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Uniqueness and Existence of the Integrated Density of States for Schrödinger Operators with Magnetic Field and Electric Potential with Singular Negative Part

By

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Abstract

We prove the coincidence of the two definitions of the integrated density of states (IDS) for Schrödinger operators with strongly singular magnetic fields and scalar potentials: the first one using the counting function of eigenvalues of the induced operator on a bounded open set with Dirichlet boundary conditions, the second one using the spectral projections of the whole space operator. Thus we generalize a result of [5], where the scalar potential was non-negative. Moreover, we prove the existence of IDS for the case of periodical magnetic field and scalar potential.

§1. Introduction

One considers the vector potential $a = (a_1, \ldots, a_n) : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$ (which is identified to the differential form $\sum_{1\leq j\leq n} a_j \, \mathrm{d}x_j$) and the scalar potential $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the following hypotheses:

i) $a_j \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \leq j \leq n$;

ii) $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $V_- := \max(0, -V)$ *belongs to the Kato class* K_n, *that is, one has*:

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$$\lim_{\varepsilon \searrow 0} \left[\sup_{x \in \mathbb{R}^n} \int\limits_{|x-y|<\varepsilon} E(x-y)\, V_-(y)\, \mathrm{d}y \right] = 0,$$

where E is the usual elementary solution of the Laplace operator Δ.

We define the sesqui-linear form $h = h(a, V)$ on $L^2(\mathbb{R}^n)$ with domain

$$D(h) = \left\{ u \in L^2(\mathbb{R}^n); (\nabla - \mathrm{i}a)u \in \left(L^2(\mathbb{R}^n)\right)^n,\ |V|^{1/2}u \in L^2(\mathbb{R}^n) \right\},$$

by

$$h(u,v) = \int\limits_{\mathbb{R}^n} (\nabla - \mathrm{i}a)u \cdot \overline{(\nabla - \mathrm{i}a)v}\, \mathrm{d}x + \int\limits_{\mathbb{R}^n} V u\, \overline{v}\, \mathrm{d}x,$$

where ∇ stands for the distributional gradient and $\mathrm{i} = \sqrt{-1}$.

It is well-known (see [14]) that h is bounded from below and closed, the space $\mathcal{C}_0^\infty(\mathbb{R}^n)$ being a core of h. Let $H = H(a, V)$ be the associated self-adjoint bounded from below operator on $L^2(\mathbb{R}^n)$, with domain

$$D(H) = \left\{ u \in D(h);\ -(\nabla - \mathrm{i}a)^2 u + Vu \in L^2(\mathbb{R}^n) \right\},$$

given by

$$Hu = -(\nabla - \mathrm{i}a)^2 u + Vu.$$

We shall also need a self-adjoint realization of the differential operator $-(\nabla - \mathrm{i}a)^2 + V$ on a open subset Ω of $\mathbb{R}^n$, corresponding to the Dirichlet boundary conditions. One identifies $L^2(\Omega)$ to the closed subspace of $L^2(\mathbb{R}^n)$ with elements which are zero on $\mathbb{R}^n \setminus \Omega$. Let P_Ω be the projection of $L^2(\mathbb{R}^n)$ onto $L^2(\Omega)$ (the multiplication operator by the characteristic function of Ω). If $H_\alpha := H + \alpha(1 - P_\Omega)$, $\alpha \geq 0$, one obtains an unique operator H_Ω, pseudo-selfadjoint on $L^2(\mathbb{R}^n)$, such that $\lim_{\alpha \to \infty} H_\alpha = H_\Omega$ in the strong resolvent sense (see Th. 4.1 in [10]). Moreover, the operator H_Ω can be considered as a self-adjoint operator on $L^2(\Omega)$ associated with the sesqui-linear form h_Ω defined by:

$$h_\Omega(u,v) := h(u,v), \quad D(h_\Omega) := \{u \in D(h); \operatorname{supp} u \subset \overline{\Omega}\},$$

identified to a form on $L^2(\Omega)$.

Remark 1.1. Usually, one works with another Dirichlet realization on Ω (see [5], for instance). More exactly, one considers the operator $\tilde{H}_\Omega$, self-adjoint on $L^2(\Omega)$, associated with the sesqui-linear form $\tilde{h}_\Omega$, which is the closure on $L^2(\Omega)$ of the form h_Ω° with domain $\mathcal{C}_0^\infty(\Omega)$, defined by

$$h_\Omega^\circ(u,v) = \int\limits_\Omega (\nabla - \mathrm{i}a)u \cdot \overline{(\nabla - \mathrm{i}a)v}\, \mathrm{d}x + \int\limits_\Omega V u\, \overline{v}\, \mathrm{d}x.$$

We shall see in §3 that $H_\Omega = \tilde{H}_\Omega$ if Ω is a "Lipschitz domain" (or, following Stein [15], a domain with "minimally smooth boundary").

In order to state the main result, we shall need a family $\mathcal{F}$ of bounded open subsets of $\mathbb{R}^n$, satisfying the conditions:

iii) *For every* $m \in \mathbb{N}^*$, *there exists* $\Omega \in \mathcal{F}$ *such that the ball* $B(0; m)$, *with centre at the origin and of radius* m, *is contained in* Ω.

iv) *For every* $\varepsilon > 0$, *there exists* $m_0 \in \mathbb{N}^*$ *such that if* $\Omega \in \mathcal{F}$ *and* $B(0; m_0) \subset \Omega$, *one has*

$$|\{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < 1\}| < \varepsilon|\Omega|.$$

Definition 1.2. Let μ, μ_Ω ($\Omega \in \mathcal{F}$) be Borel measures on $\mathbb{R}$. We say that

$$\lim_{\Omega \to \mathbb{R}^n, \Omega \in \mathcal{F}} \mu_\Omega = \mu,$$

if for every $f \in \mathcal{C}_0(\mathbb{R})$ (the space of compactly supported continuous functions on $\mathbb{R}$) and for every $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}^*$ such that if $B(0; m_0) \subset \Omega$, than one has

$$\left| \int_{\mathbb{R}} f \, \mathrm{d}\mu_\Omega - \int_{\mathbb{R}} f \, \mathrm{d}\mu \right| < \varepsilon.$$

We shall see that for every $f \in \mathcal{C}_0(\mathbb{R})$ and for every Ω bounded open subset of $\mathbb{R}^n$, the operators $f(H_\Omega)$ and $P_\Omega f(H) P_\Omega$ belong to $\mathcal{I}_1$ (the space of trace class operators). Then, using the Riesz-Markov Theorem, there exist Borel measures μ_Ω^D and μ_Ω, such that

$$|\Omega|^{-1} \mathrm{Tr}\, f(H_\Omega) = \int_{\mathbb{R}} f \, \mathrm{d}\mu_\Omega^D, \quad |\Omega|^{-1} \mathrm{Tr}\,(P_\Omega f(H) P_\Omega) = \int_{\mathbb{R}} f \, \mathrm{d}\mu_\Omega.$$

One sees that the distribution functions of those two measures satisfy the relations

$$\mu_\Omega^D((-\infty, \lambda]) = |\Omega|^{-1} N_\Omega(\lambda), \quad \mu_\Omega((-\infty, \lambda]) = |\Omega|^{-1} \mathrm{Tr}\,(P_\Omega E_\lambda(H) P_\Omega),$$

almost everywhere on $\mathbb{R}$, where $N_\Omega(\lambda)$ is the number of the eigenvalues of H_Ω which are less than λ, and $E_\lambda(H)$ is the spectral projection of H for the interval $(-\infty, \lambda]$, $\lambda \in \mathbb{R}$.

We can define the integrated density of states in two different ways.

Definition 1.3. We call density of states of H a Borel measure μ^D (respectively μ) on $\mathbb{R}$ such that

$$\lim_{\Omega\to\mathbb{R}^n,\Omega\in\mathcal{F}} \mu_\Omega^D = \mu^D \quad \left(\text{respectively } \lim_{\Omega\to\mathbb{R}^n,\Omega\in\mathcal{F}} \mu_\Omega = \mu\right).$$

The distribution function ρ^D of μ^D (respectively ρ of μ) will be the integrated density of states of H.

This definition rises two problems:

a) Prove the equivalence of the two definitions of IDS.

b) Prove the existence of IDS.

The solution of problem a) is the main result of this paper:

Theorem 1.4. *Under hypotheses* i)–iv), *the density of states μ^D exists if and only if the density of states μ exists. Moreover, if one of them exists, then $\mu^D = \mu$.*

This theorem was proved in [5] in the case where $V \geq 0$. The proof in §4 uses some ideas of [5], along with a property of comparison of resolvents, essentially proved in [4], and which requires the hypothesis $V_- \in K_n$.

Remark 1.5. An analysis of the proof of Theorem 1.2 in [5] shows that if $\mathcal{F} \subset LM(r, A, B)$ (see the notation in [5]), Theorem 1.4 remains true if the Dirichlet boundary conditions are replaced by Neumann boundary conditions.

The problem b) will be solved only in a special case. Let

$$B = \mathrm{d}a = \frac{1}{2}\sum_{1\leq j,k\leq n} B_{jk}\mathrm{d}x_j \wedge \mathrm{d}x_k, \quad B_{jk} = \partial_j a_k - \partial_k a_j$$

be the magnetic field defined by the vector potential a. (B_{jk} will be distributions on $\mathbb{R}^n$.) One considers a lattice Γ in $\mathbb{R}^n$, generated by a basis $e_1, \dots, e_n$, that is,

$$\Gamma = \left\{\sum_{j=1}^n \alpha_j e_j;\ \alpha_j \in \mathbb{Z},\ 1 \leq j \leq n\right\}.$$

One denotes by F a fundamental domain of $\mathbb{R}^n$ with respect to Γ; for instance,

$$F = \left\{\sum_{j=1}^n t_j e_j;\ 0 \leq t_j < 1,\ 1 \leq j \leq n\right\}.$$

We also remark that for every $f \in \mathcal{C}_0(\mathbb{R})$ and every Ω bounded open subset of $\mathbb{R}^n$, $P_\Omega f(h) P_\Omega$ is the product of two operators from $\mathcal{I}_2$ (the space of Hilbert-Schmidt operators). By the Fubini Theorem, the restriction of the integral kernel $K_{f(H)}$ of $f(H)$ to the diagonal set of $\mathbb{R}^n \times \mathbb{R}^n$ is well-defined and is a locally integrable function.

We suppose that the following two hypotheses hold:

iv') *For every* $\varepsilon > 0$ *there exists* $m_0 \in \mathbb{N}^*$ *such that, if* $\Omega \in \mathcal{F}$ *and* $B(0; m_0) \subset \Omega$, *then one has*

$$|\{x \in \mathbb{R}^n; \operatorname{dist}(x, \partial\Omega) < 1\} < \varepsilon |\Omega|.$$

v) V *and* B_{jk}, $1 \leq j, k \leq n$ *are* Γ*-periodic functions.*

Theorem 1.6. *Under hypotheses* i)–iii), iv') *and* v), *the IDS of* H *exists and, for every* $f \in \mathcal{C}_0(\mathbb{R})$, *one has*

$$\lim_{\Omega \to \mathbb{R}^n, \Omega \in \mathcal{F}} \frac{\operatorname{Tr}(P_\Omega f(H) P_\Omega)}{|\Omega|} = \frac{1}{|F|} \int_F K_{f(H)}(x, x) \mathrm{d}x. \tag{1.1}$$

We shall see that the integral above represents a Γ-trace of the operator $f(H)$, in the sense of Atiyah [1].

The theorem above is known in the case where B is a constant magnetic field and $V \in \mathcal{C}^\infty(\mathbb{R}^n)$ (see [6]). The case of constant magnetic fields and random electric potentials, possibly unbounded from below, was also studied (see [7]).

The plan of the paper is the following: In the second section we prove some properties of the operator H. Particularly, for the reader convenience, we give the proof of the property of comparison of resolvents. The third section is devoted to the study of the operator H_Ω: we prove the identity $H_\Omega = \tilde{H}_\Omega$ for domains with minimally smooth boundary and we generalize the aforementioned property of comparison to this case. In the last two sections we prove Theorems 1.4 and 1.6, respectively.

§2. The Operator $H = H(a, V)$

Proposition 2.1. *Under hypotheses* i) *and* ii), *for every* $\rho > 1$, *there exist* M, $\delta > 0$ *such that if* $\lambda > \max\{\delta, -\inf \sigma(H)\}$ (*where* $\sigma(H)$ *is the spectrum of* H), *then one has*

$$|(H + \lambda)^{-r} f| \leq M(\rho H_0 + \lambda - \delta)^{-r} |f| \quad a.e. \text{ on } \mathbb{R}^n \tag{2.1}$$

for every $r > 0$ *and* $f \in L^2(\mathbb{R}^n)$, *where* $H_0 := H(0, 0)$.

Proof. Firstly, one remarks that, following [2], for every $f \in L^2(\mathbb{R}^n)$ and $t > 0$, one has the inequality

$$\left|\mathrm{e}^{-tH(a,V)}f\right| \leq \mathrm{e}^{-tH(0,V)}|f|, \quad a.e \text{ on } \mathbb{R}^n. \tag{2.2}$$

Using the Feymann-Kaç formula (see, for instance, [13]), we infer that

$$\mathrm{e}^{-tH(0,V)}|f| \leq \mathrm{e}^{-tH(0,-V_-)}|f| \quad a.e \text{ on } \mathbb{R}^n. \tag{2.3}$$

It is known (see Proposition B.6.7 in [14]) that $\mathrm{e}^{-tH(0,-V_-)}$ is an integral operator whose integral kernel $k\colon \mathbb{R}_+^* \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+^*$ is a continuous function which verifies the following estimate: for every $\rho > 1$, there exist the positive constants M and δ such that, for every $t > 0$ and x, $y \in \mathbb{R}^n$, one has

$$|k(t,x,y)| \leq M\mathrm{e}^{\delta t}k_0(\rho t, x, y), \tag{2.4}$$

where k_0 is the integral kernel of e^{-tH_0}.

We also have

$$(H+\lambda)^{-r}f = \frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}\mathrm{e}^{-\lambda t}\mathrm{e}^{-tH}f\,\mathrm{d}t, \tag{2.5}$$

for every $r > 0$, $\lambda > -\inf\sigma(H)$ and $f \in L^2(\mathbb{R}^n)$, where $\Gamma(\cdot)$ is the Euler gamma function.

If $\lambda > \max\{\delta, -\inf\sigma(H)\}$, the relations (2.2)–(2.5) imply the inequalities

$$\begin{aligned}|(H+\lambda)^{-r}f| &\leq \frac{M}{\Gamma(r)}\int_0^\infty t^{r-1}\mathrm{e}^{-(\lambda-\delta)t}\mathrm{e}^{-\rho tH_0}|f|\,\mathrm{d}t \\ &= M(\rho H_0 + \lambda - \delta)^{-r}|f| \quad a.e. \text{ on } \mathbb{R}^n,\end{aligned}$$

which finish the proof. □

Remark 2.2. For the case $V \geq 0$, one proves in [9] that for $\lambda > 0$ and $f \in L^2(\mathbb{R}^n)$ one has

$$|(H+\lambda)^{-1}f| \leq (H_0+\lambda)^{-1}|f|, \quad a.e. \text{ on } \mathbb{R}^n.$$

It is this equality (or rather an extension of it at H_Ω) which is used in [5].

Remark 2.3. In [4], one gives an example which shows that the hypothesis $V_- \in K_n$ is necessary for the validity of inequality (2.1)

Proposition 2.4. *Under the hypotheses of Proposition* 2.1, *for every* $r > n/4$ *and* $\lambda > \max\{\delta, -\inf \sigma(H)\}$, *there exists a positive constant* C *such that, for every open bounded subset* Ω *of* $\mathbb{R}^n$, *we have that* $P_\Omega(H+\lambda)^{-r} \in \mathcal{I}_2$ *and the inequality*

$$\|P_\Omega(H+\lambda)^{-r}\|_{\mathcal{I}_2} \leq C|\Omega|^{1/2} \tag{2.6}$$

holds.

Proof. Using (2.1) we infer that for every $f \in L^2(\mathbb{R}^n)$,

$$|P_\Omega(H+\lambda)^{-r} f| \leq M\rho^{-r} P_\Omega\big(H_0 + \frac{\lambda-\delta}{\rho}\big)^{-r}|f| \quad \textit{a.e.} \text{ on } \mathbb{R}^n. \tag{2.7}$$

On the other hand, $\big(H_0 + \frac{\lambda-\delta}{\rho}\big)^{-r}$ is a convolution operator by a function $g \in L^2(\mathbb{R}^n)$; therefore the operator in the right hand side of (2.7) belongs to $\mathcal{I}_2$, since its integral kernel is $M\rho^{-r}\chi_\Omega(x)g(x-y)$, where χ_Ω is the characteristic function of Ω. Furthermore,

$$\left\| P_\Omega \left(H_0 + \frac{\lambda-\delta}{\rho}\right)^{-r} \right\|_{\mathcal{I}_2} = \int_\Omega \int_{\mathbb{R}^n} |g(x-y)|^2 \mathrm{d}x\, \mathrm{d}y = \|g\|^2_{L^2(\mathbb{R}^n)}|\Omega|. \tag{2.8}$$

To obtain the stated properties, it suffices to use (2.7), (2.8) and Theorem 2.13 in [12]. □

Corollary 2.5. *Under the hypotheses of Proposition* 2.1, *for every* $m > n/2$ *and* $\lambda > \max\{\delta, -\inf \sigma(H)\}$, *there exists a positive constant* C *such that, for every open bounded subset* Ω *of* $\mathbb{R}^n$, *we have that* $P_\Omega(H+\lambda)^{-m}P_\Omega \in \mathcal{I}_1$ *and the inequality*

$$\|P_\Omega(H+\lambda)^{-m}P_\Omega\|_{\mathcal{I}_1} \leq C|\Omega| \tag{2.9}$$

holds.

Proof. It suffices to use Proposition 2.4 and the inequality

$$\|P_\Omega(H+\lambda)^{-m}P_\Omega\|_{\mathcal{I}_1} \leq \|P_\Omega(H+\lambda)^{-r}\|_{\mathcal{I}_2} \cdot \|(H+\lambda)^{-s}P_\Omega\|_{\mathcal{I}_2},$$

where r, $s > n/4$ and $r + s = m$. □

Corollary 2.6. *For every* $f \in L^\infty_{\mathrm{comp}}(\mathbb{R})$, *there exists a constant* $C > 0$, *such that for every open bounded subset* Ω *of* $\mathbb{R}^n$, *we have that* $P_\Omega f(H) P_\Omega \in \mathcal{I}_1$ *and the inequality*

$$\|P_\Omega f(H) P_\Omega\|_{\mathcal{I}_1} \leq C|\Omega| \tag{2.10}$$

holds.

Proof. It suffices to write the identity

$$P_\Omega f(H)P_\Omega = P_\Omega(H+\lambda)^{-r}(H+\lambda)^{2r}f(H)(H+\lambda)^{-r}P_\Omega,$$

where $r > n/4$ and $\lambda > \max\{\delta, -\inf\sigma(H)\}$ and to apply Proposition 2.4. $\square$

In order to state the last result of this section, we denote by $\mathcal{B}(E,F)$ the space of all bounded linear operators from E to F (E and F being normed linear spaces). In particular, $\mathcal{B}(E) := \mathcal{B}(E,E)$.

Lemma 2.7. *Let $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ be such that $\partial^\alpha\varphi \in L^\infty(\mathbb{R}^n)$ if $|\alpha| \le 2$. Then φu belongs to $D(H)$ for every $u \in D(H)$. Moreover,*

$$[H,\varphi] = -2(\nabla\varphi)\cdot(\nabla - \mathrm{i}a) - \Delta\varphi \quad \text{on } D(H) \tag{2.11}$$

and

$$(H+\lambda)^{-1}[H,\varphi] \in \mathcal{B}(L^2(\mathbb{R}^n)) \quad \text{if} \quad \lambda > -\inf\sigma(H). \tag{2.12}$$

Proof. If $u \in D(H)$, then $\varphi u \in D(h)$ and for every $v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} h(\varphi u, v) &= h(u, \overline{\varphi} v) - 2\left((\nabla\varphi)\cdot(\nabla - \mathrm{i}a)u, v\right) - ((\Delta\varphi)u, v) \\ &= \left(\varphi H u - 2(\nabla\varphi)\cdot(\nabla - \mathrm{i}a)u - (\Delta\varphi)u, v\right), \end{aligned}$$

where $(\cdot,\cdot)$ denotes the scalar product of $L^2(\mathbb{R}^n)$. We deduce that $\varphi u \in D(H)$ and

$$H(\varphi u) = \varphi H u - 2(\nabla\varphi)\cdot(\nabla - \mathrm{i}a)u - (\Delta\varphi)u.$$

This yields (2.11). To get (2.12), we endow $D(H)$ with the graph topology. Then $[H,\varphi] \in \mathcal{B}(D(H), L^2(\mathbb{R}^n))$ and, by duality, $[H,\varphi] \in \mathcal{B}(L^2(\mathbb{R}^n), [D(H)]^*)$, while $(H+\lambda)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^n), D(H))$ and $(H+\lambda)^{-1} \in \mathcal{B}([D(H)]^*, L^2(\mathbb{R}^n))$ by duality. $\square$

§3. The Operator $H_\Omega = H_\Omega(a,V)$.

We fix $\gamma \in \mathbb{R}$ such that $h \ge \gamma$. Then $D(h)$ is a Hilbert space for the norm

$$\|u\|_h = [h(u,u) + (\gamma+1)\|u\|^2]^{1/2}, \quad u \in D(h),$$

where $\|\cdot\|$ denotes the norm of $L^2(\mathbb{R}^n)$.

Let $\mathcal{C}_{+\infty}(\mathbb{R}) := \{f \in \mathcal{C}(\mathbb{R}); \lim_{t\to+\infty} f(t) = 0\}$. Then for $f \in \mathcal{C}_{+\infty}(\mathbb{R})$ and Ω open subset of $\mathbb{R}^n$, we can define $f(H_\Omega) \in \mathcal{B}(L^2(\mathbb{R}^n))$ in the following way: $f(H_\Omega)\Big|_{L^2(\Omega)}$ is the operator from $\mathcal{B}(L^2(\Omega))$ associated with H_Ω, as self-adjoint operator on $L^2(\Omega)$, by the usual functional calculus, while $f(H_\Omega) = 0$ on $L^2(\Omega)^\perp$.

Proposition 3.1. *Let $f \in \mathcal{C}_{+\infty}(\mathbb{R})$, $\lambda > -\inf \sigma(H)$ and φ as in Lemma* 2.7. *Then, for every open set $\Omega \subset \mathbb{R}^n$, the operator H_Ω has the following properties*:

a) $s - \lim_{\alpha\to\infty} f(H_\alpha) = f(H_\Omega)$ *in* $\mathcal{B}(L^2(\mathbb{R}^n))$.

b) $s - \lim_{\alpha\to\infty}(H_\alpha + \lambda)^{-1} = (H_\Omega + \lambda)^{-1}$ *in* $\mathcal{B}([D(h)]^*, D(h))$.

c) $(H_\Omega + \lambda)^{-1} = P_\Omega(H_\Omega + \lambda)^{-1} = (H_\Omega + \lambda)^{-1}P_\Omega$.

d) $(H_\Omega + \lambda)^{-1}[H, \varphi]$, $[H, \varphi](H_\Omega + \lambda)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^n))$.

e) $s - \lim_{\alpha\to\infty}[H, \varphi](H_\alpha + \lambda)^{-1} = [H, \varphi](H_\Omega + \lambda)^{-1}$ *in* $\mathcal{B}(L^2(\mathbb{R}^n))$.

f) $(\partial_k - \mathrm{i}a_k)(H_\Omega + \lambda)^{-1} \in \mathcal{B}(L^2(\Omega))$, $1 \le k \le n$.

Proof. a) The property follows from [3], §3.
b) The property is a consequence of Lemma 3.7 in [3].
c) By the inequality (3.7) in [3], there exists a constant $C > 0$ such that, for every $\alpha > 0$, we have

$$\|(1 - P_\Omega)(H_\alpha + \lambda)^{-1}\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le C\alpha^{-1/2},$$

whence we get the needed inequalities.
d) The property follows from the fact that

$$(H_\Omega + \lambda)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^n), D(h)) \cap \mathcal{B}([D(h)]^*, L^2(\mathbb{R}^n))$$

(see b)) and

$$[H, \varphi] \in \mathcal{B}(D(h), L^2(\mathbb{R}^n)) \cap \mathcal{B}(L^2(\mathbb{R}^n), [D(h)]^*)$$

(see (2.11)).
e) The property follows from b) and the fact that $[H, \varphi] \in \mathcal{B}(D(h), L^2(\mathbb{R}^n))$.
f) The statement follows from $(H_\Omega + \lambda)^{-1} \in \mathcal{B}(L^2(\Omega), D(h_\Omega))$ and $\partial_k - \mathrm{i}a_k \in \mathcal{B}(D(h_\Omega), L^2(\Omega))$, where $D(h_\Omega)$ is endowed with the norm induced by the one of $D(h)$. □

We shall also need to write under a certain form the difference between the resolvent of H and the pseudo-resolvent of H_Ω.

Lemma 3.2. *Let $\lambda > -\inf \sigma(H)$ and φ be a function as in Lemma* 2.7, *$\varphi = 1$ on $\mathbb{R}^n \setminus \Omega$. Then*

$$\begin{aligned}
&(H + \lambda)^{-1} - (H_\Omega + \lambda)^{-1} \\
&\qquad = \left[(H + \lambda)^{-1} - (H_\Omega + \lambda)^{-1}\right]\left[\varphi + [H, \varphi](H_\Omega + \lambda)^{-1}\right] \\
&\qquad = \left[\varphi - (H + \lambda)^{-1}[H, \varphi]\right]\left[(H + \lambda)^{-1} - (H_\Omega + \lambda)^{-1}\right]
\end{aligned} \tag{3.1}$$

Proof. We have

$$\begin{aligned}(H+\lambda)^{-1}-(H_\Omega+\lambda)^{-1}&=s-\lim_{\alpha\to\infty}\big[(H+\lambda)^{-1}-(H_\alpha+\lambda)^{-1}\big]\\&=s-\lim_{\alpha\to\infty}(H+\lambda)^{-1}\alpha(1-P_\Omega)(H_\alpha+\lambda)^{-1}\\&=s-\lim_{\alpha\to\infty}(H+\lambda)^{-1}\alpha(1-P_\Omega)\varphi(H_\alpha+\lambda)^{-1}\\&=s-\lim_{\alpha\to\infty}\big[(H+\lambda)^{-1}\alpha(1-P_\Omega)(H_\alpha+\lambda)^{-1}\varphi\\&\qquad+(H+\lambda)^{-1}\alpha(1-P_\Omega)(H_\alpha+\lambda)^{-1}[H,\varphi](H_\alpha+\lambda)^{-1}\big]\\&=\big[(H+\lambda)^{-1}-(H_\Omega+\lambda)^{-1}\big]\varphi\\&\qquad+\big[(H+\lambda)^{-1}-(H_\Omega+\lambda)^{-1}\big][H,\varphi](H_\Omega+\lambda)^{-1},\end{aligned}$$

where in the last equality we have used the property e) from Proposition 3.1, as well as the fact that $(H_\alpha+\lambda)^{-1}$ is bounded in $\mathcal{B}(L^2(\mathbb{R}^n))$ uniformly with respect to $\alpha\geq 0$.

In the same way we prove the equality between the first and the last term of relation (3.1). □

Now we can generalize the inequality (2.1) to the operator H_Ω.

Proposition 3.3. *Under the hypotheses* i) *and* ii), *for every* $\rho>1$ *there exist* M, $\delta>0$ *such that if* $\lambda>\max\{\delta,-\inf\sigma(H)\}$, *we have*

$$|(H_\Omega+\lambda)^{-r}|\leq MP_\Omega(\rho H_0+\lambda-\delta)^{-r}P_\Omega|f|\quad a.e\ on\ \mathbb{R}^n, \tag{3.2}$$

for every $r>0$, Ω *open subset of* $\mathbb{R}^n$ *and* $f\in L^2(\mathbb{R}^n)$.

Proof. Using Proposition 3.1 a), we see that for every $t>0$

$$s-\lim_{\alpha\to 0}\mathrm{e}^{-tH_\alpha}=\mathrm{e}^{-tH_\Omega}. \tag{3.3}$$

We also note that the Feymann-Kaç formula allows us to derive the inequality

$$\mathrm{e}^{-tH(0,V+\alpha(1-\chi_\Omega))}|f|\leq \mathrm{e}^{-tH(0,-V_-)}|f|\quad a.e.\ \text{on}\ \mathbb{R}^n. \tag{3.4}$$

Hence, using (3.3), (3.4) and the first part of the proof of Proposition 2.1, we get

$$\big|\mathrm{e}^{-tH_\Omega}f\big|\leq M\mathrm{e}^{\delta t}\mathrm{e}^{-\rho tH_0}|f|\quad a.e.\ \text{on}\ \mathbb{R}^n. \tag{3.5}$$

We infer from Proposition 3.1 a) that, for every $r>0$,

$$s-\lim_{\alpha\to 0}(H_\alpha+\lambda)^{-r}=(H_\Omega+\lambda)^{-r}. \tag{3.6}$$

Hence, from the equality (2.5) for H_α, from (3.5) and (3.6), it follows that

$$\left|(H_\Omega+\lambda)^{-r}f\right| \leq M(\rho H_0+\lambda-\delta)^{-r}|f| \quad a.e. \text{ on } \mathbb{R}^n. \tag{3.7}$$

To get (3.2) it suffices to write (3.7) for $P_\Omega f$ instead of f and to use Proposition 3.1 b). □

The proof of Proposition 2.4, with (2.1) replaced by (3.2), allows us to obtain the next proposition.

Proposition 3.4. *Under the hypotheses of Proposition* 3.3, *for every* $r > n/4$ *and* $\lambda > \max\{\delta, -\inf\sigma(H)\}$, *there exists a positive constant* C *such that for every* U *and* Ω *open subsets of* $\mathbb{R}^n$, U *bounded, we have that* $P_U(H_\Omega+\lambda)^{-r} \in \mathcal{I}_2$ *and the inequality*

$$\|P_U(H_\Omega+\lambda)^{-r}\|_{\mathcal{I}_2} \leq C|\Omega\cap U|^{1/2} \tag{3.8}$$

holds.

The next two corollaries follow directly from the proposition above (see the proofs of Corollary 2.5 and 2.6).

Corollary 3.5. *Under the hypotheses of Proposition* 3.3, *for every* $m > n/2$ *and* $\lambda > \max\{\delta, -\inf\sigma(H)\}$, *there exists a positive constant* C *such that for every* Ω *open bounded subset of* $\mathbb{R}^n$, *we have that* $(H_\Omega+\lambda)^{-m} \in \mathcal{I}_1$ *and the inequality*

$$\|(H_\Omega+\lambda)^{-m}\|_{\mathcal{I}_1} \leq C|\Omega| \tag{3.9}$$

holds.

Corollary 3.6. *For every* $f \in L^\infty_{\mathrm{comp}}(\mathbb{R})$ *there exists a constant* $C > 0$ *such that, for every* Ω *open bounded subset of* $\mathbb{R}^n$, *we have that* $f(H_\Omega) \in \mathcal{I}_1$ *and the inequality*

$$\|f(H_\Omega)\|_{\mathcal{I}_1} \leq C|\Omega| \tag{3.10}$$

holds.

The last result of this section will be the equality $H_\Omega = \tilde{H}_\Omega$ for open subsets of $\mathbb{R}^n$ with minimally smooth boundary. This equality is a consequence of the following proposition.

Proposition 3.7. *Let Ω be an open subset of $\mathbb{R}^n$ with minimally smooth boundary (cf. Stein [15]). Then $\mathcal{C}_0^\infty(\Omega)$ is a core of the sesqui-linear form h_Ω.*

Proof. The proof is divided in three steps, in each of them obtaining partial results.

i) *$D(h_\Omega) \cap L^\infty(\Omega)$ is a core of h_Ω.*

We use an idea from [11]. It is obvious that the range of e^{-H_Ω} is a core of H_Ω, hence also for h_Ω. On the other hand, for every $\rho > 0$, $\mathrm{e}^{-\rho H_0}$ is a convolution operator by an $L^2(\mathbb{R}^n)$-function, hence $\mathrm{e}^{-\rho H_0} \in \mathcal{B}(L^2(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$. The inequality (3.5) implies therefore that the range of e^{-H_Ω} is contained in $L^\infty(\Omega)$.

ii) *$D(h_\Omega) \cap L^\infty_{\mathrm{comp}}(\Omega)$ is a core of h_Ω.*

There exist (see [15]) $N \in \mathbb{N}$, a sequence $(\Omega_i)_{i\geq 1}$ of open subsets of $\mathbb{R}^n$ and two sequences of functions $(\varphi_i)_{i\geq 1}$ and $(\psi_i)_{i\geq 0}$ with the following properties:

1. $\partial\Omega \subset \bigcup\limits_{i\geq 1} \Omega_i$.
2. The intersection of $N+1$ open sets Ω_i is void.
3. The functions $\varphi_i \colon \mathbb{R}^{n-1} \to \mathbb{R}$ are Lipschitz and the sequence of their Lipschitz constants is bounded.
4. We may suppose that $\Omega \cap \Omega_i = \{x \in \Omega_i; x_n > \varphi_i(x')\}$, $i \geq 1$, where $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$.
5. $\psi_i \in \mathcal{C}^\infty(\mathbb{R}^n)$, $\psi_i \geq 0$, $\operatorname{supp}\psi_0 \Subset \Omega$, and $\operatorname{supp}\psi_i \Subset \Omega_i$ for $i \geq 1$.
6. For every $\alpha \in \mathbb{N}^n$, $\partial^\alpha\psi_i$ are bounded uniformly with respect to $i \geq 0$.
7. $\sum\limits_{i\geq 0} \psi_i = 1$ on $\overline{\Omega}$.

Let $u \in D(h_\Omega) \cap L^\infty(\Omega)$. Then $\operatorname{supp} u \subset \overline{\Omega}$ and $u = \sum_{i\geq 0} \psi_i u$; the series converges in $D(h_\Omega)$, by the dominated convergence theorem. It suffices to prove that for every $i \geq 0$, $\psi_i u$ is the limit in $D(h_\Omega)$ of a sequence from $D(h_\Omega) \cap L^\infty_{\mathrm{comp}}(\Omega)$. We can construct a partition of unity on a neighborhood of $\operatorname{supp}(\psi_i u)$, that is a sequence $(\beta_j)_{j\geq 1}$, with $\beta_j \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\beta_j \geq 0$, the family $(\operatorname{supp}\beta_j)_{j\geq 0}$ being locally finite and $\sum_{j\geq 0} \beta_j = 1$ on a neighborhood of $\operatorname{supp}(\psi_i u)$. We may also suppose that for every $\alpha \in \mathbb{N}^n$, the sequence $(\partial^\alpha\beta_j)_{j\geq 0}$ is uniformly bounded. Then $\psi_i u = \sum_{j\geq 0} \beta_j(\psi_i u)$, the series being convergent in $D(h_\Omega)$. It then follows that we may henceforth suppose $\psi_i u$ to be compactly supported.

We have that $\psi_0 u \in D(h_\Omega) \cap L^\infty_{\mathrm{comp}}(\Omega)$; hence it remains to show that for every $v \in D(h_\Omega) \cap L^\infty(\Omega)$, whose support is a compact subset of $\overline{\Omega} \cap \Omega_i$, is the limit in $D(h_\Omega)$ of a sequence from $D(h_\Omega) \cap L^\infty_{\mathrm{comp}}(\Omega)$.

Let $\chi_i \colon \mathbb{R}^n \to \mathbb{R}^n$ be the homeomorphism defined by $y = \chi_i(x)$ if and only if $y' = x'$, $y_n = x_n - \varphi(x')$. It is clear that $v \in \mathcal{H}^1_{\mathrm{comp}}(\mathbb{R}^n)$ (the Sobolev space of order 1 on $\mathbb{R}^n$, whose elements are compactly supported). Then $w := v \circ \chi_i^{-1}$ belongs to $\mathcal{H}^1_{\mathrm{comp}}(\mathbb{R}^n)$ and $\operatorname{supp} w \subset \overline{\mathbb{R}^n_+}$. We consider a function $\theta \in \mathcal{C}^\infty_0(\mathbb{R}^n)$, $\theta \geq 0$, $\int\limits_{\mathbb{R}^n} \theta(y)\,\mathrm{d}y = 1$, and such that $|y| \leq 1$ and $y_n > 0$ on $\operatorname{supp}\theta$. For $0 < \varepsilon \leq 1$ we define $\theta_\varepsilon \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ by $\theta_\varepsilon(y) := \varepsilon^{-n}\theta(y/\varepsilon)$. Let w_ε be the convolution of w by θ_ε. Then $w_\varepsilon \in \mathcal{C}^\infty_0(\mathbb{R}^n)$, $\operatorname{supp} w_\varepsilon \subset \mathbb{R}^n_+$, $\sup_{0<\varepsilon\leq 1} \|w_\varepsilon\|_{L^\infty(\mathbb{R}^n)} < \infty$ and $\lim_{\varepsilon \searrow 0} w_\varepsilon = w$ in $\mathcal{H}^1(\mathbb{R}^n)$.

If $v_\varepsilon := w_\varepsilon \circ \chi_i$, then $v_\varepsilon \in \mathcal{H}^1_{\mathrm{comp}}(\mathbb{R}^n)$, $\sup_{0<\varepsilon\leq 1} \|v_\varepsilon\|_{L^\infty(\mathbb{R}^n)} < \infty$, $\lim_{\varepsilon\searrow 0} v_\varepsilon = v$ in $\mathcal{H}^1(\mathbb{R}^n)$, and there exist $\varepsilon_0 \in (0,1]$ and a compact K contained in $\overline{\Omega} \cap \Omega_i$ such that $\operatorname{supp} v_\varepsilon \subset K \cap \Omega$ for all $0 < \varepsilon \leq \varepsilon_0$. It is clear that $v_\varepsilon \in D(h_\Omega) \cap L^\infty_{\mathrm{comp}}(\Omega)$ and we easily infer the existence of a sequence $(\varepsilon_j)_{j\geq 0}$, $0 < \varepsilon_j \leq \varepsilon_0$, $\lim_{j\to\infty} \varepsilon_j = 0$ such that $\lim_{j\to\infty} v_{\varepsilon_j} = v$ in $D(h_\Omega)$.

iii) $\mathcal{C}^\infty_0(\Omega)$ *is a core of* h_Ω.

Let $u \in D(h_\Omega) \cap L^\infty_{\mathrm{comp}}(\Omega)$ and $(\theta_\varepsilon)_{0<\varepsilon\leq\varepsilon_0}$ be the sequence constructed in ii). If ε_0 is small enough, $u_\varepsilon := u * \theta_\varepsilon \in \mathcal{C}^\infty_0(\Omega)$ and there exists a compact subset M of Ω such that $\operatorname{supp} u_\varepsilon \subset M$ for all $\varepsilon \in (0, \varepsilon_0]$. Moreover, the sequence $(u_\varepsilon)_{0<\varepsilon\leq\varepsilon_0}$ is uniformly bounded and $\lim_{\varepsilon\searrow 0} u_\varepsilon = u$ in $\mathcal{H}^1(\Omega)$, since $u \in \mathcal{H}^1(\Omega)$. Hence, there exists a sequence $(\varepsilon_j)_{j\geq 0}$, $0 < \varepsilon_j \leq \varepsilon_0$, $\lim_{j\to\infty} \varepsilon_j = 0$ such that $\lim_{j\to\infty} u_{\varepsilon_j} = u$ in $D(h_\Omega)$. $\square$

§4. Proof of Theorem 1.4

The main ingredient of the proof will be the following result.

Proposition 4.1. *Under the hypotheses of Proposition* 3.3, *for every* $m \in \mathbb{N}$, $m \geq n+2$ *and* $\lambda > \max\{\delta, -\inf\sigma(H)\}$, *there exists a positive constant* C *such that we have*

$$\|P_\Omega(H+\lambda)^{-m}P_\Omega - (H_\Omega+\lambda)^{-m}\|_{\mathcal{I}_1} \leq C\,|\Omega|^{1/2}\,|\tilde{\Omega}|^{1/2} \tag{4.1}$$

for every Ω *bounded open subset of* $\mathbb{R}^n$, *where* $\tilde{\Omega} := \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < 1\}$.

Proof. We have the identity

$$\begin{aligned}(4.2)\quad P_\Omega(H+\lambda)^{-m}P_\Omega-(H_\Omega+\lambda)^{-m} &= P_\Omega\big[(H+\lambda)^{-m}-(H_\Omega+\lambda)^{-m}\big]P_\Omega\\ &=\sum_{j=0}^{m-1}P_\Omega(H+\lambda)^{j-m+1}\big[(H+\lambda)^{-1}-(H_\Omega+\lambda)^{-1}\big](H_\Omega+\lambda)^{-j}P_\Omega.\end{aligned}$$

Let $\varphi\in\mathcal{C}^\infty(\mathbb{R}^n)$ be such that $\varphi=1$ on $\mathbb{R}^n\setminus\Omega$, $\varphi=0$ on $\Omega\setminus\tilde\Omega$, and for every $\alpha\in\mathbb{N}^n$, $\partial^\alpha\varphi$ is bounded by a constant independent on Ω (we may construct it by considering the convolution of the characteristic function of a neighborhood of $\mathbb{R}^n\setminus\Omega$ by a appropriate function from $\mathcal{C}_0^\infty(\mathbb{R}^n)$).

We use Propositions 3.4, 3.1 d), Lemma 2.7 and the first equality in (3.1) to estimate the $\mathcal{I}_1$-norm of the terms in the sum in (4.2) corresponding to $j>n/2$. Everything reduces to the following two estimates:

$$\begin{aligned}&\|P_\Omega(H+\lambda)^{j-m+1}\big[(H+\lambda)^{-1}-(H_\Omega+\lambda)^{-1}\big]\varphi(H_\Omega+\lambda)^{-j-1}P_\Omega\|_{\mathcal{I}_1}\\ &\quad\le C_1\,\|\varphi(H_\Omega+\lambda)^{-j/2}\|_{\mathcal{I}_2}\|(H_\Omega+\lambda)^{-j/2}P_\Omega\|_{\mathcal{I}_2}\le C_2\,|\Omega|^{1/2}\,|\tilde\Omega|^{1/2}\end{aligned}$$

and

$$\begin{aligned}&\|P_\Omega(H+\lambda)^{j-m+1}\big[(H+\lambda)^{-1}-(H_\Omega+\lambda)^{-1}\big]\,[H,\varphi]\,(H_\Omega+\lambda)^{-j-1}P_\Omega\|_{\mathcal{I}_1}\\ &\quad\le C_3\,\|P_{\tilde\Omega}(H_\Omega+\lambda)^{-j-1}\|_{\mathcal{I}_1}\le C_4\,|\Omega|^{1/2}\,|\tilde\Omega|^{1/2},\end{aligned}$$

where the constants C_j, $1\le j\le 4$, do not depend on Ω, and we should consider the fact that the derivatives of φ have supports contained in $\tilde\Omega$.

The terms with $j\le n/2$ (hence $m-j-1>n/2$) are estimated in the same way, using the other equality of (3.1) and the identity $\varphi(H+\lambda)^{-1}=(H+\lambda)^{-1}\varphi+(H+\lambda)^{-1}H,\varphi(H+\lambda)^{-1}$. □

The assertions of Theorem 1.4 will follow from the next proposition.

Proposition 4.2. *Suppose that hypotheses* i)–iv) *hold. Then, for every* $f\in\mathcal{C}_0(\mathbb{R})$ *and* $\varepsilon>0$, *there exists* $m_0\in\mathbb{N}^*$ *such that we have*

$$(4.3)\qquad \big|\mathrm{Tr}\,(P_\Omega f(H)P_\Omega)-\mathrm{Tr}\,f(H_\Omega)\big|\le\varepsilon\,|\Omega|$$

for every $\Omega\in\mathcal{F}$ *with* $B(0;m_0)\subset\Omega$.

Proof. The proof follows [5]. For a fixed $\rho>1$, we consider $\delta>0$ as in Proposition 3.3, and let $a:=\max\{\delta,-\inf\sigma(H)\}+1$. It suffices to prove (4.3) for real functions f with $\mathrm{supp}\,f\subset[-a+1/2,\infty)$. The functions $[-a+1/2,\infty)\ni$

$t \to (a+t)^{n+2} f(t) \in \mathbb{R}$ and $[0,2] \ni \tau \to \tau^{-n-2} f(\tau^{-1}-a) \in \mathbb{R}$ are continuous. For every $\varepsilon > 0$ there exists a polynomial P_ε such that

$$|\tau^{-n-2} f(\tau^{-1}-a) - P_\varepsilon(\tau)| \le \varepsilon \quad \text{for} \quad 0 \le \tau \le 2.$$

Then

$$\left|(a+t)^{n+2} f(t) - P_\varepsilon\left(\frac{1}{a+t}\right)\right| \le \varepsilon \quad \text{for} \quad t \ge -a+1/2.$$

Let $Q_\varepsilon(t) := (a+t)^{-n-2} P_\varepsilon\left(\frac{1}{a+t}\right)$. Then, in form sense,

$$-\varepsilon(a+H)^{-n-2} \le f(H) - Q_\varepsilon(H) \le \varepsilon(a+H)^{-n-2},$$

hence

$$\begin{aligned} -\varepsilon P_\Omega(a+H)^{-n-2} P_\Omega \le P_\Omega f(H) P_\Omega - P_\Omega Q_\varepsilon(H) P_\Omega \\ \le \varepsilon P_\Omega(a+H)^{-n-2} P_\Omega. \end{aligned}$$

Using Corollary 2.5 we infer

$$\begin{aligned} &\left|\mathrm{Tr}\,(P_\Omega f(H) P_\Omega) - \mathrm{Tr}\,(P_\Omega Q_\varepsilon(H) P_\Omega)\right| \\ &\quad \le \varepsilon \mathrm{Tr}\left(P_\Omega (a+H)^{-n-2} P_\Omega\right) \le C_1\, \varepsilon\, |\Omega|, \end{aligned} \tag{4.4}$$

where C_1 is a constant independent on ε and $\Omega \in \mathcal{F}$.

Similarly we prove that there exists another constant C_2, independent on ε and $\Omega \in \mathcal{F}$, such that we have

$$\left|\mathrm{Tr}\, f(H_\Omega) - \mathrm{Tr}\, Q_\varepsilon(H_\Omega)\right| \le \varepsilon \mathrm{Tr}\,(a+H_\Omega)^{-n-2} \le C_2\, \varepsilon\, |\Omega|. \tag{4.5}$$

Therefore (4.3) follows from (4.4), (4.5), Proposition 4.1 and hypothesis iv). □

§5. Proof of Theorem 1.6

We shall identify the Γ-periodic distributions on $\mathbb{R}^n$ to the distributions on the torus $\mathbb{T}^n = \mathbb{R}^n/\Gamma$. The duality bracket for the dual pair $\left(\mathcal{D}'(\mathbb{T}^n), \mathcal{D}(\mathbb{T}^n)\right)$ is denoted by $\langle \cdot, \cdot \rangle_\Gamma$, while $\langle \cdot, \cdot \rangle$ is the scalar product of $\mathbb{R}^n$. Let

$$\Gamma^* = \{\gamma^* \in \mathbb{R}^n;\ \langle \gamma^*, \gamma \rangle \in 2\pi\mathbb{Z} \ \text{for every} \ \gamma \in \Gamma\}$$

be the dual lattice of Γ.

Proposition 5.1. *Let* $B = \frac{1}{2} \sum\limits_{1 \le j,k \le n} B_{jk}\, \mathrm{d}x_j \wedge \mathrm{d}x_k$ *be a differential 2-form whose coefficients* $B_{jk} = -B_{kj}$ *are real* Γ*-periodic distributions on*

$\mathbb{R}^n$, and such that $\mathrm{d}B = 0$. Then, there exists a differential 1-form $A = \sum_{1\le j\le n} A_j \mathrm{d}x_j$, with coefficients A_j real Γ-periodic distributions on $\mathbb{R}^n$ and such that $\mathrm{d}A = B$, if and only if

$$\langle B_{jk}, 1\rangle_\Gamma = 0, \qquad 1 \le j, k \le n. \tag{5.1}$$

Moreover, if the coefficients B_{jk} belong to the Sobolev space $\mathcal{H}^{-1}(\mathbb{T}^n)$, $1 \le j, k \le n$, we can choose $A_j \in L^2(\mathbb{T}^n)$, $1 \le j \le n$.

Proof. We may write

$$B_{jk} = \sum_{\alpha\in\Gamma^*} B_\alpha^{jk}\, \mathrm{e}^{\mathrm{i}\langle\cdot,\alpha\rangle}, \quad B_\alpha^{jk} := \frac{1}{|F|}\langle B_{jk}, \mathrm{e}^{-\mathrm{i}\langle\cdot,\alpha\rangle}\rangle_\Gamma,$$

the series being convergent in $\mathcal{D}'(\mathbb{T}^n)$, which means that there exists a constant $C > 0$ and $p \in \mathbb{Z}$ such that we have

$$|B_\alpha^{jk}| \le C(1+|\alpha|)^p, \qquad \alpha\in\Gamma^*,\ 1 \le j, k \le n. \tag{5.2}$$

The condition $\mathrm{d}B = 0$ means that $\partial_l B_{jk} + \partial_k B_{lj} + \partial_j B_{kl} = 0$, hence

$$\alpha_l B_\alpha^{jk} + \alpha_k B_\alpha^{lj} + \alpha_j B_\alpha^{kl} = 0, \qquad 1 \le j, k, l \le n, \tag{5.3}$$

where $\alpha = (\alpha_1, \dots \alpha_n)$. Similarly, we may represent A in the form

$$A_j = \sum_{\alpha\in\Gamma^*} A_\alpha^j\, \mathrm{e}^{\mathrm{i}\langle\cdot,\alpha\rangle}, \quad A_\alpha^j := \frac{1}{|F|}\langle A_j, \mathrm{e}^{-\mathrm{i}\langle\cdot,\alpha\rangle}\rangle_\Gamma, \tag{5.4}$$

and we have to find $C > 0$, $q \in \mathbb{Z}$ such that

$$|A_\alpha^j| \le C(1+|\alpha|)^q, \qquad \alpha\in\Gamma^*,\ 1 \le j \le n. \tag{5.5}$$

The equation $\mathrm{d}A = B$, that is, the differential system

$$\partial_j A_k - \partial_k A_j = B_{jk}, \qquad 1 \le j, k \le n,$$

is equivalent to the algebraic system

$$\alpha_j A_\alpha^k - \alpha_k A_\alpha^j = -\mathrm{i} B_\alpha^{jk}, \quad 1 \le j, k, l \le n,\ \alpha\in\Gamma^*. \tag{5.6}$$

The condition (5.1) means $B_0^{jk} = 0$, $1 \le j, k \le n$, and it is a necessary condition for the existence of a solution to the system (5.6). Considering (5.3), it is also sufficient, since the general solution to (5.6) is

$$A_\alpha^j = \begin{cases} C_0 & \text{for } \alpha = 0, \\ -\mathrm{i}\,|\alpha|^{-2} \sum_{1\le k\le n} \alpha_k B_\alpha^{kj} + \mathrm{i}\,\alpha_j C_\alpha & \text{for } \alpha \ne 0, \end{cases} \tag{5.7}$$

where C_α are arbitrary constants with $\overline{C_\alpha} = C_{-\alpha}$. If we choose $C_\alpha = 0$ for $\alpha \neq 0$, we may take $q = p - 1$ in (5.5). We have $\overline{B_\alpha^{kj}} = B_{-\alpha}^{kj}$, therefore also $\overline{A_\alpha^j} = A_{-\alpha}^j$, and the distribution A_j is real.

To prove the last assertion, it suffices to note the fact that $B_{jk} \in \mathcal{H}^{-1}(\mathbb{T}^n)$ means

$$\sum_\alpha (1 + |\alpha|)^{-2} \left|B_\alpha^{jk}\right|^2 < \infty.$$

Then the solution (5.7) with $C_\alpha^j = 0$ for $\alpha \neq 0$ verifies

$$\sum_\alpha \left|A_\alpha^j\right|^2 < \infty,$$

hence $A_j \in L^2(\mathbb{T}^n)$. □

Corollary 5.2. *Assume that hypotheses* i), ii) *and* v) *hold. Then there exists a vector potential* $A = \sum_{1 \leq j \leq n} A_j \, \mathrm{d}x_j$ *with* $A_j \in L^2_{\mathrm{loc}}(R^n, \mathbb{R})$ *and* Γ*-periodic*, $1 \leq j \leq n$, *and a constant magnetic field* $B^0 = \frac{1}{2} \sum_{1 \leq j,k \leq n} B_{jk}^0 \, \mathrm{d}x_j \wedge \mathrm{d}x_k$, $B_{jk}^0 = -B_{kj}^0 \in \mathbb{R}$, *such that if* $A^0 = \sum_{1 \leq j \leq n} A_j^0 \, \mathrm{d}x_j$ *with* $A_j^0(x) = \frac{1}{2} \sum_{1 \leq k \leq n} B_{jk}^0 x_k$, *the operators* $H(a, V)$ *and* $H(A + A_0, V)$ *are unitarily equivalent.*

Proof. We first choose the form B^0 with $B_{jk}^0 = -B_{kj}^0 \in \mathbb{R}$ such that we have $\langle B_{jk} - B_{jk}^0, 1 \rangle_\Gamma = 0$, $1 \leq j, k \leq n$. It is obvious that $B_{jk} \in \mathcal{H}^{-1}(\mathbb{T}^n)$, and therefore, using Proposition 5.1, we infer the existence of a 1-form $A = \sum_{1 \leq j \leq n} A_j \, \mathrm{d}x_j$ with $A_j \in L^2_{\mathrm{loc}}(R^n, \mathbb{R})$ and Γ-periodic, such that $\mathrm{d}A = B - B^0$. If A^0 is the 1-form from the statement, we shall have $\mathrm{d}(A + A_0) = B = \mathrm{d}a$. Using Lemma 1.1 in [8], we deduce the existence of a real function $g \in \mathcal{H}^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that $a - (A + A^0) = \nabla g$. If U is the multiplication operator by $\mathrm{e}^{\mathrm{i}g}$, which is unitary on $L^2(\mathbb{R}^n)$, Theorem 1.2 in [8] implies $U \, H(a, V) \, U^{-1} = H(A + A^0, V)$. □

Remark 5.3. Considering the definition of the density of states, we see that it suffices to prove the existence of this measure for $H(A + A^0, V)$. Therefore we may henceforth assume that $a = A + A^0$.

Let T_γ, $\gamma \in \Gamma$, the magnetic translations defined by B^0 in Corollary 5.2. Hence $T_\gamma = U_\gamma L_\gamma$, where L_γ are the usual translations in $\mathbb{R}^n$: $(L_\gamma u)(x) = u(x - \gamma)$, $x \in \mathbb{R}^n$, $\gamma \in \Gamma$, and $(U_\gamma u)(x) = \mathrm{e}^{\mathrm{i}\langle B^0, x \wedge \gamma \rangle / 2} u(x)$, B^0 being viewed here as a linear form on $R^n \wedge \mathbb{R}^n$.

Using the model in [1] (even if $\{T_\gamma\}_\gamma$ is not a group) we can define a Γ-trace on a class of operators from $\mathcal{B}(L^2(\mathbb{R}^n))$.

Definition 5.4. An operator $S \in \mathcal{B}(L^2(\mathbb{R}^n))$ commuting with the magnetic translations T_γ, $\gamma \in \Gamma$ is said to be of Γ-trace class if for every functions φ, $\psi \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ we have $\varphi S \psi \in \mathcal{I}_1$. We write $S \in \mathcal{I}_1^\Gamma$.

Lemma 5.5. *Let $S \in \mathcal{I}_1^\Gamma$ and φ, φ', ψ, $\psi' \in L^2_{comp}(\mathbb{R}^n)$ such that $\sum_{\gamma\in\Gamma} L_\gamma(\varphi\psi) = \sum_{\gamma\in\Gamma} L_\gamma(\varphi'\psi') = 1$. Then $\mathrm{Tr}\,(\varphi S\psi) = \mathrm{Tr}\,(\varphi' S\psi')$.*

Proof. We have

$$
\begin{aligned}
\mathrm{Tr}\,(\varphi S\psi) &= \mathrm{Tr}\left(\sum_{\gamma\in\Gamma}[L_\gamma(\varphi'\psi')]\varphi S\psi\right) = \sum_{\gamma\in\Gamma}\mathrm{Tr}\,[(L_\gamma(\varphi'\psi'))\varphi S\psi] \\
&= \sum_{\gamma\in\Gamma}\mathrm{Tr}\,[(L_\gamma\varphi')\varphi\psi S(L_\gamma\psi')] = \sum_{\gamma\in\Gamma}\mathrm{Tr}\,[T_\gamma((L_\gamma\varphi')\varphi\psi S(L_\gamma\psi'))T_\gamma^{-1}] \\
&= \sum_{\gamma\in\Gamma}\mathrm{Tr}\,([L_\gamma^{-1}(\varphi\psi)]\varphi' S\psi') = \mathrm{Tr}\left(\sum_{\gamma\in\Gamma}[L_\gamma^{-1}(\varphi\psi)]\varphi' S\psi'\right) \\
&= \mathrm{Tr}\,(\varphi' S\psi'),
\end{aligned}
$$

where one should keep in mind the fact that the sums are finite and that T_γ is a unitary operator on $L^2(\mathbb{R}^n)$. □

Thus, the following definition is justified.

Definition 5.6. If $S \in \mathcal{I}_1^\Gamma$, we call Γ-trace of S the quantity $\mathrm{Tr}_\Gamma S := \mathrm{Tr}\,(\varphi S\psi)$, where φ, $\psi \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ and $\sum_{\gamma\in\Gamma} L_\gamma(\varphi\psi) = 1$.

Lemma 5.7. *If S is a self-adjoint operator from $\mathcal{B}(L^2(\mathbb{R}^n))$ and $S \in \mathcal{I}_1^\Gamma$, then K_S, the integral kernel of S, is an L^1_{loc}-function, its restriction to the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$ is well-defined and locally integrable and, moreover,*

$$
\mathrm{Tr}_\Gamma S = \int_F K_S(x,x)\,\mathrm{d}x. \tag{5.8}
$$

Proof. The first assertions of the statement are consequences of the following remark: if φ $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we have $\varphi S\psi \in \mathcal{I}_1$, therefore there exist Φ, $\Psi \in \mathcal{I}_2$ such that $\varphi S\psi = \Phi\Psi$. But K_Φ, $K_\Psi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and

$$
K_{\varphi S\psi}(x,y) = \int_{\mathbb{R}^n} K_\Phi(x,z)\,K_\Psi(z,y)\,\mathrm{d}z,
$$

and, by the Fubini Theorem, the function $\mathbb{R}^n \ni x \to K_{\varphi S\psi}(x,x) \in \mathbb{C}$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Hence, as $K_{\varphi S\psi} = (\varphi \otimes \psi)K_S$, we infer that the functions K_S and $\mathbb{R}^n \ni x \to K_S(x,x) \in \mathbb{C}$ are locally integrable.

We now take $\varphi = \psi = \chi_F$. Then $\sum_{\gamma\in\Gamma} L_\gamma(\varphi\psi) = 1$, and

$$\begin{aligned}
\mathrm{Tr}_\Gamma S = \mathrm{Tr}\,(\varphi S\psi) &= \int_{\mathbb{R}^n} \varphi(x) K_S(x,x)\,\mathrm{d}x \\
&= \sum_{\gamma\in\Gamma} \int_F (L_\gamma\varphi)(x) K_S(x+\gamma, x+\gamma)\,\mathrm{d}x \\
&= \int_F K_S(x,x)\,\mathrm{d}x.
\end{aligned}$$

For the last equality we have used the relation $K_S(x+\gamma, x+\gamma) = K_S(x,x)$, $x \in \mathbb{R}^n$, $\gamma \in \Gamma$, a consequence of $T_\gamma S = ST_\gamma$. ☐

Now we are able to prove Theorem 1.6, having already a meaning for the integral in (1.1). It suffices to check the existence of the limit, and the relation (1.1). By Corollary 2.6, if $\Omega \in \mathcal{F}$ and $f \in \mathcal{C}_0(\mathbb{R})$, then $P_\Omega f(H) P_\Omega \in \mathcal{I}_1$ and we have

$$\frac{\mathrm{Tr}\,(P_\Omega f(H) P_\Omega)}{|\Omega|} = \frac{1}{|\Omega|} \int_\Omega K_{f(H)}(x,x)\,\mathrm{d}x.$$

Let us consider the following sets:

$$\begin{aligned}
\mathcal{M} &:= \big\{\gamma \in \Gamma; (F + \{\gamma\}) \cap \overline{\Omega} \neq \emptyset\big\}, \\
\partial\mathcal{M} &:= \big\{\gamma \in \mathcal{M}; (F + \{\gamma\}) \cap \partial\Omega \neq \emptyset\big\}, \\
\Omega_\Gamma &:= \bigcup_{\gamma\in\mathcal{M}\setminus\partial\mathcal{M}} (F + \{\gamma\}) \subset \Omega, \\
(\partial\Omega)_\Gamma &:= \bigcup_{\gamma\in\partial\mathcal{M}} (F + \{\gamma\}).
\end{aligned}$$

We remark that in iv') the inequality

$$\{x \in \mathbb{R}^n; \mathrm{dist}\,(x, \partial\Omega) < 1\} < \varepsilon|\Omega|$$

may be replaced by

$$\{x \in \mathbb{R}^n; \mathrm{dist}\,(x, \partial\Omega) < a\} < \varepsilon|\Omega|.$$

Indeed, let χ_a be the characteristic function of the set $\{x \in \mathbb{R}^n; \operatorname{dist}(x, \partial\Omega) < a\}$ and ρ_b the characteristic function of the open ball of radius b and center 0. Then

$$\rho_b * \chi_{a_1}(x) = \int \rho_b(x-y)\chi_{a_1}(y)\,\mathrm{d}y \geq \omega_{a_1}\chi_{a_2}(x)$$

whenever $b > a_1 + a_2$, where ω_c denotes the measure of the ball centred at 0 and of radius c. Hence, an integration in x in the above inequality will give

$$\omega_b\,|\{x \in \mathbb{R}^n; \operatorname{dist}(x, \partial\Omega) < a_1\} \geq \omega_{a_1}\,|\{x \in \mathbb{R}^n; \operatorname{dist}(x, \partial\Omega) < a_2\}.$$

Consequently we may suppose that $\operatorname{diam} F < 1$, and therefore

$$\Omega \setminus \Omega_\Gamma \subset (\partial\Omega)_\Gamma \subset \tilde{\Omega} = \{x \in \mathbb{R}^n; \operatorname{dist}(x, \partial\Omega) < 1\}.$$

By hypothesis iv'), we get that for every $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}^*$ such that if $B(0; m_0) \subset \Omega$, we have $\big|(\partial\Omega)_\Gamma\big| < \varepsilon|\Omega|$, and hence $|\Omega \setminus \Omega_\Gamma| \leq \varepsilon|\Omega|$. For such an Ω, we have

(5.9)
$$\frac{\operatorname{Tr}(P_\Omega f(H) P_\Omega)}{|\Omega|} = \frac{|\Omega_\Gamma|}{|\Omega|}\frac{1}{|\Omega_\Gamma|}\left[\int_{\Omega_\Gamma} K_{f(H)}(x,x)\,\mathrm{d}x + \int_{\Omega\setminus\Omega_\Gamma} K_{f(H)}(x,x)\,\mathrm{d}x\right].$$

The operator H commutes with the magnetic translations T_γ, $\gamma \in \Gamma$, whence $T_\gamma f(H) = f(H) T_\gamma$, $\gamma \in \Gamma$, and then $K_{f(H)}(x+\gamma, x+\gamma) = K_{f(H)}(x,x)$, $x \in \mathbb{R}^n$, $\gamma \in \Gamma$. Hence, we infer that

$$\frac{1}{|\Omega_\Gamma|}\int_{\Omega_\Gamma} K_{f(H)}(x,x)\,\mathrm{d}x = \frac{1}{|F|}\int_F K_{f(H)}(x,x)\,\mathrm{d}x. \tag{5.10}$$

We also have $|\Omega_\Gamma| = |\Omega| - |\Omega \setminus \Omega_\Gamma|$, hence

$$1 - \varepsilon \leq \frac{|\Omega_\Gamma|}{|\Omega|} \leq 1. \tag{5.11}$$

Finally,

$$\begin{aligned}\left|\int_{\Omega\setminus\Omega_\Gamma} K_{f(H)}(x,x)\,\mathrm{d}x\right| &\leq \int_{(\partial\Omega)_\Gamma} |K_{f(H)}(x,x)|\,\mathrm{d}x \\ &= \frac{|(\partial\Omega)_\Gamma|}{|F|}\int_F |K_{f(H)}(x,x)|\,\mathrm{d}x \\ &\leq \frac{\varepsilon|\Omega|}{|F|}\int_F |K_{f(H)}(x,x)|\,\mathrm{d}x.\end{aligned} \tag{5.12}$$

Now, the equality (1.1) follows directly from (5.9)–(5.12). □

Remark 5.8. We have that $f(H) \in \mathcal{I}_1^\Gamma$ and then, by Lemma 5.7, we see that
$$\lim_{\Omega\to\mathbb{R}^n,\Omega\in\mathcal{F}} \frac{\mathrm{Tr}\,(P_\Omega f(H) P_\Omega)}{|\Omega|} = \frac{1}{|F|}\mathrm{Tr}\,_\Gamma f(H).$$

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