On Termination of 4-fold Semi-stable Log Flips

By

Osamu FUJINO*

Abstract

In this paper, we prove the termination of 4-fold semi-stable log flips under the assumption that there always exist 4-fold (semi-stable) log flips.

§1.   Introduction

One of the most important conjectures in the (log) minimal model program ((log) MMP, for short) is *(log) Flip Conjecture* II. It claims that any sequence of (log) flips:

$$(X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots$$

$$\searrow \quad \swarrow \quad \searrow \quad \swarrow$$

$$Z_0 \qquad\qquad Z_1 \qquad\qquad ,$$

has to terminate after finitely many steps. In the non-log case, the conjecture in dimension 4 was proved for the terminal flips by Kawamata in [KMM], and for the terminal flops by Matsuki in [M1]. For the log case, we proved it for 4-fold canonical flips in [F2], which is a first step to prove the log Flip Conjecture II in dimension 4. We note that the main theorem of [F2] contains the above mentioned results of Kawamata and Matsuki. See also [F3].

Recently, Shokurov treats the log Flip Conjecture II in a much more general setting. For the details, see [S2] and [S3].

The main purpose of this paper is to prove the following theorem, which is a 4-dimensional analogue of [KM, Theorem 7.7], under the assumption that there always exist 4-fold (semi-stable) log flips (see Assumption 1.1 below). We

will prove it by the *crepant descent* technique by Kawamata and Kollár (see [Ka1], [Ka3], [Ko], and [K$^+$, Chapter 6]). For the details of the (log) semi-stable MMP, see [KM, §7.1]. Roughly speaking, $(n + 1)$-dimensional log semi-stable MMP is a kind of n-dimensional log MMP in families. So, it will play important roles in the study of the moduli of n-dimensional varieties.

We will work over $\mathbb{C}$, the complex number field, throughout this paper.

Theorem 1.1 (Termination of 4-fold semi-stable log flips)**.**   *Let (X, B) be a $\mathbb{Q}$-factorial projective 4-dimensional dlt pair, $\mu : X \longrightarrow Y$ a projective surjective morphism and $\nu : Y \longrightarrow C$ a flat morphism to a non-singular curve C such that $f := \nu \circ \mu : (X, B) \longrightarrow C$ is a dlt morphism (for the definition of dlt morphisms, see Definition 2.2 below). Then an arbitrary sequence of extremal $(K_X + B)$-flips over Y is finite.*

In the proof of Theorem 1.1, we need the following assumption: Assumption 1.1.

Assumption 1.1.   *Let (X, B) be a 4-dimensional klt pair and $f : X \longrightarrow Z$ a flipping contraction with respect to $K_X + B$. Then f has a flip.*

We note that all the flips we need here are 4-fold semi-stable (log) flips, which are special ones of klt flips in Assumption 1.1 (see Definition 2.3 and §5 Appendix). In Section 5, we will slightly generalize Theorem 1.1. We omit the details here since it is technical. Recently, Shokurov announced a proof of the existence of 4-fold log flips in [S1]. So, this assumption seems to be reasonable. We recommend the readers to see [S1].

For the proof of Theorem 1.1, we need the following two theorems. First, we recall the special termination theorem. For the details, see [S1, Section 2] and [F1].

Theorem 1.2 (4-dimensional special termination)**.**   *Let (X, B) be a $\mathbb{Q}$-factorial dlt 4-fold. Consider a sequence of extremal $(K_{X_i} + B_i)$-flips starting from $(X, B) = (X_0, B_0)$:*

$$(X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots$$
$$\searrow \quad \swarrow \ \searrow \quad \swarrow$$
$$Z_0 \qquad\qquad Z_1 \qquad\qquad\qquad .$$

Then after finitely many flips, flipping locus (and thus the flipped locus) is disjoint from $\llcorner B_i \lrcorner$.

Next, the following theorem is contained in [F2]. See also [F3, §5].

Theorem 1.3 (Termination of 4-fold terminal flips).   *Let X be a normal projective 4-fold and B an effective $\mathbb{Q}$-divisor such that (X, B) is terminal, that is*, $\mathrm{discrep}(X, B) > 0$. *Consider a sequence of $(K_{X_i} + B_i)$-flips starting from $(X, B) = (X_0, B_0)$:*

$$(X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots$$

$$\searrow \qquad \nearrow \; \searrow \qquad \nearrow$$

$$Z_0 \qquad\qquad Z_1 \qquad\qquad\qquad .$$

Then this sequence terminates after finitely many steps.

We note that we do not need Assumption 1.1 in the proofs of Theorems 1.2 and 1.3.

We summarize the contents of this paper: In Section 2, we recall some basic definitions and introduce a new notion: plt morphism. Section 3 is the preparation for the main theorem. We define a couple of invariants for plt morphisms. Section 4 is devoted to the proof of the main theorem: Theorem 1.1. Finally, Section 5 is an appendix, where we slightly generalize Theorem 1.1.

Notation.   Let $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{\geq 0}$) be a set of positive (resp. non-negative) integers. For $d \in \mathbb{Q}$, let $\llcorner d \lrcorner = \max\{t \in \mathbb{Z} \mid t \leq d\}$ and $\{d\} = d - \llcorner d \lrcorner$. Let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor such that all the D_i's are distinct prime divisors. We put $\llcorner D \lrcorner = \sum \llcorner d_i \lrcorner D_i$ (the *round down of D*) and $\{D\} = \sum \{d_i\} D_i$ (the *fractional part of D*).

§2.   Preliminaries

In this section, we collect basic properties and definitions.

2.1.   First, let us recall the definitions of discrepancies and singularities of pairs.

Definition 2.1 (Discrepancies and singularities for pairs).   Let X be a normal variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor on X, where D_i is irreducible for every i and $D_i \neq D_j$ for $i \neq j$, such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \longrightarrow X$ be a proper birational morphism from a normal variety Y. Then we can write

$$K_Y = f^*(K_X + D) + \sum a(E, X, D)E,$$

where the sum runs over all the distinct prime divisors $E \subset Y$, and $a(E, X, D)$ $\in \mathbb{Q}$. This $a(E, X, D)$ is called the *discrepancy* of E with respect to (X, D). We define

$$\mathrm{discrep}(X, D) := \inf_E \{a(E, X, D) \mid E \text{ is exceptional over } X\}.$$

From now on, we assume that $0 \leq d_i \leq 1$ for every i. We say that (X, D) is

$$\begin{cases} \text{terminal} \\ \text{canonical} \\ \text{klt} \\ \text{plt} \\ \text{lc} \end{cases} \quad \text{if} \quad \mathrm{discrep}(X, D) \quad \begin{cases} > 0, \\ \geq 0, \\ > -1 \quad \text{and} \quad \llcorner D \lrcorner = 0, \\ > -1, \\ \geq -1. \end{cases}$$

Here klt is short for *Kawamata log terminal*, plt for *purely log terminal*, and lc for *log canonical*.

If there exists a log resolution $f : Y \longrightarrow X$ of (X, D), that is, Y is non-singular, the exceptional locus $\mathrm{Exc}(f)$ is a divisor, and $\mathrm{Exc}(f) \cup f^{-1}(\mathrm{Supp} D)$ is a simple normal crossing divisor, such that $a(E_i, X, D) > -1$ for every exceptional divisor E_i on Y, then the pair (X, D) is called *dlt*. Here, dlt is short for *divisorial log terminal*.

2.2.   Next, let us recall the definition of dlt morphisms and define plt morphisms.

Definition 2.2 ([KM, Definition 7.1])**.**   Let X be a normal variety, B an effective $\mathbb{Q}$-divisor on X and $f : X \longrightarrow C$ a non-constant morphism to a non-singular curve C. We say that $f : (X, B) \longrightarrow C$ is dlt (resp. plt) if $(X, B + f^*P)$ is dlt (resp. plt) for every closed point $P \in C$. We note that if $(X, B) \longrightarrow C$ is plt, then (X, B) is klt.

The following lemma is a variant of adjunction and the inversion of adjunction. For the proof, see [KM, Theorem 5.50 (1), Proposition 5.51].

Lemma 2.1.   *Let (X, B) be a klt pair and $f : (X, B) \longrightarrow C$ a dlt morphism. Then the following four conditions are equivalent.*

(1) $f : (X, B) \longrightarrow C$ *is a plt morphism.*

(2) *every connected component of any fiber is irreducible.*

(3) $(F, B|_F)$ *is a klt pair for any fiber F.*

(4) *all the fibers of f are normal.*

The next lemma is an analogue of [KM, Lemma 7.2 (4)]. It easily follows from the definition of dlt pairs (see [KM, Definition 2.37]). We leave the details to the readers.

Lemma 2.2. *Let (X, B) be a klt pair and $f : (X, B) \longrightarrow C$ a dlt morphism. If E is an exceptional divisor over X such that the center of E on X is contained in a fiber, then the discrepancy $a(E, X, B) > 0$.*

We note the following properties, which is an easy consequence of the negativity lemma (cf. [KM, Lemma 3.38]).

Lemma 2.3 (cf. [KM, Corollary 3.44])**.** *Let $\phi : (X, B) \dashrightarrow (X^+, B^+)$ be either a $(K_X + B)$-flip over Y or a divisorial contraction of a $(K_X + B)$-negative extremal ray over Y, $f : Y \longrightarrow C$ a flat morphism onto a non-singular curve C, and $h := f \circ g : (X, B) \longrightarrow C$ a dlt (resp. plt) morphism. Then $h^+ : (X^+, B^+) \longrightarrow C$ is also a dlt (resp. plt) morphism.*

2.3.   Finally, we define *semi-stable log flips* (cf. [KM, Theorem 7.8]).

Definition 2.3.   Let (X, B) be a dlt pair and $f : X \longrightarrow W$ a flipping contraction with respect to $K_X + B$, that is, f is small and $-(K_X + B)$ is f-ample. We assume that f is *extremal*, where "extremal" means that X is $\mathbb{Q}$-factorial and the relative Picard number $\rho(X/W) = 1$. Assume that there exists a flat morphism $g : W \longrightarrow C$ to a smooth curve such that $h := g \circ f$ is dlt. Then the flip $f^+ : X^+ \longrightarrow W$ of f:

$$
\begin{array}{ccc}
X & \dashrightarrow & X^+ \\
 & \searrow \quad \swarrow & \\
 & W &
\end{array}
$$

that is,

 (i) f^+ is small,

(ii) $K_{X^+} + B^+$ is f^+-ample, where B^+ is the strict transform of B,

is called a *semi-stable* (*log*) *flip* of f. Furthermore, if (X, B) is terminal, that is, discrep$(X, B) > 0$, then we call f^+ a *semi-stable terminal flip* of f.

We treat only two examples here.

Example 1 (4-fold semi-stable flip). Let V be a projective 3-fold with $\mathbb{Q}$-factorial terminal singularities and

$$
\begin{array}{ccc}
V & \dashrightarrow & V^+ \\
 & \searrow \quad \swarrow & \\
 & Z &
\end{array}
$$

an extremal K_V-flip. We define $X := V \times \mathbb{P}^1$, $X^+ := V^+ \times \mathbb{P}^1$, and $W := Z \times \mathbb{P}^1$. We put $Y := C := \mathbb{P}^1$. Then

$$
\begin{array}{ccc}
X & \dashrightarrow & X^+ \\
 & \searrow \quad \swarrow & \\
 & W &
\end{array}
$$

is an extremal 4-fold semi-stable flip over Y. We note that the second projection $X \longrightarrow C$ is a plt morphism. It is not difficult to see that $\rho(X/W) = 1$ and X is $\mathbb{Q}$-factorial. In this case, the flipping and flipped loci are dominant onto C.

The following example is a 4-fold toric flip. We quote it from [M2, Example-Claim 14-2-8].

Example 2 (Toric 4-dimensional flip). Let $N_1 = \mathbb{Z}^4$ and $N_2 = \mathbb{Z}$. We put

$$
\begin{aligned}
v_1 &= (1,0,0,0) \\
v_2 &= (0,1,0,0) \\
v_3 &= (0,0,1,0) \\
v_4 &= (0,0,0,1) \\
v_5 &= (1,1,-1,-1)
\end{aligned}
$$

and consider the following two cones,

$$
\tau_5 = \langle v_1, v_2, v_3, v_4 \rangle, \quad \tau_4 = \langle v_1, v_2, v_3, v_5 \rangle.
$$

We define the two fans,

$$
\begin{aligned}
\Delta &= \{\tau_4, \tau_5, \text{and their faces}\}, \\
\Delta' &= \{\langle v_1, v_2, v_3, v_4, v_5 \rangle, \text{and its faces}\}.
\end{aligned}
$$

We consider the toric morphism $g : X(\Delta) \longrightarrow X(\Delta')$. This is a flipping contraction. We consider the first projection $p : N_1 \longrightarrow N_2$. This p induces

$f : X(\Delta') \longrightarrow \mathbb{A}^1 = X(\langle e \rangle)$, where $e = 1 \in N_2$. By this morphism, $X(\Delta) \longrightarrow X(\Delta')$ is a semi-stable flipping contraction. We can construct the flip of g and determine the exceptional locus of g and so on. Note that $\mathrm{Exc}(g) = V(\langle v_1, v_2 \rangle)$ and $(f \circ g)^{-1}(0) = V(\langle v_1 \rangle) \cup V(\langle v_5 \rangle)$. So, this flip is of type (B) in [Kc, Main Theorem 0.5]. For the details, see [M2, Example-Claim 14-2-8, Remark 14-2-9].

The related topics of Example 2 are [Ka2], [T1], and [T2].

§3. Preparation

In this section, we make preparations for the proof of the main theorem: Theorem 1.1.

3.1.   We write a sequence of 4-fold semi-stable flips over Y as follows:

$$(X, B) =: (X_0, B_0) \dashrightarrow (X_1, B_1) \dashrightarrow (X_2, B_2) \dashrightarrow \cdots$$
$$\searrow \qquad \swarrow \quad \searrow \qquad \swarrow$$
$$W_0 \qquad\qquad W_1 \qquad\qquad\qquad ,$$

where $\phi_i : X_i \longrightarrow W_i$ is an extremal flipping contraction with respect to $K_{X_i} + B_i$ over Y and $\phi_i^+ : X_{i+1} \longrightarrow W_i$ is the flip of ϕ_i for every i.

By the special termination theorem: Theorem 1.2, all the flipping and flipped loci are disjoint from $\llcorner B_i \lrcorner$ after finitely many flips. Therefore, we can assume that all the flipping and flipped loci are disjoint from $\llcorner B_i \lrcorner$ for every i by shifting the index i. So, we can replace B_i with its fractional part $\{B_i\}$ and assume that (X_i, B_i) is klt. From now on, we assume that (X_i, B_i) is klt for every i.

Let us recall the following definition.

Definition 3.1 ([K$^+$, 6.6 Definition]).   Let (X, B) be a klt n-fold. By [KM, Proposition 2.3.6], there are only finitely many exceptional divisors with non-positive discrepancies. The number of these divisors is denoted by $e(X, B)$. Thus (X, B) is terminal if and only if $e(X, B) = 0$ by the definition of terminal pairs.

3.2.   We prove Theorem 1.1 by induction on $e(X, B)$.

If $e(X, B) = 0$, then (X, B) is terminal. Thus a sequence of flips always terminates by Theorem 1.3. Therefore, we assume that the theorem holds for $e(X, B) \leq e - 1$, and prove it in case $e(X, B) = e$. We note that $e(X_i, B_i) \geq e(X_{i+1}, B_{i+1})$ for all i by the negativity lemma (cf. [KM, Lemma 3.38]).

3.3.   First, we add f^*P to B, where P is a closed point of C. We may regard the $(K_X + B)$-flips as the $(K_X + B + f^*P)$-flip. Then by Theorem 1.2, we can assume that all the flipping and flipped loci are not dominant onto C after finitely many flips. Thus, by shifting the index i we can assume that all the flipping and flipped loci are contained in some fibers.

So, we can assume that there are no semi-stable flips like Example 1.

3.4.   Next, we add $\sum_P f^*P$ to B, where P runs through all the closed points of C such that f^*P is not normal. By Theorem 1.2 again, we can assume that all the flipping and flipped loci are disjoint from non-normal fibers. We note that the normality of fibers are preserved by flips (see Lemmas 2.1 and 2.3). Therefore, we can assume that there exists a non-empty Zariski open set U of C such that all the flips occur over this open set U and $(X_i, B_i) \longrightarrow C$ is a plt morphism over U (see Definition 2.2).

We recall the definition of $r(X, B)$.

Definition 3.2 ([K$^+$, 6.9.8 Definition]).   Let (X, B) be a klt n-fold. We put
$$s(X, B) := \min\{a(E, X, B) > 0 \mid E \text{ is exceptional over } X\}.$$
Then we define
$$r(X, B) := (4^{\lceil s(X, B)^{-1} \rceil})! \in \mathbb{Z}_{>0}.$$

We generalize the invariants $e(X, B)$, $r(X, B)$, and $\mathrm{discrep}(X, B)$ for plt morphisms. By Lemma 2.1 (3), a plt morphism is a family of klt pairs. So, the following definition is natural.

Lemma-Definition 3.1.   *Let $f : (X, B) \longrightarrow C$ be a plt morphism. Then*
$$0 \le \max_F e(F, B|_F) < \infty,$$
$$\max_F r(F, B|_F) \in \mathbb{Z}_{>0}, \quad and$$
$$-1 < \min_F \mathrm{discrep}(F, B|_F) \le 1,$$
where F runs through all the fibers of f. We define
$$e(f; (X, B)) := \max_F e(F, B|_F),$$
$$r(f; (X, B)) := \max_F r(F, B|_F), \quad and$$
$$\mathrm{discrep}(f; (X, B)) := \min_F \mathrm{discrep}(F, B|_F).$$

Proof. We note that $K_F + B|_F := (K_X + B + F)|_F$ is klt by adjunction (see Lemma 2.1). Take a log resolution $g : Z \longrightarrow X$ of the pair (X, B) as in [KM, Proposition 2.36 (1)]. We write

$$K_Z + D - E = g^*(K_X + B),$$

where $D = \sum a_i D_i$ and $E = \sum b_j E_j$ are both effective and have no common irreducible component. Let $G = \sum G_k$ be the g-exceptional divisor such that $a(G_k, X, B) = 0$ for every k. We can assume that $\mathrm{Supp}(D \cup G)$ is non-singular. There exists a non-empty Zariski open set $U \subset C$ such that $f \circ g$ is smooth and $\mathrm{Supp}(D \cup E \cup G)$ is relatively normal crossing over U. We can assume that $g(D_i) \longrightarrow C$, $g(E_j) \longrightarrow C$, and $g(G_k) \longrightarrow C$ are flat over U for every i, j, and k after shrinking U. Over this open set U, $e(F, B|_F)$, $r(F, B|_F)$ (more precisely, $s(F, B|_F)$), and $\mathrm{discrep}(F, B|_F)$ are constant. Thus, we can check the claim. Therefore, $e(f; (X, B))$, $r(f; (X, B))$, and $\mathrm{discrep}(f; (X, B))$ are well-defined. $\qquad\square$

The next proposition will play crucial roles in the proof of the main theorem.

Proposition 3.1. *Let $f : (X, B) \longrightarrow C$ be a plt morphism and D a $\mathbb{Q}$-Cartier Weil divisor on X. Then mD is Cartier if and only if so is $mD|_F$ for every fiber F. In particular, if K_X is $\mathbb{Q}$-Cartier, then mK_X is Cartier if and only if so is mK_F for every fiber F.*

Proof. See, for example, [HL, Lemma 2.1.7]. We note that $(X, B + F)$ is plt and F is Cartier. Thus, in a neighborhood of F, $\mathrm{codim}_X(\mathrm{Sing}X \cap F) \geq 3$, where $\mathrm{Sing}X$ is the singular locus of X. So, $\mathcal{O}_X(mD)|_F \simeq \mathcal{O}_F(mD|_F)$ and $\mathcal{O}_X(m(K_X + F))|_F \simeq \mathcal{O}_F(mK_F)$ for every $m \in \mathbb{Z}_{\geq 0}$ (cf. [KM, Proposition 5.26]). $\qquad\square$

We recall the result in [K$^+$, 6.11 Theorem]. For the proof, see [K$^+$, (6.11.5)].

Theorem 3.1. *Let (V, Δ) be a klt 3-fold and E a $\mathbb{Q}$-Cartier Weil divisor on V. Then mE is Cartier for some*

$$1 \leq m \leq r(V, \Delta)^{2^{e(V,\Delta)}} \left(\frac{3}{1 + \mathrm{discrep}(V, \Delta)} \right)^{2^{e(V,\Delta)} - 1}.$$

Theorem 3.2.   *Let (X, B) be a klt 4-fold and $f : (X, B) \longrightarrow C$ a plt morphism. Let E be a $\mathbb{Q}$-Cartier Weil divisor on X. Then ME is Cartier for*

$$M := M(f; (X, B)) := (\ulcorner \varphi(f; (X, B)) \urcorner)! \in \mathbb{Z}_{>0},$$

where

$$\varphi(f; (X, B)) := r(f; (X, B))^{2^{e(f;(X,B))}} \left(\frac{3}{1 + \operatorname{discrep}(f; (X, B))} \right)^{2^{e(f;(X,B))} - 1} .$$

Let U be a non-empty Zariski open subset of C. Then the restriction

$$f|_{f^{-1}(U)} : (X, B)|_{f^{-1}(U)} \longrightarrow U$$

is a plt morphism and $M(f|_{f^{-1}(U)}; (X, B)|_{f^{-1}(U)})$ divides $M(f; (X, B))$.

Proof.   It is obvious by Theorem 3.1. We note that if E is not dominant onto C, then E is Cartier (see Lemma 2.1). The latter statement directly follows from the definition of M. $\qquad\square$

§4.   Proof of the Main Theorem

We go back to the proof of the main theorem: Theorem 1.1. Our proof is similar to that of [K$^+$, 6.11 Theorem].

Proof of Theorem 1.1.   We start the proof of the main theorem.

Step 1.   First, we take a log resolution of (X, B). We write $p : Z \longrightarrow X$ and

$$K_Z + p_*^{-1}B = p^*(K_X + B) + E - F,$$

where E and

$$F := \sum_{a_i \geq 0} a_i F_i$$

are effective exceptional divisors and have no common irreducible components. If necessary, we further blow up Z. Then we can assume that $\sum_{a_i \geq 0} F_i$ contains all the exceptional divisors whose discrepancies are non-positive, $\operatorname{Supp}(p_*^{-1}B \cup \sum F_i)$ is smooth and $\operatorname{Supp}(p_*^{-1}B \cup \sum F_i \cup (f \circ p)^*P)$ is simple normal crossing for every $P \in C$. We note that F_i is dominant onto C for every i by Lemma 2.2. By 3.2, we can assume that $\sum F_i \neq 0$, that is, $e = e(X, B) > 0$. We consider

$$f \circ p : (Z, D^{\varepsilon}) := (Z, p_*^{-1}B + F + \varepsilon \sum_{i \neq 0} F_i) \longrightarrow C.$$

It is easy to check that (Z, D^ε) is terminal and $f \circ p : (Z, D^\varepsilon) \longrightarrow C$ is a dlt morphism for $0 < \varepsilon \ll 1$. Run the log MMP over X. Then we obtain a sequence of flips and divisorial contractions over X:

$$(Z, D^\varepsilon) := (Z_0, D_0^\varepsilon) \dashrightarrow (Z_1, D_1^\varepsilon) \dashrightarrow \cdots \dashrightarrow (Z_k, D_k^\varepsilon) \dashrightarrow \cdots .$$

By Assumption 1.1, flips exist and by induction, any sequence of flips terminates since $e(Z_k, D_k^\varepsilon) < e = e(X, B)$ for every k (see Remark below). Note that each flip in the above process is a semi-stable log flip. Then we obtain a relative log minimal model $q : (Z', B') \longrightarrow X$, which satisfies the following conditions:

(1) $f \circ q : (Z', B') \longrightarrow C$ is a dlt morphism.

(2) $f \circ q : (Z', B') \longrightarrow C$ is a plt morphism over U (see 3.4).

(3) $e(Z', B') = e(X, B) - 1$.

(4) (Z', B') is a $\mathbb{Q}$-factorial klt pair.

(5) $K_{Z'} + B' = q^*(K_X + B)$, that is, q is a log crepant morphism.

(6) the relative Picard numbers $\rho(Z'/X) = 1$ and $\rho(Z'/W_0) = 2$.

We note that $\alpha : Z \dashrightarrow Z'$ is an isomorphism at the generic point of F_0 and contracts $E + \sum_{i \neq 0} F_i$.

Step 2. We put $p_0 : (Z_0^0, B_0^0) := (Z', B') \longrightarrow X =: X_0$. We construct a sequence of flips $\dashrightarrow Z_i^j \dashrightarrow Z_i^{j+1} \dashrightarrow$ over $X_i \dashrightarrow X_{i+1}$ for every i. We assume that we already have $p_i : (Z_i^0, B_i^0) \longrightarrow X_i$. Run the log MMP to (Z_i^0, B_i^0) over W_i. We obtain a sequence of flips and divisorial contractions over W_i:

$$Z_i^0 \dashrightarrow Z_i^1 \dashrightarrow \cdots \dashrightarrow Z_i^{k_i},$$

and a log minimal model $(Z_i^{k_i}, B_i^{k_i})$ for (Z_i^0, B_i^0) over W_i. This is a so-called 2 *ray games*. Note that each flip in the above process is a semi-stable log flip. Since (X_{i+1}, B_{i+1}) is the log canonical model of (Z_i^0, B_i^0) over W_i, there exists a morphism $q_i : Z_i^{k_i} \longrightarrow X_{i+1}$.

Case A. If all the steps in the above log MMP are flips, then we have $K_{Z_i^{k_i}} + B_i^{k_i} = q_i^*(K_{X_{i+1}} + B_{i+1})$. We define $p_{i+1} : (Z_{i+1}^0, B_{i+1}^0) := (Z_i^{k_i}, B_i^{k_i}) \longrightarrow X_{i+1}$. We put $c_{i+1} = 0$ in this case.

Case B. If a divisorial contraction occurs in the above log MMP, then it is not difficult to see that the final step $\beta : Z_i^{k_i - 1} \dashrightarrow Z_i^{k_i}$ is a divisorial

contraction and $q_i : Z_i^{k_i} \longrightarrow X_{i+1}$ is an isomorphism (cf. [KM, Lemma 6.39] and [K$^+$, 6.5.5 Proposition]). We note that other steps in the above log MMP are all flips. We also note that

$$K_{Z_i^{k_i-1}} + B_i^{k_i-1} = (q_i \circ \beta)^*(K_{X_{i+1}} + B_{i+1}) + c_{i+1}F_0$$

for $c_{i+1} > 0$, where F_0 is the proper transform of F_0 on $Z_i^{k_i-1}$. Then we put $p_{i+1} : (Z_{i+1}^0, B_{i+1}^0) := (Z_i^{k_i-1}, B_i^{k_i-1} - c_{i+1}F_0) \longrightarrow X_{i+1}$.

Note that $(Z_{i+1}^0, B_{i+1}^0) \longrightarrow C$ is a dlt morphism. So, $(Z_i^j, B_i^j) \longrightarrow C$ is dlt for every i, j.

Step 3.   We assume that the sequence

$$X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_i \dashrightarrow$$

is infinite. If Case B occurs only finitely many times, then we can assume that all the steps are Case A. Then we obtain an infinite sequence of flips with respect to $K_{Z_i^j} + B_i^j$. Since $e(Z_i^j, B_i^j) < e(X, B)$, it is impossible. So, Case B occurs infinitely many times. The coefficient of F_0 in B_{i+1}^0, where F_0 is the proper transform of F_0 on Z_{i+1}^0, is

$$a_0 - \sum_{0 \le j \le i} c_{j+1},$$

where $a_0 := -a(F_0, X, B) \ge 0$, that is, $a(F_0, X, B) \le 0$. Let U_{i+1} be a non-empty Zariski open set of U such that flips $(X_j, B_j) \dashrightarrow (X_{j+1}, B_{j+1})$ occur over $U \setminus U_{i+1}$ for $0 \le j \le i$. We note that it is sufficient to consider the coefficient of F_0 over U_{i+1} since F_0 is irreducible and dominant onto C. Let N be a positive integer such that NB_0 is a Weil divisor. Then NB_i is also a Weil divisor for every i. By Theorem 3.2, $MN(K_{X_i} + B_i)$ is a Cartier divisor over U_i for every i, where $M := M(h|_{(h)^{-1}(U)}; (X, B)|_{(h)^{-1}(U)})$. We note that $M(h|_{(h)^{-1}(U_i)}; (X, B)|_{(h)^{-1}(U_i)})$ divides M by Lemma 2.3 and that (X_i, B_i) is isomorphic to (X, B) over U_i. Thus MNB_i^0 is a Weil divisor over U_i for every i. So, we have that $MNc_j \in \mathbb{Z}_{\ge 0}$ for every j. Therefore, after finitely many steps, the coefficient of F_0 in B_{i+1}^0 is negative, that is, the discrepancy $a(F_0, X_{i+1}, B_{i+1}) > 0$. Thus, $e(X_{i+1}, B_{i+1}) < e = e(X, B)$. So, a sequence of flips terminates by the induction on $e(X, B)$. This is a contradiction.

We complete the proof of Theorem 1.1.   □

Remark.   Note that for the proof of the termination in case $e(X, B) = e$, we use the existence and the termination of semi-stable log flips only for $e(*, *) \le e - 1$.

§5. Appendix

It is not difficult to see that the existence of 4-dimensional semi-stable terminal flips implies that of all the 4-dimensional semi-stable log flips. It is essentially proved in the proof of Theorem 1.1.

We assume the existence of 4-dimensional semi-stable terminal flips as follows.

Assumption 5.1.   *Let (X, B) be a 4-dimensional terminal pair and $f : X \longrightarrow Z$ an extremal flipping contraction with respect to $K_X + B$. Assume that there exists a flat morphism $g : Z \longrightarrow C$ to a smooth curve such that $h := g \circ f$ is dlt. Then f has a semi-stable terminal flip.*

As mentioned above, Assumption 5.1 implies the existence of all the 4-dimensional semi-stable log flips (cf. [K$^+$, 6.4, 6.5, 6.11 Theorem]).

Proposition 5.1.   *Let (X, B) be a $\mathbb{Q}$-factorial projective 4-dimensional dlt pair and $f : X \longrightarrow W$ an extremal flipping contraction. Assume that there exists a flat morphism $g : W \longrightarrow C$ to a smooth curve such that $h := g \circ f$ is dlt. Then Assumption 5.1 implies that the semi-stable log flip of f exists.*

Proof.   We can assume that (X, B) is klt by replacing B with $(1-\varepsilon)B$ for $0 < \varepsilon \ll 1$. If $e(X, B) = 0$, then the flip exists by Assumption 5.1. Therefore, we assume that semi-stable log flips exist and any sequence of them terminates for $e(*, *) \le e - 1$, and prove the existence of the flip in case $e(X, B) = e$ (see also Remark in § 4). On this assumption, Step 1 in the proof of Theorem 1.1 works without any changes. So, we obtain $q : Z' \longrightarrow X$ as in the proof of Theorem 1.1. Run the log MMP to (Z', B') over W. We obtain a log minimal model (Z'', B'') for (Z', B') over W (see Step 2 in the proof of Theorem 1.1). Note that if a divisorial contraction occurs, then it is the final step of the above log MMP (see Case B in Step 2), and any sequence of flips in this process terminates by the assumption. Since (Z'', B'') is klt, we obtain the log canonical model (X^+, B^+) for (Z', B') over W by the relative base point free theorem. It is well-known that (X^+, B^+) is the required flip.   □

Remark.   Note that for the proof of the existence of semi-stable log flips in case $e(X, B) = e$, we use the existence and the termination of semi-stable log flips only for $e(*, *) \le e - 1$.

So, by Proposition 5.1, we can generalize Theorem 1.1 slightly. Note Remarks in § 4 and § 5.

Corollary 5.1.   *Assumption* 5.1 *implies Theorem* 1.1.

Acknowledgments

I would like to thank Professors V. V. Shokurov and Hiromichi Takagi for comments.

References

[F1] Fujino, O., On special termination and reduction theorem, to be contained in "Flips for 3-folds and 4-folds" by Corti, A., Ambro, F., Fujino, O., McKernan, J. and Takagi, H.

[F2] ———, Termination of 4-fold canonical flips, *Publ. RIMS, Kyoto Univ.*, **40** (2004), 231-237.

[F3] ———, Addendum to "Termination of 4-fold canonical flips" [*Publ. RIMS, Kyoto Univ.*, **40** (2004), 231-237]; *Publ. RIMS, Kyoto Univ.*, **41** (2005), 252-257.

[HL] Huybrechts, D. and Lehn, M., *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997.

[Kc] Kachi, Y., Flips of semi-stable 4-folds whose degenerate fibers are unions of Cartier divisors which are terminal factorial 3-folds, *Math. Ann.*, **307** (1997), 647-662.

[Ka1] Kawamata, Y., Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, *Ann. of Math.* (2), **127** (1988), 93-163.

[Ka2] ———, Small contractions of four-dimensional algebraic manifolds, *Math. Ann.*, **284** (1989), 595-600.

[Ka3] ———, Termination of log flips for algebraic 3-folds, *Internat. J. Math.*, **3** (1992), 653-659.

[KMM] Kawamata, Y., Matsuda, K. and Matsuki, K., Introduction to the Minimal Model Problem, in *Algebraic Geometry, Sendai 1985*, Adv. Stud. Pure Math., **10** (1987), Kinokuniya and North-Holland, 283-360.

[Ko] Kollár, J., Flops, *Nagoya Math. J.*, **113** (1989), 15-36.

[KM] Kollár, J. and Mori, S., *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, Vol. **134**, 1998.

[K⁺] Kollár, J., et al., Flips and Abundance for algebraic threefolds, *Astérisque*, **211** (1992).

[M1] Matsuki, K., Termination of flops for 4-folds, *Amer. J. Math.*, **113** (1991), 835-859.

[M2] ———, *Introduction to the Mori program*, Universitext. Springer-Verlag, New York, 2002.

[S1] Shokurov, V. V., Prelimiting flips, In *Birational Geometry: Linear systems and finitely generated algebras. A collection of papers.* Iskovskikh, V. A. and Shokurov, V. V. Editors, Proc. of Steklov Institute, **240** (2003), 75-213.

[S2] ———, Letters of a bi-rationalist, IV, Geometry of log flips, *Algebraic geometry*, 313–328, de Gruyter, Berlin, 2002.

[S3] ———, Letters of a bi-rationalist, V, Minimal log discrepancies and termination of log flips. (Russian) *Tr. Mat. Inst. Steklova*, **246** (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 328-351.

[T1] Takagi, H., Remarks on Gorenstein terminal fourfold flips, *J. Math. Sci. Univ. Tokyo*, **5** (1998), 149-164.

[T2] ———, The Existence of Gorenstein terminal fourfold flips, a revised version of [T1], *Preprint* 1999.