Structure of Solutions of Nonlinear Partial Differential Equations of Gérard-Tahara Type

By

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Abstract

Let us consider the following nonlinear singular partial differential equation

$$\left(t\frac{\partial}{\partial t}\right)^m u = F\left(t, x, \left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{j+|\alpha|\leq m, j<m}\right)$$

in the complex domain. When the equation is of Fuchsian type with respect to t, holomorphic and singular solutions were investigated quite well by Gérard-Tahara under some assumptions on characteristic exponents. In this paper, the same type of equations is solved in the general case without any assumption on characteristic exponents.

§1. Introduction

Let $\mathbb{C}$ be the complex plane or the set of all complex numbers, t be the variable in $\mathbb{C}_t$, and $x = (x_1, \ldots, x_n)$ be the variable in $\mathbb{C}_x^n = \mathbb{C}_{x_1} \times \cdots \times \mathbb{C}_{x_n}$. We use the notations: $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^* = \{1, 2, \ldots\}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $(\partial/\partial x)^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ we write $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$ holds for all $i = 1, \ldots, n$.

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Let $m \in \mathbb{N}^*$ be fixed and set

$$I_m = \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n \, ; \, j + |\alpha| \leq m \text{ and } j < m\},$$
$$N = \# I_m \text{ (the number of elements of } I_m),$$
$$Z = \{Z_{j,\alpha}\}_{(j,\alpha) \in I_m} \in \mathbb{C}^N.$$

Let $F(t, x, Z)$ be a function in the variables (t, x, Z) defined in a neighborhood Δ of the origin of $\mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_Z^N$. Let us consider the equation

$$\text{(E)} \qquad \left(t\frac{\partial}{\partial t}\right)^m u = F\left(t, x, \left\{\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{(j,\alpha) \in I_m}\right)$$

with the unknown function $u = u(t, x)$. Set $\Delta_0 = \Delta \cap \{t = 0, Z = 0\}$. Our main assumptions are:

    $A_1)$ $F(t, x, Z)$ is a holomorphic function on Δ;
    $A_2)$ $F(0, x, 0) \equiv 0$ on Δ_0.

In the study of singularities of solutions of nonlinear partial differential equations of the normal form, the investigation of the above type of equations has become very important (see Kobayashi [9], Lope-Tahara [12] and Tahara [15]). If we set $I_m(+) = \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n \, ; \, j + |\alpha| \leq m, j < m \text{ and } |\alpha| > 0\}$ the situation is divided into the following three cases:

Case 1 : $\dfrac{\partial F}{\partial Z_{j,\alpha}}(0, x, 0) \equiv 0$ on Δ_0 for all $(j, \alpha) \in I_m(+)$;

Case 2 : $\dfrac{\partial F}{\partial Z_{j,\alpha}}(0, 0, 0) \neq 0$ for some $(j, \alpha) \in I_m(+)$;

Case 3 : the other cases.

In Case 1, equation (E) is recently called a *Gérard-Tahara type* partial differential equation (or before it was called a *nonlinear Fuchsian type* partial differential equation) and it was studied by Gérard-Tahara [5], [6] under some assumptions on characteristic exponents. In Case 2, equation (E) is called a *spacially nondegenerate type* partial differential equation: Gérard-Tahara [7] discussed a particular class of Case 2 and proved the existence of holomorphic solutions and also singular solutions of (E). In Case 3, equation (E) is called a *nonlinear totally characteristic type* partial differential equation, and it was studied by Chen-Tahara [2],[3] and Tahara [16].

In this paper we will discuss Case 1 again and determine all the singular solutions of (E) under no assumptions on the characteristic exponents.

§2.   Main Result

We will consider only Case 1 and so we assume:

$$\text{A}_3)\quad \frac{\partial F}{\partial Z_{j,\alpha}}(0,x,0)\equiv 0 \ \text{ on } \Delta_0 \ \text{ for all } (j,\alpha)\in I_m(+).$$

Then, the indicial polynomial $C(\lambda,x)$ of (E) is defined by

$$C(\lambda,x)=\lambda^m-\sum_{j<m}\frac{\partial F}{\partial Z_{j,0}}(0,x,0)\lambda^j$$

and the characteristic exponents $\lambda_1(x),\ldots,\lambda_m(x)$ of (E) are defined by the roots of the equation $C(\lambda,x)=0$ in λ.

We denote by:

- $\mathcal{R}(\mathbb{C}\setminus\{0\})$ the universal covering space of $\mathbb{C}\setminus\{0\}$,
- S_θ the sector $\{t\in\mathcal{R}(\mathbb{C}\setminus\{0\})\,;\,|\arg t|<\theta\}$ in $\mathcal{R}(\mathbb{C}\setminus\{0\})$,
- $S(\varepsilon(s))$ the domain $\{t\in\mathcal{R}(\mathbb{C}\setminus\{0\}); 0<|t|<\varepsilon(\arg t)\}$, where $\varepsilon(s)$ is a positive-valued continuous function on $\mathbb{R}_s$,
- D_r the polydisk $\{x=(x_1,\ldots,x_n)\in\mathbb{C}^n\,;\,|x_i|<r \text{ for } i=1,\cdots,n\}$,
- $\mathbb{C}\{x\}$ the ring of convergent power series in x, or equivalently, the ring of germs of holomorphic functions at the origin of $\mathbb{C}_x^n$.

We will determine all the singular solutions of (E) belonging in the class $\widetilde{\mathcal{O}}_+$, which is defined by:

Definition 1.   We denote by $\widetilde{\mathcal{O}}_+$ the set of all $u(t,x)$ satisfying the following i) and ii):   i) $u(t,x)$ is a holomorphic function on $S(\varepsilon(s))\times D_r$ for some positive-valued continuous function $\varepsilon(s)$ on $\mathbb{R}_s$ and $r>0$;   and ii) there is an $a>0$ such that for any $0<r_1<r$ and $\theta>0$ we have

$$\max_{x\in D_{r_1}}|u(t,x)|=O(|t|^a)\quad(\text{as } t\longrightarrow 0 \text{ in } S_\theta).$$

Let us first recall the result in Gérard-Tahara [5]. Set

$$\mu=\#\{\,i\in\{1,2,\ldots,m\}\,;\,\mathrm{Re}\lambda_i(0)>0\}.$$

When $\mu=0$, this means that $\mathrm{Re}\,\lambda_i(0)\le 0$ holds for all $i=1,\ldots,m$. When $\mu\ge 1$, by a renumeration we may assume

$$(2.1)\qquad \begin{cases}\mathrm{Re}\lambda_i(0)>0 \ \text{ for } 1\le i\le\mu,\\ \mathrm{Re}\lambda_i(0)\le 0 \ \text{ for } \mu+1\le i\le m.\end{cases}$$

Theorem 1 (Gérard-Tahara (1993)). *Assume the conditions* A_1*),* A_2*) and* A_3*). Then we have the following results.*

(I)(Holomorphic solutions) *If* $\lambda_i(0) \notin \mathbb{N}^*$ *holds for all* $i = 1, \ldots, m$*, the equation* (E) *has a unique holomorphic solution* $u_0(t, x)$ *satisfying* $u_0(0, x) \equiv 0$.

(II)(Singular solutions) *Denote by* $\mathcal{S}_+$ *the set of all* $\widetilde{\mathcal{O}}_+$*-solutions of* (E)*. Then:*

(II-1) *When* $\mu = 0$*, we have*

$$\mathcal{S}_+ = \{u_0\}$$

where u_0 *is the unique holomorphic solution obtained in* (I)*.*

(II-2) *When* $\mu \geq 1$*, under* (2.1) *and the following additional conditions*:

    c-1) $\lambda_i(0) \neq \lambda_j(0)$ *for* $1 \leq i \neq j \leq \mu$,

    c-2) $C(1, 0) \neq 0$,

    c-3) $C(i + j_1\lambda_1(0) + \cdots + j_\mu\lambda_\mu(0), 0) \neq 0$ *for any* $(i, j) \in \mathbb{N} \times \mathbb{N}^\mu$
        satisfying $i + |j| \geq 2$,

we have

$$\mathcal{S}_+ = \{U(\varphi_1, \ldots, \varphi_\mu)\,;\, (\varphi_1, \ldots, \varphi_\mu) \in (\mathbb{C}\{x\})^\mu\},$$

where $U(\varphi_1, \ldots, \varphi_\mu)$ *is an* $\widetilde{\mathcal{O}}_+$*-solution of* (E) *depending on* $(\varphi_1, \ldots, \varphi_\mu) \in (\mathbb{C}\{x\})^\mu$ *and having an expansion of the following form*:

$$(2.2) \quad U(\varphi_1, \ldots, \varphi_\mu) = \sum_{1 \leq p \leq \mu} \varphi_p(x) t^{\lambda_p(x)} + \sum_{i \geq 1} u_i(x) t^i$$
$$+ \sum_{\substack{i + 2m|j| \geq k + 2m \\ |j| \geq 1,\, (i, |j|) \neq (0,1)}} \phi_{i,j,k}(x) t^{i + j_1\lambda_1(x) + \cdots + j_\mu\lambda_\mu(x)} (\log t)^k.$$

If one of the conditions c-1) $\sim$ c-3) is not satisfied, the expansion of the solution will be much more complicated as is seen in the case $m = 1$ by Yamazawa [17], and it seems difficult to describe the expansion in a concrete form. But we can still get the following theorem.

Theorem 2 (Main result). *Assume the conditions* A_1*),* A_2*),* A_3*) and* $\mu \geq 1$*. Denote by* $\mathcal{S}_+$ *the set of all* $\widetilde{\mathcal{O}}_+$*-solutions of* (E)*. Then we have*

$$(2.3) \qquad \mathcal{S}_+ = \{U(\varphi_1, \ldots, \varphi_\mu)\,;\, (\varphi_1, \ldots, \varphi_\mu) \in (\mathbb{C}\{x\})^\mu\},$$

where $U(\varphi_1, \ldots, \varphi_\mu)$ is an $\widetilde{\mathcal{O}}_+$-solution of (E) depending on $(\varphi_1, \ldots, \varphi_\mu) \in (\mathbb{C}\{x\})^\mu$ and having an expansion of the following form:

$$(2.4) \qquad U(\varphi_1, \ldots, \varphi_\mu) = \sum_{1 \le p \le \mu} \varphi_p(x) v_p(t, x) + \sum_{k \ge 1} \sum_{1 \le l \le m_k} \phi_{k,l}(x) w_{k,l}(t, x).$$

Here, $v_p(t, x)$, m_k and $w_{k,l}(t, x)$ are as follows: (1) $\{v_1(t, x), \ldots, v_\mu(t, x)\}$ is a fundamental system of solutions of $C(t\partial/\partial t, x)v = 0$ in $\widetilde{\mathcal{O}}_+$, (2) m_k ($k \ge 1$) are positive integers determined by the equation (E), and (3) $w_{k,l}(t, x)$ ($k \ge 1$ and $1 \le l \le m_k$) are functions also determined by the equation (E) satisfying the following property: there is a $\sigma > 0$ such that $w_{k,l}(t, x) = O(t^{\sigma k}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) holds for all (k, l). The coefficients $\varphi_p(x)$ and $\phi_{k,l}(x)$ are as follows: (4) $\varphi_p(x)$ ($1 \le p \le \mu$) are arbitrary holomorphic functions, and (5) $\phi_{k,l}(x)$ ($k \ge 1$ and $1 \le l \le m_k$) are holomorphic functions determined by $(\varphi_1(x), \ldots, \varphi_\mu(x))$.

In the above condition (3) the notation

$$w(t, x) = O(t^s, \widetilde{\mathcal{O}}_+) \quad (\text{as } t \longrightarrow 0)$$

means that the condition $t^{-s} w(t, x) \in \widetilde{\mathcal{O}}_+$ holds.

Remark 1. If the condition c-1) is satisfied, we can choose $\{v_1(t, x), \ldots, v_\mu(t, x)\}$ as $v_p(t, x) = t^{\lambda_p(x)}$ for $p = 1, \ldots, \mu$.

For $m = 1$, Theorem 1 was by Gérard-Tahara [4] and the general case as in Theorem 2 was by Yamazawa [17]. In this paper we will prove Theorem 2 in the general case for $m \ge 1$ by a method a little bit different from [17].

Note that our equation (E) is expanded into the form

$$(2.5) \quad C\Big(t\frac{\partial}{\partial t}, x\Big)u = b(x)t + \sum_{p+|q| \ge 2} b_{p,q}(x)t^p \prod_{(j,\alpha) \in I_m} \Big[\Big(t\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u\Big]^{q_{j,\alpha}},$$

where $b(x)$ and $b_{p,q}(x)$ ($p + |q| \ge 2$) are holomorphic functions in a common neighborhood Δ_0 of the origin of $\mathbb{C}_x^n$, and $p \in \mathbb{N}$, $q = \{q_{j,\alpha}\}_{(j,\alpha) \in I_m} \in \mathbb{N}^N$, and $|q| = \sum_{(j,\alpha) \in I_m} q_{j,\alpha}$.

The rest part of this paper is organized as follows. In the next section 3 we will define the system of functions $\{w_{k,l}(t, x); k \ge 1 \text{ and } 1 \le l \le m_k\}$ on which our formal solution (2.4) is based. The properties of these functions $w_{k,l}(t, x)$ will be investigated in Sections 4 and 5. After these preparations, we will construct a formal solution (2.4) in section 6, and prove the convergence of this formal solution in Section 7: up to this step we have a family of $\widetilde{\mathcal{O}}_+$-solutions $U(\varphi_1, \ldots, \varphi_\mu)$ of (E) and

$$\mathcal{S}_+ \supset \big\{U(\varphi_1, \ldots, \varphi_\mu) \,;\, (\varphi_1, \ldots, \varphi_\mu) \in (\mathbb{C}\{x\})^\mu\big\}.$$

In the last Section 8 we will prove the equality (2.3); that is, we will prove that every $\widetilde{\mathcal{O}}_+$-solution $u(t,x)$ of (E) is expressed in the form $u(t,x) = U(\varphi_1, \ldots, \varphi_\mu)$ for some $(\varphi_1, \ldots, \varphi_\mu) \in (\mathbb{C}\{x\})^\mu$.

§3.   Definition of the System $\{w_{k,l}\}$

Assume the conditions A$_1$), A$_2$), A$_3$) and $\mu \geq 1$. Without loss of generality we may assume:

$$(3.1) \qquad \mathrm{Re}\lambda_{0,1}(0) \leq \cdots \leq \mathrm{Re}\lambda_{0,\mu_0}(0) \leq 0 < \mathrm{Re}\lambda_{1,1}(0) = \cdots = \mathrm{Re}\lambda_{1,\mu_1}(0)$$
$$< \mathrm{Re}\lambda_{2,1}(0) = \cdots = \mathrm{Re}\lambda_{2,\mu_2}(0)$$
$$< \cdots < \mathrm{Re}\lambda_{d,1}(0) = \cdots = \mathrm{Re}\lambda_{d,\mu_d}(0),$$

where $\mu_0 = m - \mu$, $\mu_i \geq 1$ $(i = 1, \ldots, d)$ and $\mu_1 + \cdots + \mu_d = \mu$. Set

$$(3.2) \qquad a_i = \mathrm{Re}\lambda_{i,1}(0) = \cdots = \mathrm{Re}\lambda_{i,\mu_i}(0), \quad i = 1, \ldots, d.$$

We have $0 < a_1 < a_2 < \cdots < a_d < \infty$. We choose a constant σ such that $0 < \sigma < \min\{1, a_1, a_2 - a_1, \ldots, a_d - a_{d-1}\}$ and $\{\sigma k\,;\, k = 1, 2, \ldots\} \cap \{a_1, a_2, \ldots, a_d\} = \varnothing$. Then we have integers N_i $(i = 1, \ldots, d)$ such that $1 \leq N_1 < N_2 < \cdots < N_d < \infty$ and that

$$(3.3) \qquad \sigma N_i < a_i < \sigma(N_i + 1), \qquad i = 1, \ldots, d.$$

It is easy to see that for $i = 0, 1, \ldots, d$

$$(3.4) \qquad C_i(\lambda, x) = (\lambda - \lambda_{i,1}(x)) \cdots (\lambda - \lambda_{i,\mu_i}(x))$$

is a polynomial of degree μ_i in λ with coefficients being holomorphic in a neighborhood of $x = 0 \in \mathbb{C}_x^n$. We have a holomorphic decomposition

$$(3.5) \qquad C(\lambda, x) = C_0(\lambda, x)C_1(\lambda, x) \cdots C_d(\lambda, x).$$

Let $i \in \{1, \ldots, d\}$. Since the equation

$$(3.6)_i \qquad\qquad C_i\left(t\frac{\partial}{\partial t}, x\right)v = 0 \quad \text{in } \widetilde{\mathcal{O}}_+$$

is an ordinary differential equation in t of Euler type with a holomorphic parameter x and since $t = 0$ is a regular singular point, we have a fundamental

system $\{v_{i,1}(t,x), \ldots, v_{i,\mu_i}(t,x)\}$ of solutions of $(3.6)_i$ in the following sense:

v-1) $v_{i,j}(t,x)$ is an $\widetilde{\mathcal{O}}_+$-solution of $(3.6)_i$;

v-2) if $v(t,x)$ is an $\widetilde{\mathcal{O}}_+$-solution of $(3.6)_i$, $v(t,x)$ is expressed in the form

$$v(t,x) = \sum_{j=1}^{\mu_i} \varphi_j(x) v_{i,j}(t,x)$$

for some unique $(\varphi_1(x), \ldots, \varphi_{\mu_i}(x)) \in (\mathbb{C}\{x\})^{\mu_i}$.

Moreover, by the conditions (3.2) and (3.3) we have $v_{i,j}(t,x) = O(t^{\sigma N_i}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) for $j = 1, \ldots, \mu_i$. Since the equation is defined on $(\mathbb{C}_t \setminus \{0\}) \times D_R$ for some $R > 0$ we see that these $v_{i,j}(t,x)$'s are holomorphic on $\mathcal{R}(\mathbb{C}_t \setminus \{0\}) \times D_R$. For details, see Lemma 1 in Section 4. We will choose such a fundamental system $\{v_{i,1}(t,x), \ldots, v_{i,\mu_i}(t,x)\}$ and fix it from now.

For $i = 0, 1, \ldots, d$ we set

$$E_i(\tau_1, \ldots, \tau_{\mu_i}, x) = \frac{1}{\mu_i!} \sum_{\pi \in S_{\mu_i}} (\tau_1)^{-\lambda_{i,\pi(1)}(x)} \cdots (\tau_{\mu_i})^{-\lambda_{i,\pi(\mu_i)}(x)}$$

where S_{μ_i} is the group of permutations of $\{1, 2, \ldots, \mu_i\}$. By the theory of symmetric entire functions we see that $E_i(\tau_1, \ldots, \tau_{\mu_i}, x)$ is a holomorphic function on $(\mathcal{R}(\mathbb{C} \setminus \{0\}))^{\mu_i} \times D_R$. For a function $f(t,x)$ on $\mathcal{R}(\mathbb{C}_t \setminus \{0\}) \times D_R$ we define $\mathcal{R}_i[f](t,x)$ and $\mathcal{S}_i[f](t,x)$ by the following:

$$(3.7) \quad \mathcal{R}_i[f](t,x) = \int_0^t \frac{d\tau_{\mu_i}}{\tau_{\mu_i}} \int_0^{\tau_{\mu_i}} \frac{d\tau_{\mu_i-1}}{\tau_{\mu_i-1}} \times$$

$$\times \cdots \int_0^{\tau_2} \frac{d\tau_1}{\tau_1} \left[E_i\Big(\frac{\tau_1}{\tau_2}, \ldots, \frac{\tau_{\mu_i-1}}{\tau_{\mu_i}}, \frac{\tau_{\mu_i}}{t}, x\Big) f(\tau_1, x) \right],$$

$$(3.8) \quad \mathcal{S}_i[f](t,x) = (-1)^{\mu_i} \int_t^1 \frac{d\tau_{\mu_i}}{\tau_{\mu_i}} \int_{\tau_{\mu_i}}^1 \frac{d\tau_{\mu_i-1}}{\tau_{\mu_i-1}} \times$$

$$\times \cdots \int_{\tau_2}^1 \frac{d\tau_1}{\tau_1} \left[E_i\Big(\frac{\tau_1}{\tau_2}, \ldots, \frac{\tau_{\mu_i-1}}{\tau_{\mu_i}}, \frac{\tau_{\mu_i}}{t}, x\Big) f(\tau_1, x) \right].$$

The condition under which the integral (3.7) (and also (3.8)) makes sense will be investigated in Section 4. If these integrals are well defined, it is easy to see that $\mathcal{R}_i[f](t,x)$ (and also $\mathcal{S}_i[f](t,x)$) gives a solution of the equation

$$(3.9)_i \qquad\qquad C_i\Big(t\frac{\partial}{\partial t}, x\Big) w = f(t,x).$$

We define the integral $\mathcal{Q}_k[f](t, x)$ by the following:

$$(3.10) \qquad \mathcal{Q}_k[f](t,x) = \begin{cases} \mathcal{R}_0\mathcal{S}_1\mathcal{S}_2 \cdots \mathcal{S}_d[f](t,x), & \text{if } 1 \le k \le N_1; \\ \mathcal{R}_0\mathcal{R}_1\mathcal{S}_2 \cdots \mathcal{S}_d[f](t,x), & \text{if } N_1 + 1 \le k \le N_2; \\ \qquad \cdots\cdots\cdots \\ \qquad \cdots\cdots\cdots \\ \mathcal{R}_0 \cdots \mathcal{R}_{d-1}\mathcal{S}_d[f](t,x), & \text{if } N_{d-1} + 1 \le k \le N_d; \\ \mathcal{R}_0 \cdots \mathcal{R}_{d-1}\mathcal{R}_d[f](t,x), & \text{if } N_d + 1 \le k. \end{cases}$$

If every integral is well defined, by (3.5) we easily see that $\mathcal{Q}_k[f](t,x)$ gives a solution of

$$(3.11) \qquad\qquad C\left(t\frac{\partial}{\partial t}, x\right)w = f(t,x).$$

By using these integrals, we define:

Definition 2.   We define finite sets $\mathcal{F}_k$ $(k = 1, 2, \ldots)$, $\mathcal{G}_k$ $(k = 1, 2, \ldots)$ and $\mathcal{H}_k$ $(k = 1, 2, \ldots)$ of holomorphic functions on $\mathcal{R}(\mathbb{C}_t \setminus \{0\}) \times D_R$ inductively by the following procedure (1)$\sim$(3):

(1) We set $\mathcal{F}_1 = \{\mathcal{Q}_1[t]\}$. If $k \ge 2$ and if $\mathcal{H}_1, \ldots, \mathcal{H}_{k-1}$ are already defined, we set

$$\mathcal{F}_k = \bigcup_{\substack{2 \le p + |q| \le k \\ p \in \mathbb{N},\, q \in \mathbb{N}^N}} \bigcup_{\substack{k_1 + \cdots + k_{|q|} \\ = k - p \\ k_1, \ldots, k_{|q|} \in \mathbb{N}^*}} \left\{ \mathcal{Q}_k\big[t^p \psi_{k_1} \cdots \psi_{k_{|q|}}\big] \,;\, \psi_{k_\nu} \in \mathcal{H}_{k_\nu} \ (\nu = 1, \ldots, |q|) \right\},$$

where $N = \#I_m$. If $|q| = 0$ we have $p = k$; in this case $\mathcal{Q}_k[t^p \psi_{k_1} \cdots \psi_{k_{|q|}}]$ should be read as $\mathcal{Q}_k[t^k]$. Note that in the right hand side we have $1 \le k_\nu \le k - 1$ for $\nu = 1, \ldots, |q|$ and therefore $\mathcal{F}_k$ is well defined by the above formula.

(2) If $\mathcal{F}_k$ is already defined, we set

$$\mathcal{G}_k = \begin{cases} \mathcal{F}_k, & \text{if } k \neq N_1, \ldots, N_d; \\ \mathcal{F}_k \cup \{v_{i,1}, \ldots, v_{i,\mu_i}\}, & \text{if } k = N_i. \end{cases}$$

(3) If $\mathcal{G}_k$ is already defined, we set

$$\mathcal{H}_k = \bigcup_{(j,\alpha) \in I_m} \bigcup_{0 \le \beta_{j,\alpha} \le \alpha} \left\{ k^{|\beta_{j,\alpha}|} \left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^{\alpha - \beta_{j,\alpha}} W \,;\, W \in \mathcal{G}_k \right\}.$$

Definition 3.   We define the system of functions $\{w_{k,l}(t,x) \,;\, k \ge 1, 1 \le l \le m_k\}$ by the following: set $m_k = \#\mathcal{F}_k$ (the number of elements of the set

$\mathcal{F}_k$) and

$$\mathcal{F}_k = \left\{ w_{k,1}(t,x), \ldots, w_{k,m_k}(t,x) \right\} \qquad \text{for } k = 1, 2, \ldots.$$

It is clear that $m_1 = 1$ and $\mathcal{F}_1 = \{w_{1,1}(t,x)\}$ with $w_{1,1} = \mathcal{Q}_1[t]$.

Remark 2.   (1) The sets $\mathcal{H}_k$ ($k \geq 1$) are introduced only to avoid the confusion of subscripts in (1) of Definition 2.

(2) In the above finite sets $\mathcal{F}_k$, $\mathcal{G}_k$ and $\mathcal{H}_k$, every two elements with different labels (*i.e.*, $p, q, k_1, \ldots, k_{|q|}$ and so on) are regarded as different elements, even if they are the same function. Hence, if we set $M_k = \#\mathcal{G}_k$, $J_m = \{(j, \alpha, \beta) \in \mathbb{N} \times \mathbb{N}^n \times \mathbb{N}^n \, ; \, (j, \alpha) \in I_m, 0 \leq \beta \leq \alpha\}$ and

$$
\begin{aligned}
(3.12) \qquad \mathcal{Z}_k = \Big\{ &\big(p, q, \{k_i\}_{i=1}^{|q|}, \{l_i\}_{i=1}^{|q|}, \{(j_i, \alpha(i), \beta(i))\}_{i=1}^{|q|}\big) \in \mathbb{N} \times \mathbb{N}^N \times \\
&\times (\mathbb{N}^*)^{|q|} \times (\mathbb{N}^*)^{|q|} \times (\mathbb{N} \times \mathbb{N}^n \times \mathbb{N}^n)^{|q|} \, ; \, 2 \leq p + |q| \leq k, \\
&p + k_1 + \cdots + k_{|q|} = k, \, 1 \leq l_i \leq M_{k_i} \, (i = 1, \ldots, |q|), \\
&(j_i, \alpha(i), \beta(i)) \in J_m (i = 1, \ldots, |q|) \Big\}
\end{aligned}
$$

we have $m_k = \#\mathcal{Z}_k$.

The basic properties of these functions are as follows:

Proposition 1.   *Let $\sigma > 0$ be the one in (3.3). If $R > 0$ is sufficiently small, we have*:

(1) $w_{k,l}(t,x)$ ($k \geq 1, 1 \leq l \leq m_k$) *are holomorphic on* $\mathcal{R}(\mathbb{C}_t \setminus \{0\}) \times D_R$.

(2) *For any $\theta > 0$ there is a $\delta > 0$ such that*

$$(3.13) \qquad \left| \left(t \frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha w_{k,l}(t,x) \right| \leq \frac{1}{k^{m-j-|\alpha|}} |t|^{\sigma k} \quad \text{on } S_\theta(\delta) \times D_R$$

holds for any $k \geq 1$, $1 \leq l \leq m_k$ and $(j, \alpha) \in I_m$, where $S_\theta(\delta) = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}) \, ; \, |\arg t| < \theta \text{ and } 0 < |t| < \delta\}$.

In the next Section 4, we will present some preparatory lemmas which are needed in proving Proposition 1; then in Section 5 we will give a proof of this proposition.

§4.   Some Lemmas

We will present some preparatory lemmas for the proof of Proposition 1. In this section we use the following notation:

Definition 4. Let $R > 0$ and $s \in \mathbb{R}$.

(1) Let $\theta > 0$ and $\delta > 0$. We denote by $\mathcal{O}_s(S_\theta(\delta) \times D_R)$ the set of all holomorphic functions $f(t, x)$ on $S_\theta(\delta) \times D_R$ satisfying

(4.1)
$$|f(t, x)| \leq C|t|^s \quad \text{on} \quad S_\theta(\delta) \times D_R$$

for some $C > 0$.

(2) We denote by $\widetilde{\mathcal{O}}_s((\mathbb{C}_t \setminus \{0\}) \times D_R)$ the set of all functions $f(t, x)$ which satisfies the following i) and ii): i) $f(t, x)$ is a holomorphic function on $\mathcal{R}(\mathbb{C}_t \setminus \{0\}) \times D_R$, and ii) for any $\theta > 0$ and any $\delta > 0$ there is a $C > 0$ such that $|f(t, x)| \leq C|t|^s$ holds on $S_\theta(\delta) \times D_R$.

Let $p \in \mathbb{N}^*$, let $a_i(x)$ $(i = 1, \ldots, p)$ be bounded holomorphic functions on D_R, let
$$P(\xi, x) = \xi^p + a_1(x)\xi^{p-1} + \cdots + a_{p-1}(x)\xi + a_p(x),$$

denote by $\xi_1(x), \ldots, \xi_p(x)$ the roots of $P(\xi, x) = 0$ in ξ, and let us consider the following Euler type homogeneous equation:

(4.2)
$$P\Big(t\frac{\partial}{\partial t}, x\Big)v = 0$$

with the unknown function $v(t, x)$. By the theory of ordinary differential equations (or by Proposition 6.3 of Mandai [10]) we know:

Lemma 1. *Set*

$$v_k(t, x) = \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \frac{\xi^{p-k}}{P(\xi, x)} t^\xi \, d\xi, \quad k = 1, \ldots, p,$$

where Γ is a simple closed curve in the complex plane which encloses the set $\{\, \xi_i(x)\,;\, i = 1, \ldots, p \ \text{and} \ x \in D_R \,\}$. Then we have the following results.

(1) *$v_k(t, x) \in \widetilde{\mathcal{O}}_s((\mathbb{C}_t \setminus \{0\}) \times D_R)$ $(k = 1, \ldots, p)$ hold for any s satisfying $s < \inf\{\operatorname{Re}\xi_i(x)\,;\, i = 1, \ldots, p \ \text{and} \ x \in D_R\}$.*

(2) *If $\varphi_1(x), \ldots, \varphi_p(x) \in \mathbb{C}\{x\}$ satisfy*

$$\sum_{k=1}^p \varphi_k(x)v_k(t, x) = O(t^b, \widetilde{\mathcal{O}}_+) \ (\text{as} \ t \longrightarrow 0)$$

for some b with $b > \max\{\operatorname{Re}\xi_i(0)\,;\, i = 1, \ldots, p\}$, then we have $\varphi_k(x) = 0$ in $\mathbb{C}\{x\}$ for $k = 1, \ldots, p$.

(3) *If $\operatorname{Re}\xi_i(0) > 0$ holds for all $i = 1, \ldots, p$, the system $\{v_1(t, x), \ldots, v_p(t, x)\}$ is a fundamental system of $\widetilde{\mathcal{O}}_+$-solutions of (4.2).*

Next, let us consider

(4.3) $$P\Big(t\frac{\partial}{\partial t}, x\Big)u = f(t,x) \quad \text{in } \mathcal{O}_s(S_\theta(\delta) \times D_R).$$

Set

$$E(\tau_1, \dots, \tau_p, x) = \frac{1}{p!} \sum_{\pi \in S_p} (\tau_1)^{-\xi_{\pi(1)}(x)} \cdots (\tau_p)^{-\xi_{\pi(p)}(x)}$$

The following result is due to Baouendi-Goulaouic [1]:

Lemma 2.   *Let $s \in \mathbb{R}$ and $L > 0$. Assume that*

$$s - \operatorname{Re}\xi_i(x) \ge L \quad \text{on } D_R \quad \text{for } i = 1, \dots, p.$$

Then, we have:

(1) *For any $f(t,x) \in \mathcal{O}_s(S_\theta(\delta) \times D_R)$ the equation (4.3) has a unique solution $u(t,x) \in \mathcal{O}_s(S_\theta(\delta) \times D_R)$ and it is represented by the following integral formula*:

$$u(t,x) = \int_0^t \frac{d\tau_p}{\tau_p} \int_0^{\tau_p} \frac{d\tau_{p-1}}{\tau_{p-1}} \cdots \int_0^{\tau_2} \frac{d\tau_1}{\tau_1} \left[E\Big(\frac{\tau_1}{\tau_2}, \dots, \frac{\tau_{p-1}}{\tau_p}, \frac{\tau_p}{t}, x\Big) f(\tau_1, x) \right].$$

(2) *If $f(t,x)$ satisfies the estimate (4.1), the solution $u(t,x)$ satisfies*

(4.4) $$\left|\Big(t\frac{\partial}{\partial t}\Big)^j u(t,x)\right| \le \frac{A^j}{L^{p-j}} C|t|^s \quad \text{on } S_\theta(\delta) \times D_R \quad \text{for } j = 0, 1, \dots, p$$

for any constant $A > 0$ with

$$A \ge \max_{1 \le i \le p} \left[1 + \frac{\sup_{x \in D_R}|\xi_i(x)|}{L} \right].$$

Proof.   Since the equation (4.3) is written as

$$\Big(t\frac{\partial}{\partial t} - \xi_1(x)\Big) \cdots \Big(t\frac{\partial}{\partial t} - \xi_p(x)\Big)u = f(t,x),$$

(1) is obtained by integrating this directly. Let us show (2). Set

$$u_1(t,x) = \int_0^t \Big(\frac{\tau_1}{t}\Big)^{-\xi_1(x)} f(\tau_1,x) \frac{d\tau_1}{\tau_1}$$

and

$$u_i(t,x) = \int_0^t \Big(\frac{\tau_i}{t}\Big)^{-\xi_i(x)} u_{i-1}(\tau_i,x) \frac{d\tau_i}{\tau_i} \quad \text{for } i = 2, \dots, p.$$

Note that $(t\partial/\partial t - \xi_1(x))u_1 = f(t,x)$ and $(t\partial/\partial t - \xi_i(x))u_i = u_{i-1}(t,x)$ $(i = 2,\ldots,p)$ hold.

Let us first estimate $u_1(t,x)$ on $S_\theta(\delta) \times D_R$. By taking $\{\tau_1 \,;\, \tau_1 = rt, 0 \leq r \leq 1\}$ as the path of integral we have

$$(4.5) \qquad |u_1(t,x)| = \left| \int_0^1 r^{-\xi_1(x)-1} f(rt,x)\,dr \right|$$

$$\leq \int_0^1 r^{-\operatorname{Re}\xi_1(x)-1} C r^s |t|^s \,dr = C|t|^s \int_0^1 r^{s-\operatorname{Re}\xi_1(x)-1}\,dr$$

$$\leq C|t|^s \int_0^1 r^{L-1}\,dr = C|t|^s \frac{1}{L} \quad \text{on } S_\theta(\delta) \times D_R,$$

and by using $(t\partial/\partial t)u_1 = \xi_1(x)u_1 + f(t,x)$ we see

$$(4.6) \qquad \left| \left(t\frac{\partial}{\partial t} \right) u_1(t,x) \right| = |\xi_1(x)u_1(t,x) + f(t,x)|$$

$$\leq \left[\sup_{x \in D_R} |\xi_1(x)| \right] C|t|^s \frac{1}{L} + C|t|^s = \left[\frac{\sup_{x \in D_R} |\xi_1(x)|}{L} + 1 \right] C|t|^s$$

$$\leq A\, C|t|^s \quad \text{on } S_\theta(\delta) \times D_R.$$

Let us next estimate $u_2(t,x)$ on $S_\theta(\delta) \times D_R$. Since $(t\partial/\partial t - \xi_2(x))u_2 = u_1(t,x)$ holds and since $u_1(t,x)$ satisfies the estimate (4.5) on $S_\theta(\delta) \times D_R$, by the same argument as in the case u_1 we have

$$|u_2(t,x)| \leq \frac{1}{L^2} C|t|^s \text{ and } \left| \left(t\frac{\partial}{\partial t} \right) u_2(t,x) \right| \leq \frac{A}{L} C|t|^s \quad \text{on } S_\theta(\delta) \times D_R.$$

Moreover, by using $(t\partial/\partial t)^2 u_2 = \xi_2(x)(t\partial/\partial t)u_2 + (t\partial/\partial t)u_1$ and (4.6) we have

$$\left| \left(t\frac{\partial}{\partial t} \right)^2 u_2(t,x) \right| \leq |\xi_2(x)| \left| \left(t\frac{\partial}{\partial t} \right) u_2(t,x) \right| + \left| \left(t\frac{\partial}{\partial t} \right) u_1(t,x) \right|$$

$$\leq \left[\sup_{x \in D_R} |\xi_2(x)| \right] \frac{A}{L} C|t|^s + A\, C|t|^s = \left[\frac{\sup_{x \in D_R} |\xi_2(x)|}{L} + 1 \right] A\, C|t|^s$$

$$\leq A^2\, C|t|^s \quad \text{on } S_\theta(\delta) \times D_R.$$

Thus, by repeating the same argument as above we can obtain the estimate

$$\left| \left(t\frac{\partial}{\partial t} \right)^j u_p(t,x) \right| \leq \frac{A^j}{L^{p-j}} C|t|^s \quad \text{on } S_\theta(\delta) \times D_R \text{ for } j = 0,1,\ldots,p.$$

Since $u_p(t,x) = u(t,x)$ holds (by the uniqueness of the solution in (1)), this completes the proof of the part (2). $\qquad\square$

Let us also consider the Cauchy problem with initial data on $\{t = \delta\}$:

(4.7)
$$\begin{cases} P\Big(t\dfrac{\partial}{\partial t}, x\Big)u = f(t, x) & \text{in } \mathcal{O}_s(S_\theta(\delta) \times D_R), \\[2mm] \Big(t\dfrac{\partial}{\partial t}\Big)^j u\Big|_{t=\delta} = 0 & \text{for } j = 0, \ldots, p - 1. \end{cases}$$

Lemma 3.   *Let $s \in \mathbb{R}$ and $L > 0$. Assume that*

$$s - \operatorname{Re}\xi_i(x) \leq -L \quad on \ D_R \quad for \ i = 1, \ldots, p.$$

Then, we have:

(1) *For any $f(t, x) \in \mathcal{O}_s(S_\theta(\delta) \times D_R)$ the equation (4.7) has a unique solution $u(t, x) \in \mathcal{O}_s(S_\theta(\delta) \times D_R)$ and it is represented by the following integral formula:*

$$u(t, x) = (-1)^p \int_t^\delta \frac{d\tau_p}{\tau_p} \int_{\tau_p}^\delta \frac{d\tau_{p-1}}{\tau_{p-1}} \times$$

$$\times \cdots \int_{\tau_2}^\delta \frac{d\tau_1}{\tau_1} \left[E\Big(\frac{\tau_1}{\tau_2}, \ldots, \frac{\tau_{p-1}}{\tau_p}, \frac{\tau_p}{t}, x\Big) f(\tau_1, x) \right].$$

(2) *If $f(t, x)$ satisfies the estimate (4.1) and if $|\operatorname{Im}\xi_i(x)| \leq M$ $(i = 1, \ldots, p)$ hold on D_R, the solution $u(t, x)$ satisfies*

(4.8)
$$\left| \Big(t\frac{\partial}{\partial t}\Big)^j u(t, x) \right| \leq A^j \Big(e^{M\theta}\Big(\theta + \frac{1}{L}\Big) \Big)^{p-j} C|t|^s \quad on \ S_\theta(\delta) \times D_R$$

$$for \ j = 0, 1, \ldots, p$$

for any constant $A > 0$ with

$$A \geq \max_{1 \leq i \leq p} \left[1 + \Big(e^{M\theta}\Big(\theta + \frac{1}{L}\Big)\Big) \sup_{x \in D_R} |\xi_i(x)| \right].$$

Proof.   (1) is obtained by a direct integration. Let us prove (2). Set

$$u_1(t, x) = -\int_t^\delta \Big(\frac{\tau_1}{t}\Big)^{-\xi_1(x)} f(\tau_1, x)\, \frac{d\tau_1}{\tau_1}$$

and

$$u_i(t, x) = -\int_t^\delta \Big(\frac{\tau_i}{t}\Big)^{-\xi_i(x)} u_{i-1}(\tau_i, x)\, \frac{d\tau_i}{\tau_i} \quad for \ i = 2, \ldots, p.$$

Note that $(t\partial/\partial t - \xi_1(x))u_1 = f(t, x)$ and $(t\partial/\partial t - \xi_i(x))u_i = u_{i-1}(t, x)$ $(i = 2, \ldots, p)$ hold.

Let $(t, x) \in S_\theta(\delta) \times D_R$ and set $t = |t| \exp(\sqrt{-1}\psi)$. Then, by taking the path $\{\tau_1 \,;\, \tau_1 = |t| \exp(\sqrt{-1}\varphi),\ 0 \le \varphi \le \psi\} \cup \{\tau_1 \,;\, \tau_1 = r,\ |t| \le r \le \delta\}$ we have

$$|u_1(t, x)| \le \left| \int_0^\psi \left(\frac{|t|e^{\sqrt{-1}\varphi}}{|t|e^{\sqrt{-1}\psi}} \right)^{-\xi_1(x)} f(|t|e^{\sqrt{-1}\varphi}, x)\, d\varphi \right|$$

$$+ \left| \int_{|t|}^\delta \left(\frac{r}{|t|e^{\sqrt{-1}\psi}} \right)^{-\xi_1(x)} f(r, x)\, \frac{dr}{r} \right|$$

$$\le \int_0^\theta e^{M\theta} C|t|^s\, d\varphi + \int_{|t|}^\delta e^{M\theta} \left(\frac{r}{|t|} \right)^{-s-L} Cr^s\, \frac{dr}{r}$$

$$= e^{M\theta} C|t|^s\, \theta + e^{M\theta} C|t|^{s+L}\, \frac{|t|^{-L} - \delta^{-L}}{L}$$

$$\le C|t|^s \left(e^{M\theta} \left(\theta + \frac{1}{L} \right) \right),$$

and by using $(t\partial/\partial t)u_1 = \xi_1(x)u_1 + f(t, x)$ we see

$$\left| \left(t\frac{\partial}{\partial t} \right) u_1(t, x) \right| = |\xi_1(x)u_1(t, x) + f(t, x)|$$

$$\le \left[\sup_{x \in D_R} |\xi_1(x)| \right] \left(e^{M\theta} \left(\theta + \frac{1}{L} \right) \right) C|t|^s + C|t|^s$$

$$\le A\,C|t|^s \quad \text{on } S_\theta(\delta) \times D_R.$$

Thus, by the same argument as in the proof of Lemma 2 we obtain

$$\left| \left(t\frac{\partial}{\partial t} \right)^j u_p(t, x) \right| \le A^j \left(e^{M\theta} \left(\theta + \frac{1}{L} \right) \right)^{p-j} C|t|^s \quad \text{on } S_\theta(\delta) \times D_R$$

for $j = 0, 1, \ldots, p$.

This completes the proof of the part (2). $\square$

The following result is also very important in the asymptotic analysis (as $t \longrightarrow 0$):

Lemma 4.   *Let $f(t, x) \in \widetilde{\mathscr{O}}_+$ and let us consider*

(4.9) $$P\left(t\frac{\partial}{\partial t}, x \right) u = f(t, x) \quad in\ \widetilde{\mathscr{O}}_+ .$$

(1) *If $f(t, x) = O(t^s, \widetilde{\mathscr{O}}_+)$ (as $t \longrightarrow 0$) for some $s > 0$ and if $s > \max\{\operatorname{Re}\xi_i(0)\,;\, i = 1, \ldots, p\}$ holds, the equation (4.9) has a unique solution $u(t, x) \in \widetilde{\mathscr{O}}_+$ satisfying $u(t, x) = O(t^s, \widetilde{\mathscr{O}}_+)$ (as $t \longrightarrow 0$).*

(2) *If $u(t,x) \in \widetilde{\mathcal{O}}_+$ is a solution of (4.9) and if the following conditions*
   i) $u(t,x) = O(t^s, \widetilde{\mathcal{O}}_+)$ *(as $t \longrightarrow 0$),*
   ii) $f(t,x) = O(t^b, \widetilde{\mathcal{O}}_+)$ *(as $t \longrightarrow 0$), and*
   iii) $b > s > \max\{\operatorname{Re} \xi_i(0) \,;\, i = 1, \ldots, p\}$,
then we have $u(t,x) = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$).

(3) *If $f(t,x) = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) for some $s > 0$ and if $s < \min\{\operatorname{Re} \xi_i(0) \,;\, i = 1, \ldots, p\}$, every solution $u(t,x) \in \widetilde{\mathcal{O}}_+$ of the equation (4.9) satisfies $u(t,x) = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$).*

Proof. (1) is almost the same as Lemma 2. The proof of (2) is as follows. By (1) we have a solution $w(t,x) \in \widetilde{\mathcal{O}}_+$ of (4.9) satisfying $w(t,x) = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). Then, $P(t\partial/\partial t, x)(u - w) = 0$ and $(u - w)(t,x) = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). Thus, by the uniqueness part of (1) we obtain $(u - w)(t,x) = 0$ and so $u = w = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$).

Let us show (3). Assume that $f(t,x) = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) and that $0 < s < \operatorname{Re} \xi_i(0)$ holds for all $i = 1, \ldots, p$. By Lemma 3 we know that the equation (4.9) has a solution $w(t,x) \in \widetilde{\mathcal{O}}_+$ satisfying $w(t,x) = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). Moreover, since (4.9) is an ordinary differential equation with a holomorphic parameter x, we know by Lemma 1 that (4.9) with $f(t,x) = 0$ has a fundamental system $\{v_1(t,x), \ldots, v_p(t,x)\}$ of $\widetilde{\mathcal{O}}_+$-solutions such that $v_i(t,x) = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$).

Now let $u(t,x) \in \widetilde{\mathcal{O}}_+$ be any solution of (4.9). We have $P(t\partial/\partial t, x)(u - w) = 0$ and therefore $(u - w)(t,x)$ is expressed in the form $(u - w)(t,x) = \sum_{i=1}^{p} \phi_i(x) v_i(t,x)$ for some $\phi_i(x) \in \mathbb{C}\{x\}$ ($i = 1, \ldots, p$). This leads us to $u(t,x) = w(t,x) + \sum_{i=1}^{p} \phi_i(x) v_i(t,x) = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). $\qquad\square$

For a function $\phi(x)$ on D_r we define the norm $\|\phi\|_r$ by

$$\|\phi\|_r = \sup_{x \in D_r} |\phi(x)|.$$

In the proof of Proposition 1, we need also the following Nagumo's lemma (see Nagumo [11] or Lemma 5.1.3 of Hörmander [8]):

Lemma 5. *If $\phi(x)$ is a holomorphic function on D_R and if*

$$\|\phi\|_r \leq \frac{C}{(R - r)^s} \quad \text{for any } 0 < r < R$$

holds for some $C \geq 0$ and $s \geq 0$, then we have

$$\left\| \frac{\partial \phi}{\partial x_i} \right\|_r \leq \frac{(s+1)eC}{(R - r)^{s+1}} \quad \text{for any } 0 < r < R \quad \text{and } i = 1, \ldots, n.$$

§5.   Proof of Proposition 1

Let us return to the situation in Section 3. Let $0 < a_1 < a_2 < \cdots < a_d$ be the ones in (3.2), and let $\sigma > 0$ and $1 \leq N_1 < N_2 < \cdots < N_d$ be the ones in (3.3). Under these fixed constants, we choose now $a > 0$, $R > 0$, $L > 0$ and $c > 0$ so that the following properties h-1)~h-4) hold:

h-1) $\sigma < a < \min\{1, a_1\}$;

h-2) $aN_i < a_i < a(N_i + 1)$ for $i = 1, \dots, d$;

h-3) for $k = 1, 2, \dots, N_d$ we have

$$|ak - \operatorname{Re}\lambda_{i,p}(x)| \geq L \quad \text{on } D_R \text{ for all } (i, p);$$

h-4) for any $k \geq N_d + 1$ we have

$$ak - \operatorname{Re}\lambda_{i,p}(x) \geq ck \quad \text{on } D_R \text{ for all } (i, p).$$

Note that this is possible by choosing a sufficiently close to σ and by choosing R, L and c sufficiently small. Without loss of generality, we may assume that $0 < R \leq 1$ holds and so $1/(R - r) \geq 1$ holds for any $r \in (0, R)$.

Let $\mathcal{Q}_k$ be the operator defined by (3.10). By Lemmas 2 and 3 we have:

Lemma 6.   *For any $k = 1, 2, \dots$ we have the following properties $(1)_k$ and $(2)_k$, in which the constant $A_\theta > 0$ is independent of k, $f(t, x)$, r and j.*
   $(1)_k$ If $f(t, x) \in \widetilde{\mathcal{O}}_{ak}((\mathbb{C}_t \setminus \{0\}) \times D_R)$, we have $\mathcal{Q}_k[f](t, x) \in \widetilde{\mathcal{O}}_{ak}((\mathbb{C}_t \setminus \{0\}) \times D_R)$ and $C(t\partial/\partial t, x)\mathcal{Q}_k[f] = f$ on $\mathcal{R}(\mathbb{C}_t \setminus \{0\}) \times D_R$.
   $(2)_k$ Moreover, if $f(t, x)$ satisfies

$$(5.1) \qquad\qquad \|f(t)\|_r \leq C|t|^{ak} \quad \text{on } S_\theta(1)$$

for some $0 < r < R$ and $\theta > 0$, we have the estimate

$$(5.2) \quad \left\| \left(t\frac{\partial}{\partial t}\right)^j \mathcal{Q}_k[f](t) \right\|_r \leq \frac{A_\theta}{k^{m-j}}\, C|t|^{ak} \quad \text{on } S_\theta(1) \text{ for } j = 0, 1, \dots, m-1$$

Proof.   Since h-3) and h-4) hold, by applying Lemmas 2 and 3 $(d+1)$-times we easily see the property $(1)_k$. Moreover, if $k = 1, \dots, N_d$ and if $f(t, x)$ satisfies the estimate (5.1) we have

$$\left\| \left(t\frac{\partial}{\partial t}\right)^j \mathcal{Q}_k[f](t) \right\|_r \leq A_{k,j} C|t|^{ak} \quad \text{on } S_\theta(1) \text{ for } j = 0, 1, \dots, m-1$$

where the constants $A_{k,j} > 0$ ($1 \leq k \leq N_d$ and $0 \leq j \leq m-1$) depend on θ in general but are independent of $f(t, x)$ and r. If $k \geq N_d + 1$, by (3.10) we have

$Q_k[f] = \mathcal{R}_0 \mathcal{R}_1 \cdots \mathcal{R}_d[f]$; therefore by h-4) and Lemma 2 we obtain

$$\left\| \left(t \frac{\partial}{\partial t} \right)^j Q_k[f](t) \right\|_r \leq \frac{A_j}{k^{m-j}} C |t|^{ak} \quad \text{on } S_\theta(1) \text{ for } j = 0, 1, \ldots, m-1$$

where the constants $A_j > 0$ $(0 \leq j \leq m-1)$ are independent of θ, k, $f(t,x)$ and r.

Thus, by setting

$$A_\theta = \max \left[\max_{\substack{1 \leq k \leq N_d \\ 0 \leq j \leq m-1}} A_{k,j} k^{m-j}, \ \max_{0 \leq j \leq m-1} A_j \right]$$

we obtain the estimate (5.2).   □

Now, let us give a proof of Proposition 1.   Recall that $\{v_{i,1}(t,x), \ldots, v_{i,\mu_i}(t,x)\}$ is a fundamental system of solutions of $(3.6)_i$. Since $aN_i < \text{Re }\lambda_{i,p}(x)$ holds on D_R for $p = 1, \ldots, \mu_i$, by taking a smaller $R > 0$ if necessary we see the following i) and ii): i) $v_{i,h}(t,x) \in \widetilde{\mathcal{O}}_{aN_i}((\mathbb{C}_t \setminus \{0\}) \times D_R)$ $(h = 1, \ldots, \mu_i)$, and ii)   for any $\theta > 0$ there is a $B^*_{i,\theta} > 0$ such that

$$(5.3) \qquad \left\| \left(t \frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha v_{i,h}(t) \right\|_R \leq B^*_{i,\theta} |t|^{aN_i} \quad \text{on } S_\theta(1)$$

for any $(j, \alpha) \in I_m$ and $h = 1, \ldots, \mu_i$.

Therefore, by using the condition $0 < R \leq 1$ and by setting $B_{i,\theta} = B^*_{i,\theta} N_i{}^m$ we obtain the following estimates:

$$(5.4) \quad \left\| \left(t \frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha v_{i,h}(t) \right\|_r \leq \frac{1}{N_i{}^{m-j-|\alpha|}} \frac{B_{i,\theta}}{(R-r)^{m(N_i-1)}} |t|^{aN_i} \quad \text{on } S_\theta(1)$$

for any $0 < r < R$, $(j, \alpha) \in I_m$ and $h = 1, \ldots, \mu_i$.

Recall also that $\mathcal{F}_1 = \{w_{1,1}(t,x)\}$ with $w_{1,1} = Q_1[t]$, and that $0 < a < 1$ holds. Therefore, we may assume that $w_{1,1} \in \widetilde{\mathcal{O}}_a((\mathbb{C}_t \setminus \{0\}) \times D_R)$ and that for any $\theta > 0$ there is a $K_\theta > 0$ which satisfies

$$(5.5) \qquad \left\| \left(t \frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha w_{1,1}(t) \right\|_R \leq K_\theta |t|^a \quad \text{on } S_\theta(1) \text{ for any } (j, \alpha) \in I_m.$$

By induction on k we have:

Lemma 7.   *For any* $k = 1, 2, \ldots$ *we have the following properties* $(1)_k$ *and* $(2)_k$, *in which the constant* $C_\theta > 0$ *is independent of* (j, α), k *and* l.

$(1)_k$ $w_{k,l}(t,x) \in \widetilde{\mathscr{O}}_{ak}((\mathbb{C}_t \setminus \{0\}) \times D_r)$ for any $0 < r < R$ and $l = 1, \dots, m_k$.

$(2)_k$ We have the following estimates for any $\theta > 0$:

$$(5.6) \quad \left\| \left(t\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha w_{k,l}(t) \right\|_r \leq \frac{1}{k^{m-j-|\alpha|}} \frac{C_\theta^{2k-1}}{(R-r)^{m(k-1)}} |t|^{ak} \quad on\ S_\theta(1)$$

for any $0 < r < R$, $(j,\alpha) \in I_m$ and $l = 1, \dots, m_k$.

Proof. We set

$$(5.7) \quad C_\theta = \max \left[K_\theta, \left(B_{1,\theta} \right)^{1/(2N_1-1)}, \dots, \left(B_{d,\theta} \right)^{1/(2N_d-1)}, 1, (me)^m A_\theta \right],$$

where K_θ, $B_{i,\theta}$ and A_θ are the constants in (5.5), (5.4) and Lemma 6, respectively. Since $C_\theta \geq K_\theta$ holds, the case $k = 1$ is clear from (5.5). Let us prove the case $k \geq 2$ by induction on k.

Suppose that $k \geq 2$ and that $(1)_i$ and $(2)_i$ are already proved for $i = 1, \dots, k-1$. Then, by the definition of $\mathcal{F}_k = \{w_{k,1}(t,x), \dots, w_{k,m_k}(t,x)\}$ we see that $w_{k,l}(t,x)$ is expressed in the form

$$(5.8) \qquad\qquad w_{k,l}(t,x) = \mathcal{Q}_k \left[t^p \psi_{k_1} \cdots \psi_{k_{|q|}} \right]$$

where $\psi_{k_\nu} \in \mathcal{H}_{k_\nu}$ $(\nu = 1, \dots, |q|)$, $2 \leq p + |q| \leq k$ and $p + k_1 + \cdots + k_{|q|} = k$ (by Definitions 2 and 3). Also we know that each $\psi_{k_\nu}(t,x)$ is expressed as

$$\psi_{k_\nu}(t,x) = k_\nu^{|\beta_{j,\alpha}|} \left(t\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^{\alpha - \beta_{j,\alpha}} W(t,x)$$

for some $(j,\alpha) \in I_m$, $0 \leq \beta_{j,\alpha} \leq \alpha$ and $W(t,x) \in \mathcal{G}_{k_\nu}$ (by Definition 2). Since $1 \leq k_\nu \leq k-1$ holds, in the case $W(t,x) \in \mathcal{F}_{k_\nu}$ by the induction hypothesis we have

$$(5.9) \qquad \|\psi_{k_\nu}(t)\|_r = k_\nu^{|\beta_{j,\alpha}|} \left\| \left(t\frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^{\alpha - \beta_{j,\alpha}} W(t) \right\|_r$$

$$\leq k_\nu^{|\beta_{j,\alpha}|} \frac{1}{k_\nu^{m-j-|\alpha-\beta_{j,\alpha}|}} \frac{C_\theta^{2k_\nu-1}}{(R-r)^{m(k_\nu-1)}} |t|^{ak_\nu}$$

$$\leq \frac{C_\theta^{2k_\nu-1}}{(R-r)^{m(k_\nu-1)}} |t|^{ak_\nu} \quad on\ S_\theta(1)\ for\ any\ 0 < r < R.$$

In the case $k_\nu = N_i$ and $W(t,x) = v_{i,h}(t,x)$, by using (5.4) and the inequality $B_{i,\theta} \leq C_\theta^{2N_i-1}$ we can obtain the same estimate as (5.9).

Therefore, by the conditions $k_1 + \cdots + k_{|q|} = k - p$, $p + |q| \geq 2$, $0 < a < 1$, $C_\theta \geq 1$ and $1/(R - r) \geq 1$ we have

$$\left\| t^p \psi_{k_1}(t) \cdots \psi_{k_{|q|}}(t) \right\|_r \leq |t|^p \frac{C_\theta^{2k_1 - 1}}{(R - r)^{m(k_1 - 1)}} |t|^{ak_1} \cdots \frac{C_\theta^{2k_{|q|} - 1}}{(R - r)^{m(k_{|q|} - 1)}} |t|^{ak_{|q|}}$$

$$= \frac{C_\theta^{2k - 2p - |q|}}{(R - r)^{m(k - p - |q|)}} |t|^{p + a(k - p)}$$

$$\leq \frac{C_\theta^{2k - 2}}{(R - r)^{m(k - 2)}} |t|^{ak} \quad \text{on } S_\theta(1).$$

Since $\theta > 0$ is arbitrary, this implies that $t^p \psi_{k_1} \cdots \psi_{k_{|q|}} \in \widetilde{\mathcal{O}}_{ak}((\mathbb{C}_t \setminus \{0\}) \times D_r)$ for any $0 < r < R$. Thus, by $(1)_k$ of Lemma 6 we see that $w_{k,l}(t, x) \in \widetilde{\mathcal{O}}_{ak}((\mathbb{C}_t \setminus \{0\}) \times D_r)$ holds for any $0 < r < R$.

Moreover, by applying $(2)_k$ of Lemma 6 to (5.8) we have

$$\left\| \left(t \frac{\partial}{\partial t} \right)^j w_{k,l}(t) \right\|_r \leq \frac{A_\theta}{k^{m-j}} \frac{C_\theta^{2k-2}}{(R - r)^{m(k-2)}} |t|^{ak} \quad \text{on } S_\theta(1)$$

for any $0 < r < R$ and $j = 0, 1, \ldots, m - 1$

and by using Lemma 5 and $A_\theta (me)^{|\alpha|} \leq A_\theta (me)^m \leq C_\theta$ we obtain

$$\left\| \left(t \frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha w_{k,l}(t) \right\|_r$$

$$\leq \frac{A_\theta}{k^{m-j}} \frac{(m(k-2) + 1) \cdots (m(k-2) + |\alpha|) e^{|\alpha|} C_\theta^{2k-2}}{(R - r)^{m(k-2) + |\alpha|}} |t|^{ak}$$

$$\leq \frac{A_\theta}{k^{m-j-|\alpha|}} \frac{(me)^{|\alpha|} C_\theta^{2k-2}}{(R - r)^{m(k-1)}} |t|^{ak}$$

$$\leq \frac{1}{k^{m-j-|\alpha|}} \frac{C_\theta^{2k-1}}{(R - r)^{m(k-1)}} |t|^{ak} \quad \text{on } S_\theta(1)$$

for any $0 < r < R$ and $(j, \alpha) \in I_m$. Thus, we have proved $(2)_k$. $\qquad \square$

Completion of the proof of Proposition 1. We set $r = R/2$. Then we see that $w_{k,l}(t, x) \in \widetilde{\mathcal{O}}_{ak}((\mathbb{C}_t \setminus \{0\}) \times D_{R/2})$ for all (k, l), and that

$$\left\| \left(t \frac{\partial}{\partial t} \right)^j \left(\frac{\partial}{\partial x} \right)^\alpha w_{k,l}(t) \right\|_{R/2} \leq \frac{1}{k^{m-j-|\alpha|}} \frac{C_\theta^{2k-1}}{(R/2)^{m(k-1)}} |t|^{ak}$$

$$\leq \frac{1}{k^{m-j-|\alpha|}} \left(\frac{C_\theta^2}{(R/2)^m} |t|^{a-\sigma} \right)^k |t|^{\sigma k} \quad \text{on } S_\theta(1)$$

holds for any $(j, \alpha) \in I_m$ and (k, l). Thus, if we take $\delta > 0$ so that

$$\frac{C_\theta{}^2}{(R/2)^m} \delta^{a-\sigma} \leq 1$$

we obtain the estimate (3.13) with R replaced by $R/2$. This completes the proof of Proposition 1. □

By the proof of Proposition 1, we have:

Corollary to Proposition 1. *We have $v_{i,h}(t, x) = O(t^{\sigma N_i}, \widetilde{\mathscr{O}}_+)$ (as $t \longrightarrow 0$) for all (i, h), and $w_{k,l}(t, x) = O(t^{\sigma k}, \widetilde{\mathscr{O}}_+)$ (as $t \longrightarrow 0$) for all (k, l).*

§6. Construction of a Formal Solution

Let us construct a formal solution $u(t, x)$ of the equation (E) in the form

$$(6.1) \qquad u(t, x) = \sum_{k \geq 1} u_k(t, x)$$

with

$$(6.2) \quad u_k(t, x) = \begin{cases} \sum_{l=1}^{m_k} \phi_{k,l}(x) w_{k,l}(t, x), & \text{if } k \neq N_1, \ldots, N_d, \\ \sum_{j=1}^{\mu_i} \varphi_{i,j}(x) v_{i,j}(t, x) + \sum_{l=1}^{m_k} \phi_{k,l}(x) w_{k,l}(t, x), & \text{if } k = N_i, \end{cases}$$

where $\varphi_{i,j}(x)$ and $\phi_{k,l}(x)$ are suitable holomorphic functions in a common neighborhood of $x = 0$. If such a formal solution is constructed, by Corollary to Proposition 1 we have

$$(6.3) \qquad u_k(t, x) = O(t^{\sigma k}, \widetilde{\mathscr{O}}_+) \quad (\text{as } t \longrightarrow 0) \quad \text{for all } k \geq 1.$$

Let us decompose our equation (E) under the condition that the solution $u(t, x)$ is expressed in the form (6.1) with the property (6.3). By substituting (6.1) into (2.5) we have formally

$$(6.4) \quad \sum_{k \geq 1} C\left(t \frac{\partial}{\partial t}, x\right) u_k = b(x) t + \sum_{p+|q| \geq 2} b_{p,q}(x) t^p \prod_{(j,\alpha) \in I_m} \left[\sum_{k \geq 1} \mathcal{D}_{j,\alpha} u_k \right]^{q_{j,\alpha}},$$

in which we used the following notation

$$\mathcal{D}_{j,\alpha} u_k = \left(t \frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u_k.$$

Therefore by comparing the asymptotic behavior (as $t \longrightarrow 0$) of each term in the both sides of (6.4) we have:

$$C\Big(t\frac{\partial}{\partial t}, x\Big)u_1 = b(x)t + O(t^{2\sigma}, \widetilde{\mathcal{O}}_+) \quad (\text{as } t \longrightarrow 0)$$

and for $k \geq 2$

$$\sum_{1 \leq i \leq k} C\Big(t\frac{\partial}{\partial t}, x\Big)u_i$$

$$= b(x)t + \sum_{2 \leq p + |q| \leq k} b_{p,q}(x)t^p \left[\sum_{p + |k(q)| \leq k} \prod_{(j,\alpha) \in I_m} \prod_{1 \leq i \leq q_{j,\alpha}} \mathcal{D}_{j,\alpha} u_{k_{j,\alpha}(i)} \right]$$

$$+ O(t^{\sigma(k+1)}, \widetilde{\mathcal{O}}_+) \quad (\text{as } t \longrightarrow 0),$$

where

$$k(q) = \{(k_{j,\alpha}(i)) \, ; (j,\alpha) \in I_m, 1 \leq i \leq q_{j,\alpha}\}, \text{ and}$$

$$|k(q)| = \sum_{(j,\alpha) \in I_m} (k_{j,\alpha}(1) + \cdots + k_{j,\alpha}(q_{j,\alpha})).$$

Thus, from the view point of asymtotic analysis (as $t \longrightarrow 0$) the following decomposition will be reasonable:

(6.5) $$C\Big(t\frac{\partial}{\partial t}, x\Big)u_1 = b(x)t$$

and for $k \geq 2$
$(6.6)_k$

$$C\Big(t\frac{\partial}{\partial t}, x\Big)u_k = \sum_{2 \leq p + |q| \leq k} b_{p,q}(x)t^p \left[\sum_{p + |k(q)| = k} \prod_{(j,\alpha) \in I_m} \prod_{1 \leq i \leq q_{j,\alpha}} \mathcal{D}_{j,\alpha} u_{k_{j,\alpha}(i)} \right].$$

It should be remarked that in the right hand side of $(6.6)_k$ only the terms $u_1, \ldots, u_{k-1}$ and their derivatives appear and that (6.5) and $(6.6)_k$ $(k = 2, 3, \ldots)$ give a recurrent family of equations.

Now, we write $W_{k,l}(t,x) = w_{k,l}(t,x)$ for all (k,l), and in the case $k = N_i$ we set $W_{k,m_k+j}(t,x) = v_{i,j}(t,x)$ for $j = 1, \ldots, \mu_i$. Also, we set $M_k = m_k$ if $k \neq N_1, \ldots, N_d$, and set $M_k = m_k + \mu_i$ if $k = N_i$. Then we have

(6.7) $\quad \mathcal{G}_k = \{W_{k,l} \, ; 1 \leq l \leq M_k\}$

$$= \begin{cases} \{w_{k,l} \, ; 1 \leq l \leq m_k\}, & \text{if } k \neq N_1, \ldots, N_d, \\ \{w_{k,l} \, ; 1 \leq l \leq m_k\} \cup \{v_{i,1}(t,x), \ldots, v_{i,\mu_i}(t,x)\}, & \text{if } k = N_i, \end{cases}$$

and $u_k(t,x)$ in (6.2) is expressed in the form

$$(6.8)_k \qquad u_k(t,x) = \sum_{l=1}^{M_k} \phi_{k,l}(x) W_{k,l}(t,x)$$

where in the case $k = N_i$ we set $\phi_{k,m_k+j}(x) = \varphi_{i,j}(t,x)$ for $j = 1, \ldots, \mu_i$.

Recall that $v_{i,j}(t,x)$ is a homogeneous solution of $C(t\partial/\partial t, x)v = 0$ and therefore we have

$$C\Big(t\frac{\partial}{\partial t}, x\Big) \sum_{l=1}^{M_k} \phi_{k,l}(x) W_{k,l}(t,x) = \sum_{l=1}^{m_k} \phi_{k,l}(x) \Big[C\Big(t\frac{\partial}{\partial t}, x\Big) w_{k,l}(t,x) \Big]$$

for all $k \geq 1$. Since $m_1 = 1$, by substituting $(6.8)_1$ into (6.5) we have:

$$\phi_{1,1}(x) \Big[C\Big(t\frac{\partial}{\partial t}, x\Big) w_{1,1}(t,x) \Big] = b(x)t.$$

Since $w_{1,1}(t,x) = \mathcal{Q}_1[t]$ we have $C(t\partial/\partial t, x)w_{1,1} = t$ and so $\phi_{1,1}(x) = b(x)$. Thus, we obtain a solution $u_1(t,x)$ of the equation (6.5).

Let us suppose $k \geq 2$ and that a solution $u_i(t,x)$ of the equation $(6.6)_i$ is already obtained in the form $(6.8)_i$ for $i = 1, \ldots, k-1$. Under these assumptions, let us solve the equation $(6.6)_k$ and find a solution $u_k(t,x)$ in the form $(6.8)_k$. By substituting $(6.8)_1, \ldots, (6.8)_k$ into the equation $(6.6)_k$ we have

$$\sum_{l=1}^{m_k} \phi_{k,l}(x) \Big[C\Big(t\frac{\partial}{\partial t}, x\Big) w_{k,l}(t,x) \Big]$$

$$= \sum_{2 \leq p+|q| \leq k} b_{p,q}(x) t^p \Bigg[\sum_{p+|k(q)|=k} \sum_{l(q) \in L(q,k(q))} \prod_{(j,\alpha) \in I_m} \prod_{1 \leq i \leq q_{j,\alpha}} \Big\{ 1 \times$$

$$\times \mathcal{D}_{j,\alpha}\Big(\phi_{k_{j,\alpha}(i), l_{j,\alpha}(i)}(x) W_{k_{j,\alpha}(i), l_{j,\alpha}(i)}(t,x) \Big) \Big\} \Bigg],$$

where

$$l(q) = \{ (l_{j,\alpha}(i)) \, ; \, (j,\alpha) \in I_m \text{ and } 1 \leq i \leq q_{j,\alpha} \}, \text{ and}$$
$$L(q,k(q)) = \{ l(q) \, ; \, 1 \leq l_{j,\alpha}(i) \leq M_{k_{j,\alpha}(i)}$$
$$\text{holds for all } (j,\alpha) \in I_m \text{ and } 1 \leq i \leq q_{j,\alpha} \}.$$

Hence, if we set

$$\mathcal{J}_k(p,q) = \{ (k(q), l(q)) \, ; \, p + |k(q)| = k \text{ and } l(q) \in L(q,k(q)) \},$$
$$\beta(q) = \{ (\beta_{j,\alpha}(i)) \, ; \, \beta_{j,\alpha}(i) \in \mathbb{N}^n, (j,\alpha) \in I_m \text{ and } 1 \leq i \leq q_{j,\alpha} \},$$
$$\Gamma(q) = \{ \beta(q) \, ; \, \beta_{j,\alpha}(i) \leq \alpha \text{ holds for all } (j,\alpha) \in I_m \text{ and } 1 \leq i \leq q_{j,\alpha} \},$$

equation $(6.6)_k$ is expressed in the form

(6.9)

$$\sum_{l=1}^{m_k} \phi_{k,l}(x) \Big[C\Big(t\frac{\partial}{\partial t}, x\Big) w_{k,l}(t,x) \Big]$$

$$= \sum_{2 \leq p+|q| \leq k} b_{p,q}(x) \sum_{(k(q),l(q)) \in \mathcal{J}_k(p,q)} \sum_{\beta(q) \in \Gamma(q)} \psi_{k(q),l(q),\beta(q)}(x) \times$$

$$\times \left[t^p \prod_{(j,\alpha) \in I_m} \prod_{1 \leq i \leq q_{j,\alpha}} k_{j,\alpha}(i)^{|\beta_{j,\alpha}(i)|} \mathcal{D}_{j,\alpha-\beta_{j,\alpha}(i)}\Big(W_{k_{j,\alpha}(i),l_{j,\alpha}(i)}(t,x) \Big) \right]$$

and $\psi_{k(q),l(q),\beta(q)}(x)$'s are known functions; precisely they are given by

(6.10)

$$\psi_{k(q),l(q),\beta(q)}(x)$$

$$= \prod_{(j,\alpha) \in I_m} \prod_{1 \leq i \leq q_{j,\alpha}} \binom{\alpha}{\beta_{j,\alpha}(i)} \frac{1}{k_{j,\alpha}(i)^{|\beta_{j,\alpha}(i)|}} \Big(\frac{\partial}{\partial x}\Big)^{\beta_{j,\alpha}(i)} \phi_{k_{j,\alpha}(i),l_{j,\alpha}(i)}(x).$$

We remark again that in the right hand side of (6.9) (and (6.10)) the inequality $1 \leq k_{j,\alpha}(i) \leq k-1$ holds for all (j,α,i) and therefore the right hand side of (6.9) (and (6.10)) can be considered as a known part by the induction hypothesis.

Here we note the following lemma:

Lemma 8. *Let $k \geq 2$ and set $\mathcal{A}_k = \{(p,q,k(q),l(q),\beta(q)); 2 \leq p + |q| \leq k, (k(q),l(q)) \in \mathcal{J}_k(p,q)$ and $\beta(q) \in \Gamma(q)\}$. Then by a suitable injection $\pi_k : \mathcal{A}_k \longrightarrow \{1,2,\ldots,m_k\}$ we have the following equality:*

$$(6.11) \quad \mathcal{Q}_k \left[t^p \prod_{(j,\alpha) \in I_m} \prod_{1 \leq i \leq q_{j,\alpha}} k_{j,\alpha}(i)^{|\beta_{j,\alpha}(i)|} \mathcal{D}_{j,\alpha-\beta_{j,\alpha}(i)}\Big(W_{k_{j,\alpha}(i),l_{j,\alpha}(i)}(t,x) \Big) \right]$$

$$= w_{k,l}(t,x)$$

under the correspondence $\pi_k(p,q,k(q),l(q),\beta(q)) = l$.

Proof. Let $\mathcal{Z}_k$ be the set in (3.12). For $q = \{q_{j,\alpha}\}_{(j,\alpha) \in I_m} \in \mathbb{N}^N$ we set $S(q) = \{(j,\alpha,i)\,; (j,\alpha) \in I_m$ and $1 \leq i \leq q_{j,\alpha}\}$. Then we have $|q| = \#S(q)$. Therefore, by

$$\mathcal{A}_k \ni (p,q,k(q),l(q),\beta(q)) \longmapsto \big(p,q,k(q),l(q),\{(j,\alpha,\beta_{j,\alpha}(i)\}_{(j,\alpha,i) \in S(q)}\big) \in \mathcal{Z}_k$$

we have a natural injection from $\mathcal{A}_k$ into $\mathcal{Z}_k$. Since $m_k = \#\mathcal{Z}_k$ (see Remark 2), by the definition of $\mathcal{F}_k$ we easily obtain this lemma. $\square$

Thus, to solve the equation (6.9) it is sufficient to determine the coefficients $\phi_{k,l}(x)$ by

$$(6.12) \qquad \phi_{k,l}(x) = \begin{cases} 0, & \text{if } l \notin \pi_k(\mathcal{A}_k), \\ b_{p,q}(x)\psi_{k(q),l(q),\beta(q)}(x), & \text{if } l \in \pi_k(\mathcal{A}_k) \end{cases}$$

under the correspondence $l = \pi_k(p, q, k(q), l(q), \beta(q))$.

It is clear that in the case $k = N_i$ the coefficients $\varphi_{i,1}(x), \ldots, \varphi_{i,\mu_i}(x)$ of

$$u_k(t,x) = \sum_{j=1}^{\mu_i} \varphi_{i,j}(x) v_{i,j}(t,x) + \sum_{l=1}^{m_k} \phi_{k,l}(x) w_{k,l}(t,x)$$

can be chosen arbitrarily.

Thus we have proved.

Proposition 2.   *We can construct a formal solution $u(t,x)$ of the form (6.1) with (6.2). Moreover we see the following: (i) the coefficients $\varphi_{i,j}(x) \in \mathbb{C}\{x\}$ ($1 \le i \le d$ and $1 \le j \le \mu_i$) can be chosen arbitrarily, (ii) $\phi_{1,1}(x) = b(x)$, and (iii) all the other coefficients $\phi_{k,l}(x) \in \mathbb{C}\{x\}$ are determined by (6.12) with (6.10) and therefore they are all holomorphic in a common neighborhood of $x = 0 \in \mathbb{C}^n$.*

§7.   Proof of the Convergence of a Formal Solution

We will prove here the convergence of the formal solution constructed in Proposition 2.

Let a fixed constant $R > 0$ be sufficiently small with $R \le 1$. We take $B \ge 0$, $B_{p,q} \ge 0$ ($p + |q| \ge 2$) so that the coefficients $b(x)$, $b_{p,q}(x)$ ($p + |q| \ge 2$) of (2.5) satisfy

$$\|b\|_R \le B \quad \text{and} \quad \|b_{p,q}\|_R \le B_{p,q} \quad (p + |q| \ge 2),$$

and that the power series

$$B t + \sum_{p+|q|\geq 2} B_{p,q}\, t^p \prod_{(j,\alpha)\in I_m} \left[Z_{j,\alpha}\right]^{q_{j,\alpha}}$$

is convergent in a neighborhood of $(t, Z) = (0,0) \in \mathbb{C} \times \mathbb{C}^N$.

   Let

$$(7.1) \qquad u(t,x) = \sum_{k\geq 1} u_k(t,x) \quad \text{with} \quad u_k(t,x) = \sum_{l=1}^{M_k} \phi_{k,l}(x) W_{k,l}(t,x)$$

be the formal solution constructed in section 6 and assume that $\phi_{k,l}(x)$ ($k = 1, 2, \ldots$ and $1 \leq l \leq M_k$) are all holomorphic on D_R. By the construction we know that $m_1 = 1$ and $\phi_{1,1}(x) = b(x)$ and that $\phi_{k,l}(x)$ ($l = 1, \ldots, m_k$) for $k \geq 2$ are defined by (6.12). Since $W_{k,l}(t,x)$ ($k = 1, 2, \ldots$ and $1 \leq l \leq M_k$) are defined by (6.7), by Proposition 1 and (5.3) we know the following: for any $\theta > 0$ there is a $\delta > 0$ such that

$$(7.2) \qquad \|W_{k,l}(t)\|_R \leq |t|^{\sigma k} \quad \text{on } S_\theta(\delta) \text{ for all } (k,l).$$

Therefore, for any $0 < r < R$ we have

$$\|u_k(t)\|_r \leq \sum_{l=1}^{M_k} \|\phi_{k,l}\|_r \|W_{k,l}(t)\|_r \leq \sum_{l=1}^{M_k} \|\phi_{k,l}\|_r |t|^{\sigma k} \quad \text{on } S_\theta(\delta).$$

Thus, in order to estimate the term $u_k(t,x)$ in (7.1) it will be convenient to use the following norm $\|u_k\|_r^*$:

$$(7.3) \qquad \|u_k\|_r^* = \sum_{l=1}^{M_k} \|\phi_{k,l}\|_r.$$

Note that this is expressed also in the form

$$(7.4) \qquad \|u_k\|_r^* = \sum_{i=1}^{d} \delta_{k,N_i} \left(\sum_{j=1}^{\mu_i} \|\varphi_{i,j}\|_r \right) + \sum_{l=1}^{m_k} \|\phi_{k,l}\|_r,$$

where δ_{k,N_i} denotes the Kronecker's delta (that is, $\delta_{k,N_i} = 1$ if $k = N_i$, and $\delta_{k,N_i} = 0$ if $k \neq N_i$).

Moreover, in the case $k \geq 2$, by substituting (6.12) (with (6.10)) into (7.4) we have

$$
\begin{aligned}
\|u_k\|_r^* \leq & \sum_{i=1}^{d} \delta_{k,N_i} \left(\sum_{j=1}^{\mu_i} \|\varphi_{i,j}\|_r \right) + \sum_{(p,q,k(q),l(q),\beta(q)) \in \mathcal{A}_k} \|b_{p,q}\|_r \, \|\psi_{k(q),l(q),\beta(q)}\|_r \\
\leq & \sum_{i=1}^{d} \delta_{k,N_i} \left(\sum_{j=1}^{\mu_i} \|\varphi_{i,j}\|_r \right) \\
& + \sum_{(p,q,k(q),l(q),\beta(q)) \in \mathcal{A}_k} \|b_{p,q}\|_r \prod_{(j,\alpha) \in I_m} \prod_{1 \leq i \leq q_{j,\alpha}} \binom{\alpha}{\beta_{j,\alpha}(i)} \times \\
& \quad \times \frac{1}{k_{j,\alpha}(i)^{|\beta_{j,\alpha}(i)|}} \left\| \left(\frac{\partial}{\partial x} \right)^{\beta_{j,\alpha}(i)} \phi_{k_{j,\alpha}(i),l_{j,\alpha}(i)} \right\|_r ;
\end{aligned}
$$

therefore if we write

$$
(7.5) \qquad \left\| D_x^{\beta}[u_h] \right\|_r^* = \sum_{l=1}^{M_h} \left\| \left(\frac{\partial}{\partial x} \right)^{\beta} \phi_{h,l} \right\|_r
$$

we obtain

$$
\begin{aligned}
(7.6) \qquad \|u_k\|_r^* \leq & \sum_{i=1}^{d} \delta_{k,N_i} \left(\sum_{j=1}^{\mu_i} \|\varphi_{i,j}\|_r \right) \\
& + \sum_{2 \leq p+|q| \leq k} B_{p,q} \sum_{\substack{p+|k(q)|=k \\ \beta(q) \in \Gamma(q)}} \prod_{(j,\alpha) \in I_m} \prod_{1 \leq i \leq q_{j,\alpha}} \binom{\alpha}{\beta_{j,\alpha}(i)} \times \\
& \quad \times \frac{1}{k_{j,\alpha}(i)^{|\beta_{j,\alpha}(i)|}} \left\| D_x^{\beta_{j,\alpha}(i)}[u_{k_{j,\alpha}(i)}] \right\|_r^* .
\end{aligned}
$$

Since $\varphi_{i,j}(x)$ $(i = 1, \ldots, d$ and $1 \leq j \leq \mu_i)$ are known holomorphic functions on D_R, we can find $A_i > 0$ $(i = 1, \ldots, d)$ so that

$$
(7.7) \qquad A_i \geq \sum_{j=1}^{\mu_i} \|\varphi_{i,j}\|_R \quad (i = 1, \ldots, d).
$$

Now, let us consider the following analytic equation with respect to Y:

$$
\begin{aligned}
(7.8) \qquad Y = & \sum_{i=1}^{d} \frac{A_i}{(R-r)^{m(2N_i-1)}} t^{N_i} + \frac{B}{(R-r)^m} t \\
& + \frac{1}{(R-r)^m} \sum_{p+|q| \geq 2} \frac{B_{p,q}}{(R-r)^{m(2p+|q|-2)}} t^p \left(2^m (2me)^m Y \right)^{|q|}
\end{aligned}
$$

where r is a parameter with $0 < r < R$.

By the implicit function theorem we see that (7.8) has a unique holomorphic solution $Y(t)$ in a neighborhood of $t = 0 \in \mathbb{C}$ satisfying $Y(0) = 0$, and that the Taylor coefficients Y_k $(k = 1, 2, \ldots)$ of the expansion

$$(7.9) \qquad Y = \sum_{k \geq 1} Y_k t^k$$

satisfy the following recurrence formulas:

$$(7.10) \qquad Y_1 = \frac{\delta_{1,N_1} A_1 + B}{(R-r)^m}$$

and for $k \geq 2$

$$(7.11)$$

$$Y_k = \sum_{i=1}^{d} \delta_{k,N_i} \frac{A_i}{(R-r)^{m(2N_i-1)}}$$

$$+ \frac{1}{(R-r)^m} \sum_{2 \leq p+|q| \leq k} \frac{B_{p,q}}{(R-r)^{m(2p+|q|-2)}} \times$$

$$\times \sum_{p+|k(q)|=k} \prod_{(j,\alpha) \in I_m} \prod_{1 \leq i \leq q_{j,\alpha}} \left(2^m (2me)^m Y_{k_{j,\alpha}(i)} \right).$$

Moreover, by induction on k we can easily see that Y_k has the form

$$(7.12) \qquad Y_k = \frac{C_k}{(R-r)^{m(2k-1)}}, \quad k = 1, 2, \ldots$$

where $C_k \geq 0$ $(k = 1, 2, \ldots)$ are constants independent of the parameter r.

The following lemma asserts that the Taylor series (7.9) of $Y(t)$ is a majorant series of our formal solution (7.1).

Lemma 9. *For any $k = 1, 2, \ldots$ we have*

$$(7.13)_k \qquad \frac{1}{k^{|\alpha|}} \left\| D_x^\alpha [u_k] \right\|_r^* \leq (2me)^m Y_k \quad \text{for any } 0 < r < R \text{ and } |\alpha| \leq m.$$

Proof. We will prove this by induction on k. When $k = 1$ we have

$$u_1(t,x) = \delta_{1,N_1} \sum_{j=1}^{\mu_1} \varphi_{1,j}(x) v_{1,j}(t,x) + b(x) w_{1,1}(t,x);$$

therefore by (7.7), $\|b\|_R \leq B$ and by using Lemma 5 we have

$$\left\|D_x^\alpha[u_1]\right\|_r^* = \delta_{1,N_1} \sum_{j=1}^{\mu_1} \left\|\left(\frac{\partial}{\partial x}\right)^\alpha \varphi_{1,j}\right\|_r + \left\|\left(\frac{\partial}{\partial x}\right)^\alpha b\right\|_r$$

$$\leq \delta_{1,N_1} \frac{|\alpha|!e^{|\alpha|}A_1}{(R-r)^{|\alpha|}} + \frac{|\alpha|!e^{|\alpha|}B}{(R-r)^{|\alpha|}} = |\alpha|!e^{|\alpha|}\frac{\delta_{1,N_1}A_1 + B}{(R-r)^{|\alpha|}}$$

$$\leq (2me)^m \frac{\delta_{1,N_1}A_1 + B}{(R-r)^m} = (2me)^m Y_1$$

which proves $(7.13)_1$.

Let $k \geq 2$ and suppose that $(7.13)_i$ is already proved for all $i = 1, \ldots, k-1$. Then, by (7.6), (7.7) and the induction hypothesis we have

$$\|u_k\|_r^* \leq \sum_{i=1}^d \delta_{k,N_i} A_i$$

$$+ \sum_{2 \leq p+|q| \leq k} B_{p,q} \sum_{\substack{p+|k(q)|=k \\ \beta(q) \in \Gamma(q)}} \prod_{(j,\alpha) \in I_m} \prod_{1 \leq i \leq q_{j,\alpha}} \binom{\alpha}{\beta_{j,\alpha}(i)} (2me)^m Y_{k_{j,\alpha}(i)}.$$

Hence, if we note that $1/(R-r)^{m(2k-2)} \geq 1$, $1/(R-r)^{m(2p+|q|-2)} \geq 1$ and

$$\sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} = 2^{|\alpha|} \leq 2^m,$$

we see

$$(7.14) \quad \|u_k\|_r^* \leq \sum_{i=1}^d \delta_{k,N_i} \frac{A_i}{(R-r)^{m(2k-2)}}$$

$$+ \sum_{2 \leq p+|q| \leq k} \frac{B_{p,q}}{(R-r)^{m(2p+|q|-2)}} \times$$

$$\times \sum_{p+|k(q)|=k} \prod_{(j,\alpha) \in I_m} \prod_{1 \leq i \leq q_{j,\alpha}} \left(2^m (2me)^m Y_{k_{j,\alpha}(i)}\right).$$

Therefore, by comparing (7.11) and (7.14) we obtain

$$(7.15) \qquad \|u_k\|_r^* \leq (R-r)^m Y_k = \frac{C_k}{(R-r)^{m(2k-2)}} \quad \text{for any } 0 < r < R.$$

Thus, by applying Lemma 5 to (7.15) we obtain

$$\frac{1}{k^{|\alpha|}} \left\|D_x^\alpha[u_k]\right\|_r^* \leq \frac{1}{k^{|\alpha|}} \frac{(m(2k-2)+1)\cdots(m(2k-2)+|\alpha|) e^{|\alpha|} C_k}{(R-r)^{m(2k-2)+|\alpha|}}$$

$$\leq \frac{(2me)^{|\alpha|} C_k}{(R-r)^{m(2k-2)+|\alpha|}} \leq \frac{(2me)^m C_k}{(R-r)^{m(2k-1)}} = (2me)^m Y_k$$

which proves $(7.13)_k$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proof of the convergence of the formal solution (7.1). Let r be fixed; for example, we set $r = R/2$. By (7.3) and (7.13) we have

$$\sum_{k \geq 1} \|u_k(t)\|_r \leq \sum_{k \geq 1} \|u_k\|_r^* |t|^{\sigma k} \leq (2me)^m \sum_{k \geq 1} Y_k |t|^{\sigma k} \quad \text{on } S_\theta(\delta).$$

This asserts that the formal solution $u(t,x)$ in (7.1) converges on $S_\theta(\delta_1) \times D_r$, if $\delta_1 > 0$ is sufficiently small. Since $\theta > 0$ is arbitrary, we can conclude that $u(t,x)$ converges in $\widetilde{\mathcal{O}}_+$ and gives an $\widetilde{\mathcal{O}}_+$-solution of (E). $\qquad\qquad$ □

Summing up we have obtained

Theorem 3. *The equation* (E) *has a family of* $\widetilde{\mathcal{O}}_+$-*solutions which is expanded into the form*

$$(7.16) \quad u(t,x) = \sum_{k \geq 1} \left[\sum_{i=1}^{d} \delta_{k,N_i} \sum_{j=1}^{\mu_i} \varphi_{i,j}(x) v_{i,j}(t,x) + \sum_{l=1}^{m_k} \phi_{k,l}(x) w_{k,l}(t,x) \right],$$

where (i) *the coefficients* $\varphi_{i,j}(x) \in \mathbb{C}\{x\}$ $(1 \leq i \leq d$ *and* $1 \leq j \leq \mu_i)$ *can be chosen arbitrarily,* (ii) $\phi_{1,1}(x) = b(x)$, *and* (iii) *all the other coefficients* $\phi_{k,l}(x) \in \mathbb{C}\{x\}$ *are determined by the data* $\varphi_{i,j}(x)$ $(1 \leq i \leq d$ *and* $1 \leq j \leq \mu_i)$ *and they are all holomorphic in a common neighborhood of* $x = 0 \in \mathbb{C}^n$.

From now we will write the solution (7.16) as

$$u(t,x) = U\left(\vec{\varphi}_1, \ldots, \vec{\varphi}_d\right)$$

with $\vec{\varphi}_i = (\varphi_{i,1}, \ldots, \varphi_{i,\mu_i}) \in \mathbb{C}\{x\}^{\mu_i}$ for $i = 1, \ldots, d$. By the construction of $U(\vec{\varphi}_1, \ldots, \vec{\varphi}_d)$ and (2) of Lemma 1 we see:

Proposition 3. (1) $U(\vec{\varphi}_1, \ldots, \vec{\varphi}_d) = O(t^\sigma, \widetilde{\mathcal{O}}_+)$ *(as* $t \longrightarrow 0$*).*
(2) *We have*

$$U\left(\vec{\varphi}_1, \ldots, \vec{\varphi}_{p-1}, \vec{\varphi}_p, \vec{0}, \ldots, \vec{0}\right) - U\left(\vec{\varphi}_1, \ldots, \vec{\varphi}_{p-1}, \vec{0}, \vec{0}, \ldots, \vec{0}\right)$$

$$= \sum_{j=1}^{\mu_p} \varphi_{p,j}(x) v_{p,j}(t,x) + O(t^{\sigma(N_p+1)}, \widetilde{\mathcal{O}}_+) \quad \text{(as } t \longrightarrow 0)$$

where $\vec{\varphi}_p = (\varphi_{p,1}, \ldots, \varphi_{p,\mu_p}) \in \mathbb{C}\{x\}^{\mu_p}$.
(3) *If* $U(\vec{\varphi}_1, \ldots, \vec{\varphi}_d) = U(\vec{\psi}_1, \ldots, \vec{\psi}_d)$ *we have* $\vec{\varphi}_i(x) = \vec{\psi}_i(x)$ *in* $\mathbb{C}\{x\}^{\mu_i}$ *for* $i = 1, \ldots, d$.

§8.   Completion of the Proof of Theorem 2

Denote by $\mathcal{S}_+$ the set of all $\widetilde{\mathcal{O}}_+$-solutions of (E). We already proved that

$$\mathcal{S}_+ \supset \left\{ U(\vec{\varphi}_1, \ldots, \vec{\varphi}_d)\,;\, \vec{\varphi}_i \in \mathbb{C}\{x\}^{\mu_i}\ \text{for}\ i = 1, \ldots, d \right\}.$$

Therefore, to complete the proof of Theorem 2 it is enough to prove

Theorem 4.   *Every solution $u(t,x) \in \mathcal{S}_+$ is expressed in the form*

$$(8.1) \qquad\qquad u(t,x) = U\left(\vec{\varphi}_1, \ldots, \vec{\varphi}_d \right)$$

for some unique $\vec{\varphi}_i \in \mathbb{C}\{x\}^{\mu_i}$ $(i = 1, \ldots, d)$.

The proof of this theorem is almost the same as that of Théorème 4 of Gérard-Tahara [5]: but, for the sake of convenience of readers we will give here a refined version of the proof.

Let $a_1, \ldots, a_d$ be the ones in (3.2). Set $a_0 = 0$ and $a_{d+1} = \infty$. Our proof of Theorem 4 is based on the following proposition:

Proposition 4.   *Let $u_1(t,x) \in \mathcal{S}_+$ and $u_2(t,x) \in \mathcal{S}_+$.*
(1) We have $u_i(t,x) = O(t^\sigma, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) for $i = 1, 2$.
(2) If s satisfies $a_{i-1} < s < a_i$ for some $i \in \{1, \ldots, d\}$ and if $u_1 - u_2 = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) holds, we have $u_1 - u_2 = O(t^{\sigma N_i}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$).
(3) If $u_1 - u_2 = O(t^{\sigma N_i}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) holds for some $i \in \{1, \ldots, d\}$, we have

$$u_1 - u_2 = \sum_{j=1}^{\mu_i} \varphi_{i,j}(x) v_{i,j}(t,x) + O(t^{\sigma(N_i+1)}, \widetilde{\mathcal{O}}_+) \quad (\text{as}\ t \longrightarrow 0)$$

for some $\varphi_{i,j}(x) \in \mathbb{C}\{x\}$ $(j = 1, \ldots, \mu_i)$.
(4) If $s > a_d$ holds and if $u_1 - u_2 = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$), then we have $u_1 = u_2$ in $\widetilde{\mathcal{O}}_+$.

Let us admit this proposition for a moment. By using this result we can give a proof of Theorem 4 as follows.

Proof of Theorem 4.   Let $u(t,x) \in \mathcal{S}_+$. Set $u_0 = U(\vec{0}, \ldots, \vec{0})$. Then, by the definition we see that $u - u_0 = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) for some $a_0 = 0 < s < a_1$. Therefore, by (2),(3) of Proposition 4 we have

$$(8.2) \qquad u - u_0 = \sum_{j=1}^{\mu_1} \varphi_{1,j}(x) v_{1,j}(t,x) + O(t^{\sigma(N_1+1)}, \widetilde{\mathcal{O}}_+) \quad (\text{as}\ t \longrightarrow 0)$$

for some $\varphi_{1,j}(x) \in \mathbb{C}\{x\}$ $(j = 1, \ldots, \mu_1)$. Using this $\vec{\varphi}_1 = (\varphi_{1,1}, \ldots, \varphi_{1,\mu_1})$ we set $u_1 = U(\vec{\varphi}_1, \vec{0}, \ldots, \vec{0})$. Then, by (8.2) and (2) of Proposition 3 we have

$$
\begin{aligned}
u - u_1 &= (u - u_0) - (u_1 - u_0) \\
&= (u - u_0) - \big(U(\vec{\varphi}_1, \vec{0}, \ldots, \vec{0}) - U(\vec{0}, \vec{0}, \ldots, \vec{0})\big) \\
&= \left[\sum_{j=1}^{\mu_1} \varphi_{1,j}(x) v_{1,j}(t,x) + O(t^{\sigma(N_1+1)}, \widetilde{\mathcal{O}}_+) \right] \\
&\quad - \left[\sum_{j=1}^{\mu_1} \varphi_{1,j}(x) v_{1,j}(t,x) + O(t^{\sigma(N_1+1)}, \widetilde{\mathcal{O}}_+) \right] \\
&= O(t^{\sigma(N_1+1)}, \widetilde{\mathcal{O}}_+) \quad (\text{as } t \longrightarrow 0).
\end{aligned}
$$

Since $a_1 < \sigma(N_1 + 1) < a_2$ holds, by using (2),(3) of Proposition 4 again we see that $u - u_1$ is expressed in the form

$$(8.3) \qquad u - u_1 = \sum_{j=1}^{\mu_2} \varphi_{2,j}(x) v_{2,j}(t,x) + O(t^{\sigma(N_2+1)}, \widetilde{\mathcal{O}}_+) \quad (\text{as } t \longrightarrow 0)$$

for some $\varphi_{2,j}(x) \in \mathbb{C}\{x\}$ $(j = 1, \ldots, \mu_2)$. Using this $\vec{\varphi}_2 = (\varphi_{2,1}, \ldots, \varphi_{2,\mu_2})$ we set $u_2 = U(\vec{\varphi}_1, \vec{\varphi}_2, \vec{0}, \ldots, \vec{0})$. Then, by (8.3) and (2) of Proposition 3 we have

$$
\begin{aligned}
u - u_2 &= (u - u_1) - (u_2 - u_1) \\
&= (u - u_1) - \big(U(\vec{\varphi}_1, \vec{\varphi}_2, \vec{0}, \ldots, \vec{0}) - U(\vec{\varphi}_1, \vec{0}, \vec{0}, \ldots, \vec{0})\big) \\
&= \left[\sum_{j=1}^{\mu_1} \varphi_{2,j}(x) v_{2,j}(t,x) + O(t^{\sigma(N_2+1)}, \widetilde{\mathcal{O}}_+) \right] \\
&\quad - \left[\sum_{j=1}^{\mu_1} \varphi_{2,j}(x) v_{2,j}(t,x) + O(t^{\sigma(N_2+1)}, \widetilde{\mathcal{O}}_+) \right] \\
&= O(t^{\sigma(N_2+1)}, \widetilde{\mathcal{O}}_+) \quad (\text{as } t \longrightarrow 0).
\end{aligned}
$$

Repeating the same argument as above we can find $\vec{\varphi}_i = (\varphi_{i,1}, \ldots, \varphi_{i,\mu_i}) \in \mathbb{C}\{x\}^{\mu_i}$ $(i = 1, \ldots, d)$ so that if we set

$$u_p = U(\vec{\varphi}_1, \ldots, \vec{\varphi}_p, \vec{0}, \ldots, \vec{0}), \quad p = 1, \ldots, d$$

we have the asymptotics

$$u - u_p = O(t^{\sigma(N_p+1)}, \widetilde{\mathcal{O}}_+) \quad (\text{as } t \longrightarrow 0), \quad p = 1, \ldots, d.$$

Thus, if we take the case $p = d$ we have $u - u_d = O(t^{\sigma(N_d+1)}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$): then, by the condition $\sigma(N_d + 1) > a_d$ and by (4) of Proposition 4 we obtain the result $u = u_d$ in $\widetilde{\mathcal{O}}_+$.

This proves that $u(t,x) \in \mathcal{S}_+$ is expressed in the form (8.1) for some $\vec{\varphi}_i(x) \in \mathbb{C}\{x\}^{\mu_i}$ $(i = 1, \ldots, d)$. The uniqueness of $\vec{\varphi}_i(x)$ $(i = 1, \ldots, d)$ follows from (3) of Proposition 3. $\qquad \square$

Now, let us give a proof of Proposition 4. We note:

Lemma 10. *Assume that $w(t,x) \in \widetilde{\mathcal{O}}_+$ and $f(t,x) \in \widetilde{\mathcal{O}}_+$ satisfy the equation*

$$(8.4) \qquad C\Big(t\frac{\partial}{\partial t}, x\Big)w = f(t,x).$$

(1) *If $a_{i-1} < s < b < a_i$ holds for some $i \in \{1, \ldots, d\}$, if $w(t,x) = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) and if $f(t,x) = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$), then we have $w(t,x) = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$).*

(2) *If $a_{i-1} < s < a_i < b < a_{i+1}$ holds for some $i \in \{1, \ldots, d\}$, if $w(t,x) = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) and if $f(t,x) = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$), then we have*

$$w(t,x) = \sum_{j=1}^{\mu_i} \varphi_{i,j}(x) v_{i,j}(t,x) + O(t^b, \widetilde{\mathcal{O}}_+) \quad (as \ t \longrightarrow 0)$$

for some $\varphi_{i,j}(x) \in \mathbb{C}\{x\}$ $(j = 1, \ldots, \mu_i)$.

Proof. Set

$$P_1 = \prod_{k=0}^{i-1} C_k\Big(t\frac{\partial}{\partial t}, x\Big), \quad P_2 = \prod_{k=i}^{d} C_k\Big(t\frac{\partial}{\partial t}, x\Big),$$

and $w_2 = P_1 w$. Then we have $C(t\partial/\partial t, x) = P_2 P_1$ and $P_2 w_2 = f$. Since $f(t,x) = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) and since $b < a_k$ holds for all $k = i, \ldots, d$, by applying (3) of Lemma 4 to the equation $P_2 w_2 = f$ we have $w_2(t,x) = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). Since $w(t,x) = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) and since $a_k < s$ holds for all $k = 0, \ldots, i-1$, by applying (2) of Lemma 4 to the equation $P_1 w = w_2$ we obtain $w(t,x) = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). This proves the part (1).

Next let us prove (2). Assume that $a_{i-1} < s < a_i < b < a_{i+1}$ holds for some $i \in \{1, \ldots, d\}$. Set

$$P_3 = \prod_{k=i+1}^{d} C_k\Big(t\frac{\partial}{\partial t}, x\Big) \quad \text{if } i < d,$$

and $P_3 = identity$ if $i = d$. Set also $w_1 = C_i(t\partial/\partial t, x)w$ and $w_3 = P_1 w_1$. Then we have $C(t\partial/\partial t, x) = P_3 P_1 C_i$ and $P_3 w_3 = f$. Since $a_{i-1} < s < a_i < b < a_{i+1}$

holds, by the same argument as above we see that $w_3(t,x) = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) and $w_1(t,x) = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$).

Now let us consider the relation $C_i w = w_1$. We already know that $w_1(t,x) = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) and $s < a_i < b$ hold. Hence, by (1) of Lemma 4 we have a unique solution $W(t,x)$ of $C_i W = w_1$ satisfying $W = O(t^b, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). Then, we have $C_i(w-W) = 0$ and therefore by using the fundamental solutions $\{v_{i,1}(t,x), \ldots, v_{i,\mu_i}(t,x)\}$ we can express $(w-W)(t,x)$ in the form

$$w - W = \sum_{j=1}^{\mu_i} \varphi_{i,j}(x) v_{i,j}(t,x)$$

for some $\varphi_{i,j}(x) \in \mathbb{C}\{x\}$ ($j = 1, \ldots, \mu_i$). This leads us to the conclusion of the part (2). $\qquad\Box$

Proof of Proposition 4. First we will prove (1) only in the case $i = 1$. Note that our equation for u_1 is written in the form

$$(8.5) \qquad C\left(t\frac{\partial}{\partial t}, x\right) u_1 = H[u_1]$$

where

$$H[u_1] = b(x)\,t + \sum_{p+|q|\geq 2} b_{p,q}(x) t^p \prod_{(j,\alpha)\in I_m} \left[\mathcal{D}_{j,\alpha} u_1\right]^{q_{j,\alpha}}.$$

It is easy to see that the operator $H[f]$ with $f \in \widetilde{\mathcal{O}}_+$ satisfies the following properties: γ-1) $H[\cdot]$ is a mapping from $\widetilde{\mathcal{O}}_+$ to $\widetilde{\mathcal{O}}_+$, and γ-2) if $f = O(t^\varepsilon, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) for some $\varepsilon > 0$ we have $H[f] = O(t^r, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) for any $r < \min\{1, 2\varepsilon\}$.

Let $u_1(t,x) \in \mathcal{S}_+$. By the definition we have $u_1(t,x) = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) for some $s > 0$. If $s \geq \sigma$ we have nothing to prove. If $s < \sigma$ we choose a sequence $s_0, s_1, \ldots, s_p$ such that $s_0 = s < s_1 < \cdots < s_p = \sigma$ (< 1) and that $s_k < \min\{1, 2s_{k-1}\}$ holds for $k = 1, \ldots, p$; then we can prove the property $u_1(t,x) = O(t^\sigma, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) in the following way.

Note that $a_0 = 0 < s_k < a_1$ for $k = 0, 1, \ldots, p$ and that $u_1 = O(t^{s_0}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) is known. Therefore, by γ-2) we have $H[u_1] = O(t^{s_1}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) and by applying (1) of Lemma 10 to (8.5) we have $u_1 = O(t^{s_1}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). Then, by γ-2) we have $H[u_1] = O(t^{s_2}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) and by applying (1) of Lemma 10 to (8.5) again we obtain $u_1 = O(t^{s_2}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). Thus, by repeating the same argument we obtain $u_1 = O(t^{s_p}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). Since $s_p = \sigma$, this completes the proof of the part (1).

Next let us show (2). By (1) we have $u_i(t,x) = O(t^\sigma, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) for $i = 1, 2$. Set $w = u_1 - u_2$. Then we have $w(t,x) = O(t^\sigma, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) and we see that $w(t,x)$ satisfies the equation

$$(8.6) \qquad C\left(t\frac{\partial}{\partial t}, x\right) w = G[w]$$

where

$$G[w] = \sum_{p+|q|\geq 2,\, |q|\geq 1} b_{p,q}(x) t^p \left[\prod_{(j,\alpha) \in I_m} \left(\mathcal{D}_{j,\alpha} w + \mathcal{D}_{j,\alpha} u_2(t,x)\right)^{q_{j,\alpha}} \right.$$
$$\left. - \prod_{(j,\alpha) \in I_m} \left(\mathcal{D}_{j,\alpha} u_2(t,x)\right)^{q_{j,\alpha}} \right].$$

We see easily that $G[\cdot]$ is a mapping from $\widetilde{\mathcal{O}}_+$ to $\widetilde{\mathcal{O}}_+$ and that if $f = O(t^\varepsilon, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) for some $\varepsilon > 0$ we have $G[f] = O(t^r, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) for any $r \leq \min\{\sigma + \varepsilon, 2\varepsilon\}$.

Assume that $a_{i-1} < s < a_i$ holds for some $i \in \{1, \ldots, d\}$ and that $w = u_1 - u_2 = O(t^s, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). If $s \geq \sigma N_i$ we have nothing to prove in (2). If $s < \sigma N_i$ we choose a sequence $s_0, s_1, \ldots, s_p$ such that $s_0 = s < s_1 < \cdots < s_p = \sigma N_i$ and that $s_k \leq \min\{\sigma + s_{k-1}, 2s_{k-1}\}$ holds for $k = 1, \ldots, p$; then, applying the same argument as in the proof of (1) to the equation (8.6) we obtain $w(t,x) = O(t^{s_p}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). This proves the part (2).

The proof of (3) is as follows. Assume that $w = u_1 - u_2 = O(t^{\sigma N_i}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$). Then we have $G[w] = O(t^{\sigma(N_i+1)}, \widetilde{\mathcal{O}}_+)$ (as $t \longrightarrow 0$) and therefore by applying (2) of Lemma 10 to the equation (8.6) we obtain the conclusion of the part (3).

Lastly, we note that the part (4) is the same as Proposition 3 of Gérard-Tahara [5] and so we omit the proof. We remark that the part (4) can be proved also by the same argument as in Tahara [13],[14]. □

Remark 3.   Let $\{v_{i,1}(t,x), \ldots, v_{i,\mu_i}(t,x)\}$ be a fundamental system of $\widetilde{\mathcal{O}}_+$-solutions of $(3.6)_i$ as before. The following assertion will be easily verified: the system $\{v_{i,j}(t,x)\,;\, 1 \leq i \leq d,\, 1 \leq j \leq \mu_i\}$ is a fundamental system of $\widetilde{\mathcal{O}}_+$-solutions of the equation $C(t\partial/\partial t, x)v = 0$. See the condition (1) in Theorem 2.

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