{R}^d)'$. Let σ be the canonical 2-form on $\mathbf{R}^{2d}$

$$\sigma(X,Y)=\xi\cdot y-\eta\cdot x,$$

where $X=(x,\xi),\ Y=(y,\eta)\in\mathbf{R}^{2d}$. Let g be a Riemannian metric on $\mathbf{R}^{2d}$. The Riemannian metric g^σ on $\mathbf{R}^{2d}$ is defined by

$$g^\sigma_X(Y)=\sup_{Y'\neq 0}\frac{\sigma(Y,Y')^2}{g_X(Y')}.$$

We consider three conditions on g.

(G1) (slow variation). There are $c, C>0$ such that for every $X,Y,Z\in\mathbf{R}^{2d}$

$$g_X(Y)\le c\Rightarrow C^{-1}g_X(Z)\le g_{X+Y}(Z)\le Cg_X(Z).$$

(G2) (σ temperance). There are C, $N>0$ such that for every $X,Y,Z\in\mathbf{R}^{2d}$

$$g_Y(Z)\le Cg_X(Z)(1+g^\sigma_Y(X-Y))^N.$$

(G3) (uncertainty principle). For every $X\in\mathbf{R}^{2d}$

$$\gamma(X)=\sup_{Y\in\mathbf{R}^{2d},Y\neq 0}(g_X(Y)/g^\sigma_X(Y))^{1/2}\le 1.$$

In the rest of this section, we fix a Riemannian metric g satisfying (G1)–(G3). A positive function $m:\mathbf{R}^{2d}\to(0,\infty)$ is said to be a g weight if it satisfies the following conditions.

(M1) (g continuity). There are $c, C>0$ such that for every $X,Y\in\mathbf{R}^{2d}$

$$g_X(Y)\le c\Rightarrow C^{-1}\le m(X+Y)/m(X)\le C.$$

(M2) (σ, g temperance). There are C, $N>0$ such that for every $X,Y\in\mathbf{R}^{2d}$

$$m(Y)\le Cm(X)(1+g^\sigma_Y(X-Y))^N.$$

Remark. For every nonzero $Y \in \mathbf{R}^{2d}$, $g_X(Y)$ is a g weight as a function of X. In particular, if $g = \varphi^2|dx|^2 + \Phi^2|d\xi|^2$ for positive functions φ and Φ, then φ and Φ are g weights. For a g weight m, m^s is a g weight for every $s \in \mathbf{R}$, and so is $\log m$ if $\inf m > 1$.

As a symbol-to-operator correspondence, we adopt the Weyl quantization. For $a \in \mathcal{S}'(\mathbf{R}^{2d})$, the operator $a^w = a^w(x,D) \in L(\mathcal{S}(\mathbf{R}^d), \mathcal{S}'(\mathbf{R}^d))$ is defined by

$$a^w u(x) = a^w(x,D)u\,(x) = \frac{1}{(2\pi)^d} \iint a\left(\frac{x+y}{2}, \xi\right) e^{i(x-y)\cdot\xi} u(y)dyd\xi,$$
$$u \in \mathcal{S}(\mathbf{R}^d),$$

where the integral is in the sense of temperate distribution. Then the correspondence $\mathrm{Op} : \mathcal{S}'(\mathbf{R}^{2d}) \ni a \mapsto \mathrm{Op}(a) = a^w \in L(\mathcal{S}(\mathbf{R}^d), \mathcal{S}'(\mathbf{R}^d))$ is an isomorphism. For $A \in L(\mathcal{S}(\mathbf{R}^d), \mathcal{S}'(\mathbf{R}^d))$, set $\sigma(A) = (\mathrm{Op})^{-1}(A)$, called the Weyl symbol of A.

If $a_1, a_2 \in \mathcal{S}(\mathbf{R}^{2d})$, then $a_1^w a_2^w = (a_1 \# a_2)^w$ with

$$\begin{aligned}(a_1\#a_2)(X) &= \exp\left(\frac{i\sigma(D_X, D_Y)}{2}\right) a_1(X)a_2(Y)|_{Y=X} \\ &= \sum_{j=0}^{N-1} \frac{1}{j!} \left(\frac{i\sigma(D_X, D_Y)}{2}\right)^j a_1(X)a_2(Y)|_{Y=X} + r_N(a_1,a_2)(X);\end{aligned}$$

$$r_N(a_1,a_2)(X)$$
$$= \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} \exp\left(\frac{i\theta\sigma(D_X, D_Y)}{2}\right) \left(\frac{i\sigma(D_X, D_Y)}{2}\right)^N a_1(X)a_2(Y)|_{Y=X} d\theta.$$

Here $N \in \mathbf{N}$. Set $r_0(a_1,a_2) = a_1\#a_2$.

Now we recall fundamental theorems due to Hörmander.

Theorem 3.1 [8, Theorem 18.5.4]. *Let m_1, m_2 be g weights and $N \in \mathbf{N}_0$. Then the map $\mathcal{S}(\mathbf{R}^{2d}) \times \mathcal{S}(\mathbf{R}^{2d}) \ni (a_1,a_2) \mapsto r_N(a_1,a_2) \in \mathcal{S}(\mathbf{R}^{2d})$ can be extended to a weakly continuous bilinear map from $S(m_1,g) \times S(m_2,g)$ to $S(\gamma^N m_1 m_2, g)$, denoted by the same symbol. Moreover, the extended bilinear map is bounded from $S(m_1,g) \times S(m_2,g)$ to $S(\gamma^N m_1 m_2, g)$.*

Theorem 3.2 [8, Theorems 18.6.2, 18.6.3, and 18.6.14]. (1) *Let m be a g weight. Then $S(m,g) \ni a \mapsto a^w \in L(\mathcal{S}(\mathbf{R}^d))$ (resp. $L(\mathcal{S}'(\mathbf{R}^d))$) is continuous. Moreover, if $a_n \to a$ weakly in $S(m,g)$, then $a_n^w u \to a^w u$ in $\mathcal{S}(\mathbf{R}^d)$ (resp. $\mathcal{S}'(\mathbf{R}^d)$) for all $u \in \mathcal{S}(\mathbf{R}^d)$ (resp. $\mathcal{S}'(\mathbf{R}^d)$). Here $\mathcal{S}'(\mathbf{R}^d)$ is endowed with the weak* topology.*

(2) *The map* $S(1,g) \ni a \mapsto a^w \in L(L^2(\mathbf{R}^d))$ *is continuous. Moreover, if* $a_n \to a$ *weakly in* $S(1,g)$, *then* $a_n^w u \to a^w u$ *in* $L^2(\mathbf{R}^d)$ *for all* $u \in L^2(\mathbf{R}^d)$.

(3) (*The sharp Gårding inequality*). *If* $a \in S(\gamma^{-1}, g; M_n(\mathbf{C}))$ *satisfies* $\Re a = (a + a^*)/2 \geq 0$, *then there exists a continuous seminorm* $C(\cdot)$ *on* $S(\gamma^{-1}, g; M_n (\mathbf{C}))$ *such that*

$$\Re(a^w u, u) \geq -C(a)\|u\|^2, \quad u \in \mathcal{S}(\mathbf{R}^d, \mathbf{C}^n).$$

Here $(u,v) = (u,v)_{L^2(\mathbf{R}^d,\mathbf{C}^n)}$, $\|u\| = \sqrt{(u,u)}$.

(4) *Let* m_j *be a* g *weight, and* $a_j \in S(m_j, g)$ $(j = 1,2)$. *Then* $a_1^w a_2^w = (a_1 \# a_2)^w$.

Example. Let us reconsider $\mathcal{B}^s(\mathbf{R}^d)$. Since $H_{\mathrm{osc}}^{s/2} \in S(\langle X\rangle^s, \langle X\rangle^{-2}|dX|^2)$ with $\sigma(H_{\mathrm{osc}}^{s/2}) - \langle X\rangle^s \in S(\langle X\rangle^{s-2}, \langle X\rangle^{-2}|dX|^2)$ (see [7]), it follows

$$\mathcal{B}^s(\mathbf{R}^d) = \{u \in \mathcal{S}'(\mathbf{R}^d); Pu \in L^2(\mathbf{R}^d) \text{ for all } P \in \operatorname{Op} S(\langle X\rangle^s, \langle X\rangle^{-2}|dX|^2)\}. \ \square$$

The following lemma is useful for obtaining better estimates of the remainder term of a symbol product.

Lemma 3.3. *For* g *weights* m_1 *and* m_2, *the maps*

$$Q_\theta : \mathcal{S}(\mathbf{R}^{2d}) \times \mathcal{S}(\mathbf{R}^{2d}) \ni (a_1, a_2) \mapsto \exp\left(\frac{i\theta\sigma(D_X, D_Y)}{2}\right) a_1(X) a_2(Y)|_{Y=X} \in \mathcal{S}(\mathbf{R}^{2d})$$

extend to weakly continuous bilinear maps from $S(m_1, g) \times S(m_2, g)$ *to* $S(m_1 m_2, g)$ *for all* $\theta \in [0,1]$, *denoted by the same symbol. Moreover, for every* $j \in \mathbf{N}_0$ *there are* $C > 0$ *and* $k \in \mathbf{N}_0$ *such that for all* $(\theta, a_1, a_2) \in [0,1] \times S(m_1, g) \times S(m_2, g)$

$$\|Q_\theta(a_1, a_2)\|_{j, S(m_1 m_2, g)} \leq C \|a_1\|_{k, S(m_1, g)} \|a_2\|_{k, S(m_2, g)}.$$

In particular, if $(a_1, a_2) \in S(m_1, g) \times S(m_2, g)$ *satisfies that*

$$\sigma(D_X, D_Y)^N a_1(X) a_2(Y) = \sum_{k=1}^n a_{1,k}(X) a_{2,k}(Y)$$

with some $N \in \mathbf{N}_0$, g *weights* $m_{j,k}$, *and symbols* $a_{j,k} \in S(m_{j,k}, g)$, $j = 1, 2$, $k = 1, \ldots, n$, *then* $r_N(a_1, a_2) \in S(\sum_{k=1}^n m_{1,k} m_{2,k}, g)$.

Proof. The first part follows from the chapters 18.4-5 of [8] if uniformity in θ is considered. The second part is valid because

$$r_N(a_1, a_2)(X) = \frac{i^N}{(N-1)! 2^N} \sum_{k=1}^n \int_0^1 (1-\theta)^{N-1} Q_\theta(a_{1,k}, a_{2,k})(X) d\theta. \quad \square$$

Next we prepare a series of lemmas.

Lemma 3.4. *Assume that g is of the form*

$$g_X = \varphi(X)^2|dx|^2 + \Phi(X)^2|d\xi|^2, \quad X \in \mathbf{R}^{2d},$$

where φ and Φ are positive functions. Let φ_0 be a g weight such that $\varphi_0 \le \varphi$ on $\mathbf{R}^{2d}$, and set $\gamma_0 = \varphi_0\Phi$ (recall that $\gamma = \varphi\Phi$ in this case). For $N \in \mathbf{N}_0$ and a g weight m, denote by $S_N(m, \varphi_0, g)$ the set of all $a \in S(m, g)$ satisfying $\partial_x^\alpha a \in S(\varphi_0^{|\alpha|} m, g)$ for all $\alpha \in \mathbf{N}_0^d$ with $|\alpha| \le N$, which has a natural Fréchet space structure.

(1) *$S_N(m, \varphi_0, g) \subset S_{N+1}(m/\gamma_0, \varphi_0, g)$.*

(2) *If $(a_1, a_2) \in S_N(m_1, \varphi_0, g) \times S_N(m_2, \varphi_0, g)$, then*

$$r_k(a_1, a_2) \in S_{N-k}(\gamma_0^k m_1 m_2, \varphi_0, g), \quad k \le N;$$
$$r_k(a_1, a_2) \in S(\gamma^{k-N}\gamma_0^N m_1 m_2, g), \quad k \ge N.$$

(3) *If $(a_1, a_2) \in S_1(m_1, \varphi_0, g) \times S_1(m_2, \varphi_0, g)$, then*

$$a_1 a_2 \in S_1(m_1 m_2, \varphi_0, g); \quad \{a_1, a_2\} \in S(\gamma_0 m_1 m_2, g);$$
$$a_1\#a_2 \in S_1(m_1 m_2, \varphi_0, g); \quad r_k(a_1, a_2) \in S(\gamma^{k-1}\gamma_0 m_1 m_2, g), \quad k \ge 1;$$
$$a_1\#a_2 - a_1 a_2 - \{a_1, a_2\}/(2i) = r_2(a_1, a_2) \in S(\gamma\gamma_0 m_1 m_2, g);$$
$$a_1\#a_2 + a_2\#a_1 - 2a_1 a_2 = r_2(a_1, a_2) + r_2(a_2, a_1) \in S(\gamma\gamma_0 m_1 m_2, g);$$
$$a_1\#a_2 - a_2\#a_1 - \{a_1, a_2\}/i = r_3(a_1, a_2) - r_3(a_2, a_1) \in S(\gamma^2\gamma_0 m_1 m_2, g).$$

Proof. (1) If $a \in S_N(m, \varphi_0, g)$, then

$$\partial_x^\alpha a \in S(\varphi_0^k m, g) \subset S(\varphi_0^k m/\gamma_0, g) \quad (|\alpha| = k \le N),$$
$$\partial_x^\alpha a \in S(\varphi_0^N \varphi m, g) = S(\varphi_0^{N+1} m\gamma/\gamma_0, g) \subset S(\varphi_0^{N+1} m/\gamma_0, g) \quad (|\alpha| = N+1).$$

This implies $a \in S_{N+1}(m/\gamma_0, \varphi_0, g)$.

(2) By assumption,

$$\frac{1}{k!}(i\sigma(D_X, D_Y)/2)^k a_1(X)a_2(Y) = \sum_{|\alpha|+|\beta|=k} \frac{i^k(-1)^{|\alpha|}}{2^k\alpha!\beta!}\partial_\xi^\alpha\partial_x^\beta a_1(X)\partial_\eta^\beta\partial_y^\alpha a_2(Y);$$
$$\partial_\xi^\alpha\partial_x^\beta a_j \in S_{N-|\beta|}(\varphi_0^{|\beta|}\Phi^{|\alpha|} m_j, \varphi_0, g), \quad |\beta| \le N \quad (j = 1, 2);$$
$$\partial_\xi^\alpha\partial_x^\beta a_j \in S(\varphi_0^N \varphi^{|\beta|-N}\Phi^{|\alpha|} m_j, g), \quad |\beta| \ge N \quad (j = 1, 2).$$

If $k \geq N$, we have $r_k(a_1, a_2) \in S(\gamma^{k-N}\gamma_0^N m_1 m_2, g)$ by Lemma 3.3. If $k \leq N$, we have

$$r_k(a_1,a_2) = \sum_{j=k}^{N-1} \frac{1}{j!}(i\sigma(D_X, D_Y)/2)^j a_1(X)a_2(Y)|_{Y=X} + r_N(a_1,a_2)$$
$$\in \sum_{j=k}^{N} S_{N-j}(\gamma_0^j m_1 m_2, \varphi_0, g) \subset S_{N-k}(\gamma_0^k m_1 m_2, \varphi_0, g)$$

by virtue of (1). □

Lemma 3.5. *Let m be a g weight such that*

$$m(X) \leq \langle X \rangle^{-c}, \; X \in \mathbf{R}^{2d},$$

with some $c > 0$. If $r \in S(m, g)$ satisfies $\|r^w\| < 1$, then $(1-r^w)^{-1} \in L(L^2(\mathbf{R}^d))$ belongs to $\operatorname{Op} S(1, g)$ with $(1 - r^w)^{-1} - \sum_{j=0}^{N-1}(r^w)^j \in \operatorname{Op} S(m^N, g)$ for every $N \in \mathbf{N}$.

Proof. Let $N \in \mathbf{N}$. For every $k \in \mathbf{N}_0$, there are $s \geq 0$ and $C > 0$ such that

$$\|\sigma(A)\|_{k,S(m^N,g)} \leq C\|A\|_{L(\mathcal{B}^{-s}(\mathbf{R}^d),\mathcal{B}^s(\mathbf{R}^d))}$$

for all $A \in L(\mathcal{B}^{-s}(\mathbf{R}^d), \mathcal{B}^s(\mathbf{R}^d))$. Take $M \in \mathbf{N}$ such that $2M \geq N$ and $cM \geq s$. Since

$$(1 - r^w)^{-1} - \sum_{j=0}^{N-1}(r^w)^j = \sum_{N \leq j \leq 2M-1}(r^w)^j + (r^w)^M(1 - r^w)^{-1}(r^w)^M$$
$$\in \operatorname{Op} S(m^N, g) + L(\mathcal{B}^{-s}(\mathbf{R}^d), \mathcal{B}^s(\mathbf{R}^d)),$$

we have

$$\Big\|\sigma((1 - r^w)^{-1} - \sum_{j=0}^{N-1}(r^w)^j)\Big\|_{k,S(m^N,g)} < \infty,$$

which completes the proof. □

Lemma 3.6. *Let $a \in S(\gamma^{-1}, g)$ be real scalar, and let $b \in S(1, g; M_n(\mathbf{C}))$ such that $b = b^* \geq cI_n$ for a constant $c > 0$. Then for every $0 < c_0 < c$ there is $C > 0$ such that*

$$(a^2 b)^w \geq c_0 (a^w)^2 I_n - C I_n$$

as a form on $\mathcal{S}(\mathbf{R}^d, \mathbf{C}^n)$.

Proof. Set $b_0 = b - c_0 I_n$, $p_0 = b_0^{1/2} \in S(1, g; M_n(\mathbf{C}))$ and $b_1 = \sigma(b_0^w - (p_0^w)^2) \in S(\gamma, g; M_n(\mathbf{C}))$. If an Hermitian matrix $h \in M_n(\mathbf{C})$ has the eigenvalues $\lambda_1, \dots, \lambda_n$, then the real linear map $u \mapsto hu + uh$ on the real vector space of Hermitian matrices has the eigenvalues $\lambda_j + \lambda_k$ $(1 \le j, k \le n)$. So $p_0 p_1 + p_1 p_0 = b_1$ has a unique solution $p_1 = p_1^* \in S(\gamma, g; M_n(\mathbf{C}))$. Put $p = p_0 + p_1$. Then

$$b_2^w = b_0^w - (p^w)^2 = b_1^w - (p_0^w p_1^w + p_1^w p_0^w + (p_1^w)^2) \in \operatorname{Op} S(\gamma^2, g; M_n(\mathbf{C})).$$

On the other hand, since a is scalar,

$$\begin{aligned} a^w b^w a^w &= ((ab)^w + \{a, b\}^w/(2i)) a^w + r_1^w \\ &= (a^2 b)^w + \{ab, a\}^w/(2i) + (\{a, b\}a)^w/(2i) + r_2^w = (a^2 b)^w + r_2^w, \end{aligned}$$

where $r_j \in S(1, g; M_n(\mathbf{C}))$. Therefore

$$\begin{aligned} (a^2 b)^w &= a^w b^w a^w - r_2^w \\ &= c_0 (a^w)^2 I_n + a^w (p^w)^2 a^w + a^w b_2^w a^w - r_2^w \ge c_0 (a^w)^2 I_n - C I_n. \end{aligned}$$

Here $C > 0$. □

Lemma 3.7. *Let m_1, m_2, and m_3 be g weights. If (a_1, a_2, a_3) varies in a bounded subset of $S(m_1, g) \times S(m_2, g) \times S(m_3, g)$ in such a way that $a_1 a_3 = 0$, then $a_1 \# a_2 \# a_3$ remains bounded in $S(\gamma^N m_1 m_2 m_3, g)$ for every $N \in \mathbf{N}_0$. Here γ is defined in* (G3).

Proof. Since $a_1 \# a_2 \# a_3 = r_N(a_1 \# a_2 - r_N(a_1, a_2), a_3) + r_N(a_1, a_2) \# a_3$ for every $N \in \mathbf{N}_0$, the proof is complete. □

In application, we shall use a parameter-dependent version of the calculus above. Let Λ be an index set, and let m_λ be a g weight with the constants in $(M1)$ and $(M2)$ independent of $\lambda \in \Lambda$. We say that $a_\lambda \in S(m_\lambda, g)$ uniformly in $\lambda \in \Lambda$ if $\sup_{\lambda \in \Lambda} \|a_\lambda\|_{k, S(m_\lambda, g)} < \infty$ for every $k \in \mathbf{N}_0$. Similarly, we say that $a_\lambda \in S_N(m_\lambda, q_0, g)$ uniformly in $\lambda \in \Lambda$ if $\sup_{\lambda \in \Lambda} \|a_\lambda\|_{k, S_N(m_\lambda, q_0, g)} < \infty$ for every $k \in \mathbf{N}_0$. Then all the statements in this section have the natural parameter dependent version, which will be used later.

Finally, we define time dependent symbol classes.

Definition 3.8. For an interval $I \subset \mathbf{R}$ and a symbol space S the space $B(I, S)$ consists of all $p : I \to S$ such that $p(K)$ is bounded in S for every compact subset K of I and that $I \ni t \mapsto p(t) \in C^\infty(\mathbf{R}^{2d}, M_n(\mathbf{C}))$ is continuous.

§4. Well-posedness of the Cauchy Problem

In this section, we assume (H1)–(H4). Define the Riemannian metric g_δ on $\mathbf{R}^{2d}$ by

$$(g_\delta)_X = \langle x\rangle^{2\delta}|dx|^2 + \langle X\rangle^{-2}|d\xi|^2, \quad X = (x,\xi) \in \mathbf{R}^{2d},$$

which satisfies (G1)–(G3). We shall use (the time dependent version of) Lemma 3.4 with

$$g = g_\delta,\ \varphi_0 = \langle x\rangle^{-1},\ \gamma = \langle x\rangle^{\delta}/\langle X\rangle,\ \gamma_0 = 1/(\langle x\rangle\langle X\rangle).$$

By the definitions,

$$\begin{aligned} &h_0 \in S_1(\langle X\rangle^2, \langle x\rangle^{-1}, g_\delta),\\ &h_j(\cdot) \in B(\mathbf{R}, S_1(\langle X\rangle^{2-j}\langle x\rangle^j, \langle x\rangle^{-1}, g_\delta; M_n(\mathbf{C}))), \quad j = 1, 2. \end{aligned}$$

Fix a compact interval $I = [t_1, t_2]$ $(t_1 < t_2)$.

After preparing Lemmas 4.1–4.4, we shall prove two well-posedness theorems of the Cauchy problem for Schrödinger equations.

Lemma 4.1. *Let $s \in \mathbf{R}$. For $L \geq 1$, set*

$$e_{s,L} = (h_0(x,\xi) + |x|^2 + L^2)^{s/2} \in S_1(\langle X\rangle^s, \langle x\rangle^{-1}, g_\delta).$$

Then there exists $L(s) \geq 1$ such that for every $L \geq L(s)$

$$(e_{s,L}^w)^{-1} = e_{-s,L}^w(1 - r_{s,L}{}^w)^{-1} \in \operatorname{Op} S(\langle X\rangle^{-s}, g_\delta)$$

with $r_{s,L} \in S(\langle x\rangle^{\delta-1}\langle X\rangle^{-2}, g_\delta)$ satisfying $\|r_{s,L}^w\| \leq 1/2$.

Remark. Setting $e_s = e_{s,L(s)}$ and $E_s = e_s^w$, we can use $\|E_s \cdot\|$ as a norm of $\mathcal{B}^s(\mathbf{R}^d)$ or $\mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$ (see the example after Theorem 3.2).

Proof. Set $\langle X\rangle_L = (L^2 + |X|^2)^{1/2}$. Since

$$\partial_{\xi_j} e_{s,L} \in S_1(\langle X\rangle\langle X\rangle_L^{s-2}, \langle x\rangle^{-1}, g_\delta),\ \partial_{x_j} e_{s,L} \in S(\langle x\rangle^{-1}\langle X\rangle^2\langle X\rangle_L^{s-2}, g_\delta),$$

all uniformly in $L \geq 1$, it follows that

$$\begin{aligned} &\sigma(D_X, D_Y)^2 e_{s,L}(X)e_{-s,L}(Y) = \sum_{k=1}^{3} a_{k,L}(X)b_{k,L}(Y);\\ &a_{1,L} \in S(\langle x\rangle^{\delta-1}\langle X\rangle^2\langle X\rangle_L^{s-2}, g_\delta), \quad b_{1,L} \in S(\langle X\rangle_L^{-s-2}, g_\delta),\\ &a_{2,L} \in S(\langle x\rangle^{-1}\langle X\rangle\langle X\rangle_L^{s-2}, g_\delta), \quad b_{2,L} \in S(\langle x\rangle^{-1}\langle X\rangle\langle X\rangle_L^{-s-2}, g_\delta),\\ &a_{3,L} \in S(\langle X\rangle_L^{s-2}, g_\delta), \quad b_{3,L} \in S(\langle x\rangle^{\delta-1}\langle X\rangle^2\langle X\rangle_L^{-s-2}, g_\delta), \end{aligned}$$

all uniformly in $L \geq 1$. This implies, by Lemma 3.3, that

$$r_{s,L} := -r_2(e_{s,L}, e_{-s,L}) = 1 - e_{s,L} \# e_{-s,L} \in S(\langle x\rangle^{\delta-1} \langle X\rangle^2 \langle X\rangle_L^{-4}, g_\delta)$$

uniformly in $L \geq 1$. Take $L(s) \geq 1$ so that $\|r_{s,L}^w\| \leq 1/2$ for every $L \geq L(s)$. Fix $L \geq L(s)$. By Lemma 3.5, we have $(1 - r_{s,L}^w)^{-1} \in \operatorname{Op} S(1, g_\delta)$. Therefore, $(e_{s,L}^w)^{-1} = e_{-s,L}^w (1 - r_{s,L}{}^w)^{-1} = (1 - \overline{r_{s,L}}{}^w)^{-1} e_{-s,L}^w \in \operatorname{Op} S(\langle X\rangle^{-s}, g_\delta)$. □

Lemma 4.2. *$E_s H(t) E_s^{-1} = H(t) + B_s(t)$ with $B_s(\cdot) \in \operatorname{Op} B(I, S(1, g_\delta; M_n(\mathbf{C})))$.*

Proof. Since $h_1 + h_2 \in B(I, S_1(\langle X\rangle\langle x\rangle, \langle x\rangle^{-1}, g_\delta; M_n(\mathbf{C})))$ and $e_s \in S_1(\langle X\rangle^s, \langle x\rangle^{-1}, g_\delta)$, Lemma 3.4 (3) gives

$$\sigma([E_s, h_1^w(\cdot) + h_2^w(\cdot)]) \in B(I, S(\langle X\rangle^s, g_\delta; M_n(\mathbf{C}))).$$

Since $h_0 \in S_1(\langle X\rangle^2, \langle x\rangle^{-1}, g_\delta)$, Lemma 3.4 (3) implies

$$\sigma([E_s, h_0^w]) - \{e_s, h_0\}/i \in S(\langle x\rangle^{2\delta-1} \langle X\rangle^{s-1}, g_\delta) \subset S(\langle X\rangle^s, g_\delta).$$

Thanks to the special form of e_s, we have

$$\{e_s, h_0\} = \frac{s}{2} e_{s-2} \{|x|^2, h_0\} \in S(\langle X\rangle^s, g_\delta).$$

Therefore, $[E_s, h_0^w] \in \operatorname{Op} S(\langle X\rangle^s, g_\delta)$. In conclusion, $E_s H(t) E_s^{-1} = H(t) + B_s(t)$ with $B_s(\cdot) = [E_s, H(\cdot)] E_s^{-1} \in \operatorname{Op} B(I, S(1, g_\delta; M_n(\mathbf{C})))$. □

Lemma 4.3. *Let $j \in C_0^\infty(\mathbf{R})$ such that $j(0) = 1$ and $j \geq 0$. Set $j_\varepsilon(X) = j(\varepsilon e_1(X))$ and $J_\varepsilon = j_\varepsilon^w$ for $0 < \varepsilon \leq 1$. Then $(j_\varepsilon)_{0<\varepsilon\leq 1}$ is bounded in $S_1(1, \langle x\rangle^{-1}, g_\delta)$, and so is $(\sigma([H(t), J_\varepsilon]))_{t\in I, 0<\varepsilon\leq 1}$ in $S(1, g_\delta; M_n(\mathbf{C}))$. Moreover, $\sigma([H(t), J_\varepsilon]) \to 0$ weakly in $S(1, g_\delta; M_n(\mathbf{C}))$ as $\varepsilon \to +0$ for each $t \in I$.*

Proof. By direct calculation, $(j_\varepsilon)_{0<\varepsilon\leq 1}$ is bounded in $S_1(1, \langle x\rangle^{-1}, g_\delta)$. By Lemma 3.4 (3),

$$\begin{aligned}
&\sigma([h_1^w(t) + h_2^w(t), J_\varepsilon]) \in S(1, g_\delta; M_n(\mathbf{C})),\\
&\sigma([h_0^w, J_\varepsilon]) - \{h_0, j_\varepsilon\}/i \in S(\langle x\rangle^{2\delta-1} \langle X\rangle^{-1}, g_\delta) \subset S(1, g_\delta),\\
&\{h_0, j_\varepsilon\} = \varepsilon j'(\varepsilon e_1) \{h_0, e_1\} = \varepsilon e_1 j'(\varepsilon e_1) \{h_0, e_1\}/e_1 \in S(1, g_\delta),
\end{aligned}$$

all uniformly in $0 < \varepsilon \leq 1$ and $t \in I$. Since $[H(t), J_\varepsilon]u \to 0$ in $\mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n)$ as $\varepsilon \to +0$ for all $u \in \mathcal{S}(\mathbf{R}^d, \mathbf{C}^n)$, the proof is complete. □

Lemma 4.4. *Let* $s \in \mathbf{R}$ *and* $t_0 \in I$. *Set* $\gamma = \sup_{t\in I} \|B_s(t)\|$, *where* $E_s H(t) E_s^{-1} = H(t) + B_s(t)$. *Then*

$$(4.1)\quad e^{-\gamma|t-t_0|}\| E_s u(t) \| \leq \| E_s u(t_0) \| + \left| \int_{t_0}^{t} e^{-\gamma|\tau-t_0|}\| E_s f(\tau) \| \, d\tau \right|, \quad t \in I,$$

for all $u \in C(I, \mathcal{B}^{s+2}(\mathbf{R}^d, \mathbf{C}^n)) \cap C^1(I, \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$. *Here* $f(t) = (\partial_t + iH(t))$ $u(t)$.

Proof. Since $v = E_s u \in C(I, \mathcal{B}^2(\mathbf{R}^d, \mathbf{C}^n)) \cap C^1(I, \mathcal{B}^0(\mathbf{R}^d, \mathbf{C}^n))$ satisfies

$$E_s f(t) = (\partial_t + iH(t) + iB_s(t))v(t),$$

we obtain

$$\begin{aligned}\partial_t \|v(t)\|^2 &= 2\Re(-(iH(t) + iB_s(t))v(t) + E_s f(t), v(t)) \\ &\leq 2\|v(t)\|(\,\gamma\|v(t)\| + \|E_s f(t)\|\,), \quad t \in I,\end{aligned}$$

which implies

$$\partial_t \|v(t)\| \leq \gamma\|v(t)\| + \|E_s f(t)\|, \quad \text{a.e. } t \in I.$$

By a Gronwall-type inequality, we get (4.1) if $t \geq t_0$. We can deal with the case $t \leq t_0$ similarly. □

Theorem 4.5. *Let* $s \in \mathbf{R}$ *and* $t_0 \in I$. *For every* $u_0 \in \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$ *and* $f \in L^1(I, \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$, *there exists* $u \in C(I, B^s(\mathbf{R}^d, \mathbf{C}^n))$ *satisfying*

$$(4.2)\qquad (\partial_t + iH(\cdot))u = f \text{ in } \mathcal{D}'((t_1, t_2) \times \mathbf{R}^d, \mathbf{C}^n), \quad u(t_0) = u_0,$$

which is unique in $C(I, \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n))$. *Moreover, the estimate* (4.1) *holds.*

Proof. Uniqueness. Suppose that $u \in C(I, \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n))$ is a solution of (4.2) with $u_0 = 0$ and $f = 0$. Since $\{u(t); t \in I\}$ is bounded in some $\mathcal{B}^{s+4}(\mathbf{R}^d, \mathbf{C}^n)$, it follows from the equation that $u \in C(I, \mathcal{B}^{s+2}(\mathbf{R}^d, \mathbf{C}^n))$ (in fact, Lipshitz continuous) and hence $u \in C^1(I, \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$. By (4.1), we get $u = 0$.

Existence. We treat the case $t_1 = t_0$ (we can treat the case $t_2 = t_0$ similarly and hence the remaining case by combining the both cases). For simplicity, we assume $t_0 = 0$ and $t_2 = T > 0$.

First, assume $u_0 \in \mathcal{B}^{s+4}(\mathbf{R}^d, \mathbf{C}^n)$ and $f \in C(I, \mathcal{B}^{s+4}(\mathbf{R}^d, \mathbf{C}^n))$. If there is a solution $u \in C(I, \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n))$, then $u \in C^1(I, \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n))$ and it satisfies

$$(4.3)\qquad \int_0^T (-(\partial_t + iH(t))v(t), u(t))dt = (v(0), u_0) + \int_0^T (v(t), f(t))dt$$

for every $v \in \mathcal{Y} = \{v \in C^1(I, \mathcal{S}(\mathbf{R}^d, \mathbf{C}^n)); v(T) = 0\}$. Set

$$\mathcal{X} = \{\phi(\cdot) = -(\partial_t + iH(\cdot))v(\cdot); v \in \mathcal{Y}\}.$$

By Lemma 4.4 we have $\sup_{t\in I} \|E_{-s-4}v(t)\| \leq C\|\phi\|_{\in L^1(I,\mathcal{B}^{-s-4}(\mathbf{R}^d,\mathbf{C}^n))}$, and the functional

$$\mathcal{X} \ni \phi(\cdot) = -(\partial_t + iH(\cdot))v(\cdot) \mapsto (v(0), u_0) + \int_0^T (v(t), f(t))dt \in \mathbf{C}$$

is bounded if $\mathcal{X}$ is regarded as a subspace of $L^1(I, \mathcal{B}^{-s-4}(\mathbf{R}^d, \mathbf{C}^n))$. By the Hahn-Banach theorem, there is $u \in L^\infty(I, \mathcal{B}^{s+4}(\mathbf{R}^d, \mathbf{C}^n))$ such that (4.3) holds for all $v \in \mathcal{Y}$. (In fact, the Hahn-Banach theorem is not necessary, because we can prove that $\mathcal{X}$ is dense in $L^1(I, \mathcal{B}^{-s-4}(\mathbf{R}^d, \mathbf{C}^n))$). Taking $v \in C_0^\infty((0,T) \times \mathbf{R}^d, \mathbf{C}^n)$, we obtain

$$(\partial_t + iH(\cdot))u = f \text{ in } \mathcal{D}'((0,T) \times \mathbf{R}^d, \mathbf{C}^n),$$

which implies $u \in C^1(I, \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$. By integrating (4.3) by parts, we have $(v(0), u(0)) = (v(0), u_0)$ for all $v \in \mathcal{Y}$; hence $u(0) = u_0$. So $u \in C^1(I, \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$ is the solution of (2.1).

Next, assume $u_0 \in \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$ and $f \in L^1(I, \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$. Take $u_{0,j} \in \mathcal{B}^{s+4}(\mathbf{R}^d, \mathbf{C}^n)$ and $f_j \in C(I, \mathcal{B}^{s+4}(\mathbf{R}^d, \mathbf{C}^n))$ such that $u_{0,j} \to u_0$ in $\mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$ and $f_j \to f$ in $L^1(I, \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$ as $j \to \infty$. Let $u_j \in C^1(I, \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$ be the solution of (2.1) with u_0 and f replaced by $u_{0,j}$ and f_j. Then (u_j) is a Cauchy sequence in $C(I, \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$ by Lemma 4.4, and its limit u satisfies (4.1) and (4.2). □

For the proof of Theorem 2.5, we need to generalize Theorem 4.5 so that it can allow a nonsymmetric perturbation of lower order. For simplicity, we treat only the forward Cauchy problem with $I = [0,T]$ and $t_0 = 0$.

Theorem 4.6. *Let* $p(t) = ih_0 I_n + ip_1(t) + p_2(t) + p_3(t) \quad (t \in [0,T])$ *such that*

(H6) $p_1 = p_1^* \in B([0,T], S_1(\langle x\rangle\langle X\rangle, \langle x\rangle^{-1}, g_\delta; M_n(\mathbf{C})))$;

(H7) $p_2 = \sum_{j=0}^N \alpha_j^2 \beta_j$, *where* $\alpha_j \in B([0,T], S(\langle X\rangle/\langle x\rangle^\delta, g_\delta))$ *is real scalar, and* $\beta_j = \beta_j^* \in B([0,T], S(1, g_\delta; M_n(\mathbf{C})))$ *satisfies* $\beta_j \geq I_n$ $(j = 0, 1, \ldots, N)$;

(H8) $p_3 \in B([0,T], S(\langle X\rangle/\langle x\rangle^\delta, g_\delta; M_n(\mathbf{C})))$ *such that* $\Re p_3 \geq -CI_n$ *with* $C > 0$.

(1) *For every* $s \in \mathbf{R}$, *there are* $C_1, C_2 > 0$ *such that*

$$\| E_s u(t) \| \leq C_1 \| E_s u(0) \| + C_1 \int_0^t \| E_s f(\tau) \| \, d\tau, \tag{4.4}$$

$$\sum_{j=0}^{N} \int_0^t \|\alpha_j^w(\tau) E_s u(\tau)\|^2 d\tau \leq C_2 \left(\| E_s u(0) \| + \int_0^t \| E_s f(\tau) \| \, d\tau \right)^2, \tag{4.5}$$

for all $t \in [0,T]$ *and* $u \in C([0,T], \mathcal{B}^{s+2}(\mathbf{R}^d, \mathbf{C}^n)) \cap C^1([0,T], \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$. *Here* $f = (\partial_t + p^w(t))u$.

(2) *Let* $s \in \mathbf{R}$. *For every* $u_0 \in \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$ *and* $f \in L^1([0,T], \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$, *there is* $u \in C([0,T], \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$ *satisfying*

$$(\partial_t + p^w(t))u = f \text{ in } \mathcal{D}'((0,T) \times \mathbf{R}^d, \mathbf{C}^n), \quad u(0) = u_0, \tag{4.6}$$

which is unique in $C([0,T], \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n))$. *Moreover, for every* $j = 0, \ldots, N$, $\alpha_j^w(\cdot) E_s u(\cdot) \in L^2([0,T], L^2(\mathbf{R}^d, \mathbf{C}^n))$, *and the estimates* (4.4) *and* (4.5) *hold.*

Proof. (1) Let $s \in \mathbf{R}$. By Lemma 4.1 and Theorem 3.1,

$$[E_s, p_2^w(t) + p_3^w(t)] E_s^{-1} = -i(H_{e_s} p_2(t)/e_s)^w + r_1^w(t)$$

with $r_1 \in B([0,T], S(1, g_\delta; M_n(\mathbf{C})))$. Similarly to the proof of Lemma 4.2, we obtain

$$[E_s, ih_0^w I_n + ip_1^w(t)] \, E_s^{-1} = r_2^w(t)$$

with $r_2 \in B([0,T], S(1, g_\delta; M_n(\mathbf{C})))$. To sum up,

$$\tilde{p}^w(t) = E_s p^w(t) E_s^{-1} = (p(t) - iH_{e_s} p_2(t)/e_s + r_1(t) + r_2(t))^w.$$

By Theorem 3.2 and Lemma 3.6,

$$p_2^w(t) + p_3^w(t) \geq 2^{-1} \sum_{j=0}^{N} \alpha^w(t)^2 I_n - C_1 I_n$$

with $C_1 > 0$. Since $v = E_s u \in C([0,T], \mathcal{B}^2(\mathbf{R}^d, \mathbf{C}^n)) \cap C^1([0,T], \mathcal{B}^0(\mathbf{R}^d, \mathbf{C}^n))$ satisfies $E_s f = (\partial_t + \tilde{p}^w(t))v$, we obtain

$$\begin{aligned} \partial_t \|v(t)\|^2 &= 2\Re\,(-(p_2(t) + p_3(t) + r_1(t) + r_2(t))^w v(t) + E_s f(t), v(t) \,) \\ &\leq 2\|v(t)\|(\, C_2\|v(t)\| + \|E_s f(t)\| \,) - \sum_{j=0}^{N} \|\alpha_j^w(t) v(t)\|^2, \quad t \in [0,T], \end{aligned}$$

which implies

$$\partial_t \|v(t)\| \leq C_2 \|v(t)\| + \|E_s f(t)\|, \quad \text{a.e. } t \in [0, T].$$

By a Gronwall-type inequality, we get (4.4). Since

$$\sum_{j=0}^{N} \int_0^t \|\alpha_j^w(\tau) v(\tau)\|^2 d\tau$$
$$\leq \|v(0)\|^2 + 2C_2 t \sup_{\tau \in [0,t]} \|v(\tau)\|^2 + 2 \sup_{\tau \in [0,t]} \|v(\tau)\| \cdot \int_0^t \|E_s f(\tau)\| d\tau,$$

we obtain (4.5) by virtue of (4.4).

(2) The proof of (1) shows that (4.4) and (4.5) hold also when f is defined as $f(t) = (\partial_t + p(T-t)^{*w})u(t)$. By taking $u(\cdot) = v(T - \cdot)$, we obtain the following: for every $s \in \mathbf{R}$, there is $C > 0$ such that

$$\| E_s v(t) \| \leq C \| E_s v(T) \| + C \int_t^T \| E_s f(\tau) \| d\tau, \ 0 \leq t \leq T,$$

for all $v \in C([0, T], \mathcal{B}^{s+2}(\mathbf{R}^d, \mathbf{C}^n)) \cap C^1([0, T], \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n))$ with $f(t) = (-\partial_t + p(t)^{*w})v(t)$. After this preparation, we can prove the first part of (2) similarly to Theorem 4.5 if we define

$$\mathcal{X} = \{\phi(\cdot) = (-\partial_t + p(\cdot)^{*w})v(\cdot) \in L^1(I, \mathcal{B}^{-s-4}(\mathbf{R}^d, \mathbf{C}^n)); v \in \mathcal{Y}\}.$$

We can prove the second part, additional regularities of solutions, by approximation argument in view of (4.5). □

§5. Transformation of the Schrödinger Operator

This section shows how the Schrödinger operator transforms when conjugated by an invertible pseudodifferential operator. The result will be used in the next section.

Let g be a Riemannian metric on $\mathbf{R}^{2d}$ satisfying (G1)–(G3). We assume that g is of the form

$$g_X = \varphi(X)^2 |dx|^2 + \Phi(X)^2 |d\xi|^2,$$

where φ and Φ are positive functions. Then φ and Φ are g weights by (G1)–(G3). Let φ_0 be a g weight such that $\varphi_0 \leq \varphi$, and set $\gamma = \varphi\Phi \leq 1$ and $\gamma_0 = \varphi_0 \Phi$. Let $(\phi_L)_{L \geq 1}$ be a bounded family of $S(1, g)$ such that $1 - \phi_L \in C_0^\infty(\mathbf{R}^{2d})$, $0 \leq \phi_L \leq 1$, and $\operatorname{supp} \phi_L \subset \{X \in \mathbf{R}^{2d}; |X| \geq L\}$.

Lemma 5.1. *Let W be a g weight such that $c_0 \le W \le \gamma^{-1}$ with some $c_0 > 0$, and define $G = Wg$. Then G satisfies* (G1)–(G3). *Moreover, every g weight is a G weight.*

Proof. There are $0 < c < 1$ and $C > 0$ such that if $g_X(X-Y) \le c$, then $1/C \le W(X)/W(Y) \le C$ and $1/C \le g_X/g_Y \le C$, which gives $1/C^2 \le G_X/G_Y \le C^2$. Thus $G_X(X-Y) \le cc_0$ implies $1/C^2 \le G_X/G_Y \le C^2$. By definition, $\sup_{Y\neq 0} G_X(Y)/G_X^\sigma(Y) = (W(X)\gamma(X))^2 \le 1$. We now consider the σ temperance of G. Since $g_Y(X-Y) \le c$ implies $G_Y \le C^2 G_X$, we assume $g_Y(X-Y) \ge c$. Then $g_Y^\sigma(X-Y) \le c^{-1} g_Y(X-Y) g_Y^\sigma(X-Y) \le c^{-1}\gamma(Y)^2 g_Y^\sigma(X-Y)^2 \le c^{-1} G_Y^\sigma(X-Y)^2$; therefore,

$$1 + g_Y^\sigma(X-Y) \le c^{-1}(1 + G_Y^\sigma(X-Y))^2. \tag{5.1}$$

On the other hand, there are $C_1 > 0$ and $N > 0$ such that $g_Y \le C_1 g_X(1 + g_Y^\sigma(X-Y))^N$ and $W(Y) \le C_1 W(X)(1+g_Y^\sigma(X-Y))^N$. Thus, $G_Y \le C_1^2 c^{-2N}(1+G_Y^\sigma(X-Y))^{4N} G_X$.

Let m be a g weight. Then G continuity of m follows from g continuity; σ, G temperance from (5.1). □

We recall that the symbol $r_j(\cdot,\cdot)$ (the j-th remainder term of the symbol product) is defined just before Theorem 3.1.

Lemma 5.2. *Assume $\gamma \le C\langle X\rangle^{-c}$ with some $c > 0$ and $C > 0$. Let W be a g weight such that $W \ge c_0$ with some $c_0 > 1$ and that $(\log W)^2 \le \gamma^{-1}$, and define $G = (\log W)^2 g$.*

(1) If $\lambda \in S_1(\log W, \varphi_0, g)$ and $\lambda \le m \log W + C$ with $m, C \in \mathbf{R}$, then $e^\lambda \in S_1(W^m, \varphi_0 \log W, G)$.

(2) Let W_j be g weights, $p_j \in S_1(W_j, \varphi_0, g)$, and $\lambda_j \in S_1(\log W, \varphi_0, g)$ $(j = 1, 2)$. Then

$$\begin{aligned} &r_0(e^{\lambda_1}p_1, e^{\lambda_2}p_2)e^{-(\lambda_1+\lambda_2)} - p_1p_2 \in S(W_1W_2\gamma_0(\log W)^2, g);\\ &r_N(e^{\lambda_1}p_1, e^{\lambda_2}p_2)e^{-(\lambda_1+\lambda_2)} \in S(W_1W_2\gamma_0\gamma^{N-1}(\log W)^{2N}, g),\ N \in \mathbf{N}. \end{aligned}$$

If in addition $\lambda_1 = 0$ or $\lambda_2 = 0$, then

$$\begin{aligned} &r_0(e^{\lambda_1}p_1, e^{\lambda_2}p_2)e^{-(\lambda_1+\lambda_2)} - p_1p_2 \in S(W_1W_2\gamma_0 \log W, g);\\ &r_N(e^{\lambda_1}p_1, e^{\lambda_2}p_2)e^{-(\lambda_1+\lambda_2)} \in S(W_1W_2\gamma_0\gamma^{N-1}(\log W)^{N}, g),\ N \in \mathbf{N}. \end{aligned}$$

(3) Let $\lambda \in S_1(\log W, \varphi_0, g)$. Set $\lambda_L = \lambda\phi_L$ and $r_L = r_2(e^{\lambda_L}, e^{-\lambda_L})$. Then $r_L \in S(\gamma\gamma_0(\log W)^4, g)$ for each $L \ge 1$ and there is $L_0 \ge 1$ such that $\|r_L^w\| \le 1/2$ for every $L \ge L_0$. In particular, $((e^{\lambda_L})^w)^{-1} = (e^{-\lambda_L})^w(1 + r_L^w)^{-1}$.

(4) *Let* λ_L *be the symbol in* (3) *with* $L \geq L_0$ *being fixed. Let* W_0 *be a* g *weight, and* $a \in S_1(W_0, \varphi_0, g)$. *Then*

$$(e^{\lambda_L})^w a^w(\,(e^{\lambda_L})^w\,)^{-1} = (a + H_\lambda a/i - H_\lambda^2 a/2)^w + r^w$$

with $r \in S(W_0\gamma_0\gamma^2(\log W)^5, g)$.

Remark. The function $\log W$ is a g weight because so is W and inf $W > 1$.

Remark. If $\varphi_0 = \varphi$, then the claims (1)–(4) are simplified: $\gamma_0 = \gamma$ in (1)–(4); $\lambda \in S(\log W, g)$ and $e^\lambda \in S(W^m, G)$ replace $\lambda \in S_1(\log W, \varphi_0, g)$ and $e^\lambda \in S_1(W^m, \varphi_0 \log W, G)$ respectively in (1); $p_j \in S(W_j, g)$ and $\lambda_j \in S(\log W, g)$ replace $p_j \in S_1(W_j, \varphi_0, g)$ and $\lambda_j \in S_1(\log W, \varphi_0, g)$ respectively in (2); $\lambda \in S(\log W, g)$ replaces $\lambda \in S_1(\log W, \varphi_0, g)$ in (3); $a \in S(W_0, g)$ replaces $a \in S_1(W_0, \varphi_0, g)$ in (4).

Proof. (1) This is by simple calculation.

(2) Choose $m_j, C_j \geq 0$ so that $|\lambda_j| \leq m_j \log W + C_j$ $(j = 1, 2)$. Let $N \in \mathbf{N}$. For every $k \in \mathbf{N}_0$ there are $M \in \mathbf{N}$, $M > N$, and $C > 0$ such that

$$\|a\|_{k, S(W_1W_2\gamma_0\gamma^{N-1}(\log W)^{2N}, g)} \leq C\|a\|_{k, S(W^{2m_1+2m_2}W_1W_2\gamma^M(\log W)^{2M}, G)}$$

for all $a \in S(W^{2m_1+2m_2}W_1W_2\gamma^M(\log W)^{2M}, G)$ by the assumption $\gamma \leq C\langle X\rangle^{-c}$, because every g weight is polynomially bounded. Since $r_N(e^{\lambda_1}p_1, e^{\lambda_2}p_2) = e^{\lambda_1+\lambda_2}\sum_{j=N}^M q_j$ with

$$\begin{aligned} q_j(X) &= e^{-(\lambda_1(X)+\lambda_2(X))}\frac{1}{j!}(i\sigma(D_X, D_Y)/2)^j e^{\lambda_1(X)}p_1(X)e^{\lambda_2(Y)}p_2(Y)|_{Y=X} \\ &\in S(W_1W_2\gamma_0\gamma^{j-1}(\log W)^{2j}, g) \quad (j = N, N+1, \ldots, M-1) \\ q_M(X) &= r_M(e^{\lambda_1}p_1, e^{\lambda_2}p_2)e^{-(\lambda_1+\lambda_2)} \in S(W^{2m_1+2m_2}W_1W_2\gamma^M(\log W)^{2M}, G), \end{aligned}$$

we have

$$\|r_N(e^{\lambda_1}p_1, e^{\lambda_2}p_2)e^{-(\lambda_1+\lambda_2)}\|_{k, S(W_1W_2\gamma_0\gamma^{N-1}(\log W)^{2N}, g)} < \infty.$$

This implies

$$\begin{aligned} &r_N(e^{\lambda_1}p_1, e^{\lambda_2}p_2)e^{-(\lambda_1+\lambda_2)} \in S(W_1W_2\gamma_0\gamma^{N-1}(\log W)^{2N}, g), \\ &r_0(e^{\lambda_1}p_1, e^{\lambda_2}p_2)e^{-(\lambda_1+\lambda_2)} - p_1p_2 = r_1(e^{\lambda_1}p_1, e^{\lambda_2}p_2)e^{-(\lambda_1+\lambda_2)}. \end{aligned}$$

The other statements can be proved similarly.

(3) By (2) we have $r_L \in S(\gamma\gamma_0(\log W)^4, g)$ for each $L \geq 1$. In the rest of the proof of (3), all statements are uniform in $L \geq 1$. Take $m, C \geq 0$ such that $|\lambda| \leq m \log W + C$. Take $N \in \mathbf{N}$ such that $\sup W^{2m}(\log W)^{2N}\gamma^{N-1} < \infty$. By definition, $r_L = r_2(e^{\lambda_L}, e^{-\lambda_L}) = \sum_{j=2}^{N-1} c_{j,L} + r_N(e^{\lambda_L}, e^{-\lambda_L})$, where

$$c_{j,L}(X) = \frac{1}{j!}(i\sigma(D_X, D_Y)/2)^j e^{\lambda_L}(X)e^{-\lambda_L}(Y)|_{Y=X} \in S(\gamma^j(\log W)^{2j}, g).$$

Since $\operatorname{supp} c_{j,L} \subset \operatorname{supp} \lambda_L$, we have $\sum_{j=2}^{N-1} c_{j,L} \in S(L^{-c}, g)$. On the other hand,

$$\sigma(D_X, D_Y)^N e^{\lambda_L}(X)e^{-\lambda_L}(Y) = \sum_{k=0}^{N} a_{1,k,L}(X)a_{2,k,L}(Y),$$

where $a_{1,k,L} \in S(W^m\varphi^k\Phi^{N-k}(\log W)^N, G)$, $a_{2,k,L} \in S(W^m\varphi^{N-k}\Phi^k (\log W)^N, G)$, and $\operatorname{supp} a_{j,k,L} \subset \operatorname{supp} \lambda_L$. Thus $a_{1,k,L} \in S(L^{-c}\gamma^{-1}W^m\varphi^k\Phi^{N-k} (\log W)^N, G)$. By Lemma 3.3 we get $r_N(e^{\lambda_L}, e^{-\lambda_L}) \in S(L^{-c}W^{2m}(\log W)^{2N} \gamma^{N-1}, G) \subset S(L^{-c}, G)$. Therefore, $\|r_L^w\| = O(L^{-c})$ as $L \to \infty$.

(4) Fix $L \geq L_0$. Since $e^{\pm\lambda_L} - e^{\pm\lambda} \in C_0^\infty(\mathbf{R}^{2d})$, we have

$$\begin{aligned}&(e^{\lambda_L})^w a^w((e^{\lambda_L})^w)^{-1} = a^w + [(e^{\lambda_L})^w, a^w](e^{-\lambda_L})^w(1 + r_L^w)^{-1}\\ &= a^w + [(e^{\lambda})^w, a^w](e^{-\lambda})^w + c_1^w + c_2^w\end{aligned}$$

where $c_1 \in \mathcal{S}(\mathbf{R}^{2d})$ and $c_2^w = [(e^\lambda)^w, a^w](e^{-\lambda})^w(\,(1 + r_L^w)^{-1} - 1\,)$. By (2), we have

$$\begin{gathered}\sigma(\,[(e^\lambda)^w, a^w]\,) = e^\lambda(H_\lambda a/i + b),\\ b = e^{-\lambda}(\,r_3(e^\lambda, a) - r_3(a, e^\lambda)\,) \in S(W_0\gamma_0\gamma^2(\log W)^3, g);\\ r_0(e^\lambda(H_\lambda a/i + b),\, e^{-\lambda}) = H_\lambda a/i - H_\lambda^2 a/2 + c_3 \in S(W_0\gamma_0 \log W, g),\\ c_3 = r_2(e^\lambda H_\lambda a/i, e^{-\lambda}) + r_0(e^\lambda b,\, e^{-\lambda}) \in S(W_0\gamma_0\gamma^2(\log W)^5, g).\end{gathered}$$

Since $r_L \in S(\gamma_0\gamma(\log W)^4, g)$, we have $c_2 \in S(W_0\gamma_0^2\gamma(\log W)^5, g)$. Therefore,

$$(e^{\lambda_L})^w a^w((e^{\lambda_L})^w)^{-1} = a^w + (H_\lambda a/i - H_\lambda^2 a/2)^w + c_1^w + c_2^w + c_3^w$$

with $c_1 + c_2 + c_3 \in S(W_0\gamma_0\gamma^2(\log W)^5, g)$. □

Lemma 5.3. *Assume $\gamma \leq C\langle X\rangle^{-c}$ with some $c > 0$ and $C > 0$. Let W be a g weight such that $W \geq c_0$ with some $c_0 > 1$ and that $(\log W)^2 \leq \gamma^{-1}$, and define $G = (\log W)^2 g$. Let $\lambda \in B([0, T], S_1(\log W, \varphi_0, g))$, and set $\lambda_L(t, X) = \lambda(t, X)\phi_L(X)$.*

(1) *There is* $L_0 \geq 1$ *such that if* $L \geq L_0$ *then*

$$((e^{\lambda_L(t)})^w)^{-1} = (e^{-\lambda_L(t)})^w(1 + r_L^w(t))^{-1},$$

where $r_L(t) \in B([0,T], S(\gamma\gamma_0(\log W)^4, g))$ *with* $\sup_{t\in[0,T]} \|r_L^w(t)\| \leq 1/2$.

(2) *Let* W_0 *be a* g *weight, and assume* $\partial_t\lambda \in B([0,T], S(W_0\gamma_0 \log W, g))$. *If* $L \geq L_0$ *and* $h \in B([0,T], S_1(W_0, \varphi_0, g))$, *then*

$$\begin{aligned}&(e^{\lambda_L(t)})^w(\partial_t + ih^w(t))(\,(e^{\lambda_L(t)})^w\,)^{-1}\\ =&\partial_t + i(\,h(t) - H^2_{\lambda(t)}h(t)/2 + \{\lambda(t), \partial_t\lambda(t)\}/2\,)^w - (\partial_t\lambda(t)\\ &+ H_{h(t)}\lambda(t))^w + c^w(t)\end{aligned}$$

with $c \in B([0,T], S(W_0\gamma_0\gamma^2(\log W)^5, g))$.

Proof. The proof of (1) is similar to that of Lemma 5.2. Since $\partial_t\lambda \in B([0,T], S(W_0\gamma_0 \log W, g))$, it follows that

$$\begin{aligned}&(e^{\lambda_L(t)})^w\ \partial_t\ (\,(e^{\lambda_L(t)})^w\,)^{-1} = \partial_t - (e^{\lambda_L(t)}\partial_t\lambda_L(t))^w(\,(e^{\lambda_L(t)})^w\,)^{-1}\\ &= \partial_t - (e^{\lambda(t)}\partial_t\lambda(t))^w(e^{-\lambda(t)})^w + c_1^w(t) + c_2^w(t)\\ &= \partial_t - (\partial_t\lambda(t) + \{\lambda(t), \partial_t\lambda(t)\}/(2i)\,)^w + c_1^w(t) + c_2^w(t) + c_3^w(t).\end{aligned}$$

Here $c_1 \in B([0,T], \mathcal{S}(\mathbf{R}^{2d}))$,

$$\begin{aligned}c_2^w(t) &= -(e^{\lambda(t)}\partial_t\lambda(t))^w(e^{-\lambda(t)})^w(\,(1 + r_L^w(t))^{-1} - 1\,)\\ &\in \operatorname{Op} B([0,T], S(W_0\gamma_0^2\gamma(\log W)^5, g)),\\ c_3(t) &= -r_2(e^{\lambda(t)}\partial_t\lambda(t), e^{-\lambda(t)}) \in B([0,T], S(W_0\gamma_0\gamma^2(\log W)^5, g)).\end{aligned}$$

The remaining proof of (2) is similar to that of Lemma 5.2. □

§6. Smoothing Effects

In this section, we assume (H1)–(H5). We use our main assumption (H5) only in the part (d) of the proof of Lemma 6.1. We apply the results in Section 5 to the following case

$$\begin{aligned}g &= g_\delta = \langle x\rangle^{2\delta}|dx|^2 + \langle X\rangle^{-2}|d\xi|^2,\\ \varphi_0 &= 1/\langle x\rangle, \quad \gamma = \langle x\rangle^\delta/\langle X\rangle, \quad \gamma_0 = 1/(\langle x\rangle\langle X\rangle),\\ W &= \langle X\rangle_e^{\delta_1}, \quad G = (\log W)^2 g = (\delta_1 \log \langle X\rangle_e)^2 g_\delta,\end{aligned}$$

where $\langle X\rangle_e = (e^2 + |X|^2)^{1/2}$ and $\delta_1 = \inf(\,(\langle X\rangle/\langle x\rangle^\delta)^{1/2}/\log\langle X\rangle_e\,) > 0$; δ_1 is chosen so that the condition $(\log W)^2 \leq 1/\gamma$, or $(\delta_1 \log\langle X\rangle_e)^2 \leq \langle X\rangle/\langle x\rangle^\delta$, holds.

Let $T > 0$, $R < R_1 < R_2$ and $0 < \sigma_0 < \sigma_1 < \sigma_2 < \sigma$, where R and σ are the constants in (H5). Take $\phi, \psi, \chi \in C^\infty(\mathbf{R})$ such that

(i) $\operatorname{supp}\phi \subset (R_1, \infty)$, $\phi(t) = 1$ if $t \geq R_2$, $\phi' \geq 0$, $\sqrt{\phi}, \sqrt{\phi'} \in C^\infty(\mathbf{R})$,

(ii) $\operatorname{supp}\psi \subset (-\infty, -\sigma_1)$, $\psi(t) = 1$ near $(-\infty, -\sigma_2]$, $\psi' \leq 0$, $\sqrt{\psi}, \sqrt{-\psi'} \in C^\infty(\mathbf{R})$,

(iii) $\operatorname{supp}\chi \subset (-\infty, 2T)$, $\chi(t) = 1$ if $t \leq 3T/2$, $\chi' \leq 0$, $\sqrt{\chi}, \sqrt{-\chi'} \in C^\infty(\mathbf{R})$.

For $\rho \geq 0$ and $0 < \nu \ll 1$, we define

$$\begin{aligned} q &= \sqrt{h_0}, \quad \theta = H_{h_0} r/q, \\ w(t) &= (r + \sigma_0 t q)^\rho (2 - r^{-\nu}), \\ \lambda(t) &= \phi(r)\psi(\theta)\chi(t + r/(Mq)) \log w(t) \quad (t \in [0, T]). \end{aligned}$$

Here $M = 2\sup|\theta| + 1$. Observing that

$$\begin{aligned} r/q &\leq 2MT && \text{on} \quad \overline{\cup_{0\leq t\leq T} \operatorname{supp}\chi(t + r/(Mq))}, \\ MT/2 \leq r/q &\leq 2MT && \text{on} \quad \overline{\cup_{0\leq t\leq T} \operatorname{supp}\chi'(t + r/(Mq))}, \end{aligned}$$

where the support is as functions in $(x, \xi) \in T^*\mathbf{R}^d$, we have

$$\begin{aligned} \lambda &\in B([0, T], S_1(\log\langle X\rangle_e, \langle x\rangle^{-1}, g_\delta)), \\ \partial_t \lambda(\cdot) &\in B([0, T], S(\langle x\rangle^{-1}\langle X\rangle \log\langle X\rangle_e, g_\delta)). \end{aligned}$$

Take $\phi_1, \psi_1, \chi_1 \in C^\infty(\mathbf{R})$ such that $0 \leq \phi_1, \psi_1, \chi_1 \leq 1$, $\operatorname{supp}\phi_1 \subset (R_1, \infty)$, $\phi_1(t) = 1$ on $\operatorname{supp}\phi$, $\operatorname{supp}\psi_1 \subset (-\infty, -\sigma_1)$, $\psi_1(t) = 1$ on $\operatorname{supp}\psi$, $\operatorname{supp}\chi_1 \subset (-\infty, 2T)$, $\chi_1(t) = 1$ on $\operatorname{supp}\chi$. Since

$$\begin{aligned} r/q &\leq 2MT && \text{on } \operatorname{supp}\chi_1(r/(Mq)), \\ 3MT/2 \leq r/q &\leq 2MT && \text{on } \operatorname{supp}\chi_1'(r/(Mq)), \end{aligned}$$

we have

$$\lambda_1 = \phi_1(r)\psi_1(\theta)\chi_1(r/(Mq)) \in S_1(1, \langle x\rangle^{-1}, g_\delta).$$

Clearly, $\lambda_1 = 1$ on $\overline{\cup_{t\in[0,T]} \operatorname{supp}\lambda(t)}$. Take $\psi_2 \in C_0^\infty(\mathbf{R})$ such that $\psi_2 = 1$ in a neighborhood of $\operatorname{supp}\psi'$, $\operatorname{supp}\psi_2 \subset (-\sigma_2, -\sigma_1)$ and $0 \leq \psi_2 \leq 1$.

By direct calculation, we have

$$
\begin{aligned}
&-(\partial_t + H_{h(t)})\lambda(t) = \sum_{j=0}^{4} \alpha_j(t)^2 \beta_j(t);\\
&\alpha_0(t)^2 = \nu q r^{-1-\nu}(2-r^{-\nu})^{-1}\phi(r)\psi(\theta)\chi(t+r/(Mq)), \quad \beta_0(t) = -q^{-1}H_{h(t)}r,\\
&\alpha_1(t)^2 = \rho q(r+\sigma_0 tq)^{-1}\phi(r)\psi(\theta)\chi(t+r/(Mq)),\\
&\qquad\qquad\qquad\qquad \beta_1(t) = -(\sigma_0 I_n + q^{-1}H_{h(t)}(r+\sigma_0 tq)),\\
&\alpha_2(t)^2 = -\chi'(t+r/(Mq))\phi(r)\psi(\theta)\log w(t), \quad \beta_2(t) = I_n + H_{h(t)}(r/Mq),\\
&\alpha_3(t)^2 = q\phi'(r)\psi(\theta)\chi(t+r/(Mq))\log w(t), \quad \beta_3(t) = \beta_0(t) = -q^{-1}H_{h(t)}r,\\
&\alpha_4(t)^2 = -r^{-1}q\phi(r)\psi'(\theta)\chi(t+r/(Mq))\log w(t), \quad \beta_4(t) = rq^{-1}(H_{h(t)}\theta).
\end{aligned}
$$

Here $\alpha_j(t) \geq 0$. By modifying $\beta_j(t)$ outside $\operatorname{supp}\alpha_j(t)$, we define $\tilde{\beta}_j(t)$:

$$
\begin{aligned}
&\tilde{\beta}_j(t) = \lambda_1\beta_j(t) + (1-\lambda_1)I_n \quad (j=0,1,2,3),\\
&\tilde{\beta}_4(t) = \lambda_1\psi_2(\theta)\beta_4(t) + (1-\lambda_1\psi_2(\theta))I_n.
\end{aligned}
$$

By the definitions,

$$
\begin{aligned}
&\alpha_j \in B([0,T], S((\langle x\rangle^{-1}\langle X\rangle \log\langle X\rangle_e)^{1/2}, g_\delta)) \subset B([0,T], S(\langle X\rangle/\langle x\rangle^{\delta}, g_\delta)),\\
&\tilde{\beta}_j = \tilde{\beta}_j^* \in B([0,T], S(1, g_\delta; M_n(\mathbf{C}))).
\end{aligned}
$$

Set $\mu_1(T,R_1) = \mu_1([0,T],R_1) + \mu_1'([0,T],R_1)$, $\mu_2(T,R_1) = \mu_2'([0,T],R_1)$, and $\mu(T,R_1) = \mu_1(T,R_1)T + \mu_2(T,R_1)T^2$. Then $\lim_{R_1\to\infty}\mu(T,R_1) \leq 2\mu_1([0,T])T + \mu_2([0,T])T^2$ (see Subsection 2.2).

Lemma 6.1. *There are $\mu_0 > 0$ and $c_j > 0$, depending only on d, h_0 and r (not on T or R_1), such that if $\mu(T,R_1) \leq \mu_0$, then $\tilde{\beta}_j \geq c_j I_n \quad (j=0,\dots,4)$.*

Proof. In this proof, we denote by $C_1, C_2, \dots$ positive constants depending only on d, h_0 and r. We derive estimates on $[0,T]\times\operatorname{supp}\lambda_1$ in (a), (b), (c), and on $[0,T]\times\operatorname{supp}(\lambda_1\psi_2)$ in (d). The claims for $\tilde{\beta}_0(=\tilde{\beta}_3), \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_4$ follow from (a), (b), (c), (d) in this order.

(a) Since

$$
\begin{aligned}
-H_{h_0}r &\geq \sigma_1 q,\\
|H_{h_1(t)}r| &\leq C_1\mu_1(T,R_1)r \leq 2MTC_1\mu_1(T,R_1)q,\\
|H_{h_2(t)}r| &= 0,
\end{aligned}
$$

it follows

$$
-(H_{h(t)}r)/q \geq (\sigma_1 - 2MTC_1\mu_1(T,R_1))I_n \geq c_0 I_n
$$

with $c_0 = \sigma_1/2 > 0$ if $\mu(T,R_1)$ is small enough.

(b) Since

$$|\sigma_0 t H_{h_1(t)} q| \leq C_2 \sigma_0 T \mu_1(T, R_1) q,$$
$$|\sigma_0 t H_{h_2(t)} q| \leq C_3 \sigma_0 T \mu_2(T, R_1) r \leq 2MT^2 C_3 \sigma_0 \mu_2(T, R_1) q,$$

we obtain

$$\begin{aligned}
&-(\sigma_0 I_n + (H_{h(t)} r)/q + \sigma_0 t (H_{h(t)} q)/q) \\
&\geq (\sigma_1 - \sigma_0 - 2MTC_1 \mu_1(T, R_1) - C_2 \sigma_0 T \mu_1(T, R_1) \\
&\quad - 2MT^2 C_3 \sigma_0 \mu_2(T, R_1)) I_n \\
&\geq c_1 I_n
\end{aligned}$$

with $c_1 = (\sigma_1 - \sigma_0)/2 > 0$ if $\mu(T, R_1)$ is small enough.

(c) Since

$$|H_{h_0}(r/q)| = |\theta| \leq M/2,$$
$$|H_{h_1(t)}(r/q)| \leq C_4 \mu_1(T, R_1) r/q \leq 2MTC_4 \mu_1(T, R_1),$$
$$|H_{h_2(t)}(r/q)| \leq C_5 \mu_2(T, R_1) r^2/q^2 \leq (2MT)^2 C_5 \mu_2(T, R_1),$$

we have

$$\begin{aligned}
I_n + H_{h(t)}(r/Mq) \geq (1 - 1/2 - 2TC_4 \mu_1(T, R_1) \\
-4MT^2 C_5 \mu_2(T, R_1)) I_n \geq c_2 I_n
\end{aligned}$$

with $c_2 = 1/4 > 0$ if $\mu(T, R_1)$ is small enough.

(d) By virtue of (H5), we have

$$H_{h_0} \theta = (2^{-1} H_{h_0}^2 (r^2) - (H_{h_0} r)^2)/(rq) \geq (\sigma^2 - \sigma_2^2) q/r.$$

Moreover,

$$|H_{h_1(t)} \theta| \leq C_6 \mu_1(T, R_1) \leq 2MTC_6 \mu_1(T, R_1) q/r,$$
$$|H_{h_2(t)} \theta| \leq C_7 \mu_2(T, R_1) r/q \leq (2MT)^2 C_7 \mu_2(T, R_1) q/r.$$

Therefore, we obtain

$$rq^{-1} H_{h(t)} \theta \geq (\sigma^2 - \sigma_2^2 - 2MTC_6 \mu_1(T, R_1) - (2MT)^2 C_7 \mu_2(T, R_1)) I_n \geq c_4 I_n$$

with $c_4 = (\sigma^2 - \sigma_2^2)/2 > 0$ if $\mu(T, R_1)$ is small enough. □

Hereafter in this section, we assume $\mu(T, R_1) \leq \mu_0$ so that the conclusion of Lemma 6.1 *holds.*

Let $(\phi_L)_{L\geq 1}$ be a bounded family of $S(1,g_\delta)$ such that $1-\phi_L \in C_0^\infty(\mathbf{R}^{2d})$, $0 \leq \phi_L \leq 1$, and $\operatorname{supp}\phi_L \subset \{X \in \mathbf{R}^{2d}; |X| \geq L\}$. Set $\lambda_L(t,X) = \lambda(t,X)\phi_L(X)$. By Lemma 5.3 there exists $L_0 \geq 1$ such that if $L \geq L_0$ then

$$((e^{\lambda_L(t)})^w)^{-1} = (e^{-\lambda_L(t)})^w(1+r_L^w(t))^{-1},$$

where $r_L(t) \in B([0,T], S(\gamma\gamma_0(\log W)^4, g_\delta))$ with $\sup_{t\in[0,T]} \|r_L^w(t)\| \leq 1/2$. Fix $L \geq L_0$ and set $K(t) = k^w(t) = (e^{\lambda_L(t)})^w$. Then

$$K \in \operatorname{Op} B([0,T], S_1(\langle X\rangle^\rho, \langle x\rangle^{-1}\log\langle X\rangle_e, G)), \quad K^{-1} \in \operatorname{Op} B([0,T], S(1,G)).$$

By the definition, $\operatorname{supp}\lambda(t,\cdot) \subset \operatorname{cone}(S_-(R_1,\sigma_1))$; and

$$\alpha_0(t,x,\xi)^2 = \frac{\nu q(x,\xi)}{r(x)^{1+\nu}(2-r(x)^{-\nu})}, \quad e^{\lambda(t,x,\xi)} = (r(x)+\sigma_0 t q(x,\xi))^\rho(2-r(x)^{-\nu})$$

if $(x,\xi) \in \operatorname{cone}(S_-(R_2,\sigma_2))$ and $r(x) \leq MTq(x,\xi)/2$.

The following theorem means that the solution of the Schrödinger equation gains the regularity in $S_-(R_2,\sigma_2)$ if the initial data decays in $S_-(R_1,\sigma_1)$.

Theorem 6.2. *For every $u_0 \in \mathcal{B}^s(\mathbf{R}^d,\mathbf{C}^n)$ and $f \in L^1([0,T],\mathcal{B}^s(\mathbf{R}^d,\mathbf{C}^n))$, let $u \in C([0,T],B^s(\mathbf{R}^d,\mathbf{C}^n))$ be the solution of*

$$(\partial_t + iH(\cdot))u = f \text{ in } \mathcal{D}'((0,T)\times\mathbf{R}^d,\mathbf{C}^n), \quad u(0) = u_0. \tag{6.1}$$

Assume that $K(0)u_0 \in \mathcal{B}^s(\mathbf{R}^d,\mathbf{C}^n)$ and $K(\cdot)f(\cdot) \in L^1([0,T],\mathcal{B}^s(\mathbf{R}^d,\mathbf{C}^n))$. Then $v = K(\cdot)u(\cdot) \in C([0,T],B^s(\mathbf{R}^d,\mathbf{C}^n))$ and $\alpha_j^w(\cdot)v(\cdot) \in L^2([0,T],B^s(\mathbf{R}^d,\mathbf{C}^n))$ $(j = 0,1,\ldots,4)$. Moreover, there are $C_1, C_2 > 0$, independent of u_0, f, and u, such that the following estimates hold: for all $0 \leq t \leq T$

$$\|E_sK(t)u(t)\| \leq C_1\|E_sK(0)u_0\| + C_1\int_0^t \|E_sK(\tau)f(\tau)\|\,d\tau,$$

$$\sum_{j=0}^4 \int_0^t \|\alpha_j^w(\tau)E_sK(\tau)u(\tau)\|^2 d\tau$$
$$\leq C_2\left(\|E_sK(0)u_0\| + \int_0^t \|E_sK(\tau)f(\tau)\|\,d\tau\right)^2.$$

Proof. By Lemma 5.3,

$$\begin{aligned} &K(t)(\partial_t + iH(t))K(t)^{-1} \\ &\quad= \partial_t + i(\,h(t) - H_{\lambda(t)}^2 h(t)/2 + \{\lambda(t),\partial_t\lambda(t)\}/2\,)^w \\ &\qquad - (\partial_t\lambda(t) + H_{h(t)}\lambda(t))^w + c^w(t) \\ &\quad= \partial_t + P(t) \end{aligned}$$

with $c \in B([0,T], S(1, g_\delta; M_n(\mathbf{C})))$. The conditions (H6)–(H8) in Theorem 4.6 are valid if we set

$$p_1(t) = h_1(t) + h_2(t) - H^2_{\lambda(t)} h(t)/2 + \{\lambda(t), \partial_t\lambda(t)\}/2,$$
$$p_2(t) = -(\partial_t\lambda(t) + H_{h(t)}\lambda(t)), \quad p_3(t) = c(t).$$

Since $v = K(\cdot)u(\cdot) \in C([0,T], B^{s-\rho}(\mathbf{R}^d, \mathbf{C}^n)) \subset C([0,T], \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n))$ is the solution of

$$(\partial_t + iP(\cdot))v = K(\cdot)f \text{ in } \mathcal{D}'((0,T)\times\mathbf{R}^d, \mathbf{C}^n), \quad v(0) = K(0)u_0,$$

Theorem 4.6 completes the proof. □

The next task is to prove increase in regularity at every point that is not trapped backward by Φ_t if the initial data decays in an incoming region. To express this property, it is convenient to introduce

Definition 6.3. For an open subset U of $S^*\mathbf{R}^d$, $S^\mu_{cpt}(U)$ is the set of all $p \in S(\langle\xi\rangle^\mu, |dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2)$ satisfying $\operatorname{supp} p \subset \operatorname{cone}(K)$ for some compact set $K \subset U$.

Lemma 6.4. *Let U be a relatively-compact open subset of $S^*\mathbf{R}^d$, and set $\Gamma = \cup_{0\le t\le t_0}\Phi_t(U)$, where $t_0 > 0$ is an arbitrarily fixed constant. Let $s \in \mathbf{R}$ and $\rho \ge 0$. Then for every $a \in S^0_{cpt}(\Gamma)$, there are $b \in S^0_{cpt}(U)$ and a constant $C > 0$ such that the a priori estimate below holds*:

$$\|w_t(D)^\rho\langle D\rangle^s a^w u(t)\|^2 + \int_0^t \|w_\tau(D)^\rho\langle D\rangle^{s+1/2} a^w u(\tau)\|^2 d\tau \tag{6.2}$$
$$\le C\int_0^t \|w_\tau(D)^\rho\langle D\rangle^{s+1/2} b^w u(\tau)\|^2 d\tau + C\|E_s u_0\|^2, \quad 0 \le t \le T,$$

for all $u_0 \in \mathcal{S}(\mathbf{R}^d, \mathbf{C}^n)$ with $u(t) = S(t,0)u_0$. Here $w_t(\xi) = 1 + t\langle\xi\rangle$ $(t \ge 0)$.

Lemma 6.4 is a little modification of [5, Theorem 2.1] and will be proved at the end of this section. Admitting this lemma, we shall prove

Theorem 6.5. *Let V be a relatively-compact open subset of $S^*\mathbf{R}^d$ such that $\overline{V} \cap T_- = \emptyset$. Let $s \in \mathbf{R}$ and $\rho \ge 0$.*

(1) *For every $a \in S^0_{cpt}(V)$, there is $C > 0$ such that the estimate below holds*:

$$\|w_t(D)^\rho\langle D\rangle^s a^w u(t)\|^2 + \int_0^t \|w_\tau(D)^\rho\langle D\rangle^{s+1/2} a^w u(\tau)\|^2 d\tau \tag{6.3}$$
$$\le C\| E_s K(0)u_0\|^2, \quad 0 \le t \le T,$$

for all $u_0 \in \mathcal{S}(\mathbf{R}^d, \mathbf{C}^n)$ with $u(t) = S(t,0)u_0$.

(2) *Let* $u_0 \in \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$ *satisfy* $K(0)u_0 \in \mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$. *Then for every* $a \in S^0_{cpt}(V)$,

$$w_t(D)^\rho \langle D\rangle^s a^w u(t) \in C([0,T], L^2(\mathbf{R}^d, \mathbf{C}^n)),$$
$$w_t(D)^\rho \langle D\rangle^{s+1/2} a^w u(t) \in L^2([0,T], L^2(\mathbf{R}^d, \mathbf{C}^n)).$$

Moreover, there is $C > 0$, *independent of* u_0, *such that the estimate* (6.3) *holds.*

Proof. (1) By Proposition 2.4, there is $t_0 > 0$ such that $U = \Phi_{-t_0}(V) \subset S_-(R_2, \sigma_2)$, because $R_2 > R$ and $0 < \sigma_2 < \sigma$. Set $\Gamma = \cup_{0\le t\le t_0}\Phi_t(U)$. Let $a \in S^0_{cpt}(V) \subset S^0_{cpt}(\Gamma)$. By Lemma 6.4, there are $b \in S^0_{cpt}(U)$ and a constant $C > 0$ such that the a priori estimate (6.2) holds. So it is sufficient to prove the claim below in view of Theorem 6.2. For simplicity, we set $S(m) = S(m, |dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2)$ and define $B([0,T], S(\langle\xi\rangle^s w_t^r))$ as the set of all $a \in C([0,T], C^\infty(\mathbf{R}^{2d}))$ such that $a(t) \in S(\langle\xi\rangle^s w_t^r)$ uniformly in $t \in [0,T]$. □

Claim. There are $c_1 \in B([0,T], S(1))$ with $\operatorname{supp} c_1(t,\cdot) \subset \operatorname{supp} b$, and $c_2 \in B([0,T], \mathcal{S})$ such that

$$w_t(D)^\rho \langle D\rangle^{s+1/2} b^w = c_1^w(t)\alpha_0^w(t) E_s K(t) + c_2^w(t), \quad 0 \le t \le T.$$

Proof of the claim. Note that

$$\alpha_0(t) = (\,\nu q r^{-1-\nu}/(2 - r^{-\nu})\,)^{1/2}, \;\; k(t) = (r + \sigma_0 t q)^\rho (2 - r^{-\nu})$$

for all $X \in \operatorname{cone}(U)$ and $t \in [0,T]$ if $q(X) \gg 1$. Take $b_1, b_2 \in S_{cpt}(U)$ such that $b_1 = 1$ in a neighborhood of $\operatorname{supp} b$ and $b_2 = 1$ in a neighborhood of $\operatorname{supp} b_1$. Since $b_2\alpha_0(t) \in B([0,T], S(\langle\xi\rangle^{1/2}))$, $b_2 e_s \in S(\langle\xi\rangle^s)$, and $b_2 k(t) \in B([0,T], S(w_t(\xi)^\rho))$, we have

$$\begin{aligned} a_1^w(t) &:= b_1^w \alpha_0^w(t) E_s K(t) = b_1^w (b_2\alpha_0(t))^w (b_2 e_s)^w (b_2 k(t))^w + r_1^w(t) \\ &\in \operatorname{Op} B([0,T], S(\langle\xi\rangle^{s+1/2} w_t(\xi)^\rho)) \end{aligned}$$

with $r_1 \in B([0,T], \mathcal{S})$. Moreover, $\Re a_1(t,X) \ge C\langle\xi\rangle^{s+1/2} w_t(\xi)^\rho$ for all $X \in \operatorname{supp} b$ and $t \in [0,T]$ if $q(X) \gg 1$. Write $w_t(D)^\rho \langle D\rangle^{s+1/2} b^w = a_0^w(t) + r_2^w(t)$ with $a_0 \in B([0,T], S(\langle\xi\rangle^{s+1/2} w_t(\xi)^\rho))$, $\operatorname{supp} a_0(t) \subset \operatorname{supp} b$, and $r_2 \in B([0,T], \mathcal{S})$. Take $c_j \in B([0,T], S(\langle\xi\rangle^{-j}))$, $\operatorname{supp} c_j(t) \subset \operatorname{supp} b$, such that

$$c_0(t,X) = a_0(t,X)/a_1(t,X),$$
$$c_j(t,X) = -\sum_{k=1}^{j} \frac{1}{k!}\left(\frac{i\sigma(D_X, D_Y)}{2}\right)^k c_{j-k}(t,X) a_1(t,Y)|_{Y=X}/a_1(t,X) \qquad (j \ge 1)$$

when $q(X) \gg 1$. Choose $c \in B([0,T], S(1))$, $\operatorname{supp} c(t) \subset \operatorname{supp} b$, such that $c - \sum_{j<N} c_j \in B([0,T], S(\langle\xi\rangle^{-N}))$ for all $N \in \mathbf{N}$. Then $a_0^w(t) = c^w(t) a_1^w(t) + r_3(t)$ with $r_3 \in B([0,T], \mathcal{S})$. Therefore

$$w_t(D)^\rho \langle D\rangle^{s+1/2} b^w = c^w(t) b_1^w \alpha_0^w(t) E_s K(t) + r_4^w(t) = \tilde{c}^w(t)\alpha_0^w(t) E_s K(t) + r_5^w(t)$$

with $r_4, r_5 \in B([0,T], \mathcal{S})$ and $\tilde{c} \in B([0,T], S(1))$, $\operatorname{supp} \tilde{c}(t) \subset \operatorname{supp} b$.

(2) Take a sequence $(v_k)_{k\in\mathbf{N}}$ in $\mathcal{S}(\mathbf{R}^d, \mathbf{C}^n)$ which converges to $K(0)u_0$ in $\mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$. Put $u_k(t) = S(t,0)u_{0,k}$ with $u_{0,k} = K(0)^{-1} v_k \in \mathcal{S}(\mathbf{R}^d, \mathbf{C}^n)$. Since $(K(0)u_{0,k})_{k\in\mathbf{N}}$ converges to $K(0)u_0$ in $\mathcal{B}^s(\mathbf{R}^d, \mathbf{C}^n)$, it follows from (6.3) that $(\langle D\rangle^s w_t(D)^\rho a^w u_k(\cdot))_{k\in\mathbf{N}}$ is a Cauchy sequence in $C([0,T], L^2(\mathbf{R}^d, \mathbf{C}^n))$, and so is $(\langle D\rangle^{s+1/2} w_t(D)^\rho a^w u_k(\cdot))_{k\in\mathbf{N}}$ in $L^2([0,T], L^2(\mathbf{R}^d, \mathbf{C}^n))$. On the other hand, $(u_k)_{k\in\mathbf{N}}$ converges to u in $C([0,T], \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n))$. This completes the proof. □

Proof of Lemma 6.4. We first localize the problem. Take $\phi \in C_0^\infty(\mathbf{R}^d)$ such that $\phi = 1$ in a neighborhood of the base projection of $\overline{\Gamma}$. Take $\phi_1 \in C_0^\infty(\mathbf{R}^d, \mathbf{R})$ such that $\phi_1 = 1$ in a neighborhood of $\operatorname{supp}\phi$, and set $\tilde{h}(t,x,\xi) = h(t, \phi_1(x)x, \xi)$ and $\tilde{h}_j(t,x,\xi) = h_j(t, \phi_1(x)x, \xi)$ $(j = 0,1,2)$. Then $\tilde{h}_0 \in S(\langle\xi\rangle^2)$ and $\tilde{h}_j \in B([0,T], S(\langle\xi\rangle^{2-j}; M_n(\mathbf{C})))$ $(j = 1,2)$. Put $\tilde{H}(t) = \tilde{h}^w(t)$.

Apply Lemma A.3 to the case where $m = 2$ and $h(t) = \tilde{h}(t)$, $h_0 = \tilde{h}_0$, $h_1(t) = \tilde{h}_1(t) + \tilde{h}_2(t)$. Let $u_0 \in \mathcal{S}(\mathbf{R}^d, \mathbf{C}^n)$ and set $u(t) = S(t,0)u_0$. Put $v(t) = \phi u(t)$. Since $(\partial_t + i\tilde{H}(t))v(t) = [\, iH(t), \phi I_n\,] u(t) =: f(t)$, we have

$$\begin{aligned}
&\|w_t(D)^\rho \langle D\rangle^s a^w v(t)\|^2 + \int_0^t \|w_\tau(D)^\rho \langle D\rangle^{s+1/2} a^w v(\tau)\|^2 d\tau \\
&\le C\int_0^t \|w_\tau(D)^\rho \langle D\rangle^{s+1/2} b^w v(\tau)\|^2 d\tau + C\int_0^t \|w_\tau(D)^\rho \langle D\rangle^s \tilde{a}^w f(\tau)\|^2 d\tau \\
&+C\|\langle D\rangle^s \tilde{a}^w v(0)\|^2 + C\sup_{0\le\tau\le t} \|\langle D\rangle^{s-L} v(\tau)\|^2 + C\int_0^t \|\langle D\rangle^{s-L} f(\tau)\|^2 d\tau
\end{aligned}$$

for all $0 \le t \le T$. This completes the proof, because $a^w(1-\phi)$, $\tilde{a}^w(1-\phi)$, $b^w(1-\phi) \in \operatorname{Op}\mathcal{S}$, $\tilde{a}^w[\, iH(t), \phi I_n\,] \in \operatorname{Op} B([0,T], \mathcal{S})$, and $\|E_s S(t,0)u_0\| \le C \|E_s u_0\|$. □

§7. Proofs for Section 2

Proof of Theorem 2.1. Theorem 2.1 is contained in Theorem 4.5. □

Proof of Theorem 2.2. Since (1)–(4) follows directly from Theorem 2.1, we prove only (5). Let H_1 be the operator H with domain $C_0^\infty(\mathbf{R}^d, \mathbf{C}^n)$. If $u \in L^2(\mathbf{R}^d, \mathbf{C}^n)$ satisfies $Hu \in L^2(\mathbf{R}^d, \mathbf{C}^n)$, then $J_\varepsilon u \in \mathcal{S}(\mathbf{R}^d, \mathbf{C}^n)$ and $HJ_\varepsilon u \to$

Hu in $L^2(\mathbf{R}^d, \mathbf{C}^n)$ as $\varepsilon \to +0$ by Lemma 4.3. This implies that $\mathcal{S}(\mathbf{R}^d, \mathbf{C}^n)$, hence $C_0^\infty(\mathbf{R}^d, \mathbf{C}^n)$, is a core for H_1^*. Thus, H_1 is essentially self-adjoint. Let $t_0 \in \mathbf{R}$ and $u_0 \in L^2(\mathbf{R}^d, \mathbf{C}^n)$. Then $u(t) = e^{-i(t-t_0)H}u_0 \in C(\mathbf{R}, L^2(\mathbf{R}^d, \mathbf{C}^n))$ is a solution of (2.3). By uniqueness, $e^{-i(t-t_0)H}u_0 = S(t, t_0)u_0$ for every $t \in \mathbf{R}$. □

Proof of Lemma 2.3. Take $\rho \in C_0^\infty(\mathbf{R}^d)$ such that $\rho \geq 0$, $\int_{\mathbf{R}^d} \rho(x)dx = 1$, $\operatorname{supp} \rho \subset \{x \in \mathbf{R}^d; |x| < 1\}$, $\rho(-x) = \rho(x)$ $(x \in \mathbf{R}^d)$; set $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(x/\varepsilon)$ $(0 < \varepsilon < 1)$. Define $f_\varepsilon = \rho_\varepsilon * f$. By the definition, $f_\varepsilon \geq 1$ and $\lim_{|x|\to\infty} f_\varepsilon(x) = \infty$. For every $\alpha \in \mathbf{N}_0^d$ with $|\alpha| \geq 2$, write $\alpha = \beta + \gamma$ with $|\beta| = 2$. Then $\partial^\alpha f_\varepsilon = (\partial^\gamma \rho_\varepsilon) * (\partial^\beta f) \in L^\infty(\mathbf{R}^d)$. Since $\int_{\mathbf{R}^d} y_j \rho(y)dy = 0$, we have

$$\begin{aligned}|f_\varepsilon(x) - f(x)| &= \left| \int_{\mathbf{R}^d} \rho_\varepsilon(y)(f(x-y) - f(x) + y \cdot \nabla_x f(x))dy \right| \\ &\leq \varepsilon^2 \sum_{|\alpha|=2} \frac{1}{\alpha!} \int_{\mathbf{R}^d} |y^\alpha|\rho(y)dy \sup_{y\in\mathbf{R}^d} |\partial^\alpha f(y)|.\end{aligned}$$

More directly,

$$|\partial^\alpha(f_\varepsilon(x) - f(x))| \leq \varepsilon \int_{\mathbf{R}^d} |y|\rho(y)dy \sup_{y\in\mathbf{R}^d} |\nabla\partial^\alpha f(y)|$$

if $|\alpha| = 1$, and

$$|\partial^\alpha(f_\varepsilon(x) - f(x))| \leq \sup_{x,h\in\mathbf{R}^d, |h|\leq\varepsilon} |\partial_x^\alpha(f(x+h) - f(x))|$$

if $|\alpha| = 2$. So for every $0 < \sigma < \tilde{\sigma}$ and $R > \tilde{R}$, there exists $0 < \varepsilon_0 \ll 1$ such that for every $0 < \varepsilon \leq \varepsilon_0$

$$H_{h_0}^2 f_\varepsilon \geq 2\sigma^2 h_0 \quad \text{if } f_\varepsilon(x) \geq R^2.$$

Set $f_{cv} = f_{\varepsilon_0}$. □

Proof of Theorem 2.5. The continuity of the forward propagator follows from Theorem 6.5 because $K(0)\langle x\rangle^{-\rho} \in \operatorname{Op} S(1, G)$. If $u \in C(I, \mathcal{S}'(\mathbf{R}^d, \mathbf{C}^n))$ satisfies $(\partial_t + ih^w(t, x, D))u(t) = f(t)$, then $v(t) = \overline{u(-t)}$ satisfies $(\partial_t + ih^w(-t, x, -D))v(t) = -\overline{f(-t)}$. Moreover, $T_+ = \{(x, \xi); (x, -\xi) \in T_-\}$. So the continuity of the backward propagator for $\partial_t + ih^w(t, x, D)$ follows from that of the forward propagator for $\partial_t + ih^w(-t, x, -D)$. □

Proof of Corollary 2.6. This follows easily from Theorem 2.5. □

Proof of Corollary 2.7. Let $I = [t_1, t_2]$ be an interval satisfying the condition. Let $t_1 \leq t_0 \leq t \leq t_2$. If A is a compactly supported pseudodifferential operator of order 0 such that its essential support has no intersection with T_- (resp. T_+), then $AS(t,t_0)$ (resp. $AS(t_0,t)$) has a C^∞ distribution kernel by Theorem 2.5; hence

$$WF(K(t,t_0)) \subset (T_- \times T^*\mathbf{R}^d) \cup (0 \times T^*\mathbf{R}^d \setminus 0), \tag{7.1}$$

$$WF(K(t_0,t)) \subset (T_+ \times T^*\mathbf{R}^d) \cup (0 \times T^*\mathbf{R}^d \setminus 0). \tag{7.2}$$

Since $K(t,t_0)(x,x') = \overline{K(t_0,t)(x',x)}$, we have

$$WF(K(t,t_0)) = \{(x,\xi;x',\xi');(x',-\xi';x,-\xi) \in WF(K(t_0,t))\}. \tag{7.3}$$

Further, $T_- = \{(x,\xi);(x,-\xi) \in T_+\}$. Combining these with (7.2), we get

$$WF(K(t,t_0)) \subset (T^*\mathbf{R}^d \times T_-) \cup (T^*\mathbf{R}^d \setminus 0 \times 0). \tag{7.4}$$

The upper estimate of $WF(K(t,t_0))$ follows from (7.1) and (7.4), and that of $WF(K(t_0,t))$ follows consequently. □

§8. Smoothing Effect of Order Half

This section discusses the smoothing effect of order half for the Schrödinger equation in Section 1. We assume (H1)–(H4) throughout this section. We shall use Lemma 3.4 (3) with

$$g = g_\delta = \langle x\rangle^{2\delta}|dx|^2 + \langle X\rangle^{-2}|d\xi|^2,$$
$$\varphi_0 = 1/\langle x\rangle, \quad \gamma = \langle x\rangle^\delta/\langle X\rangle, \quad \gamma_0 = 1/(\langle x\rangle\langle X\rangle).$$

Set $q = \sqrt{h_0}$. Consider several conditions on the principal symbol h_0.

(H9) (Global escape function). There exists $a \in C^\infty(\mathbf{R}^{2d},\mathbf{R})$ such that for every $\alpha,\beta \in \mathbf{N}_0^d$

$$|\partial_x^\beta\partial_\xi^\alpha a(x,\xi)| \le C_{\alpha,\beta}\langle x\rangle^{1-|\beta|}\langle\xi\rangle^{-|\alpha|}, \qquad x,\xi \in \mathbf{R}^d, \quad \text{if } |\beta| = 0,1; \tag{8.1}$$

$$|\partial_x^\beta\partial_\xi^\alpha a(x,\xi)| \le C_{\alpha,\beta}\langle x\rangle^{\delta(|\beta|-1)}\langle\xi\rangle^{-|\alpha|}, \quad x,\xi \in \mathbf{R}^d, \quad \text{if } |\beta| \ge 1; \tag{8.2}$$

and that for some $c > 0$ and $C > 0$

$$H_{h_0}a(x,\xi) \ge cq(x,\xi) - C, \quad x,\xi \in \mathbf{R}^d.$$

(H10) (Escape function near infinity). There exists $a \in C^\infty(\mathbf{R}^{2d},\mathbf{R})$ such that for every $\alpha,\beta \in \mathbf{N}_0^d$, (8.1) and (8.2) hold and that for some $c > 0$, $C > 0$, and $R > 0$

$$H_{h_0}a(x,\xi) \ge cq(x,\xi) - C \quad \text{for } |x| \ge R,\ \xi \in \mathbf{R}^d.$$

(H11) (Finite escape time). For every compact set $K \subset S^*\mathbf{R}^d$, there exists $t_K > 0$ such that $\Phi_t(K) \cap K = \emptyset$ for all $t \ge t_K$.

Remark. If $|\nabla g^{jk}(x)| = o(|x|^{-1})$ as $|x| \to \infty$ for all j, k, then (H10) holds with $a = x \cdot \xi/(1+h_0)^{1/2}$ or $a = H_{h_0}|x|^2/(1+h_0)^{1/2}$.

Remark. The condition (H5) implies (H10): we can choose a in (H10) as $a = H_{h_0} f_{cv}/(1+h_0)^{1/2}$.

Lemma 8.1. *Under* (H5), *all conditions* (H11), $T_{cpt} = \emptyset$, $T_+ = \emptyset$, *and* $T_- = \emptyset$ *are equivalent.*

Proof. If $X_0 \in T_{cpt,+}$, then the positive limit set of X_0 is relatively compact, and hence the total orbit of each positive limit point of X_0 is relatively compact. Thus $T_{cpt,+} \neq \emptyset$ implies $T_{cpt} \neq \emptyset$. Similarly, $T_{cpt,-} \neq \emptyset$ implies $T_{cpt} \neq \emptyset$. By Proposition 2.4, $T_{cpt,+} = T_+$ and $T_{cpt,-} = T_-$. So $T_{cpt} = \emptyset$, $T_+ = \emptyset$, and $T_- = \emptyset$ are equivalent. Clearly, (H11) implies $T_{cpt} = \emptyset$. For the assertion that $T_{cpt} = \emptyset$ implies (H11), see the proof of the lemma 1.3 of [2]. □

Lemma 8.2. (H9) *is equivalent to* (H10) *and* (H11).

Proof. Suppose (H9). Then for all $X \in T^*\mathbf{R}^d$ with $q(X) = L \gg 1$

$$\frac{d}{dt} a(\Phi_t(X)) = H_{h_0} a(\Phi_t(X)) \geq 1, \quad t \geq 0,$$

which implies $a(\Phi_t(X)) \geq t + a(X)$, $t \geq 0$. Therefore for every compact set $K \subset \{X \in T^*\mathbf{R}^d; q(X) = L\}$, $\Phi_t(K) \cap K = \emptyset$ if $t \geq 2\sup_{X \in K} |a(X)| + 1$. This gives (H11).

The proof of the converse is similar to that of the lemma 1.5 of [2]. □

Remark. We summarize the relations among the conditions above:

$$\begin{aligned}
&(\text{H5}) \Rightarrow (\text{H10}),\\
&(\text{H5}) + (T_{cpt} = \emptyset) \Leftrightarrow (\text{H5}) + (T_+ = \emptyset) \Leftrightarrow (\text{H5}) + (T_- = \emptyset) \Leftrightarrow (\text{H5}) + (\text{H11}),\\
&(\text{H5}) + (\text{H11}) \Rightarrow (\text{H10}) + (\text{H11}) \Leftrightarrow (\text{H9}).
\end{aligned}$$

Lemma 8.3 (non-trapping case)**.** *Assume* (H9). *For every* $0 < \nu \ll 1$ *there exist a real-valued symbol* $\lambda \in S_1(1, \langle x \rangle^{-1}, g_\delta)$ *and constants* $c, C > 0$ *such that*

$$-H_{h_0}\lambda(X) \geq c\langle x \rangle^{-1-\nu}\langle X \rangle - C, \quad X = (x, \xi) \in T^*\mathbf{R}^d.$$

Proof. This lemma is a minor modification of the lemma 2.3 of [2].

Let $0 < \varepsilon \ll 1$ be a parameter to be fixed later. Take $\psi, \chi \in C^\infty(\mathbf{R})$ such that

(i) $\operatorname{supp}\psi \subset (\varepsilon, \infty)$, $\psi(t) = 1$ near $[2\varepsilon, \infty)$, $\psi' \geq 0$,

(ii) $\operatorname{supp}\chi \subset (-\infty, 1)$, $\chi(t) = 1$ if $t \leq 1/2$, $0 \leq \chi \leq 1$.

Set $\psi_+(t) = \psi(t)$, $\psi_-(t) = \psi(-t)$, $\psi_0(t) = 1 - \psi_+(t) - \psi_-(t)$. Define

$$r(x) = \langle x\rangle, \quad \theta = a/\langle x\rangle,$$
$$\lambda = \Big(-\theta\psi_0(\theta) + (M_0 - (1+|a|)^{-\nu})\big(\psi_-(\theta) - \psi_+(\theta)\big)\Big)\chi(r/q).$$

Here $M_0 = 2 + 2\varepsilon$. Since $q \geq r$ on $\operatorname{supp}\chi(r/q)$ and $|a| \geq \varepsilon r$ on $\operatorname{supp}(\psi_+(\theta) + \psi_-(\theta))$, we have $\lambda \in S_1(1, \langle x\rangle^{-1}, g_\delta)$. By (H9) we have

$$H_{h_0}\theta = \big(H_{h_0}a - \theta H_{h_0}r\big)/r \geq c_0 q/r - C_0 \quad \text{on } \operatorname{supp}\psi_0(\theta)$$

with constants c_0, $C_0 > 0$ if ε is small enough. Fix such ε. By direct calculation, we obtain

$$\begin{aligned}
-H_{h_0}\lambda =& \Big((H_{h_0}\theta)\psi_0(\theta) + \nu(1+|a|)^{-1-\nu}(H_{h_0}a)\big(\psi_-(\theta) + \psi_+(\theta)\big)\Big)\chi(r/q)\\
&+ (H_{h_0}\theta)\big(M_0 - (1+|a|)^{-\nu} - |\theta|\big)\big(-\psi_-'(\theta) + \psi_+'(\theta)\big)\chi(r/q)\\
&- \Big(-\theta\psi_0(\theta) + (M_0 - (1+|r\theta|)^{-\nu})\big(\psi_-(\theta) - \psi_+(\theta)\big)\Big)\chi'(r/q)H_{h_0}r/q\\
\geq& \Big((H_{h_0}\theta)\psi_0(\theta) + \nu(1+|a|)^{-1-\nu}(H_{h_0}a)\big(\psi_-(\theta) + \psi_+(\theta)\big)\Big)\chi(r/q) - C_1\\
\geq& \Big((c_0 q/r)\psi_0(\theta) + c_1(1+|a|)^{-1-\nu}q\left(\psi_-(\theta) + \psi_+(\theta)\right)\Big)\chi(r/q) - C_2\\
\geq& c_2 r^{-1-\nu} q\,\chi(r/q) - C_2\\
\geq& c_3\langle x\rangle^{-1-\nu}\langle X\rangle - C_3,
\end{aligned}$$

with constants $c_1, c_2, c_3, C_1, C_2, C_3 > 0$. □

Lemma 8.4 (general case). *Assume* (H5). *Let* $\phi_R \in C^\infty(\mathbf{R})$ *such that* $\phi_R(t) = 1$ *if* $t \geq R'$ *for some* $R' > R$, $\operatorname{supp}\phi_R \subset (R, \infty)$, *and* $\phi_R' \geq 0$. *Let* $0 < \nu \ll 1$. *Then there exist a real-valued symbol* $\lambda \in S_1(1, \langle x\rangle^{-1}, g_\delta)$ *and constants* $c, C > 0$ *such that*

$$-H_{h_0}\lambda(X) \geq c\phi_R(r)\langle x\rangle^{-1-\nu}\langle X\rangle - C, \quad X = (x, \xi) \in T^*\mathbf{R}^d.$$

Proof. This lemma is a minor modification of the lemma 2.4 of [2]. Set $a = H_{h_0}f_{cv}/(1+h_0)^{1/2} = 2rH_{h_0}r/(1+h_0)^{1/2}$. By (H5)

$$H_{h_0}a = H_0^2 f_{cv}/(1+h_0)^{1/2} \geq c_0 q - C_0 \quad \text{on } \{(x,\xi) \in T^*\mathbf{R}^d; r(x) \geq R\}$$

for some $c_0, C_0 > 0$. For every $0 < \nu \ll 1$, define λ as in the proof of Lemma 8.3:

$$\lambda = \Big(-\theta\psi_0(\theta) + (M_0 - (1+|a|)^{-\nu})\big(\psi_-(\theta) - \psi_+(\theta)\big)\Big)\chi(r/q).$$

Put $\tilde{\lambda} = \phi_R(r)\lambda$. Then $\tilde{\lambda} \in S_1(1, \langle x\rangle^{-1}, g_\delta)$ and $a\lambda \leq 0$. Therefore

$$\begin{aligned}-H_{h_0}\tilde{\lambda} &= -\phi_R'(r)(H_{h_0}r)\lambda - \phi_R(r)H_{h_0}\lambda \\ &\geq -\phi_R(r)H_{h_0}\lambda \geq c\phi_R(r)\langle x\rangle^{-1-\nu}\langle X\rangle - C\end{aligned}$$

with constants $c, C > 0$. □

Theorem 8.5 (smoothing effect of order half). *Let* $s \in \mathbf{R}$ *and* $0 < \nu \ll 1$. *Let* $I = [t_1, t_2]$ $(t_1 < t_2)$ *and* $t_0 \in I$.

(1) *Assume* (H9). *Then the two estimates in Theorem* 2.8 (1) *holds.*

(2) *Assume* (H5). *Then the assertion of Theorem* 2.8 (2) *holds.*

Proof. For simplicity, we consider only the case where $I = [0, T]$ and $t_0 = 0$. Let $u \in C^1([0, T], \mathcal{S}(\mathbf{R}^d, \mathbf{C}^n))$ with $f(t) = (\partial_t + iH(t))u(t)$. By Lemma 4.2, $E_s H(t) E_s^{-1} = H(t) + B_s(t)$ with $B_s(\cdot) \in \operatorname{Op} B([0, T], S(1, g_\delta; M_n(\mathbf{C})))$. Then $v = E_s u \in C^1([0, T], \mathcal{S}(\mathbf{R}^d, \mathbf{C}^n))$ satisfies

$$E_s f(t) = (\partial_t + iH(t) + iB_s(t))v(t).$$

We shall denote by $C_1, C_2, \ldots$ several constants independent of $t \in [0, T]$ and u.

(1) Suppose (H9). Take λ satisfying Lemma 8.3. We may assume $\|\lambda^w\| \leq 1/2$. Then

$$i[\, H(t), \lambda^w\,] + iB_s(t)^*\lambda^w - i\lambda^w B_s(t) = (H_{h_0}\lambda)^w I_n + b^w(t)$$

with $b \in B([0, T], S(1, g_\delta; M_n(\mathbf{C})))$. By Lemma 8.3, the Sharp Gårding inequality gives

$$-(H_{h_0}\lambda)^w \geq C_1^{-1} E_{1/2}\langle x\rangle^{-1-\nu} E_{1/2} - C_2$$

as a quadratic form on $\mathcal{S}(\mathbf{R}^d)$. Define a norm $N(v)$ of $L^2(\mathbf{R}^d, \mathbf{C}^n)$ by $N(v)^2 = ((1 + \lambda^w)v, v)$. (We define $N(v) = ((1 - \lambda^w)v, v)$ if $I = [-T, 0]$ and $t_0 = 0$.) Then

$$\begin{aligned}\frac{d}{dt}N(v(t))^2 &= ((H_{h_0}\lambda I_n + b(t))^w v(t), v(t)) + (i(B_s(t)^* - B_s(t))v(t), v(t)) \\ &\quad + 2\Re((1 + \lambda^w)E_s f(t), v(t)) \\ &\leq -C_1^{-1}\|\langle x\rangle^{-(1+\nu)/2} E_{1/2} v(t)\|^2 + C_3 N(v(t))^2 \\ &\quad + 2\Re((1 + \lambda^w)E_s f(t), v(t)).\end{aligned}$$

Since

$$|((1 + \lambda^w)E_s f(t), v(t))| \leq N(E_s f(t)) \cdot N(v(t)),$$

we have

$$
\begin{aligned}
&C_1^{-1}\int_0^t \|\langle x\rangle^{-(1+\nu)/2}E_{1/2}v(\tau)\|^2 d\tau\\
&\le N(v(0))^2 + C_3 t \sup_{\tau\in[0,t]} N(v(\tau))^2 + 2\sup_{\tau\in[0,t]} N(v(\tau))\int_0^t N(E_s f(\tau))d\tau.
\end{aligned}
$$

Applying Theorem 2.1, we obtain the first estimate. Since

$$
\begin{aligned}
&2|((1+\lambda^w)E_s f(t), v(t))|\\
&\quad \le C_4\|\langle x\rangle^{(1+\nu)/2}E_{s-1/2}f(\tau)\|\cdot\|\langle x\rangle^{-(1+\nu)/2}E_{1/2}v(\tau)\|\\
&\quad \le (2C_1)^{-1}\|\langle x\rangle^{-(1+\nu)/2}E_{1/2}v(\tau)\|^2 + C_5\|\langle x\rangle^{(1+\nu)/2}E_{s-1/2}f(\tau)\|^2,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{d}{dt}N(v(t))^2 \le &-(2C_1)^{-1}\|\langle x\rangle^{-(1+\nu)/2}E_{1/2}v(t)\|^2\\
&+ C_6 N(v(t))^2 + C_5\|\langle x\rangle^{(1+\nu)/2}E_{s-1/2}f(t)\|^2.
\end{aligned}
$$

By a Gronwall-type inequality, we get

$$
\begin{aligned}
&e^{-C_6 t}N(v(t))^2 + (2C_1)^{-1}\int_0^t e^{-C_6\tau}\|\langle x\rangle^{-(1+\nu)/2}E_{1/2}v(\tau)\|^2 d\tau\\
&\le N(v(0))^2 + C_5\int_0^t e^{-C_6\tau}\|\langle x\rangle^{(1+\nu)/2}E_{s-1/2}f(\tau)\|^2 d\tau.
\end{aligned}
$$

This implies the second estimate.

(2) Suppose (H5). Applying Lemma 8.4 and imitating the proof of the theorem 1.2 of [2], we can construct a real-valued symbol $\lambda_0 \in S_1(1, \langle x\rangle^{-1}, g_\delta)$ such that

$$
-H_{h_0}\lambda_0 \ge \langle x\rangle^{-1-\nu}\langle X\rangle|a|^2 - C_7, \quad x,\xi\in\mathbf{R}^d.
$$

By the sharp Gårding inequality, we have

$$
-(H_{h_0}\lambda_0)^w \ge C_8^{-1}|\langle x\rangle^{-(1+\nu)/2}E_{s+1/2}a^w E_s^{-1}|^2 - C_9
$$

as a quadratic form on $\mathcal{S}(\mathbf{R}^d)$. Then the rest of the proof goes similarly to the first part of the proof of (1). □

A. Estimate along the Hamilton Flow for a Dispersive Equation

This appendix, independent of Sections 1, 2, 4–8, aims at deriving an energy estimate along the Hamilton flow of the principal symbol for a general dispersive operator $\partial_t + iH(t)$ by slightly modifying the proof in the section 6 of [5]. Here $g = |dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2$, $\sigma(H(t)) = h(t) = h_0 I_n + h_1(t)$, $h_0 \in S(\langle\xi\rangle^m, g)$, and $h_1(\cdot) \in B([0,T], S(\langle\xi\rangle^{m-1}, g; M_n(\mathbf{C})))$; h_0 satisfies that for a constant $C > 0$, $h_0(x,\xi) \geq C^{-1}|\xi|^m - C$, $(x,\xi) \in T^*\mathbf{R}^d$, and that $h_0(x,\xi)$ is homogeneous of degree m in ξ if $h_0(x,\xi) \geq 1/4$.

Let $\check{h}_0 \in C^\infty(T^*\mathbf{R}^d \setminus 0)$ be the homogeneous function of degree m in ξ such that $\check{h}_0 = h_0$ if $h_0 \geq 1/4$. Let Φ_t be the $H_{\check{h}_0}$-flow. Set $q = \check{h}_0^{1/m}$. Let U be a relatively compact, open subset of $S^*\mathbf{R}^d = \{z \in T^*\mathbf{R}^d \setminus 0; h_0(z) = 1\}$, and put $\Gamma = \cup_{0\leq t\leq t_0}\Phi_t(U)$ for an arbitrarily fixed $t_0 > 0$.

Lemma A.1. (1) *For every $f \in C_0^\infty(\Gamma)$, there is $u \in C_0^\infty(\Gamma)$ such that $H_{h_0}u + f \in C_0^\infty(U)$.*

(2) *For every nonnegative function $f \in C_0^\infty(\Gamma)$, there is a nonnegative function $u \in C_0^\infty(\Gamma)$ such that $u > 0$ on $\{X \in S^*\mathbf{R}^d; f(X) > 0\}$ and that $H_{h_0}u + f \in C_0^\infty(U)$.*

Proof. (1) By compactness, there exist $t_1, \ldots, t_J \in [0, t_0]$ satisfying supp $f \subset \cup_{j=1}^J \Phi_{t_j}(U)$. Take $\phi_j \in C_0^\infty(\Phi_{t_j}(U))$ such that $\phi_j \geq 0$, $\sum_{j=1}^J \phi_j = 1$ in a neighborhood of supp f. Set

$$u(X) := \sum_{j=1}^J \int_0^{t_j} (\phi_j f) \circ \Phi_t(X) dt.$$

Then $u \in C_0^\infty(\Gamma)$ and $H_{h_0}u + f \in C_0^\infty(U)$.

(2) The function $u \in C_0^\infty(\Gamma)$ constructed as above satisfies all the properties. □

Set $w_t(\xi) = 1 + t\langle\xi\rangle^{m-1}$ $(t \geq 0)$ and denote by $B([0,T], S(\langle\xi\rangle^b w_t^\rho, g))$ the set of all $p(\cdot) \in C([0,T], C^\infty(\mathbf{R}^{2d}))$ such that $\{p(t)w_t^{-\rho}\}_{0\leq t\leq T}$ is bounded in $S(\langle\xi\rangle^b, g)$ $(b, \rho \in \mathbf{R})$. Let $\theta \in C^\infty(\mathbf{R})$ such that $0 \leq \theta \leq 1$, $\theta(t) = 0$ if $t < 1/4$, and $\theta(t) = 1$ if $t \geq 1/2$.

Lemma A.2. *Let $s \in \mathbf{R}$, $\rho \geq 0$, $N \in \mathbf{N}$. Then for every compact set K of Γ, there are $f_j \in C^\infty(T^*\mathbf{R}^d \setminus 0)$, $f_j \geq 0$, homogeneous of degree 0 in ξ $(j = 0, 1, \ldots, N)$, and $v(t) \in B([0,T], S(w_t, g))$, $v(t) > 0$, satisfying* (i)–(iii).

(i) $v(t)^{-1} \in B([0,T], S(w_t^{-1}, g))$, $\quad \partial_t v \in B([0,T], S(\langle\xi\rangle^{m-1}w_t, g))$.

(ii) $f_0 > 0$ *on* K, $f_j > 0$ *on* $\operatorname{supp} f_{j-1} \cap S^*\mathbf{R}^d$ $(j = 1, 2, \ldots, N)$, $\operatorname{supp} f_N \cap S^*\mathbf{R}^d \subset \Gamma$.

(iii) *There exists a constant* $\lambda_0 > 0$ *such that for each* $\lambda \geq \lambda_0$, *we can find* $C > 0$ *and* $\alpha \in B([0,T]; S(\langle\xi\rangle^{2s+m-1} w_t^{2\rho}, g))$, $\operatorname{supp} \alpha(t,\cdot,\cdot) \subset \operatorname{cone}(K')$ *for a compact set* K' *of* U, *such that*

$$-(\partial_t P(t) + iH(t)^* P(t) - iP(t)H(t))$$
$$\geq \frac{1}{2}\sum_{j=0}^{N} \lambda^j ((q^{(m-1)/2} q_j(t))^w)^2 I_n - \alpha^w(t) I_n - C w_t(D)^{2\rho} \langle D\rangle^{2s+m-2-N} I_n$$

for all $t \in [0,T]$ *as a quadratic form on* $H^{(\infty)}(\mathbf{R}^d, \mathbf{C}^n)$. *Here*

$$P(t) = \sum_{j=0}^{N} \lambda^j q_j^w(t)^2, \qquad q_j(t) = q^{s-j/2} f_j v(t)^\rho \theta(q) \in B([0,T], S(\langle\xi\rangle^{s-j/2} w_t^\rho, g)).$$

Proof. By Lemma A.1 we can choose $a_j, b \in C_0^\infty(\Gamma)$, $a_j \geq 0$ $(j = 0, 1, \ldots, N)$ so that

(a) $a_0 > 0$ on K, $\quad a_j > 0$ on $\operatorname{supp} a_{j-1}$ $(j = 1, 2, \ldots, N)$;

(b) $-H_{h_0} a_j = b_j - \alpha_j$ with $b_j \in C_0^\infty(\Gamma)$, $b_j \geq 0$, and $\alpha_j \in C_0^\infty(U)$ $(j = 0, 1, \ldots, N)$;

(c) $-H_{h_0} b = 1 - \beta$ near $\operatorname{supp} a_N$ with $\beta \in C_0^\infty(U)$.

In fact, let $a_{-1} : S^*\mathbf{R}^d \to \mathbf{R}$ such that $a_{-1} = 1$ on K and $a_{-1} = 0$ outside K. Take a nonnegative function $b_j \in C_0^\infty(\Gamma)$ such that $b_j = 1$ near $\operatorname{supp} a_{j-1}$, and choose a nonnegative function $a_j \in C_0^\infty(\Gamma)$ such that $a_j > 0$ on $\{X \in S^*\mathbf{R}^d; b_j(X) > 0\}$ and that $\alpha_j := H_{h_0} a_j + b_j \in C_0^\infty(U)$, inductively in $j = 0, 1, \ldots, N+1$, and set $b = a_{N+1}$ and $\beta = \alpha_{N+1}$.

Take $M > 1$ such that $\|h_1(t,x,\xi)\|_{L(\mathbf{C}^n)} \leq M q^{m-1}/4$ for all $t \in [0,T]$ and $x, \xi \in \mathbf{R}^d$ if $h_0 \geq 1/4$. Take $\varepsilon > 0$ such that $e^b \geq \varepsilon$. Extend $a_j, \alpha_j, b_j, b, \beta$ as homogeneous functions of degree 0. Set $f(t) = e^b + \varepsilon t q^{m-1}$. For $j = 0, 1, \ldots, N$, put

$$q_j(t) = q^{s-j/2} a_j e^{Mb} f(t)^\rho \theta(q) \in B([0,T], S(\langle\xi\rangle^{s-j/2} w_t^\rho(\xi), g)).$$

Define $v(t) \in B([0,T], S(w_t, g))$ as a modification of $f(t)$ outside $\operatorname{supp}\theta(q)$ so that (i) is valid. Set $f_j = a_j e^{Mb}$. Then (ii) is valid. By calculation,

$$-(\partial_t + H_{h_0})\, q_j(t)^2 = 2Mq^{m-1} q_j(t)^2 + 2q^{m-1} q_j(t)^2 \rho(e^b - \varepsilon)/f(t)$$
$$+2q^{2s-j+m-1} a_j b_j e^{2Mb} f(t)^{2\rho} \theta(q)^2 - \beta_j(t).$$

Here $\beta_j \in B([0,T], S(\langle\xi\rangle^{2s-j+m-1} w_t^{2\rho}, g))$, $\operatorname{supp}\beta_j(t,\cdot,\cdot) \subset \operatorname{supp}\alpha_j \cup \operatorname{supp}\beta$.

By the product formula and the (sharp) Gårding inequality,

$$
\begin{aligned}
\text{(A.1)}\quad & -(\partial_t\, q_j^w(t)^2 I_n + iH(t)^* q_j^w(t)^2 - iq_j^w(t)^2 H(t)) \\
&= (-(\partial_t + H_{h_0}) q_j(t)^2 I_n + 2q_j(t)^2 \Im h_1(t)\,)^w - \gamma_{j1}^w(t) \\
&= \big(2(Mq^{m-1} I_n + \Im h_1(t)) q_j(t)^2 + c_j(t) v(t)^{2\rho} I_n\big)^w - \gamma_{j1}^w(t) - \beta_j^w(t) I_n \\
&= 2(q^{(m-1)/2} q_j(t))^w \,(M I_n + \Im h_1(t) q^{1-m}\theta(2q))^w\, (q^{(m-1)/2} q_j(t))^w \\
&\quad + (v(t)^\rho)^w c_j^w(t) (v(t)^\rho)^w I_n - \gamma_{j2}^w(t) - \beta_j^w(t) I_n \\
&\geq ((q^{(m-1)/2} q_j(t))^w)^2 I_n - C_1 \langle D\rangle^{2s-j+m-2} w_t(D)^{2\rho} I_n - \beta_j^w(t) I_n.
\end{aligned}
$$

Here $c_j \in B([0,T], S(\langle\xi\rangle^{2s-j+m-1}, g))$, $c_j \geq 0$, was estimated from below by $-C\langle D\rangle^{2s-j+m-2} w_t(D)^{2\rho} I_n$ by using the sharp Gårding inequality; $\gamma_{jk} \in B([0,T], S(\langle\xi\rangle^{2s-j+m-2} w_t^{2\rho}, g; M_n(\mathbf{C})))$ $(k=1,2)$, and $C_1 > 0$.

Take $\tilde{a}_j \in C_0^\infty(\Gamma)$ such that $\tilde{a}_j \geq 0$, $\tilde{a}_j = 1$ in a neighborhood of $\operatorname{supp} a_j$, and $a_{j+1} > 0$ on $\operatorname{supp} \tilde{a}_j$. Extend $\tilde{a}_j$ as a homogeneous function of degree 0, and set $d_j = \tilde{a}_j \theta(2q)$. By microlocal ellipticity, there are $e_j(\cdot) \in B([0,T], S(1,g))$ and $r_{0,j}(\cdot) \in B([0,T]; S(\langle\xi\rangle^{-\infty}, g))$ such that

$$
e_j^w(t) (q^{(m-1)/2} q_{j+1}(t))^w + r_{0,j}^w(t) = \langle D\rangle^{s+(m-j-2)/2} w_t(D)^\rho d_j^w
$$

for $j = 0, 1, \ldots, N$. Here $q_{N+1}(t,x,\xi) = \langle\xi\rangle^{s-(N+1)/2} w_t(\xi)^\rho$. Multiplying (A.1) by d_j^w from both sides, we have

$$
\begin{aligned}
&- (\partial_t\, q_j^w(t)^2 I_n + iH(t)^* q_j^w(t)^2 - iq_j^w(t)^2 H(t)) \\
\geq& ((q^{(m-1)/2} q_j(t))^w)^2 I_n - C_2 |\langle D\rangle^{s+(m-j-2)/2} w_t(D)^\rho d_j^w|^2 I_n \\
&- d_j^w \beta_j(t)^w d_j^w I_n - r_{1,j}^w(t) I_n \\
\geq& ((q^{(m-1)/2} q_j(t))^w)^2 I_n - C((q^{(m-1)/2} q_{j+1}(t))^w)^2 I_n - \tilde{\beta}_j^{\,w}(t) I_n - r_{2,j}^w(t) I_n.
\end{aligned}
$$

Here $r_{1,j}, r_{2,j} \in B([0,T], S(\langle\xi\rangle^{-\infty}, g))$, $\tilde{\beta}_j \in B([0,T], S(\langle\xi\rangle^{2s-j+m-1} w_t^{2\rho}, g))$ with $\operatorname{supp} \tilde{\beta}_j(t,\cdot,\cdot) \subset \operatorname{supp} \alpha_j \cup \operatorname{supp} \beta$, and $C, C_2 > 0$.

Define $P(t) = \sum_{j=0}^N \lambda^j q_j^w(t)^2$ with a parameter $\lambda > 0$. Then for every $\lambda \geq \lambda_0 := 2C$ there are $C' > 0$ and $\alpha \in B([0,T], S(\langle\xi\rangle^{2s+m-1} w_t^{2\rho}, g))$, satisfying $\operatorname{supp} \alpha(t,\cdot,\cdot) \subset (\cup_{j=0}^N \operatorname{supp} \alpha_j) \cup \operatorname{supp} \beta$, such that

$$
\begin{aligned}
&-(\partial_t\, P(t) + iH(t)^* P(t) - iP(t) H(t)) \\
&\geq \frac{1}{2} \sum_{j=0}^N \lambda^j ((q^{(m-1)/2} q_j(t))^w)^2 I_n - \alpha^w(t) I_n - C' \langle D\rangle^{2s+m-N-2} w_t(D)^{2\rho} I_n.
\end{aligned}
$$

□

Lemma A.3. *Let $s \in \mathbf{R}$, $\rho \geq 0$ and $L \gg 1$. Then for every $a \in S^0_{cpt}(\Gamma)$, there are $\tilde{a} \in S^0_{cpt}(\Gamma)$, $b \in S^0_{cpt}(U)$, and a constant $C > 0$ such that the a priori estimate below holds*:

$$\begin{aligned}
&\|w_t(D)^\rho \langle D\rangle^s a^w u(t)\|^2 + \int_0^t \|w_\tau(D)^\rho \langle D\rangle^{s+(m-1)/2} a^w u(\tau)\|^2 d\tau \\
&\leq C\int_0^t \|w_\tau(D)^\rho \langle D\rangle^{s+(m-1)/2} b^w u(\tau)\|^2 d\tau + C\int_0^t \|w_\tau(D)^\rho \langle D\rangle^s \tilde{a}^w f(\tau)\|^2 d\tau \\
&\quad + C\|\langle D\rangle^s \tilde{a}^w u(0)\|^2 + C\sup_{0\leq\tau\leq t} \|\langle D\rangle^{s-L} u(\tau)\|^2 + C\int_0^t \|\langle D\rangle^{s-L} f(\tau)\|^2 d\tau
\end{aligned}$$

for all $t \in [0,T]$ and $u \in C^1([0,T], H^\infty(\mathbf{R}^d, \mathbf{C}^n))$ with $f(t) = (\partial_t + iH(t))u(t)$.

Proof. For $L \gg 1$, take $N \in \mathbf{N}$ such that $N + 2 - m > 2L + 2\rho(m-1)$. Take a compact set $K \subset \Gamma$ such that $\operatorname{supp} a \subset \operatorname{cone} K$, apply Lemma A.2 with this compact set K, and take $\tilde{a} \in S^0_{cpt}(\Gamma)$ such that $\tilde{a}(X) = 1$ if $X \in \operatorname{supp} f_N$ and $h_0(X) \geq 1/4$. Fix $\lambda \geq \lambda_0$ and define two seminorms

$$N_k(t,u) = \left(\sum_{j=0}^N \lambda^j \|(q^{k(m-1)/2} q_j(t))^w u\|^2\right)^{1/2} \qquad (k = 0, 1).$$

Note that $(P(t)u, u) = N_0(t,u)^2$. Then there exists $C > 0$ such that

$$\|w_t(D)^\rho \langle D\rangle^{s+k(m-1)/2} a^w u\|^2 \leq CN_k(t,u)^2 + C\|\langle D\rangle^{s-L} u\|^2, \tag{A.2}$$

$$N_k(t,u)^2 \leq C\|w_t(D)^\rho \langle D\rangle^{s+k(m-1)/2} \tilde{a}^w u\|^2 + C\|\langle D\rangle^{s-L} u\|^2 \tag{A.3}$$

for all $u \in H^\infty(\mathbf{R}^d, \mathbf{C}^n)$ and $0 \leq t \leq T$.

Let $u \in C^1([0,T], H^\infty(\mathbf{R}^d, \mathbf{C}^n))$ and set $f(t) = (\partial_t + iH(t))u(t)$. Then we have

$$\begin{aligned}
&\frac{d}{dt} N_0(t, u(t))^2 \\
&= ((\partial_t P(t) + iH(t)^* P(t) - iP(t)H(t))u(t), u(t)) + 2\Re(P(t)f(t), u(t)) \\
&\leq -N_1(t,u(t))^2/2 + (\alpha^w(t)u(t), u(t)) + C\|w_t(D)^\rho \langle D\rangle^{s+(m-2-N)/2} u(t)\|^2 \\
&\quad + N_0(t,u(t))^2 + N_0(t,f(t))^2.
\end{aligned}$$

By a Gronwall-type inequality, we get

$$\begin{aligned}
&e^{-t} N_0(t,u(t))^2 + \int_0^t e^{-\tau} N_1(\tau, u(\tau))^2 d\tau/2 \\
&\leq N_0(0,u(0)) + \int_0^t e^{-\tau}\Big((\alpha^w(\tau)u(\tau), u(\tau)) \\
&\qquad + C\|w_\tau(D)^\rho \langle D\rangle^{s+(m-2-N)/2} u(\tau)\|^2 + N_0(\tau, f(\tau))^2\Big) d\tau.
\end{aligned}$$

By (A.2), (A.3), and a similar estimate about the term containing $\alpha(\cdot)$, we can complete the proof. □

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