

Level Zero Fundamental Representations over Quantized Affine Algebras and Demazure Modules

By

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Abstract

Let $W(\varpi_k)$ be the finite-dimensional irreducible module over a quantized affine algebra $U'_q(\mathfrak{g})$ with the fundamental weight ϖ_k as an extremal weight. We show that its crystal $B(W(\varpi_k))$ is isomorphic to the Demazure crystal $B^-(-\Lambda_0 + \varpi_k)$. This is derived from the following general result: for a dominant integral weight λ and an integral weight μ , there exists a unique homomorphism $U_q^-(\mathfrak{g})(u_\lambda \otimes u_\mu) \rightarrow V(\lambda + \mu)$ that sends $u_\lambda \otimes u_\mu$ to $u_{\lambda+\mu}$. Here $V(\lambda)$ is the extremal weight module with λ as an extremal weight, and $u_\lambda \in V(\lambda)$ is the extremal weight vector of weight λ .

§1. Introduction

The finite-dimensional representations of quantized affine algebras $U'_q(\mathfrak{g})$ are extensively studied in connection with exactly solvable models. It is expected that there exists a “good” finite-dimensional $U'_q(\mathfrak{g})$ -module $W(m\varpi_k)$ with a multiple $m\varpi_k$ of a fundamental weight ϖ_k as an extremal weight. This module is good in the sense that it is irreducible and it has a crystal base and moreover a global basis.

In the untwisted case, its conjectural character formula is given by Kirillov–Reshetikhin ([17], see also [16]), and its conjectural fusion construction is given by Kuniba–Nakanishi–Suzuki ([18]). It is proved by Nakajima ([22]) that the

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fusion construction gives irreducible modules with the expected character in the simply laced case, and by Chari ([3]) in some cases.

It is also expected that any “good” finite-dimensional $U'_q(\mathfrak{g})$ -module is a tensor product of modules of the above type.

It is also conjectured in [4, 5] that the $U'_q(\mathfrak{g})$ -modules $W(m\varpi_k)$ has a perfect crystal of level ℓ if and only if $m = \ell c_k^\vee$ ($c_k^\vee := \max(1, 2/(\alpha_k, \alpha_k))$). Moreover it is conjectured that the crystal base $B(W(\ell c_k^\vee \varpi_k))$ is isomorphic to the Demazure crystal $B^-(-\ell\Lambda_0 + \ell c_k^\vee \varpi_k)$ if we forget the 0-arrows. Here, for an integral weight λ , $B^\pm(\lambda)$ denotes the crystal for the $U_q^\pm(\mathfrak{g})$ -module generated by the extremal vector with weight λ . They are proved in certain cases ([7, 8]). More general relations of perfect crystals and Demazure crystals are discussed in [6].

In this paper we show that $B(W(\varpi_k))$ is isomorphic to the Demazure crystal $B^-(-\Lambda_0 + \varpi_k)$, or equivalently $B(W(-\varpi_k))$ is isomorphic to the Demazure crystal $B^+(\Lambda_0 - \varpi_k)$ (Corollary 4.8).

The main ingredient is the following theorem, which the author started to study in order to answer a question raised by Miwa et al:

Theorem 3.3. *Let $U_q^-(\mathfrak{g})$ be a quantized affine algebra. Let $\lambda \in P^+$ be a dominant integral weight and $\mu \in P$ an integral weight. Then there exists a unique homomorphism $V(\lambda) \otimes V(\mu) \supset U_q^-(\mathfrak{g})(u_\lambda \otimes u_\mu) \longrightarrow V(\lambda + \mu)$ that sends $u_\lambda \otimes u_\mu$ to $u_{\lambda+\mu}$. Moreover this morphism is compatible with global bases.*

Here $V(\lambda)$ is the extremal weight module with λ as an extremal weight, and $u_\lambda \in V(\lambda)$ is the extremal weight vector of weight λ .

§2. Review on Crystal Bases and Global Bases

In this section, we shall review briefly the quantized universal enveloping algebras and crystal bases. We refer the reader to [9, 10, 13, 14, 15, 19].

§2.1. Quantized universal enveloping algebras

We shall define the quantized universal enveloping algebra $U_q(\mathfrak{g})$. Assume that we are given the following data.

P : a free \mathbb{Z} -module (called a weight lattice),

I : an index set (for simple roots),

$$\begin{aligned} \alpha_i &\in P \text{ for } i \in I \text{ (called a simple root),} \\ h_i &\in P^* := \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \text{ (called a simple coroot),} \\ (\cdot, \cdot) &: P \times P \rightarrow \mathbb{Q} \text{ a bilinear symmetric form.} \end{aligned}$$

We shall denote by $\langle \cdot, \cdot \rangle : P^* \times P \rightarrow \mathbb{Z}$ the canonical pairing.

The data above are assumed to satisfy the following axioms.

$$(2.1) \quad \begin{aligned} (\alpha_i, \alpha_i) &> 0 \quad \text{for any } i \in I, \\ (\alpha_i, \alpha_j) &\leq 0 \quad \text{for any } i, j \in I \text{ with } i \neq j, \\ \langle h_i, \lambda \rangle &= \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{for any } i \in I \text{ and } \lambda \in P. \end{aligned}$$

Let us take a positive integer d such that $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}d^{-1}$ for any $i \in I$. Now let q be an indeterminate and set

$$(2.2) \quad K = \mathbb{Q}(q_s) \text{ where } q_s = q^{1/d}.$$

We define its subrings $\mathbf{A}_0, \mathbf{A}_\infty$ and \mathbf{A} as follows.

$$(2.3) \quad \begin{aligned} \mathbf{A}_0 &= \{f/g; f, g \in \mathbb{Q}[q_s], g(0) \neq 0\}, \\ \mathbf{A}_\infty &= \{f/g; f, g \in \mathbb{Q}[q_s^{-1}], g(0) \neq 0\}, \\ \mathbf{A} &= \mathbb{Q}[q_s, q_s^{-1}]. \end{aligned}$$

Definition 2.1. The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is the algebra over K generated by the symbols e_i, f_i ($i \in I$) and $q(h)$ ($h \in d^{-1}P^*$) with the following defining relations.

- (1) $q(h_1)q(h_2) = q(h_1 + h_2)$ for $h_1, h_2 \in d^{-1}P^*$, and $q(h) = 1$ for $h = 0$.
- (2) $q(h)e_i q(h)^{-1} = q^{\langle h, \alpha_i \rangle} e_i$ and $q(h)f_i q(h)^{-1} = q^{-\langle h, \alpha_i \rangle} f_i$ for any $i \in I$ and $h \in d^{-1}P^*$.
- (3) $[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$. Here $q_i = q^{(\alpha_i, \alpha_i)/2}$ and $t_i = q^{(\frac{\alpha_i, \alpha_i}{2} h_i)}$.
- (4) (Serre relation) For $i \neq j$,

$$\sum_{k=0}^b (-1)^k e_i^{(k)} e_j e_i^{(b-k)} = \sum_{k=0}^b (-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0.$$

Here $b = 1 - \langle h_i, \alpha_j \rangle$ and

$$\begin{aligned} e_i^{(k)} &= e_i^k / [k]_i!, \quad f_i^{(k)} = f_i^k / [k]_i!, \\ [k]_i &= (q_i^k - q_i^{-k}) / (q_i - q_i^{-1}), \quad [k]_i! = [1]_i \cdots [k]_i. \end{aligned}$$

For $i \in I$, we denote by $U_q^-(\mathfrak{g})_i$ the subalgebra of $U_q^-(\mathfrak{g})$ generated by e_i, f_i and $q(h)$ ($h \in d^{-1}P^*$).

Let us denote by W the Weyl group, the subgroup of $GL(P)$ generated by the simple reflections $s_i: s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$.

Let $\Delta \subset Q := \sum_i \mathbb{Z}\alpha_i$ be the set of roots. Let $\Delta^\pm := \Delta \cap Q_\pm$ be the set of positive and negative roots, respectively. Here $Q_\pm := \pm \sum_i \mathbb{Z}_{\geq 0}\alpha_i$. Let Δ^{re} be the set of real roots, and set $\Delta_\pm^{\text{re}} := \Delta_\pm \cap \Delta^{\text{re}}$.

§2.2. Braid group action on integrable modules

The q -analogue of the action of the Weyl group is introduced in [19, 23]. We define a q -analogue of the exponential function by

$$(2.4) \quad \exp_q(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} x^n}{[n]!}.$$

This satisfies the following equations:

$$(2.5) \quad \begin{aligned} \exp_q(x) \exp_q(y) &= \exp_q(x+y) \quad \text{if } xy = q^2yx, \\ \exp_q(x) \exp_{q^{-1}}(y) &= \sum_{n=0}^{\infty} \frac{1}{[n]!} \prod_{\nu=0}^{n-1} (q^\nu x + q^{-\nu} y) \quad \text{if } [x, y] = 0, \\ \exp_q(x) \exp_{q^{-1}}(-x) &= 1, \\ \exp_q(x) &= \left(1 + (1 - q^2)x\right) \exp_q(q^2x), \\ \exp_q(x) &= \prod_{n=0}^{\infty} \left(1 + q^{2n}(1 - q^2)x\right) \quad \text{for } |q| < 1, \end{aligned}$$

For $i \in I$, we set

$$(2.6) \quad \begin{aligned} S_i &= \exp_{q_i^{-1}}(q_i^{-1}e_i t_i^{-1}) \exp_{q_i^{-1}}(-f_i) \exp_{q_i^{-1}}(q_i e_i t_i) q_i^{h_i(h_i+1)/2} \\ &= \exp_{q_i^{-1}}(-q_i^{-1}f_i t_i) \exp_{q_i^{-1}}(e_i) \exp_{q_i^{-1}}(-q_i f_i t_i^{-1}) q_i^{h_i(h_i+1)/2}. \end{aligned}$$

We regard S_i as an endomorphism of integrable $U_q(\mathfrak{g})$ -modules, and $q_i^{h_i(h_i+1)/2}$ acts on the weight space of weight λ by the multiplication of $q_i^{\langle h_i, \lambda \rangle (\langle h_i, \lambda \rangle + 1)/2}$.

On the $(l+1)$ -dimensional irreducible representation of $U_q(\mathfrak{g})_i$ with a highest weight vector $u_0^{(l)}$ and $u_k^{(l)} = f_i^{(k)} u_0^{(l)}$,

$$(2.7) \quad S_i(u_k^{(l)}) = (-1)^{l-k} q_i^{(l-k)(k+1)} u_{l-k}^{(l)},$$

Hence, S_i sends the weight space of weight λ to the weight space of weight $s_i \lambda$. By the above formula, we have

$$(2.8) \quad S_i u_l^{(l)} = u_0^{(l)} \quad \text{and} \quad S_i u_0^{(l)} = (-q_i)^l u_l^{(l)}.$$

Since $\{S_i\}$ satisfies the braid relations, we can extend the actions of S_i on integrable modules to the action of the braid group by

$$\begin{aligned} S_{ww'} &= S_w \circ S_{w'} \quad \text{if } l(ww') = l(w) + l(w'), \\ S_{s_i} &= S_i. \end{aligned}$$

§2.3. Braid group action on $U_q^-(\mathfrak{g})$

We define the ring automorphism T_i of $U_q(\mathfrak{g})$ by

$$(2.9) \quad T_i(q) = q$$

$$(2.10) \quad T_i(q(h)) = q(s_i h),$$

$$(2.11) \quad T_i(e_i) = -f_i t_i,$$

$$(2.12) \quad T_i(f_i) = -t_i^{-1} e_i,$$

$$(2.13) \quad T_i(e_j) = \sum_{k=0}^{-\langle h_i, \alpha_j \rangle} (-1)^k q_i^{-k} e_i^{(-\langle h_i, \alpha_j \rangle - k)} e_j e_i^{(k)},$$

$$(2.14) \quad T_i(f_j) = \sum_{k=0}^{-\langle h_i, \alpha_j \rangle} (-1)^k q_i^k f_i^{(k)} f_j f_i^{(-\langle h_i, \alpha_j \rangle - k)} \text{ for } i \neq j.$$

Then it is well-defined, and it satisfies

$$(2.15) \quad T_i(P)u = S_i P S_i^{-1} u$$

for any $P \in U_q(\mathfrak{g})$ and any element u of an integrable $U_q(\mathfrak{g})$ -module.

The operator T_i is invertible and its inverse is given as follows.

$$(2.16) \quad T_i^{-1}(q(h)) = q(s_i h),$$

$$(2.17) \quad T_i^{-1}(e_i) = -t_i^{-1} f_i,$$

$$(2.18) \quad T_i^{-1}(f_i) = -e_i t_i,$$

$$(2.19) \quad T_i^{-1}(e_j) = \sum_{k=0}^{-\langle h_i, \alpha_j \rangle} (-1)^k q_i^{-k} e_i^{(k)} e_j e_i^{(-\langle h_i, \alpha_j \rangle - k)},$$

$$(2.20) \quad T_i^{-1}(f_j) = \sum_{k=0}^{-\langle h_i, \alpha_j \rangle} (-1)^k q_i^k f_i^{(-\langle h_i, \alpha_j \rangle - k)} f_j f_i^{(k)}.$$

We can extend the action T_i to the action of the braid group by

$$\begin{aligned} T_{ww'} &= T_w \circ T_{w'} \quad \text{if } l(ww') = l(w) + l(w'), \\ T_{s_i} &= T_i. \end{aligned}$$

The following proposition is proved in [19].

Proposition 2.2. For $w \in W$ and $i, j \in I$ such that $w\alpha_i = \alpha_j$, we have

$$T_w e_i = T_{w^{-1}}^{-1} e_i = e_j \quad \text{and} \quad T_w f_i = T_{w^{-1}}^{-1} f_i = f_j.$$

§2.4. Crystals

We shall not review the notion of crystals, but refer the reader to [9, 10, 13, 15]. For a subset J of I , let us denote by $U_q(\mathfrak{g}_J)$ the subalgebra of $U_q^-(\mathfrak{g})$ generated by e_j, f_j ($j \in J$) and $q(h)$ ($h \in d^{-1}P^*$). We say that a crystal B over $U_q^-(\mathfrak{g})$ is a *regular crystal* if, for any $J \subset I$ of finite-dimensional type, B is, as a crystal over $U_q(\mathfrak{g}_J)$, isomorphic to a crystal base associated with an integrable $U_q(\mathfrak{g}_J)$ -module.

By [13], the Weyl group W acts on any regular crystal. This action S is given by

$$S_{s_i} b = \begin{cases} \tilde{f}_i^{\langle h_i, \text{wt}(b) \rangle} b & \text{if } \langle h_i, \text{wt}(b) \rangle \geq 0, \\ \tilde{e}_i^{-\langle h_i, \text{wt}(b) \rangle} b & \text{if } \langle h_i, \text{wt}(b) \rangle \leq 0. \end{cases}$$

Let us denote by $U_q^-(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$) the subalgebra of $U_q^-(\mathfrak{g})$ generated by the f_i 's (resp. by the e_i 's). Then $U_q^-(\mathfrak{g})$ has a crystal base denoted by $B(\infty)$ ([10]). A unique vector of $B(\infty)$ with weight 0 is denoted by u_∞ . Similarly $U_q^+(\mathfrak{g})$ has a crystal base denoted by $B(-\infty)$, and a unique vector of $B(-\infty)$ with weight 0 is denoted by $u_{-\infty}$.

Let ψ be the ring automorphism of $U_q^-(\mathfrak{g})$ that sends q_s, e_i, f_i and $q(h)$ to q_s, f_i, e_i and $q(-h)$. It induces bijections $U_q^-(\mathfrak{g}) \xrightarrow{\sim} U_q^+(\mathfrak{g})$ and $B(\infty) \xrightarrow{\sim} B(-\infty)$ by which $u_\infty, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \text{wt}$ correspond to $u_{-\infty}, \tilde{f}_i, \tilde{e}_i, \varphi_i, \varepsilon_i, -\text{wt}$.

Let $\tilde{U}_q(\mathfrak{g})$ be the modified quantized universal enveloping algebra $\bigoplus_{\lambda \in P} U_q^-(\mathfrak{g})a_\lambda$ (see [13]). The elements a_λ , the projectors to the weight λ -space, satisfy $a_\lambda \cdot a_\mu = \delta_{\lambda, \mu} a_\lambda$ and $a_\lambda P = P a_{\lambda - \text{wt}(P)}$ for $P \in U_q^-(\mathfrak{g})$.

Then $\tilde{U}_q(\mathfrak{g})$ has a crystal base $(L(\tilde{U}_q(\mathfrak{g})), B(\tilde{U}_q(\mathfrak{g})))$. As a crystal, $B(\tilde{U}_q(\mathfrak{g}))$ is regular and isomorphic to

$$\bigsqcup_{\lambda \in P} B(\infty) \otimes T_\lambda \otimes B(-\infty).$$

Here, T_λ is the crystal consisting of a single element t_λ with $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$ and $\text{wt}(t_\lambda) = \lambda$.

Let $*$ be the anti-involution of $U_q^-(\mathfrak{g})$ that sends $q(h)$ to $q(-h)$, and q_s, e_i, f_i to themselves. The involution $*$ of $U_q^-(\mathfrak{g})$ induces an involution $*$ on $B(\infty), B(-\infty), B(\tilde{U}_q(\mathfrak{g}))$. Then $\tilde{e}_i^* = * \circ \tilde{e}_i \circ *$, etc. give another crystal structure

on $B(\infty)$, $B(-\infty)$, $B(\tilde{U}_q(\mathfrak{g}))$. We call it *the star crystal structure*. These two crystal structures on $B(\tilde{U}_q(\mathfrak{g}))$ are compatible, and $B(\tilde{U}_q(\mathfrak{g}))$ may be considered as a crystal over $\mathfrak{g} \oplus \mathfrak{g}$, which corresponds to the $U_q^-(\mathfrak{g})$ -bimodule structure on $\tilde{U}_q(\mathfrak{g})$. Hence, for example, S_w^* , the Weyl group action on $B(\tilde{U}_q(\mathfrak{g}))$ with respect to the star crystal structure is a crystal automorphism of $B(\tilde{U}_q(\mathfrak{g}))$ with respect to the original crystal structure. In particular, the two Weyl group actions S_w and S_w^* commute with each other.

§2.5. Global bases

Recall that $\mathbf{A}_0 \subset K$ is the subring of K consisting of rational functions in q_s without pole at $q_s = 0$. Let $-$ be the automorphism of K sending q_s to q_s^{-1} . Then $\overline{\mathbf{A}}_0$ coincides with the ring \mathbf{A}_∞ of rational functions regular at $q_s = \infty$. Set $\mathbf{A} := \mathbb{Q}[q_s, q_s^{-1}]$. Let V be a vector space over K , L_0 an \mathbf{A} -submodule of V , L_∞ an \mathbf{A}_∞ -submodule, and $V_{\mathbf{A}}$ a \mathbf{A} -submodule. Set $E := L_0 \cap L_\infty \cap V_{\mathbf{A}}$.

Definition 2.3 [10]. We say that $(L_0, L_\infty, V_{\mathbf{A}})$ is *balanced* if each of L_0 , L_∞ and $V_{\mathbf{A}}$ generates V as a K -vector space, and if one of the following equivalent conditions is satisfied.

- (i) $E \rightarrow L_0/q_s L_0$ is an isomorphism.
- (ii) $E \rightarrow L_\infty/q_s^{-1} L_\infty$ is an isomorphism.
- (iii) $(L_0 \cap V_{\mathbf{A}}) \oplus (q_s^{-1} L_\infty \cap V_{\mathbf{A}}) \rightarrow V_{\mathbf{A}}$ is an isomorphism.
- (iv) $\mathbf{A}_0 \otimes_{\mathbb{Q}} E \rightarrow L_0$, $\mathbf{A}_\infty \otimes_{\mathbb{Q}} E \rightarrow L_\infty$, $\mathbf{A} \otimes_{\mathbb{Q}} E \rightarrow V_{\mathbf{A}}$ and $K \otimes_{\mathbb{Q}} E \rightarrow V$ are isomorphisms.

Let $-$ be the ring automorphism of $U_q^-(\mathfrak{g})$ sending $q_s, q(h), e_i, f_i$ to $q_s^{-1}, q(-h), e_i, f_i$.

Let $U_q^-(\mathfrak{g})_{\mathbf{A}}$ be the \mathbf{A} -subalgebra of $U_q^-(\mathfrak{g})$ generated by $e_i^{(n)}, f_i^{(n)}$ and $q(h)$ ($h \in d^{-1}P^*$).

Let M be a $U_q^-(\mathfrak{g})$ -module. Let $-$ be an involution of M satisfying $(au)^- = \bar{a}\bar{u}$ for any $a \in U_q^-(\mathfrak{g})$ and $u \in M$. We call in this paper such an involution a *bar involution*. Let $(L(M), B(M))$ be a crystal base of an integrable $U_q^-(\mathfrak{g})$ -module M .

Let $M_{\mathbf{A}}$ be a $U_q^-(\mathfrak{g})_{\mathbf{A}}$ -submodule of M such that

$$(2.21) \quad (M_{\mathbf{A}})^- = M_{\mathbf{A}}, \text{ and } (u - \bar{u}) \in (q_s - 1)M_{\mathbf{A}} \text{ for every } u \in M_{\mathbf{A}}.$$

Definition 2.4. A $U_q^-(\mathfrak{g})$ -module M endowed with $(L(M), B(M), M_{\mathbf{A}}, -)$ as above is called with a *global basis*, if $(L(M), L(M)^-, M_{\mathbf{A}})$ is balanced,

In such a case, let $G: L(M)/q_s L(M) \xrightarrow{\sim} E := L(M) \cap L(M)^- \cap M_{\mathbf{A}}$ be the inverse of $E \xrightarrow{\sim} L(M)/q_s L(M)$. Then $\{G(b); b \in B(M)\}$ forms a basis of M . We call this basis a (lower) *global basis*. The global basis enjoys the following properties (see [10, 11]):

- (i) $\overline{G(b)} = G(b)$ for any $b \in B(M)$.
- (ii) For any $n \in \mathbb{Z}_{\geq 0}$, $\{G(b); \varepsilon_i(b) \geq n\}$ is a basis of the \mathbf{A} -submodule $\sum_{m \geq n} f_i^{(m)} M_{\mathbf{A}}$.
- (iii) for any $i \in I$ and $b \in B(M)$, we have

$$f_i G(b) = [1 + \varepsilon_i(b)]_i G(\tilde{f}_i b) + \sum_{b'} F_{b, b'}^i G(b').$$

Here the sum ranges over $b' \in B(M)$ such that $\varepsilon_i(b') > 1 + \varepsilon_i(b)$. The coefficient $F_{b, b'}^i$ belongs to $q_s q_i^{1 - \varepsilon_i(b')} \mathbb{Q}[q_s]$. Similarly for $e_i G(b)$.

Let M and N be $U_q^-(\mathfrak{g})$ -modules with global bases. We say that a $U_q^-(\mathfrak{g})$ -morphism $f: M \rightarrow N$ is *compatible with global bases* if it satisfies the following conditions:

- (i) If u is a global basis vector of M , then $f(u)$ is a global basis vector of N or 0.
- (ii) If a pair of global basis vectors u and v of M satisfies $f(u) = f(v) \neq 0$, then $u = v$.

These conditions are equivalent to the following set of conditions:

- (a) f commutes with the bar involutions.
- (b) f sends $L(M)$ to $L(N)$ and $M_{\mathbf{A}}$ to $N_{\mathbf{A}}$.
- (c) The induced morphism $\bar{f}: L(M)/q_s L(M) \rightarrow L(N)/q_s L(N)$ sends $B(M)$ to $B(N) \cup \{0\}$.
- (d) $\text{Ker}(f)$ is generated by a part of the global basis of M .

In such a case, $f(M)$ has a global basis, and we have

$$B(M) \supset B(f(M)) \subset B(N).$$

If f is a monomorphism then $B(M) \simeq B(f(M)) \subset B(N)$, and if f is an epimorphism then $B(M) \supset B(f(M)) \simeq B(N)$.

§2.6. Extremal vectors

Let M be an integrable $U_q^-(\mathfrak{g})$ -module. A non-zero vector $u \in M$ of weight $\lambda \in P$ is called *extremal* (see [13]), if we can find a subset F of non-zero weight vectors in M containing u and satisfying the following properties:

$$(2.22) \quad \begin{aligned} &\text{if } v \in F \text{ and } i \text{ satisfy } \langle h_i, \text{wt}(v) \rangle \geq 0, \text{ then } e_i v = 0 \text{ and } f_i^{\langle h_i, \text{wt}(v) \rangle} v \in F, \\ &\text{if } v \in F \text{ and } i \text{ satisfy } \langle h_i, \text{wt}(v) \rangle \leq 0, \text{ then } f_i v = 0 \text{ and } e_i^{-\langle h_i, \text{wt}(v) \rangle} v \in F, \end{aligned}$$

The Weyl group W acts on the set of extremal vectors by

$$(2.23) \quad \begin{aligned} &\text{if } \langle h_i, \text{wt}(u) \rangle \geq 0, \text{ then } S_{s_i}^{\text{norm}} u = f_i^{\langle h_i, \text{wt}(u) \rangle} u, \\ &\text{if } \langle h_i, \text{wt}(u) \rangle \leq 0, \text{ then } S_{s_i}^{\text{norm}} u = e_i^{-\langle h_i, \text{wt}(u) \rangle} u. \end{aligned}$$

We have $\text{wt}(S_w^{\text{norm}} u) = w \text{wt}(u)$ for $w \in W$. Note that, by (2.7), $S_w^{\text{norm}} u$ is equal to $S_w u$ up to a non-zero constant multiple.

Similarly, for a vector b of a regular crystal B with weight λ , we say that b is an extremal vector if it satisfies the following similar conditions:

$$(2.24) \quad \begin{aligned} &\text{if } w \in W \text{ and } i \in I \text{ satisfy } \langle h_i, w\lambda \rangle \geq 0, \text{ then } \tilde{e}_i S_w b = 0, \\ &\text{if } w \in W \text{ and } i \in I \text{ satisfy } \langle h_i, w\lambda \rangle \leq 0 \text{ then } \tilde{f}_i S_w b = 0. \end{aligned}$$

For $\lambda \in P$, let us denote by $V(\lambda)$ the $U_q^-(\mathfrak{g})$ -module generated by u_λ with the defining relation that u_λ is an extremal vector of weight λ . This is in fact infinitely many linear relations on u_λ .

For a dominant weight λ , $V(\lambda)$ is an irreducible highest weight module with highest weight λ , and $V(-\lambda)$ is an irreducible lowest weight module with lowest weight $-\lambda$.

We proved in [13]¹ that $V(\lambda)$ has a global basis $(L(\lambda), B(\lambda))$. We denote by the same letter u_λ the element of $B(\lambda)$ corresponding to $u_\lambda \in V(\lambda)$. Moreover $U_q^-(\mathfrak{g})a_\lambda \rightarrow V(\lambda)$ ($a_\lambda \mapsto u_\lambda$) is compatible with global bases. Hence the crystal

¹In [13], it is denoted by $V^{\max}(\lambda)$, because I thought there would be a natural $U_q^-(\mathfrak{g})$ -module whose crystal base is the connected component of $B(\lambda)$.

$B(\lambda)$ is isomorphic to the subcrystal of $B(\infty) \otimes t_\lambda \otimes B(-\infty)$ consisting of vectors b such that b^* is an extremal vector of weight $-\lambda$. By this embedding, $u_\lambda \in B(\lambda)$ corresponds to $u_\infty \otimes t_\lambda \otimes u_{-\infty}$.

Note that

$$(2.25) \quad U_q^+(\mathfrak{g})u_\lambda = \bigoplus_{b \in B(\lambda) \cap (u_\infty \otimes t_\lambda \otimes B(-\infty))} KG(b).$$

For any $w \in W$, $u_\lambda \mapsto S_w^{\text{norm}} u_{w\lambda}$ gives an isomorphism of $U_q^-(\mathfrak{g})$ -modules:

$$V(\lambda) \xrightarrow{\sim} V(w\lambda).$$

This is compatible with global bases. Similarly, letting S_w^* be the Weyl group action on $B(\tilde{U}_q(\mathfrak{g}))$ with respect to the star crystal structure and regarding $B(\lambda)$ as a subcrystal of $B(\tilde{U}_q(\mathfrak{g}))$, $S_w^* : B(\tilde{U}_q(\mathfrak{g})) \xrightarrow{\sim} B(\tilde{U}_q(\mathfrak{g}))$ induces an isomorphism of crystals

$$(2.26) \quad S_w^* : B(\lambda) \xrightarrow{\sim} B(w\lambda).$$

This coincides with the crystal isomorphism induced by $V(\lambda) \xrightarrow{\sim} V(w\lambda)$. Note that we have

$$S_w S_w^*(u_\infty \otimes t_\lambda \otimes u_{-\infty}) = u_\infty \otimes t_{w\lambda} \otimes u_{-\infty}.$$

§2.7. Global bases of tensor products

Let us recall the following results proved by Lusztig ([19]). Let \mathcal{O}_{int} be the category of integrable $U_q^-(\mathfrak{g})$ -modules which are a direct sum of $V(\lambda)$'s ($\lambda \in P^+$). Similarly let $\mathcal{O}_{\text{int}}^-$ be the category of integrable $U_q^-(\mathfrak{g})$ -modules which are a direct sum of $V(\lambda)$'s ($\lambda \in P^-$). Let M and N be $U_q^-(\mathfrak{g})$ -modules. Assume that M and N have bar involutions, and that either $M \in \mathcal{O}_{\text{int}}$ or $N \in \mathcal{O}_{\text{int}}^-$. Then there exists a unique bar involution on $M \otimes N$ such that

$$(u \otimes v)^- = \bar{u} \otimes \bar{v} \quad \text{for every } u \in M \text{ and } v \in N \text{ such that either}$$

$$u \text{ is a highest weight vector or } v \text{ is a lowest weight vector.}$$

Assume further that M and N have a global basis. Then $M \otimes N$ has a crystal base $(L(M \otimes N), B(M \otimes N)) := (L(M) \otimes_{\mathbf{A}_0} L(N), B(M) \otimes B(N))$, and an \mathbf{A} -form $(M \otimes N)_{\mathbf{A}} = M_{\mathbf{A}} \otimes_{\mathbf{A}} N_{\mathbf{A}}$. Then $M \otimes N$ has a global basis; namely $(L(M \otimes N), L(M \otimes N)^-, (M \otimes N)_{\mathbf{A}})$ is balanced. In particular, $V(\lambda) \otimes V(\mu)$ has a global basis either if λ is dominant or if $-\mu$ is dominant.

Let $\lambda \in P$. Then for any pair of dominant integral weights ξ and η such that $\lambda = \xi - \eta$, $U_q^-(\mathfrak{g})a_\lambda \rightarrow V(\xi) \otimes V(-\eta)$ ($a_\lambda \mapsto u_\xi \otimes u_{-\eta}$) is compatible with global bases. Conversely the global basis of $U_q^-(\mathfrak{g})a_\lambda$ is characterized by the above property.

Lemma 2.5. For $\lambda \in P^+$ and $\mu \in P$,

$$(2.27) \quad U_q^-(\mathfrak{g})a_{\lambda+\mu} \rightarrow V(\lambda) \otimes U_q^-(\mathfrak{g})a_\mu \quad (a_{\lambda+\mu} \mapsto u_\lambda \otimes a_\mu)$$

is compatible with global bases.

Proof. For dominant integral weights ξ and η such that $\mu = \xi - \eta$, we have a diagram of morphisms compatible with crystal bases except the dotted arrow:

$$\begin{array}{ccc} U_q(\mathfrak{g})a_{\lambda+\mu} & \longrightarrow & V(\lambda + \xi) \otimes V(-\eta) \\ \vdots & & \downarrow \\ V(\lambda) \otimes U_q^-(\mathfrak{g})a_\mu & \longrightarrow & V(\lambda) \otimes V(\xi) \otimes V(-\eta) \end{array}$$

Hence the dotted arrow is compatible with crystal bases. □

This morphism (2.27) induces an embedding of crystals

$$B(U_q^-(\mathfrak{g})a_{\lambda+\mu}) \hookrightarrow B(\lambda) \otimes B(U_q^-(\mathfrak{g})a_\mu) \text{ for } \lambda \in P^+ \text{ and } \mu \in P.$$

There exists an embedding $B(\infty) \hookrightarrow B(\lambda) \otimes B(\infty) \otimes T_{-\lambda}$, and the above morphism coincides with the composition

$$\begin{aligned} B(U_q^-(\mathfrak{g})a_{\lambda+\mu}) &\simeq B(\infty) \otimes T_{\lambda+\mu} \otimes B(-\infty) \hookrightarrow B(\lambda) \otimes B(\infty) \\ &\quad \otimes T_{-\lambda} \otimes T_{\lambda+\mu} \otimes B(-\infty) \\ &\simeq B(\lambda) \otimes B(\infty) \otimes T_\mu \otimes B(-\infty) \simeq B(\lambda) \otimes B(U_q^-(\mathfrak{g})a_\mu). \end{aligned}$$

§2.8. Demazure modules

Let M be an integrable $U_q^-(\mathfrak{g})$ -module with a global basis $(L(M), B(M), M_{\mathbf{A}}, -)$. Let N be a $U_q^+(\mathfrak{g})$ -submodule of M . We say that N is *compatible with the global basis of M* if there exists a subset $B(N)$ of $B(M)$ such that $N = \bigoplus_{b \in B(N)} KG(b)$.

It is shown in [12] that

$$(2.28) \quad \tilde{e}_i B(N) \subset B(N) \cup \{0\}, \text{ and } U_q^-(\mathfrak{g})N = U_q^-(\mathfrak{g})N \text{ is also compatible with the global basis.}$$

Namely there exists a subset $B(U_q^-(\mathfrak{g})N)$ of $B(M)$ such that

$$U_q^-(\mathfrak{g})N = \bigoplus_{b \in B(U_q^-(\mathfrak{g})N)} KG(b).$$

Moreover we have

$$B(U_q^-(\mathfrak{g})N) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_m} b; m \geq 0, i_1, \dots, i_m \in I, b \in B(N) \right\} \setminus \{0\}.$$

For $\lambda \in P$, the $U_q^\pm(\mathfrak{g})$ -submodule $U_q^\pm(\mathfrak{g})u_\lambda$ of $V(\lambda)$ is compatible with the global basis of $V(\lambda)$ (see (2.25)).

We set

$$B^\pm(\lambda) = B(U_q^\pm(\mathfrak{g})u_\lambda).$$

Regarding $B(\lambda)$ as a subset of $B(U_q^-(\mathfrak{g})a_\lambda) = B(\infty) \otimes t_\lambda \otimes B(-\infty)$, we have

$$B^+(\lambda) = B(\lambda) \cap (u_\infty \otimes t_\lambda \otimes B(-\infty)) \quad \text{and} \quad B^-(\lambda) = B(\lambda) \cap (B(\infty) \otimes t_\lambda \otimes u_{-\infty}).$$

The subset $B^+(\lambda)$ satisfies the following properties.

Lemma 2.6.

- (i) $\tilde{e}_i B^+(\lambda) \subset B^+(\lambda) \cup \{0\}$.
- (ii) For any $b \in B^+(\lambda)$, if $\varepsilon_i(b) > 0$, then $\tilde{f}_i b \in B^+(\lambda) \cup \{0\}$. Or equivalently, for any i -string S of $B(\lambda)$, $S \cap B^+(\lambda)$ is either S itself, the empty set or the set consisting of the highest weight vector of S . Here an i -string is a connected component with respect to the crystal structure over $U_q^-(\mathfrak{g})_i$.

This is a consequence of the following lemma. Note that $B(U_q^+(\mathfrak{g})a_\lambda) = u_\infty \otimes T_\lambda \otimes B(-\infty)$.

Lemma 2.7.

- (i) $\tilde{e}_i B(U_q^+(\mathfrak{g})a_\lambda) \subset B(U_q^+(\mathfrak{g})a_\lambda) \cup \{0\}$.
- (ii) For any $b \in B(U_q^+(\mathfrak{g})a_\lambda)$, if $\varepsilon_i(b) > 0$, then $\tilde{f}_i b \in B(U_q^+(\mathfrak{g})a_\lambda) \cup \{0\}$. Or equivalently, for any i -string S of $B(U_q^-(\mathfrak{g})a_\lambda)$, $S \cap B(U_q^+(\mathfrak{g})a_\lambda)$ is either S itself, the empty set or the set consisting of the highest weight vector of S .

Proof. The first property is evident. In order to prove (ii), write $b = u_\infty \otimes t_\lambda \otimes b'$ with $b' \in B(-\infty)$. Then $\varepsilon_i(b) = \max(0, \varepsilon_i(t_\lambda \otimes b'))$, and hence $0 = \varphi_i(u_\infty) < \varepsilon_i(t_\lambda \otimes b')$. We have therefore $\tilde{f}_i b = u_\infty \otimes t_\lambda \otimes \tilde{f}_i b'$. \square

Similar results hold for $B^-(\lambda)$ and $B(U_q^-(\mathfrak{g})a_\lambda)$.

Proposition 2.8. For $\beta \in \Delta_+^{\text{re}}$ and $\lambda \in P$, assume $(\beta, \lambda) \geq 0$. Then we have

$$S_{s_\beta} u_\lambda \in U_q^-(\mathfrak{g})u_\lambda \quad \text{and} \quad S_{s_\beta} u_\lambda \in B^-(\lambda).$$

Proof. We shall argue by the induction of $\text{ht}(\beta)$. Let us take $i \in I$ such that $\langle h_i, \beta \rangle > 0$. If $\beta = \alpha_i$ then the assertion is trivial. Otherwise we have $\gamma := s_i(\beta) \in \Delta_+^{\text{re}}$. Since $(\gamma, s_i \lambda) = (\beta, \lambda) \geq 0$, the induction hypothesis implies that

$$(2.29) \quad S_{s_\gamma} S_i u_\lambda \in U_q^-(\mathfrak{g})S_i u_\lambda.$$

If $\langle h_i, \lambda \rangle \geq 0$, we have

$$U_q^-(\mathfrak{g})u_\lambda \supset U_q^-(\mathfrak{g})S_i u_\lambda \supset U_q^-(\mathfrak{g})S_{s_\gamma} S_i u_\lambda = S_i S_{s_\beta} u_\lambda$$

Since $U_q^-(\mathfrak{g})u_\lambda$ is an $U_q(\mathfrak{g})_i$ -module, it contains $S_{s_\beta} u_\lambda$.

Now assume that $\langle h_i, \lambda \rangle < 0$. Then $\langle h_i, s_\beta \lambda \rangle = \langle h_i, \lambda \rangle - \langle \beta^\vee, \lambda \rangle \langle h_i, \beta \rangle < 0$. By (2.29), we have

$$\begin{aligned} S_i S_{s_\beta} S_i (u_\infty \otimes t_{s_i \lambda} \otimes u_{-\infty}) &= S_{s_\gamma} (u_\infty \otimes t_{s_i \lambda} \otimes u_{-\infty}) \\ &\in B(\infty) \otimes t_{s_i \lambda} \otimes u_{-\infty}. \end{aligned}$$

Applying $\tilde{e}_i^{*\max}$, we have (here $\tilde{e}_i^{*\max} b = \tilde{e}_i^{*\varepsilon_i^*(b)} b$ and $\tilde{f}_i^{\max} b = \tilde{f}_i^{\varphi_i(b)} b$)

$$\begin{aligned} S_i S_{s_\beta} (u_\infty \otimes t_\lambda \otimes u_{-\infty}) &= \tilde{e}_i^{*\max} S_i S_{s_\beta} S_i (u_\infty \otimes t_{s_i \lambda} \otimes u_{-\infty}) \\ &\in \tilde{e}_i^{*\max} (B(\infty) \otimes t_{s_i \lambda} \otimes u_{-\infty}), \end{aligned}$$

and hence

$$\begin{aligned} S_{s_\beta} (u_\infty \otimes t_\lambda \otimes u_{-\infty}) &= \tilde{f}_i^{\max} S_i S_{s_\beta} (u_\infty \otimes t_\lambda \otimes u_{-\infty}) \\ &\in \tilde{f}_i^{\max} \tilde{e}_i^{*\max} (B(\infty) \otimes t_{s_i \lambda} \otimes u_{-\infty}) \\ &\subset \tilde{f}_i^{\max} \left(\bigcup_{n \geq 0} B(\infty) \otimes t_\lambda \otimes \tilde{e}_i^n u_{-\infty} \right) \\ &\subset B(\infty) \otimes t_\lambda \otimes u_{-\infty}. \end{aligned}$$

The last inclusion follows from

$$\tilde{f}_i^{\max} (b_1 \otimes t_\lambda \otimes b_2) = b'_1 \otimes t_\lambda \otimes \tilde{f}_i^{\max} b_2 \quad \text{for some } b'_1 \in B(\infty).$$

□

§2.9. Affine case

Until now, we have assumed that \mathfrak{g} is a symmetrizable Kac-Moody algebra. From now on, we assume further that $U_q^-(\mathfrak{g})$ is a quantized *affine* algebra.

2.9.1. Extended Weyl groups We take a weight lattice P of rank $\text{rk}(\mathfrak{g})+1$ and an inner product on P as in [14]. We set $\mathfrak{t}^* = \mathbb{Q} \otimes P$, which is canonically determined by the Dynkin diagram.

Let us define $\delta \in \sum_i \mathbb{Z}_{\geq 0} \alpha_i$ and $c \in \sum_i \mathbb{Z}_{\geq 0} h_i$ by

$$(2.30) \quad \begin{aligned} & \{\lambda \in \sum_i \mathbb{Z} \alpha_i ; \langle h_i, \lambda \rangle = 0 \text{ for every } i \in I\} = \mathbb{Z} \delta, \\ & \{h \in \sum_i \mathbb{Z} h_i ; \langle h, \alpha_i \rangle = 0 \text{ for every } i \in I\} = \mathbb{Z} c. \end{aligned}$$

By the inner product of \mathfrak{t}^* , we sometimes identify \mathfrak{t}^* and its dual. Note that the inner product on \mathfrak{t}^* is so normalized that δ and c correspond by this identification.

For $\alpha \in \Delta^{\text{re}}$, we set $c_\alpha := \max(1, (\alpha, \alpha)/2) \in \mathbb{Z}$. Then we have

$$(\alpha + \mathbb{Z} \delta) \cap \Delta = \alpha + c_\alpha \mathbb{Z}.$$

Let us denote by P_{cl} the quotient space $P/(P \cap \mathbb{Q} \delta)$, and let us denote by $\text{cl}: P \rightarrow P_{\text{cl}}$ the canonical projection. Let us denote by P_{cl}^* the dual lattice of P_{cl} , i.e. $P_{\text{cl}}^* = \text{Ker}(\delta: P^* \rightarrow \mathbb{Z}) = (\sum_i \mathbb{Q} h_i) \cap P^*$.

Similarly to P_{cl} , we define $\mathfrak{t}_{\text{cl}}^* := \mathfrak{t}^*/\mathbb{Q} \delta$, and let $\text{cl}: \mathfrak{t}^* \rightarrow \mathfrak{t}_{\text{cl}}^*$ be the canonical projection. Define $\mathfrak{t}^{*0} := \text{Ker}(c: \mathfrak{t}^* \rightarrow \mathbb{Q})$, and $\mathfrak{t}_{\text{cl}}^{*0} = \text{cl}(\mathfrak{t}^{*0})$. The dimension of $\mathfrak{t}_{\text{cl}}^{*0}$ is equal to $\text{rk}(\mathfrak{g}) - 1$. The inner product of \mathfrak{t}^* induces a positive definite inner product on $\mathfrak{t}_{\text{cl}}^{*0}$.

Let us denote by $O(\mathfrak{t}^*)$ the orthogonal group, and $O(\mathfrak{t}^*)_\delta := \{g \in O(\mathfrak{t}^*) ; g\delta = \delta\}$ the isotropy subgroup at δ . Then there is an exact sequence

$$1 \longrightarrow \mathfrak{t}_{\text{cl}}^{*0} \xrightarrow{t} O(\mathfrak{t}^*)_\delta \xrightarrow{\text{cl}_0} O(\mathfrak{t}_{\text{cl}}^{*0}) \longrightarrow 1.$$

Here $t: \mathfrak{t}_{\text{cl}}^{*0} \rightarrow O(\mathfrak{t}^*)_\delta$ is given by

$$t(\text{cl}(\xi))(\lambda) = \lambda + (\lambda, \delta)\xi - (\lambda, \xi)\delta - \frac{(\xi, \xi)}{2}(\lambda, \delta)\delta \quad \text{for } \xi \in \mathfrak{t}^{*0} \text{ and } \lambda \in \mathfrak{t}^*.$$

Let us set $W_{\text{cl}} = \text{cl}_0(W)$. Then W_{cl} is the Weyl group of the root system $\Delta_{\text{cl}} := \text{cl}(\Delta^{\text{re}}) \subset \mathfrak{t}_{\text{cl}}^{*0}$. We define the extended Weyl group \widetilde{W} by

$$\widetilde{W} := \{w \in O(\mathfrak{t}^*)_\delta ; w\Delta = \Delta \text{ and } \text{cl}_0(w) \in W_{\text{cl}}\}.$$

Then we have a commutative diagram with the exact rows:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \tilde{Q} & \longrightarrow & W & \longrightarrow & W_{\text{cl}} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \parallel \\
 1 & \longrightarrow & \tilde{P} & \longrightarrow & \tilde{W} & \longrightarrow & W_{\text{cl}} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathfrak{t}_{\text{cl}}^{*0} & \xrightarrow{t} & \mathcal{O}(\mathfrak{t}^*)_{\delta} & \xrightarrow{\text{cl}_0} & \mathcal{O}(\mathfrak{t}_{\text{cl}}^{*0}) \longrightarrow 1
 \end{array}$$

Here \tilde{P} and \tilde{Q} are given by

$$\tilde{P} = P_{\text{cl}}^0 \cap P_{\text{cl}}^{0\vee} \quad \text{and} \quad \tilde{Q} = Q_{\text{cl}} \cap Q_{\text{cl}}^{\vee},$$

where

$$\begin{aligned}
 P_{\text{cl}}^0 &:= \{ \lambda \in \mathfrak{t}_{\text{cl}}^{*0} ; \langle h_i, \lambda \rangle \in \mathbb{Z} \text{ for every } i \in I \}, \\
 P_{\text{cl}}^{0\vee} &:= \{ \lambda \in \mathfrak{t}_{\text{cl}}^{*0} ; \langle \alpha_i, \lambda \rangle \in \mathbb{Z} \text{ for every } i \in I \}, \\
 Q_{\text{cl}} &:= \sum_{i \in I} \mathbb{Z} \text{cl}(\alpha_i), \\
 Q_{\text{cl}}^{\vee} &:= \sum_{i \in I} \mathbb{Z} \text{cl}(h_i).
 \end{aligned}$$

The Weyl group W is a normal subgroup of \tilde{W} , and \tilde{W} is a semi-direct product of W and $\text{Aut}_0(\text{Dyn}) := \{ \iota ; \iota \text{ is a Dynkin diagram automorphism such that } \text{cl}_0(\iota) \in W_{\text{cl}} \}$.

$$\tilde{P}/\tilde{Q} \xrightarrow{\sim} \tilde{W}/W \xrightarrow{\sim} \text{Aut}_0(\text{Dyn}).$$

Remark 2.9.

- (i) If \mathfrak{g} is untwisted, then $(\alpha, \alpha)/2 \leq 1$ for every $\alpha \in \Delta^{\text{re}}$ and

$$\tilde{P} = P_{\text{cl}}^{0\vee} \subset P_{\text{cl}}^0, \quad \tilde{Q} = Q_{\text{cl}}^{\vee} \subset Q_{\text{cl}}.$$

- (ii) If \mathfrak{g} is the dual of an untwisted affine algebra, then $(\alpha, \alpha)/2 \geq 1$ for every $\alpha \in \Delta^{\text{re}}$ and

$$\tilde{P} = P_{\text{cl}}^0 \subset P_{\text{cl}}^{0\vee}, \quad \tilde{Q} = Q_{\text{cl}} \subset Q_{\text{cl}}^{\vee}.$$

- (iii) If $\mathfrak{g} = A_{2n}^{(2)}$, then we have $(\alpha, \alpha)/2 = 1/2, 1$ or 2 , and

$$\tilde{P} = \tilde{Q} = P_{\text{cl}}^0 = P_{\text{cl}}^{0\vee} = Q_{\text{cl}} = Q_{\text{cl}}^{\vee} = \sum_{\alpha \in \Delta^{\text{re}}} \mathbb{Z} \text{cl}(\alpha) = \sum_{\alpha \in \Delta^{\text{re}}, (\alpha, \alpha)/2=1} \mathbb{Z} \text{cl}(\alpha).$$

2.9.2. Peter-Weyl theorem Let us recall some of the results by Nakajima and Beck-Nakajima.

The following theorem is conjectured in [14] by the author and proved in [2] by Beck-Nakajima.

Theorem 2.10 (a version of Peter-Weyl theorem).

$$B(\tilde{U}_q(\mathfrak{g})) \simeq \left(\bigsqcup_{\lambda \in P} B(\lambda) \times B_0(-\lambda) \right) / W.$$

Here $B_0(\lambda)$ is the connected component of $B(\lambda)$ containing u_λ . Note that $B_0(\lambda) = B(\lambda)$ when the level of λ does not vanish. The Weyl group acts on $\bigsqcup_{\lambda \in P} B(\lambda) \times B_0(-\lambda)$ by $W \ni w: B(\lambda) \times B_0(-\lambda) \rightarrow B(w\lambda) \times B_0(-w\lambda)$ via the action given in (2.26). The left crystal structure $(\tilde{e}_i, \tilde{f}_i)$ on $B(\tilde{U}_q(\mathfrak{g}))$ is compatible with the crystal structure of $B(\lambda)$, the first factor of $B(\lambda) \times B_0(-\lambda)$, and the right crystal structure $(\tilde{e}_i^*, \tilde{f}_i^*)$ on $B(\tilde{U}_q(\mathfrak{g}))$ is compatible with the crystal structure of $B_0(-\lambda)$, the second factor of $B(\lambda) \times B_0(-\lambda)$.

For $\lambda \in P$, there exists a unique symmetric bilinear form (\cdot, \cdot) on $V(\lambda)$ that satisfies:

$$\begin{aligned} (u_\lambda, G(b)) &= \delta_{b, u_\lambda} \text{ for every } b \in B(\lambda), \\ (e_i u, v) &= (u, f_i v) \text{ for every } u, v \in V(\lambda), \\ (q(h)u, v) &= (u, q(h)v) \text{ for every } u, v \in V(\lambda) \text{ and } h \in d^{-1}P^*. \end{aligned}$$

The following theorem is trivial for non-zero level case, and proved in [21, 2] by Nakajima and Beck-Nakajima for the zero level case.

Theorem 2.11.

- (i) *This symmetric bilinear form on $V(\lambda)$ is non-degenerate.*
- (ii) *$(G(b), G(b')) \in q^{(\mu, \mu) - (\lambda, \lambda)} (\delta_{b, b'} + q_s \mathbf{A}_0)$ for any $\mu \in P$ and $b, b' \in B(\lambda)_\mu$.*
- (iii) *For $b, b' \in B(\lambda)_\lambda$, we have $(G(b), G(b')) = \delta_{b, b'}$.*

In particular if v is a non-zero vector of $V(\lambda)$, then there exists $P \in U_q^-(\mathfrak{g})$ such that (u_λ, Pv) does not vanish. Note that (u_λ, Pv) coincides with the coefficient of u_λ when we write Pv as a linear combination of the global basis.

Conjecture 2.12. Theorem 2.11 holds for an arbitrary symmetrizable Kac-Moody Lie algebra \mathfrak{g} .

§3. Extremal Vectors

We assume that $U_q^-(\mathfrak{g})$ is a quantized affine algebra. Let M be an integrable $U_q^-(\mathfrak{g})$ -module with a global basis. Let N be a $U_q^+(\mathfrak{g})$ -submodule of M compatible with the global basis of M . Then, for $\lambda \in P^+$, $u_\lambda \otimes N$ is also a $U_q^+(\mathfrak{g})$ -submodule of $V(\lambda) \otimes M$ compatible with the global basis. Hence $U_q^-(\mathfrak{g})(u_\lambda \otimes N)$ is a $U_q^-(\mathfrak{g})$ -module compatible with the global basis of $V(\lambda) \otimes M$.

Proposition 3.1. *Assume that for any $b \in B(N)$ if $\varepsilon_i(b) > 0$, then $\tilde{f}_i b \in B(N) \cup \{0\}$. Or equivalently, for any i -string S of $B(M)$, $S \cap B(N)$ is either S itself, the empty set or the set consisting of the highest weight vector of S . Assume further that $f_i N \subset N$ whenever $\langle h_i, \lambda \rangle = 0$. Then we have*

$$U_q^-(\mathfrak{g})(u_\lambda \otimes N) \cap u_\lambda \otimes M = u_\lambda \otimes N.$$

Proof. It is enough to show that

$$B(U_q^-(\mathfrak{g})(u_\lambda \otimes N)) \cap u_\lambda \otimes B(M) = u_\lambda \otimes B(N).$$

We have

$$B(U_q^-(\mathfrak{g})(u_\lambda \otimes N)) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_m} b ; m \geq 0, i_1, \dots, i_m \in I, b \in u_\lambda \otimes B(N) \right\} \setminus \{0\}.$$

Hence it is enough to show that, for $b \in B(N)$ such that $\tilde{f}_i(u_\lambda \otimes b) = u_\lambda \otimes \tilde{f}_i b$, we have $\tilde{f}_i b \in B(N) \cup \{0\}$. Since $\tilde{f}_i(u_\lambda \otimes b) = u_\lambda \otimes \tilde{f}_i b$ if and only if $\langle h_i, \lambda \rangle \leq \varepsilon_i(b)$, and $\tilde{f}_i B(N) \subset B(N) \cup \{0\}$ if $\langle h_i, \lambda \rangle = 0$, the assertion follows. \square

Corollary 3.2. *For $\lambda \in P^+$ and $\mu \in P$ such that $\langle h_i, \mu \rangle \leq 0$ whenever $\langle h_i, \lambda \rangle = 0$, we have*

$$U_q^-(\mathfrak{g})(u_\lambda \otimes u_\mu) \cap u_\lambda \otimes V(\mu) = u_\lambda \otimes U_q^+(\mathfrak{g})u_\mu,$$

and

$$B(U_q^-(\mathfrak{g})(u_\lambda \otimes u_\mu)) \cap u_\lambda \otimes B(\mu) = u_\lambda \otimes B^+(\mu).$$

Indeed, Lemma 2.6 allows us to apply the proposition above to $M = V(\mu)$ and $N = U_q^+(\mathfrak{g})u_\mu$.

Let $\lambda \in P^+$ be a dominant integral weight and $\mu \in P$ an integral weight. Then we have a chain of morphisms compatible with global bases

$$U_q^-(\mathfrak{g})a_{\lambda+\mu} \rightarrow V(\lambda) \otimes U_q^-(\mathfrak{g})a_\mu \rightarrow V(\lambda) \otimes V(\mu).$$

Theorem 3.3. *Let $\lambda \in P^+$ be a dominant integral weight and $\mu \in P$ an integral weight. Then there exist a unique homomorphism $U_q^-(\mathfrak{g})(u_\lambda \otimes u_\mu) \rightarrow V(\lambda + \mu)$ that sends $u_\lambda \otimes u_\mu$ to $u_{\lambda+\mu}$. Moreover this morphism is compatible with global bases.*

Proof. We have a commutative diagram

$$\begin{array}{ccccc}
 U_q^-(\mathfrak{g})a_{\lambda+\mu} & \longrightarrow & U_q^-(\mathfrak{g})(u_\lambda \otimes u_\mu) & \hookrightarrow & V(\lambda) \otimes V(\mu) \\
 & \searrow & \vdots & & \\
 & & V(\lambda + \mu) & &
 \end{array}$$

All the solid arrows are compatible with global bases. Hence, in order to show the theorem, it is enough to show the existence of the dotted arrow.

Correspondingly, we obtain the following diagram of crystal bases.

$$\begin{array}{ccc}
 B(U_q^-(\mathfrak{g})a_{\lambda+\mu}) & \longleftarrow & B((U_q^-(\mathfrak{g})(u_\lambda \otimes u_\mu)) \\
 & \nwarrow & \\
 & & B(\lambda + \mu)
 \end{array}$$

Let $G(b) \in U_q^-(\mathfrak{g})a_{\lambda+\mu}$ be the global basis vector corresponding to $b \in B(U_q^-(\mathfrak{g})a_{\lambda+\mu})$. Then $b \in B(\lambda + \mu)$ means that $G(b)u_{\lambda+\mu} \neq 0$ (we regard $V(\lambda + \mu)$ as an $U_q^-(\mathfrak{g})$ -module), and $b \in B(U_q^-(\mathfrak{g})(u_\lambda \otimes u_\mu))$ means $G(b)(u_\lambda \otimes u_\mu) \neq 0$.

Hence we have reduced the problem to the following proposition:

$$B(\lambda + \mu) \subset B(U_q^-(\mathfrak{g})(u_\lambda \otimes u_\mu)) \text{ as subsets of } B(U_q^-(\mathfrak{g})a_{\lambda+\mu}).$$

Taking $b \in B(\lambda + \mu) \subset B(U_q^-(\mathfrak{g})a_{\lambda+\mu})$, let us show that $b \in B(U_q^-(\mathfrak{g})(u_\lambda \otimes u_\mu))$.

Since $G(b)u_{\lambda+\mu} \neq 0$, Theorem 2.11 implies that there exists $P \in U_q^-(\mathfrak{g})$ such that, when we write $PG(b)u_{\lambda+\mu}$ as a linear combinations of the global basis of $V(\lambda + \mu)$, the coefficient of $u_{\lambda+\mu}$ does not vanish. Hence, if we write $PG(b)$ as a linear combinations of the global basis of $U_q^-(\mathfrak{g})a_{\lambda+\mu}$ the coefficient of $a_{\lambda+\mu}$ does not vanish. Hence $PG(b)(u_\lambda \otimes u_\mu)$ as a linear combinations of the global basis of $V(\lambda) \otimes V(\mu)$ the coefficient of $u_\lambda \otimes u_\mu$ does not vanish. We conclude then that $PG(b)(u_\lambda \otimes u_\mu) \neq 0$. Hence $b \in B(U_q^-(\mathfrak{g})(u_\lambda \otimes u_\mu))$. \square

Remark 3.4. Theorem 3.3 holds also for any finite-dimensional \mathfrak{g} , because Theorem 2.11 trivially holds in such a case.

Corollary 3.5. *If $\lambda \in P^+$ and $\mu \in P$, then $u_\lambda \otimes u_\mu \in B(\lambda) \otimes B(\mu)$ is an extremal vector, and we have an inclusion $B(\lambda + \mu) \subset B(\lambda) \otimes B(\mu)$ as subsets of $B(\lambda) \otimes B(U_q^-(\mathfrak{g})a_\mu)$.*

Note that the first statement holds for an arbitrary Kac-Moody Lie algebra.

§4. Fundamental Representations

Write the smallest positive imaginary root δ and the smallest positive imaginary coroot c as

$$\delta = \sum_i a_i \alpha_i, \quad c = \sum_i a_i^\vee h_i.$$

Then we have

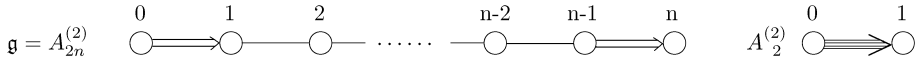
$$a_i^\vee = \frac{(\alpha_i, \alpha_i)}{2} a_i.$$

We choose $0 \in I$ such that

- (i) Setting $I_0 = \{i \in I; i \neq 0\}$ and $W_0 := \langle s_i; i \in I_0 \rangle \subset W$, the composition $W_0 \hookrightarrow W \xrightarrow{c|_0} W_{cl}$ is an isomorphism.
- (ii) $a_0 = 1$.

Such a 0 exists and is unique up to a Dynkin diagram automorphism.

If $\mathfrak{g} = A_{2n}^{(2)}$, α_0 is the longest simple root and $a_0^\vee = 2$.



$$\delta = \alpha_0 + 2(\alpha_1 + \dots + \alpha_n),$$

$$c = 2(h_0 + \dots + h_{n-1}) + h_n.$$

$$(\alpha_0, \alpha_0) = 4, (\alpha_n, \alpha_n) = 1, (\alpha_i, \alpha_i) = 2 \text{ for } 0 < i < n$$

Figure 1. $\mathfrak{g} = A_{2n}^{(2)}$

If \mathfrak{g} is not of type $A_{2n}^{(2)}$, then $a_0^\vee = 1$. Note that $\delta - \alpha_0 \in \Delta^{re}$ if \mathfrak{g} is not of type $A_{2n}^{(2)}$, and $(\delta - \alpha_0)/2 \in \Delta^{re}$ if \mathfrak{g} is of type $A_{2n}^{(2)}$. Hence one has always $s_{\delta - \alpha_0} \in W$.

Let us denote by $U_q(\mathfrak{g}_0)$ the subalgebra of $U_q^-(\mathfrak{g})$ generated by e_i, f_i ($i \in I_0$). This is the quantized universal enveloping algebra associated with a finite-dimensional simple Lie algebra.

Let ϖ_k be a fundamental weight of level 0. That is, $\{\varpi_k\}_{k \in I_0}$ is a set of vectors such that $\langle h_j, \varpi_k \rangle = 0$ for $j \in I_0$ with $j \neq k$, and

$$\begin{aligned} P_{cl}^{0+} &:= \{\lambda \in \mathfrak{t}_{cl}^*; \langle c, \lambda \rangle = 0 \text{ and } \langle h_i, \lambda \rangle \in \mathbb{Z}_{\geq 0} \text{ for every } i \in I_0\} \\ &= \sum_{k \in I_0} \mathbb{Z}_{\geq 0} \varpi_k. \end{aligned}$$

A fundamental weight of level 0 is unique up to $\mathbb{Q}\delta$. We can take

$$(4.1) \quad \varpi_k = \begin{cases} \Lambda_k - a_k^\vee \Lambda_0 & \text{when } a_0^\vee = 1, \\ \frac{2}{(\alpha_k, \alpha_k)} \Lambda_k - \Lambda_0 & \text{when } \mathfrak{g} = A_{2n}^{(2)}. \end{cases}$$

Here Λ_k is a vector in P satisfying $\langle h_i, \Lambda_k \rangle = \delta_{ik}$ for $i \in I$.

Let $k \in I \setminus \{0\}$. Set $c_k = \max(1, (\alpha_k, \alpha_k)/2) \in \mathbb{Z}$. Then we have

$$\{n \in \mathbb{Z}; \alpha_k + n\delta \in \Delta\} = \mathbb{Z}c_k,$$

and

$$W\varpi_k \cap (\varpi_k + \mathbb{Z}\delta) = \varpi_k + \mathbb{Z}c_k\delta.$$

We have $\varpi_k + n\delta \in \text{Wt}(V(\varpi_k))$ if and only if $n \in c_k\mathbb{Z}$.

For any $\xi = w\lambda \in W\lambda$, we use the notation u_ξ for the extremal vector $S_w^{\text{norm}}u_\lambda \in V(\lambda)$. Note that $S_w^{\text{norm}}u_\lambda$ is a unique global basis vector of weight ξ .

We denote by $U'_q(\mathfrak{g})$ the subalgebra of $U_q^-(\mathfrak{g})$ generated by e_i, f_i ($i \in I$) and $q(h)$ ($h \in d^{-1}P_{\text{cl}}^* \subset d^{-1}P^*$).

Then there exists a unique $U'_q(\mathfrak{g})$ -morphism $z_k: V(\varpi_k) \rightarrow V(\varpi_k)$ sending u_{ϖ_k} to $u_{\varpi_k + c_k\delta}$. The operator z_k has weight $c_k\delta$. The global basis of $V(\varpi_k)$ is stable by z_k . We have

$$z_k u_\xi = u_{\xi + c_k\delta} \quad \text{for every } \xi \in W \cdot \lambda.$$

The quotient $W(\varpi_k) := V(\varpi_k)/(z_k - 1)V(\varpi_k)$ is an irreducible $U'_q(\mathfrak{g})$ -module with a global basis. The morphism $V(\varpi_k) \rightarrow W(\varpi_k)$ sends the members of the global basis of $V(\varpi_k)$ to the one of $W(\varpi_k)$.

In this section, we set

$$(4.2) \quad \lambda := \varpi_k, \quad \mu := w_0\varpi_k \quad \text{where } w_0 \text{ be the longest element of } W_0.$$

Then $\mu \equiv -\varpi_{k'} \pmod{\mathbb{Z}\delta}$ for some $k' \in I_0$.

Then $U_q(\mathfrak{g}_0)u_\lambda = U_q(\mathfrak{g}_0)u_\mu$ is an irreducible $U_q(\mathfrak{g}_0)$ -module with highest weight λ and lowest weight μ . Note that $U_q^+(\mathfrak{g})u_\mu$ is a $U_q(\mathfrak{g}_0)$ -module. We have

$$u_\xi \in U_q^+(\mathfrak{g})u_\mu \text{ for any } \xi \in W_0 \cdot \lambda = W \cdot \lambda \cap (\lambda + \sum_{i \in I_0} \mathbb{Z}\alpha_i).$$

Lemma 4.1. $z_k u_\mu \in U_q^+(\mathfrak{g})u_\mu$.

Proof. Assume $a_0^\vee = 1$. Then we have $\tilde{\alpha}_{k'} := c_k\alpha_{k'}^\vee \in \tilde{Q}$ and $t(\tilde{\alpha}_{k'})(\mu) = \mu + c_k\delta$. Hence $z_k u_\mu = S_{t(\tilde{\alpha}_{k'})}^{\text{norm}}u_\mu$. Since we have $c_k\delta - \alpha_{k'} \in \Delta_+^{\text{re}}$, $t(\tilde{\alpha}_{k'}) =$

$s_{c_k\delta - \alpha_{k'}} s_{\alpha_{k'}}$, and $(\alpha_{k'}, \mu) = (c_k\delta - \alpha_{k'}, s_{k'}\mu) < 0$, Proposition 2.8 implies that $S_{t(\tilde{\alpha}_{k'})}^{\text{norm}} u_\mu \in U_q^+(\mathfrak{g})u_\mu$.

Now assume that $\mathfrak{g} = A_{2n}^{(2)}$. Then $t(-\alpha_0/2)\mu = \mu + \delta$ and $t(-\alpha_0/2) = s_{\alpha_0} s_{(\delta - \alpha_0)/2}$, and $(\delta - \alpha_0, \mu) = (\alpha_0, s_{(\delta - \alpha_0)/2}\mu) < 0$. Note that $(\delta - \alpha_0)/2 \in \Delta_+^{\text{re}}$. Hence Proposition 2.8 implies that $S_{t(\tilde{\alpha}_0)}^{\text{norm}} u_\mu \in U_q^+(\mathfrak{g})u_\mu$. \square

Lemma 4.2.

- (i) $B^+(\mu) \setminus z_k B^+(\mu) \simeq B(W(\lambda))$ as a crystal over \mathfrak{g}_0 .
- (ii) $U_q^+(\mathfrak{g})u_\mu / U_q^+(\mathfrak{g})z_k u_\mu$ is isomorphic to $W(\lambda)$ as a $U_q(\mathfrak{g}_0)$ -module

Proof. The crystal $B^+(\mu)$ is a regular crystal over \mathfrak{g}_0 . The crystal $B^+(\mu)$ is invariant by z_k , and $\bigcup_{n \in \mathbb{Z}} z_k^n B^+(\mu) = B(\mu)$, $\bigcap_{n \in \mathbb{Z}} z_k^n B^+(\mu) = \emptyset$. On the other hand, the crystal $B(W(\lambda))$ is isomorphic to the quotient of $B(\mu)$ by the action of \mathbb{Z} given by z_k . Hence $B^+(\mu) \setminus z_k B^+(\mu) \rightarrow B(W(\lambda))$ is bijective. (ii) follows from (i). \square

Lemma 4.3. Assume that $\langle h_0, \lambda \rangle = -1$ (i.e. $a_k^\vee = 1$ or $\mathfrak{g} = A_{2n}^{(2)}$).

- (i) $c_k = 1$.
- (ii) $W(\lambda)$ is an irreducible $U_q(\mathfrak{g}_0)$ -module.

Proof. (i) if $\mathfrak{g} = A_{2n}^{(2)}$, then $(\alpha_k, \alpha_k)/2 \leq 1$ for $k \neq 0$, and hence $c_k = 1$. If $a_k^\vee = 1$, then $1 = a_k^\vee = \frac{(\alpha_k, \alpha_k)}{2} a_k \geq \frac{(\alpha_k, \alpha_k)}{2}$, and hence $c_k = 1$.
 (ii) By Lemma 4.2, it is enough to show that $N := U_q^+(\mathfrak{g})u_\mu / U_q^+(\mathfrak{g})z_k u_\mu$ is irreducible as a $U_q(\mathfrak{g}_0)$ -module. Note that $e_0 u_\lambda = u_{s_0 \lambda} = u_{\lambda + \delta} = z_k u_\lambda \in U_q^+(\mathfrak{g})z_k u_\mu$ and $e_i u_\lambda = 0$ for $i \neq 0$. Since $U_q^+(\mathfrak{g})u_\mu = U_q^+(\mathfrak{g})U_q(\mathfrak{g}_0)u_\lambda = U_q(\mathfrak{g}_0)U_q^+(\mathfrak{g})u_\lambda$, we have $N = U_q(\mathfrak{g}_0)u_\lambda \pmod{z_k U_q^+(\mathfrak{g})u_\mu}$. \square

Two vectors $u_{\Lambda_0} \otimes u_\lambda$ and $u_{\Lambda_0} \otimes u_\mu$ are extremal vectors in the same connected component of $B(\Lambda_0) \otimes B(\lambda)$, because u_{Λ_0} belongs to the trivial representation with respect to \mathfrak{g}_0 and $\mu \in W_0 \lambda$. Since the level of $\Lambda_0 + \mu$ is equal to one, there exists a unique dominant weight ξ_0 of level one such that $\xi_0 \in W \cdot (\Lambda_0 + \lambda) = W \cdot (\Lambda_0 + \mu)$. By Corollary 3.5, the connected component of $B(\Lambda_0) \otimes B(\lambda)$ containing $u_{\Lambda_0} \otimes u_\lambda$ is isomorphic to $B(\xi_0)$. Set $M = V(\Lambda_0) \otimes V(\mu)$ and $M_n = U_q^-(\mathfrak{g})(u_{\Lambda_0} \otimes z_k^n u_\lambda) = U_q^-(\mathfrak{g})(u_{\Lambda_0} \otimes z_k^n u_\mu) \subset M$. Note that Corollary 3.2 implies

$$M_n \cap (u_{\Lambda_0} \otimes V(\lambda)) = u_{\Lambda_0} \otimes U_q^+(\mathfrak{g})z_k^n u_\mu.$$

Lemma 4.4.

- (i) $M_{n+1} \subset M_n$.
- (ii) $M = \bigcup_{n \in \mathbb{Z}} M_n$.
- (iii) $\bigcap_{n \in \mathbb{Z}} M_n = 0$.

Proof. (i) is obvious. Since $W(\mu)$ is generated by u_μ as an $U_q^+(\mathfrak{g})$ -module (by [1, Proposition 1.16]), we have $V(\mu) = \bigcup_n U_q^+(\mathfrak{g})z_k^n u_\mu$, which implies (ii).

In order to prove (iii), it is enough to show that $\bigcap_n B(M_n) = \emptyset$. Any vector $b \in \bigcap_n B(M_n)$ is connected with a vector in $u_{\Lambda_0} \otimes B(\mu)$. Since $B(M_n) \cap (u_{\Lambda_0} \otimes B(\mu)) = u_{\Lambda_0} \otimes z_k^n B^+(\mu)$, the result follows from $\bigcap_n z_k^n B^+(\mu) = \emptyset$, which is an immediate consequence of $\bigcap_n (\mu + n\delta + Q_+) = \emptyset$. □

Lemma 4.5. $B(M_0) \setminus B(M_1) \simeq B(\Lambda_0) \otimes B(W(\lambda))$.

The proof is similar to the one of Lemma 4.2.

Proposition 4.6.

- (i) *The vector $u_{\Lambda_0} \otimes u_{\lambda + nc_k \delta}$, regarded as a vector of M_n/M_{n+1} , is an extremal vector.*
- (ii)

$$\xi_0 \equiv \begin{cases} \Lambda_0 + \lambda & \text{if } \langle h_0, \lambda \rangle = -1, \text{ i.e. } a_k^\vee = 1 \text{ or } \mathfrak{g} = A_{2n}^{(2)}, \\ \iota^{-1}(\Lambda_0) & \text{if } a_0^\vee = 1, \frac{(\alpha_k, \alpha_k)}{2} \geq 1, \text{ and } \iota \text{ is the Dynkin} \\ & \text{diagram automorphism such that } t(\lambda) \in W\iota, \\ \Lambda_4 & \text{if } \mathfrak{g} = F_4^{(1)} \text{ and } k = 3, \end{cases}$$

mod $\mathbb{Q}\delta$. For the last case, see Figure 2 in the proof.

Proof. We may assume that $n = 0$. If $u_{\Lambda_0} \otimes u_\lambda \in M_1$, then $u_\lambda \in U_q^+(\mathfrak{g})z_k u_\mu$, which is a contradiction. Hence $u_{\Lambda_0} \otimes u_\lambda \pmod{M_1}$ is a non-zero vector.

We divide the proof into three cases.

Case 1) $a_k^\vee = 1$ or $\mathfrak{g} = A_{2n}^{(2)}$

In this case $\langle h_0, \lambda \rangle = -1$ and $c_k = 1$ by Lemma 4.3. We shall show that $u_{\Lambda_0} \otimes u_\lambda$ is a highest weight vector of M_0/M_1 . We have $e_i(u_{\Lambda_0} \otimes u_\lambda) = 0$ for

$i \neq 0$. We have $e_0(u_{\Lambda_0} \otimes u_\lambda) = u_{\Lambda_0} \otimes e_0 u_\lambda$, and $e_0 u_\lambda = S_0 u_\lambda = u_{s_{\alpha_0 - \delta} \lambda + \delta} = z_k u_{s_{\alpha_0 - \delta} \lambda} \in z_k U_q^+(\mathfrak{g}) u_\mu$.

Case 2) $a_0^\vee = 1$ and $\frac{(\alpha_k, \alpha_k)}{2} \geq 1$

We have $\lambda \in \tilde{P}$ and hence $t(\lambda) \in \tilde{W}$. We have then $t(-\lambda)(\Lambda_0 + \lambda) \equiv \Lambda_0 \pmod{\mathbb{Z}\delta}$. Hence if we write $t(\lambda) = w\iota$ with $w \in W$ and a Dynkin diagram automorphism ι , then we have $\xi_0 = w^{-1}(\Lambda_0 + \lambda) = \iota^{-1}t(-\lambda)(\Lambda_0 + \lambda) \equiv \iota^{-1}(\Lambda_0) \pmod{\mathbb{Q}\delta}$.

Set $v = S_{t(\lambda)}^{-1}(u_{\Lambda_0} \otimes u_\lambda)$. Here we regard $S_{\iota^{-1}}$ as an isomorphism $V(\Lambda_0) \otimes V(\lambda) \rightarrow V(\iota^{-1}\Lambda_0) \otimes V(\iota^{-1}\lambda)$ such that $S_{\iota^{-1}}(au) = \iota^{-1}(a)S_{\iota^{-1}}(u)$ for $a \in U_q^-(\mathfrak{g})$ and $u \in V(\Lambda_0) \otimes V(\lambda)$. Hence, v is regarded as a vector in $V(\iota^{-1}\Lambda_0) \otimes V(\iota^{-1}\lambda)$. We shall show that $e_i v \in S_{\iota^{-1}} M_1$ for every $i \in I$.

(i) $i \neq 0, k$

In this case $t(\lambda)\alpha_i = \alpha_i$ holds. Hence we have $T_{t(\lambda)}e_i = e_i$, and

$$S_{t(\lambda)}e_i v = (T_{t(\lambda)}e_i)(u_{\Lambda_0} \otimes u_\lambda) = e_i(u_{\Lambda_0} \otimes u_\lambda) = 0.$$

(ii) $i = k$

Since $\langle h_k, \text{wt}(v) \rangle = 0$, it is enough to show that $e_k S_k^{-1} S_{t(\lambda)}^{-1}(u_{\Lambda_0} \otimes u_\lambda) \in S_{\iota^{-1}} M_1$. This is equivalent to saying that $(T_{t(\lambda)s_k} e_k)(u_{\Lambda_0} \otimes u_\lambda) \in M_1$. Since $t(\lambda)s_k \alpha_k = c_k \delta - \alpha_k \in \Delta^+$, we have $T_{t(\lambda)s_k} e_k \in U_q^+(\mathfrak{g})$ and $(T_{t(\lambda)s_k} e_k)(u_{\Lambda_0} \otimes u_\lambda) = u_{\Lambda_0} \otimes (T_{t(\lambda)s_k} e_k)u_\lambda$. The last factor is calculated as $(T_{t(\lambda)s_k} e_k)u_\lambda = S_{t(\lambda)s_k} e_k S_{t(\lambda)s_k}^{-1} u_\lambda = S_{t(\lambda)s_k} e_k u_{s_k t(-\lambda)\lambda}$ up to a non-zero constant multiple. Since we have $\langle h_k, s_k t(-\lambda)\lambda \rangle = -\langle h_k, \lambda \rangle = -1$, we obtain $e_k u_{s_k t(-\lambda)\lambda} = u_{t(-\lambda)\lambda}$. Thus we obtain $(T_{t(\lambda)s_k} e_k)u_\lambda = S_{t(\lambda)s_k} u_{t(-\lambda)\lambda} = u_{t(\lambda)s_k t(-\lambda)\lambda}$ up to a non-zero constant multiple. Since $t(\lambda)s_k t(-\lambda)\lambda = s_{\alpha_k - c_k \delta} \lambda = s_k \lambda + c_k \delta$, we have

$$u_{t(\lambda)s_k t(-\lambda)\lambda} = u_{s_k \lambda + c_k \delta} = z_k u_{s_k \lambda} \in z_k U_q^+(\mathfrak{g}) u_\mu.$$

(iii) $i = 0$

Let us first show that

$v =_{t(w\lambda)}^{-1}(u_{\Lambda_0} \otimes u_{w\lambda})$ up to a non-zero constant multiple for every $w \in W_0$, by the induction of the length of w . Assuming that the assertion is true, we shall show that it is true for $s_i w$ for $i \in I_0$. According that $s_i t(w\lambda) \geq t(w\lambda)$, we have $S_{t(s_i w\lambda)} = S_i^\pm S_{t(w\lambda)} S_i^\mp$. Hence we have

$$\begin{aligned} S_{t(s_i w\lambda)}^{-1}(u_{\Lambda_0} \otimes u_{s_i w\lambda}) &= S_i^\pm S_{t(w\lambda)}^{-1} S_i^\mp(u_{\Lambda_0} \otimes u_{s_i w\lambda}) \\ &= S_i^\pm S_{t(w\lambda)}^{-1}(u_{\Lambda_0} \otimes u_{w\lambda}) = S_i^\pm v = v \end{aligned}$$

up to a non-zero constant multiple.

Now we divide the proof into two cases.

(a) $(\alpha_k, \alpha_k)/2 = 1$

In this case, $c_k = 1$. There exists $w \in W_0$ such that

$$w^{-1}\alpha_0 \equiv -\alpha_k \pmod{\mathbb{Z}\delta}.$$

Hence we have $\langle h_0, w\lambda \rangle = -1$, which implies that $t(w\lambda)\alpha_0 = \alpha_0 + \delta$ and $\langle t(w\lambda)\alpha_0, w\lambda \rangle = -1$. Then we have

$$\begin{aligned} S_{t(w\lambda)}e_0v &= S_{t(w\lambda)}e_0S_{t(w\lambda)}^{-1}(u_{\Lambda_0} \otimes u_{w\lambda}) \\ &= u_{\Lambda_0} \otimes S_{t(w\lambda)}e_0S_{t(w\lambda)}^{-1}u_{w\lambda} \\ &= u_{\Lambda_0} \otimes S_{t(w\lambda)}S_0S_{t(w\lambda)}^{-1}u_{w\lambda} \\ &= u_{\Lambda_0} \otimes S_{\alpha_0+\delta}u_{w\lambda} \\ &= u_{\Lambda_0} \otimes u_{s_{\alpha_0+\delta}w\lambda}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} s_{\alpha_0+\delta}w\lambda &= w\lambda - (\alpha_0 + \delta, w\lambda)(\alpha_0 + \delta) \\ &= w\lambda - (\alpha_0 - \delta, w\lambda)(\alpha_0 - \delta) + 2\delta \\ &= s_{\delta-\alpha_0}w\lambda + 2\delta. \end{aligned}$$

This implies that

$$u_{s_{\alpha_0+\delta}w\lambda} = z_k^2 u_{s_{\delta-\alpha_0}w\lambda} \in z_k U_q^+(\mathfrak{g})u_\mu.$$

Hence $e_0(u_{\Lambda_0} \otimes u_\lambda) \in S_{t(w\lambda)}^{-1}M_1 = S_{t^{-1}}M_1$.

(b) $(\alpha_k, \alpha_k)/2 > 1$

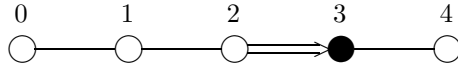
In this case, by the classification of affine Dynkin diagrams, there exists $i \neq 0, k$ such that $(\alpha_i, \alpha_i)/2 = 1$. Let us take $w \in W_0$ such that $w^{-1}\alpha_0 \equiv \alpha_i \pmod{\mathbb{Z}\delta}$. Hence $\langle w^{-1}\alpha_0, \lambda \rangle = 0$, which implies $t(w\lambda)\alpha_0 = \alpha_0$. Hence we have $S_{t(w\lambda)}e_0S_{t(w\lambda)}^{-1} = e_0$ and

$$\begin{aligned} S_{t(w\lambda)}e_0v &= S_{t(w\lambda)}e_0S_{t(w\lambda)}^{-1}(u_{\Lambda_0} \otimes u_{w\lambda}) \\ &= u_{\Lambda_0} \otimes e_0u_{w\lambda}. \end{aligned}$$

Since $\langle \alpha_0, w\lambda \rangle = \langle w^{-1}\alpha_0, \lambda \rangle = 0$, $e_0u_{w\lambda}$ must vanish.

Case 3) the remaining case (i.e. $a_0^\vee = 1$, $a_k^\vee > 1$ and $(\alpha_k, \alpha_k)/2 < 1$)

By the classification of affine Dynkin diagrams, there is only one remaining case, namely $\mathfrak{g} = F_4^{(1)}$ and $k = 3$:



$$\begin{aligned} \delta &= \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \\ c &= h_0 + 2h_1 + 3h_2 + 2h_3 + h_4, \end{aligned}$$

Figure 2. $\mathfrak{g} = F_4^{(1)}$

We have in this case $c_3 = 1$, $\lambda = \Lambda_3 - 2\Lambda_0$ and $\mu = \lambda - 4\alpha_1 - 8\alpha_2 - 12\alpha_3 - 6\alpha_4$.

We have

$$s_4s_3s_2s_1s_0(\Lambda_0 + \lambda) = \Lambda_0 + \lambda + \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \equiv \Lambda_4 \pmod{\mathbb{Z}\delta}.$$

Set $x = s_0s_1s_2s_3s_4$ and $v = S_x^{-1}(u_{\Lambda_0} \otimes u_\lambda)$. Let us show that v is a highest weight vector of M_0/M_1 , i.e. $e_i v \in M_1$ for $i \in I$.

(1) $i = 0$

$$\text{Since } x\alpha_0 = \alpha_1, \text{ we have } S_x e_0 v = (T_x e_0)(u_{\Lambda_0} \otimes u_\lambda) = e_1(u_{\Lambda_0} \otimes u_\lambda) = 0.$$

(2) $i = 1$

Since $x\alpha_1 = \alpha_2$, we can conclude $e_1 v = 0$ by the same argument as above.

(3) $i = 2$

We have $x\alpha_2 = \alpha_0 + \alpha_1 + \alpha_2 + 2\alpha_3$. Hence we have

$$\begin{aligned} S_x e_2 v &= S_x e_2 S_x^{-1}(u_{\Lambda_0} \otimes u_\lambda) \\ &= u_{\Lambda_0} \otimes (S_x e_2 S_x^{-1} u_\lambda), \end{aligned}$$

and since $\langle xh_2, \lambda \rangle = -1$, we have

$$\begin{aligned} S_x e_2 S_x^{-1} u_\lambda &= S_x S_2 S_x^{-1} u_\lambda = u_{s_x \alpha_2 \lambda} \\ &= u_{\lambda + \alpha_0 + \alpha_1 + \alpha_2 + 2\alpha_3} \\ &= u_{\lambda + \delta - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4} \\ &= z_k u_{\lambda - \alpha_1 - 2\alpha_2 - 2\alpha_3 - 2\alpha_4} \in z_k U_q^+(\mathfrak{g}) u_\mu. \end{aligned}$$

This implies that $e_2 v \in M_1$.

(4) $i = 3$

$e_3v = 0$ follows from $x\alpha_3 = \alpha_4$.

(5) $i = 4$

We have seen that $v \in M_0/M_1$ is invariant by S_2 and S_3 . Hence it is enough to show that $e_4S_3^{-1}S_2^{-1}v = 0$. Since $xs_2s_3\alpha_4 = \alpha_3$, we have

$$S_{xs_2s_3}e_4S_3^{-1}S_2^{-1}v = (T_{xs_2s_3}e_4)(u_{\Lambda_0} \otimes u_\lambda) = e_3(u_{\Lambda_0} \otimes u_\lambda) = 0.$$

□

Theorem 4.7.

$$U_q^-(\mathfrak{g})(u_{\Lambda_0} \otimes u_\lambda)/U_q^-(\mathfrak{g})(u_{\Lambda_0} \otimes u_{\lambda+c_k\delta}) \simeq V(\Lambda_0 + \lambda).$$

By this isomorphism, $u_{\Lambda_0} \otimes u_\lambda$ corresponds to $u_{\Lambda_0+\lambda}$.

Proof. By the preceding proposition, there exists a morphism $g: V(\Lambda_0 + \lambda) \rightarrow M_0/M_1$, sending $u_{\Lambda_0+\lambda}$ to $u_{\Lambda_0} \otimes u_\lambda \bmod M_1$. On the other hand, Theorem 3.3 implies the existence of a morphism $\psi: M_0 \rightarrow V(\Lambda_0 + \lambda)$. Since $\Lambda_0 + \lambda + c_k\delta$ is not a weight of $V(\Lambda_0 + \lambda)$, ψ factors through M_0/M_1 and thus we obtain a morphism $M_0/M_1 \rightarrow V(\Lambda_0 + \lambda)$ sending $u_{\Lambda_0} \otimes u_\lambda$ to $u_{\Lambda_0+\lambda}$. Obviously it is an inverse of g . □

Note that the theorem holds if we replace λ with μ .

Corollary 4.8. $u_{\Lambda_0} \otimes (B^+(\mu) \setminus B^+(\mu + c_k\delta)) \simeq B^+(\Lambda_0 + \mu)$. In particular $B(W(\lambda)) \simeq B^+(\Lambda_0 + \mu)$ as a crystal over \mathfrak{g}_0 .

Proof. By the preceding theorem, we have

$$U_q^+(\mathfrak{g})u_{\Lambda_0+\mu} \simeq U_q^+(\mathfrak{g})(u_{\Lambda_0} \otimes u_\mu)/(U_q^-(\mathfrak{g})(u_{\Lambda_0} \otimes u_{\mu+c_k\delta}) \cap U_q^+(\mathfrak{g})(u_{\Lambda_0} \otimes u_\mu))$$

On the other hand, Corollary 3.2 implies that $U_q^-(\mathfrak{g})(u_{\Lambda_0} \otimes u_{\mu+c_k\delta}) \cap U_q^+(\mathfrak{g})(u_{\Lambda_0} \otimes u_\mu) \subset U_q^-(\mathfrak{g})(u_{\Lambda_0} \otimes u_{\mu+c_k\delta}) \cap u_{\Lambda_0} \otimes V(\mu) = u_{\Lambda_0} \otimes U_q^+(\mathfrak{g})u_{\mu+c_k\delta}$, which implies that $U_q^-(\mathfrak{g})(u_{\Lambda_0} \otimes u_{\mu+c_k\delta}) \cap U_q^+(\mathfrak{g})(u_{\Lambda_0} \otimes u_\mu) = u_{\Lambda_0} \otimes U_q^+(\mathfrak{g})u_{\mu+c_k\delta}$. Hence we have

$$U_q^+(\mathfrak{g})u_{\Lambda_0+\mu} \simeq (u_{\Lambda_0} \otimes U_q^+(\mathfrak{g})u_\mu)/(u_{\Lambda_0} \otimes U_q^+(\mathfrak{g})u_{\mu+c_k\delta}).$$

Thus we obtain the desired result. □

Lemma 4.5 and Theorem 4.7 imply the following result.

Corollary 4.9. $B(\Lambda_0) \otimes B(W(\lambda)) \simeq B(\Lambda_0 + \lambda)$.

Corollary 4.10. *There exists a unique vector $b \in B(W(\lambda))$ such that $\varepsilon_i(b) \leq \delta_{i,0}$ for all i .*

Proof. The condition is equivalent to saying that $u_{\Lambda_0} \otimes b$ is a highest weight vector, and the preceding corollary implies that $B(\Lambda_0) \otimes B(W(\lambda))$ has a unique highest weight vector. \square

Corollary 4.11.

- (i) *If $b \in B(\mu)$ satisfies $\text{wt}(b) \notin \xi_0 - \Lambda_0 - c_k\delta + Q_-$, then $b \in B^+(\mu)$.*
- (ii) *If an integral weight η satisfies $\eta \notin \xi_0 - \Lambda_0 - c_k\delta + Q_-$, then $V(\lambda)_\eta = (U_q^+(\mathfrak{g})u_\mu)_\eta$.*

Proof. There exist $b' \in B^+(\mu) \setminus B(\mu + c_k\delta)$ and $n \in \mathbb{Z}$ such that $b = z_k^n b'$. By Corollary 4.8, we have $\text{wt}(u_{\Lambda_0} \otimes b') = \Lambda_0 + \text{wt}(b) - nc_k\delta \in \xi_0 + Q_-$. Hence the assumption implies $n \geq 0$. Thus we conclude (i), and (ii) follows from (i). \square

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