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The b-functions for Prehomogeneous Vector Spaces of Commutative Parabolic Type and Universal Generalized Verma Modules

By

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Abstract

We shall give a new elementary proof of the uniform expression for the b-functions of prehomogeneous vector spaces of commutative parabolic type obtained by Muller, Rubenthaler and Schiffmann [5] by using micro-local analysis. Our method is similar to Kashiwara's approach using the universal Verma modules. We shall also give a new proof for the criterion of the irreducibility of the generalized Verma module in terms of b-functions due to Suga [10], Gyoja [1], Wachi [13].

§1. Introduction

In this paper we deal with the b-functions of the invariants on the flag manifolds G/P. In the case where P is a Borel subgroup, Kashiwara [3] determined the b-functions by using the universal Verma modules. For general parabolic subgroups P we show that b-functions are regarded as generators of ideals defined by universal generalized Verma modules. When the unipotent radical of P is commutative, we determine the generator.

Let $\mathfrak{g}$ be a simple Lie algebra over the complex number field $\mathbb{C}$, and let G be a connected simply-connected simple algebraic group with Lie algebra $\mathfrak{g}$. Fix a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. We denote the reductive part of $\mathfrak{p}$ and the nilpotent part of $\mathfrak{p}$ by $\mathfrak{l}$ and $\mathfrak{n}$ respectively. Let L be the subgroup of G

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corresponding to $\mathfrak{l}$. Let R be the symmetric algebra of the commutative Lie algebra $\mathfrak{p}/[\mathfrak{p},\mathfrak{p}]$. For a Lie algebra $\mathfrak{a}$ we set $U_R(\mathfrak{a}) = R \otimes_{\mathbb{C}} U(\mathfrak{a})$ where $U(\mathfrak{a})$ denotes the enveloping algebra of $\mathfrak{a}$. The canonical map $c : \mathfrak{p} \to R$ induces a one-dimensional $U_R(\mathfrak{p})$-module R_c. Let $\mathbb{C}_\mu$ be the one-dimensional $\mathfrak{p}$-module with weight μ. Set $R_{c+\mu} = R_c \otimes_{\mathbb{C}} \mathbb{C}_\mu$. Then $R_{c+\mu}$ is a one-dimensional $U_R(\mathfrak{p})$-module.

For a character μ of $\mathfrak{p}$ we regard μ as a weight of $\mathfrak{g}$, and let $V(\mu)$ be the irreducible $\mathfrak{g}$-module with highest weight μ. We assume that the weight μ of $\mathfrak{g}$ is dominant integral. We define a $U_R(\mathfrak{g})$-module homomorphism

$$\iota : U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} R_{c+\mu} \to U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} (R_c \otimes_{\mathbb{C}} V(\mu))$$

by $\iota(1 \otimes 1) = 1 \otimes 1 \otimes v_\mu$, where v_μ is the highest weight vector of $V(\mu)$. For a $U_R(\mathfrak{g})$-module homomorphism ψ from $U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} (R_c \otimes_{\mathbb{C}} V(\mu))$ to $U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} R_{c+\mu}$ the composite $\psi\iota$ is the multiplication on $U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} R_{c+\mu}$ by an element ξ of R:

$$\begin{array}{ccc} U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} R_{c+\mu} & = & U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} R_{c+\mu} \\ \iota \downarrow & & \downarrow \xi\,\mathrm{id} \\ U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} (R_c \otimes_{\mathbb{C}} V(\mu)) & \xrightarrow[\psi]{} & U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} R_{c+\mu}. \end{array} \tag{1.1}$$

The set Ξ_μ consisting of all $\xi \in R$ induced by $U_R(\mathfrak{g})$-module homomorphisms from $U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} (R_c \otimes_{\mathbb{C}} V(\mu))$ to $U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} R_{c+\mu}$ as above is an ideal of R. We can construct a particular homomorphism $\psi_\mu : U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} (R_c \otimes_{\mathbb{C}} V(\mu)) \to U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} R_{c+\mu}$ by considering the irreducible decomposition of $V(\mu)$ as a $\mathfrak{p}$-module (see Section 3 below). However, there is an example where $\xi_\mu \in \Xi_\mu$ is not a generator of the ideal Ξ_μ (cf. Remark 1). Note that Kashiwara [3] gave the generator of Ξ_μ when P is a Borel subgroup.

Let $\psi \in \mathrm{Hom}_{U_R(\mathfrak{g})}(U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} (R_c \otimes_{\mathbb{C}} V(\mu)), U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{p})} R_{c+\mu})$ and let $\xi \in \Xi_\mu$ be the corresponding element. Then as in Kashiwara [3] we can define a differential operator $P(\psi)$ on G satisfying

$$P(\psi) f^{\lambda+\mu} = \xi(\lambda) f^\lambda$$

for any character λ of $\mathfrak{p}$ which can be regarded as a dominant integral weight of $\mathfrak{g}$. Here, f^λ denotes the invariant on G corresponding to λ (see Section 4 below) and ξ is regarded as a function on $\mathrm{Hom}(\mathfrak{p}, \mathbb{C})$.

In the rest of Introduction we assume that the nilpotent radical $\mathfrak{n}$ of $\mathfrak{p}$ is commutative. Then the pair $(L, \mathfrak{n})$ is a prehomogeneous vector space via the adjoint action of L. In this case there exists exactly one simple root α_0 such that the root space $\mathfrak{g}_{\alpha_0}$ is in $\mathfrak{n}$. We denote the fundamental weight corresponding to α_0 by ϖ_0.

We define an element $\xi_0 \in R$ by

$$\xi_0(\lambda) = \prod_{\eta \in Wt(\varpi_0)\setminus\{\varpi_0\}} \big((\lambda+\rho+\varpi_0, \lambda+\rho+\varpi_0) - (\lambda+\rho+\eta, \lambda+\rho+\eta)\big)$$

where $\lambda \in \mathbb{C}\varpi_0$, $Wt(\varpi_0)$ is the set of the highest weights of irreducible $\mathfrak{l}$-submodules of $V(\varpi_0)$ and ρ is the half sum of positive roots of $\mathfrak{g}$.

Theorem 1.1. *We have $\xi_0 = \xi_{\varpi_0}$, and the ideal Ξ_{ϖ_0} of R is generated by ξ_0.*

We denote by ψ_0 the homomorphism satisfying $\psi_0\iota = \xi_0\mathrm{id}$.

Let $\mathfrak{n}^-$ be the nilpotent part of the parabolic subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{p}$. We can define a constant coefficient differential operator $P'(\psi_0)$ on $\mathfrak{n}^- \simeq \exp(\mathfrak{n}^-)$ by

$$(P(\psi_0)f)|_{\exp(\mathfrak{n}^-)} = P'(\psi_0)(f|_{\exp(\mathfrak{n}^-)}).$$

Theorem 1.2. *If the prehomogeneous vector space $(L, \mathfrak{n})$ is regular, then $P'(\psi_0)$ is coincide with the differential operator defining the b-function $b(s)$ of the unique irreducible relative invariant of $(L, \mathfrak{n})$, and $b(s) = \xi_0(s\varpi_0)$.*

Note that the uniform expression of the b-function of $(L, \mathfrak{n})$ given in Theorem 1.2 was already obtained by Muller, Rubenthaler and Schiffmann [5] by using the micro-local analysis.

Moreover, using the commutative diagram (1.1) for ξ_0 and ψ_0 we give a new proof of the following criterion of the irreducibility of the generalized Verma module due to Suga [10], Gyoja [1], Wachi [13]:

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_{s_0\varpi_0} \text{ is irreducible } \iff \xi_0((s_0 - m)\varpi_0) \neq 0 \text{ for any } m \in \mathbb{Z}_{>0}.$$

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§2. Prehomogeneous Vector Spaces

In this section we recall some basic facts on prehomogeneous vector spaces (see Sato and Kimura [8]).

Definition 2.1.

(i) For a connected algebraic group G over the complex number field $\mathbb{C}$ and a finite dimensional G-module V, the pair (G, V) is called a prehomogeneous vector space if there exists a Zariski open orbit O in V.

(ii) We denote the ring of polynomial functions on V by $\mathbb{C}[V]$. A nonzero element $f \in \mathbb{C}[V]$ is called a relative invariant of a prehomogeneous vector space (G, V) if there exists a character χ of G such that $f(gv) = \chi(g)f(v)$ for any $g \in G$ and $v \in V$.

(iii) A prehomogeneous vector space is called regular if there exists a relative invariant f such that the Hessian $H_f = \det(\partial^2 f/\partial x_i \partial x_j)$ is not identically zero, where $\{x_i\}$ is a coordinate system of V.

For a prehomogeneous vector space (G, V) with open orbit O, we set $S = V \setminus O$. Let $S_i = \{v \in V | f_i(v) = 0\}$ $(1 \leq i \leq l)$ be the one-codimensional irreducible components of S. Then all f_i are relative invariants, and for any relative invariant f there exist $m_i \in \mathbb{Z}_{\geq 0}$ such that $f \in \mathbb{C}f_1^{m_1} \cdots f_l^{m_l}$ (see Sato-Kimura [8]). These irreducible polynomials $f_1, \ldots, f_l$ are called basic relative invariants.

In the remainder of this section we assume that G is reductive. Then we have the following proposition.

Proposition 2.1 (see [8]). *The prehomogeneous vector space (G, V) with open orbit O is regular if and only if $S = V \setminus O$ is a hypersurface.*

Let V^* be the dual space of V. The pair (L, V^*) is also a prehomogeneous vector space by $\langle gv^*, v\rangle = \langle v^*, g^{-1}v\rangle$, where $\langle\ ,\ \rangle$ is the natural pairing of V^* and V. If $f \in \mathbb{C}[V]$ is a relative invariant of (G, V) with character χ, then there exists a relative invariant f^* of (G, V^*) with character χ^{-1}. For $h \in \mathbb{C}[V^*]$ we define a constant coefficient differential operator $h(\partial)$ by

$$h(\partial) \exp\langle v^*, v\rangle = h(v^*) \exp\langle v^*, v\rangle,$$

where $v \in V$ and $v^* \in V^*$. Then there exists a polynomial $b(s) \in \mathbb{C}[s]$ such that

$$f^*(\partial) f^{s+1} = b(s) f^s.$$

This polynomial is called the b-function of f. It is known that $\deg b = \deg f = \deg f^*$ (see [6]).

§3. Universal Generalized Verma Modules

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system and $W \subset \mathrm{GL}(\mathfrak{h})$ the Weyl group. For $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. We denote the set of positive roots by Δ^+ and the set of simple roots by $\{\alpha_i\}_{i \in I_0}$, where I_0 is an index set. Let ρ be the half sum of positive roots of $\mathfrak{g}$. We set

$$\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\pm\alpha}, \qquad \mathfrak{b}^{\pm} = \mathfrak{h} \oplus \mathfrak{n}^{\pm}.$$

For $i \in I_0$ let $h_i \in \mathfrak{h}$ be the simple coroot and $\varpi_i \in \mathfrak{h}^*$ the fundamental weight corresponding to i. We denote the longest element of W by w_0. Let $(\ , \ )$ be the W-invariant nondegenerate symmetric bilinear form on $\mathfrak{h}^*$. We denote the irreducible $\mathfrak{g}$-module with highest weight $\mu \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \varpi_i$ by $V(\mu)$ and its highest weight vector by v_μ. For a Lie algebra $\mathfrak{a}$ we denote the enveloping algebra of $\mathfrak{a}$ by $U(\mathfrak{a})$.

For a subset $I \subset I_0$ we set

$$\begin{aligned}
\Delta_I &= \Delta \cap \sum_{i \in I} \mathbb{Z}\alpha_i, & \mathfrak{l}_I &= \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha \right), \\
\mathfrak{n}_I^{\pm} &= \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_{\pm\alpha}, & \mathfrak{p}_I^{\pm} &= \mathfrak{l}_I \oplus \mathfrak{n}_I^{\pm}, \\
\mathfrak{h}_I &= \mathfrak{h} / \sum_{i \in I} \mathbb{C} h_i, & \mathfrak{h}_I^* &= \sum_{i \in I_0 \setminus I} \mathbb{C}\varpi_i.
\end{aligned}$$

Let W_I be the subgroup of W generated by the simple reflections corresponding to $i \in I$. We denote the longest element of W_I by w_I. Let $\mathfrak{h}_{I,+}^*$ be the set of dominant integral weights in $\mathfrak{h}_I^*$. For $\mu \in \mathfrak{h}_I^*$ we define a one-dimensional $U(\mathfrak{p}_I^+)$-module $\mathbb{C}_{I,\mu}$ by

$$\mathbb{C}_{I,\mu} = U(\mathfrak{p}_I^+) \Big/ \Big(U(\mathfrak{p}_I^+)\mathfrak{n}^+ + \sum_{h \in \mathfrak{h}} U(\mathfrak{p}_I^+)(h - \mu(h)) + U(\mathfrak{p}_I^+)(\mathfrak{n}^- \cap \mathfrak{l}_I) \Big).$$

We denote the canonical generator of $\mathbb{C}_{I,\mu}$ by 1_μ. Set $M_I(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I^+)} \mathbb{C}_{I,\mu}$, which is called the scalar generalized Verma module with highest weight μ. We denote the irreducible $\mathfrak{p}_I^+$-module with highest weight $\mu \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \varpi_i + \sum_{j \notin I} \mathbb{Z}\varpi_j$ by $W(\mu)$.

Let G be a connected simply-connected simple algebraic group with Lie algebra $\mathfrak{g}$. We denote the subgroups of G corresponding to $\mathfrak{h}, \mathfrak{b}^{\pm}, \mathfrak{n}^{\pm}, \mathfrak{l}_I, \mathfrak{n}_I^{\pm}$ by $T, B^{\pm}, N^{\pm}, L_I, N_I^{\pm}$ respectively.

Let R_I be the symmetric algebra of $\mathfrak{h}_I$, and define a linear map $c : \mathfrak{h} \to R_I$ as the composite of the natural projection from $\mathfrak{h}$ to $\mathfrak{h}_I$ and the natural injection from $\mathfrak{h}_I$ to R_I. Set $U_{R_I}(\mathfrak{a}) = R_I \otimes_{\mathbb{C}} U(\mathfrak{a})$ for a Lie algebra $\mathfrak{a}$.

We set for $\mu \in \mathfrak{h}_I^*$

$$J_{I,c+\mu} = U_{R_I}(\mathfrak{p}_I^+)\mathfrak{n}^+ + \sum_{h\in\mathfrak{h}} U_{R_I}(\mathfrak{p}_I^+)(h - c(h) - \mu(h)) + U_{R_I}(\mathfrak{p}_I^+)(\mathfrak{n}^- \cap \mathfrak{l}_I),$$

$$R_{I,c+\mu} = U_{R_I}(\mathfrak{p}_I^+)\Big/ J_{I,c+\mu}.$$

We denote the canonical generator of $R_{I,c+\mu}$ by $1_{c+\mu}$.

Definition 3.1. For $\mu \in \mathfrak{h}_I^*$ set $M_{R_I}(c+\mu) = U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} R_{I,c+\mu}$. We call this $U_{R_I}(\mathfrak{g})$-module the universal scalar generalized Verma module.

Note that $M_{R_\emptyset}(c)$ is the universal Verma module in Kashiwara [3]. For $\lambda \in \mathfrak{h}_I^*$ we regard $\mathbb{C}$ as an R_I-module by $c(h_i)1 = \lambda(h_i)$. Then we have

$$\mathbb{C} \otimes_{R_I} M_{R_I}(c+\mu) = M_I(\lambda+\mu).$$

The next lemma is obvious.

Lemma 3.1. $\mathrm{End}_{U_{R_I}(\mathfrak{g})}(M_{R_I}(c+\mu)) = R_I$.

For $\mu \in \mathfrak{h}_I^*$ we define a $U_{R_I}(\mathfrak{g})$-module homomorphism

$$\iota_\mu : M_{R_I}(c+\mu) \longrightarrow U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} V(\mu))$$

by $\iota_\mu(1\otimes 1_{c+\mu}) = 1\otimes 1_c \otimes v_\mu$. We denote by Ξ_μ the ideal of R_I consisting of ξ such that there exists $\psi \in \mathrm{Hom}_{U_{R_I}(\mathfrak{g})}(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} V(\mu)), M_{R_I}(c+\mu))$ satisfying $\psi\iota_\mu = \xi\,\mathrm{id}$. Let us give a particular element ξ_μ of Ξ_μ for $\mu \in \mathfrak{h}_{I,+}^*$.

Lemma 3.2. *For $\mu_1, \mu_2 \in \sum_{i\in I} \mathbb{Z}_{\geq 0}\varpi_i + \sum_{j\notin I} \mathbb{Z}\varpi_j$ we define a function p_{μ_1,μ_2} on $\mathfrak{h}_I^*$ by*

$$p_{\mu_1,\mu_2}(\lambda) = (\lambda+\rho+\mu_1, \lambda+\rho+\mu_1) - (\lambda+\rho+\mu_2, \lambda+\rho+\mu_2),$$

which is regarded as an element of R_I. Then we have

$$p_{\mu_1,\mu_2}\ \mathrm{Ext}^1_{U_{R_I}(\mathfrak{g})}\big(W_1,\ W_2\big) = 0,$$

where $W_i = U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} W(\mu_i))$.

Proof. The action of the Casimir element of $U(\mathfrak{g})$ on $U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} W(\mu))$ is given by the multiplication by $p_\mu \in R_I$, where $p_\mu(\lambda) = (\lambda+\rho+\mu, \lambda+\rho+\mu) - (\rho,\rho)$ for $\lambda \in \mathfrak{h}_I^*$. Using this action, we can easily check that $p_{\mu_1,\mu_2} = p_{\mu_1} - p_{\mu_2}$ is an annihilator. □

Lemma 3.3. *For any $\mu \in \mathfrak{h}^*_{I,+}$ there exist $\mathfrak{p}_I^+$-submodules $F_1, F_2, \dots, F_r$ of $V(\mu)$ and weights $\eta_1, \eta_2, \dots, \eta_{r-1} \in \sum_{i\in I} \mathbb{Z}_{\geq 0}\varpi_i + \sum_{i \in I_0 \setminus I} \mathbb{Z}\varpi_i$ satisfying the following conditions*:

(i) $\mathbb{C}v_\mu = F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r = V(\mu)$.

(ii) $F_{i+1}/F_i \simeq W(\eta_i)^{\oplus N_i}$ *for some positive integer* N_i.

(iii) $\eta_i \neq \eta_j$ *for* $i \neq j$.

Proof. For a non-negative integer m we set

$$P(m) = \left\{ \lambda \in \mathfrak{h}^* \mid \mu - \lambda = \sum_{i \in I_0} m_i \alpha_i \text{ and } \sum_{i \notin I} m_i = m \right\},$$

$$V_m = \bigoplus_{\lambda \in P(m)} V(\mu)_\lambda,$$

where $V(\mu)_\lambda$ is the weight space of $V(\mu)$ with weight λ. Then V_m is an $\mathfrak{l}_I$-module, and we have the irreducible decomposition

$$V_m = \tilde{W}(\eta_{m,1})^{\oplus N_{m,1}} \oplus \cdots \oplus \tilde{W}(\eta_{m,t_m})^{\oplus N_{m,t_m}}$$

where $\tilde{W}(\eta)$ is the irreducible $\mathfrak{l}_I$-module with highest weight η, and $\eta_{m,i} \neq \eta_{m,j}$ for $i \neq j$. For $1 \leq i \leq t_m$ we define a $\mathfrak{p}_I^+$-submodule $F_{m,i}$ of $V(\mu)$ by

$$F_{m,i} = V_0 \oplus \cdots \oplus V_{m-1} \oplus \tilde{W}(\eta_{m,1})^{\oplus N_{m,1}} \oplus \cdots \oplus \tilde{W}(\eta_{m,i})^{\oplus N_{m,i}}.$$

Then we have the sequence

$$\mathbb{C}v_\mu = F_{0,1} \subsetneq \cdots \subsetneq F_{m-1,t_{m-1}} \subsetneq F_{m,1} \subsetneq F_{m,2} \subsetneq \cdots \subsetneq F_{r,t_r} = V(\mu).$$

It is clear that the above sequence satisfies the conditions (ii) and (iii). □

For $\mu \in \mathfrak{h}^*_{I,+}$, we fix the sequence $\{F_1, F_2, \dots, F_r\}$ of $\mathfrak{p}_I^+$-submodules of $V(\mu)$ satisfying the conditions of Lemma 3.3, and set $\xi_\mu = \prod_{i=1}^{r-1} p_{\mu,\eta_i} \in R_I$.

Theorem 3.4. *For $\mu \in \mathfrak{h}^*_{I,+}$ we have $\xi_\mu \in \Xi_\mu$.*

Proof. It is clear that $U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_1) \simeq M_{R_I}(c+\mu)$. Let $\iota_j : U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_j) \to U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_{j+1})$ be the canonical injection. We show that there exists a commutative diagram

$$\begin{array}{ccc} U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_1) & = & M_{R_I}(c+\mu) \\ \downarrow{\scriptstyle \iota_{j-1}\cdots\iota_1} & & \downarrow{\scriptstyle \prod_{i=1}^{j-1} p_{\mu,\eta_i}} \\ U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_j) & \xrightarrow[\psi_j]{} & M_{R_I}(c+\mu) \end{array} \tag{3.1}$$

by the induction on j. Assume that there exists a commutative diagram (3.1) for j (≥ 1). From the exact sequence

$$\begin{aligned} 0 \longrightarrow U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_j) &\xrightarrow{\iota_j} U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_{j+1}) \\ &\longrightarrow U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_{j+1}/F_j) \longrightarrow 0, \end{aligned}$$

we have a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \operatorname{Hom}_{U_{R_I}(\mathfrak{g})}\big(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_{j+1}/F_j),\ M_{R_I}(c+\mu)\big) \\ &\longrightarrow \operatorname{Hom}_{U_{R_I}(\mathfrak{g})}\big(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_{j+1}),\ M_{R_I}(c+\mu)\big) \\ &\longrightarrow \operatorname{Hom}_{U_{R_I}(\mathfrak{g})}\big(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_j),\ M_{R_I}(c+\mu)\big) \\ &\xrightarrow{\delta} \operatorname{Ext}^1_{U_{R_I}(\mathfrak{g})}\big(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_{j+1}/F_j),\ M_{R_I}(c+\mu)\big) \\ &\longrightarrow \cdots . \end{aligned}$$

By Lemma 3.2 we have $\delta(p_{\mu,\eta_j}\psi_j) = p_{\mu,\eta_j}\delta(\psi_j) = 0$. Hence there exists an element $\psi_{j+1} \in \operatorname{Hom}_{U_{R_I}(\mathfrak{g})}\big(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_{j+1}),\ M_{R_I}(c+\mu)\big)$ such that $\psi_{j+1}\iota_j = p_{\mu,\eta_j}\psi_j$. Hence we have the commutative diagram

$$\begin{array}{ccc} U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_1) & = & M_{R_I}(c+\mu) \\ \Big\downarrow{\scriptstyle \iota_{j-1}\cdots\iota_1} & & \Big\downarrow{\scriptstyle \prod_{i=1}^{j-1} p_{\mu,\eta_i}} \\ U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_j) & \xrightarrow[\psi_j]{} & M_{R_I}(c+\mu) \\ \Big\downarrow{\scriptstyle \iota_j} & & \Big\downarrow{\scriptstyle p_{\mu,\eta_j}} \\ U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} F_{j+1}) & \xrightarrow[\psi_{j+1}]{} & M_{R_I}(c+\mu). \end{array}$$

In particular $\psi_r\iota_\mu = \xi_\mu$. Therefore $\xi_\mu \in \Xi_\mu$. □

Let $\psi_\mu \in \operatorname{Hom}_{U_{R_I}(\mathfrak{g})}\big(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} V(\mu)), M_{R_I}(c+\mu)\big)$ satisfying $\psi_\mu\iota_\mu = \xi_\mu$. Note that ψ_μ is non-zero since $\xi_\mu \neq 0$.

Remark 1. In general ξ_μ is not a generator of the ideal Ξ_μ. For example let $\mathfrak{g}$ be a simple Lie algebra of type G_2. We take the simple roots α_1 and α_2 such that α_1 is short. If $I = \{2\}$ and $\mu = \varpi_1$, then we have

$$\xi_\mu = (c(h_1)+1)(c(h_1)+2)(c(h_1)+3)(2c(h_1)+5)$$

up to constant multiple. But $(c(h_1)+1)(2c(h_1)+5) \in \Xi_\mu$.

Remark 2. For $I=\emptyset$ it is shown in Kashiwara [3] that Ξ_μ is generated by

$$\xi_\mu^0 = \prod_{\alpha\in\Delta^+}\prod_{j=0}^{\mu(h_\alpha)-1}(c(h_\alpha)+\rho(h_\alpha)+j),$$

where h_α is the coroot corresponding to α. Now we have $c(h_\alpha)+\rho(h_\alpha)+j = p_{\mu,\mu-(\mu(h_\alpha)-j)\alpha}$ up to constant multiple. Let $Wt(\mu)$ be the set of weights of $V(\mu)$. Then we have $\xi_\mu = \prod_{\eta\in Wt(\mu)\setminus\{\mu\}} p_{\mu,\eta}$, so $\xi_\mu^0 \notin \mathbb{C}^\times \xi_\mu$ in general.

§4. Semi-invariants

Let λ be a dominant integral weight. We regard the dual space $V(\lambda)^*$ as a left $\mathfrak{g}$-module via $\langle xv^*, v\rangle = \langle v^*, -xv\rangle$ for $x\in\mathfrak{g}$, $v^*\in V(\lambda)^*$ and $v\in V(\lambda)$. We denote the lowest weight vector of $V(\lambda)^*$ by v_λ^*. We normalize v_λ^* by $\langle v_\lambda^*, v_\lambda\rangle = 1$.

Definition 4.1. We define a regular function f^λ on G by $f^\lambda(g) = \langle v_\lambda^*, gv_\lambda\rangle$.

For $b^\pm\in B^\pm$ and $g\in G$ we have

$$f^\lambda(b^-gb^+) = \lambda^-(b^-)\lambda^+(b^+)f^\lambda(g),$$

where $\lambda^\pm$ is the character of $B^\pm$ corresponding to λ. This function f^λ is called $B^-\times B^+$-semi-invariant. Note that $f^{\lambda_1+\lambda_2} = f^{\lambda_1}f^{\lambda_2}$.

Fix $\mu\in\mathfrak{h}_{I,+}^*$. We take a basis $\{v_{\mu,j}\}_{0\le j\le n}$ of $V(\mu)$ consisting of weight vectors such that $v_{\mu,0}=v_\mu$ is the highest weight vector and $v_{\mu,n}$ is the lowest. We denote the dual basis of $V(\mu)^*$ by $\{v_{\mu,j}^*\}$. For a $U_{R_I}(\mathfrak{g})$-module homomorphism

$$\psi : U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c}\otimes_{\mathbb{C}} V(\mu)) \longrightarrow M_{R_I}(c+\mu)$$

we define elements $Y_j'\in U_{R_I}(\mathfrak{n}_I^-)$ for $0\le j\le n$ by

$$\psi(1\otimes 1_c\otimes v_{\mu,j}) = Y_j'\otimes 1_{c+\mu},$$

and define an element $\xi\in\Xi_\mu$ by $\xi=\psi\iota_\mu$. Note that $Y_0'=\xi$. Let $\pi_\mu=\pi : R_I\to U(\sum_{i\notin I}\mathbb{C}h_i)$ be the algebra isomorphism defined by $\pi(c(h_i)) = h_i-\mu(h_i)$ for $i\notin I$. Set $\pi(\sum_j a_j\otimes y_j) = \sum_j y_j\pi(a_j)$ for $a_j\in R_I$ and $y_j\in U(\mathfrak{n}_I^-)$. Clearly we have $y\otimes 1_{c+\mu} = \pi(y)\otimes 1_{c+\mu}\in M_{R_I}(c+\mu)$ $(y\in U_{R_I}(\mathfrak{n}_I^-))$. We set $Y_j=\pi(Y_j')$.

We define differential operators $P_\mu(\psi)$ and $\tilde{P}_\mu(\psi)$ on G by

$$(P_\mu(\psi)\varphi)(g) = \sum_{j=0}^{n} \langle g v^*_{\mu,j}, v_{\mu,0}\rangle (R(Y_j)\varphi)(g),$$
$$(\tilde{P}_\mu(\psi)\varphi)(g) = \sum_{j=0}^{n} \langle g v^*_{\mu,j}, v_{\mu,n}\rangle (R(Y_j)\varphi)(g),$$

where $R(y)$ $(y \in U(\mathfrak{g}))$ is the left invariant differential operator induced by the right action of G on itself. Then we have the following theorem.

Theorem 4.1. *Let $\mu \in \mathfrak{h}^*_{I,+}$ and*

$$\psi \in \mathrm{Hom}_{U_{R_I}(\mathfrak{g})}(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} V(\mu)), M_{R_I}(c+\mu)).$$

Then we have

$$P_\mu(\psi) f^{\lambda+\mu} = \xi(\lambda) f^\lambda$$

for any $\lambda \in \mathfrak{h}^*_{I,+}$. Here ξ is the element of Ξ_μ defined by $\xi = \psi \iota_\mu$.

Proof. We can prove this theorem similarly to Kashiwara [3, Theorem 2.1]. We give the sketch of the proof. First we can show that $P_\mu(\psi) f^{\lambda+\mu}$ is right N^+-invariant. Since $B^- N^+$ is an open dense subset of G, it is sufficient to show the statement on B^-. Next we can show that $(R(Y_j) f^{\lambda+\mu})(B^-) = 0$ for $j \geq 1$. By definitions we have $(R(Y_0) f^{\lambda+\mu})(g) = \xi(\lambda) f^{\lambda+\mu}(g)$ and $\langle g v^*_{\mu,0}, v_{\mu,0}\rangle = f^\mu(g)^{-1}$ for $g \in B^-$. So we have $P_\mu(\psi) f^{\lambda+\mu} = \xi(\lambda) f^\lambda$ on B^-. □

For a dominant integral weight λ we define a function $\tilde{f}^\lambda$ on G by

$$\tilde{f}^\lambda(g) = \langle v^*_{w_0\lambda}, g v_\lambda \rangle,$$

where $v^*_{w_0\lambda}$ is the highest weight vector which is normalized by $\langle v^*_{w_0\lambda}, \dot{w}_0 v_\lambda\rangle = 1$ and $\dot{w}_0 \in N_G(T)$ is a representative element of $w_0 \in W = N_G(T)/T$. Since $\tilde{f}^\lambda(\dot{w}_0 g) = f^\lambda(g)$, we obtain the following lemma.

Lemma 4.2. *Let $\lambda, \mu \in \mathfrak{h}^*_{I,+}$. For any $g \in G$ we have*

$$(\tilde{P}_\mu(\psi)\tilde{f}^\lambda)(\dot{w}_0 g) = (P_\mu(\psi) f^\lambda)(g).$$

By Theorem 4.1 we have the following corollary.

Corollary 4.3. *Let $\mu \in \mathfrak{h}^*_{I,+}$. We have*

$$\tilde{P}_\mu(\psi)\tilde{f}^{\lambda+\mu} = \xi(\lambda)\tilde{f}^\lambda$$

for any $\lambda \in \mathfrak{h}^*_{I,+}$. Here ξ is the element of Ξ_μ defined by $\xi = \psi \iota_\mu$.

§5. Commutative Parabolic Type

In the remainder of this paper we assume that

$$I = I_0 \setminus \{i_0\}$$

and that the highest root θ of $\mathfrak{g}$ is in $\alpha_{i_0} + \sum_{i \neq i_0} \mathbb{Z}_{\geq 0}\alpha_i$. Then it is known that $[\mathfrak{n}_I^{\pm}, \mathfrak{n}_I^{\pm}] = \{0\}$ and the pairs $(L_I, \mathfrak{n}_I^{\pm})$ are prehomogeneous vector spaces via the adjoint action, which are called of commutative parabolic type. The all pairs $(\mathfrak{g}, i_0)$ of commutative parabolic type are given by the Dynkin diagrams of Figure 1. Here the white vertex corresponds to i_0.

Since $\mathfrak{n}_I^-$ is identified with the dual space of $\mathfrak{n}_I^+$ via the Killing form, the symmetric algebra $S(\mathfrak{n}_I^-)$ is isomorphic to $\mathbb{C}[\mathfrak{n}_I^+]$. By the commutativity of $\mathfrak{n}_I^-$ we have $S(\mathfrak{n}_I^-) = U(\mathfrak{n}_I^-)$. Hence $\mathbb{C}[\mathfrak{n}_I^+]$ is identified with $U(\mathfrak{n}_I^-)$.

Set $\gamma_1 = \alpha_{i_0}$. For $i \geq 1$ we take γ_{i+1} as the lowest root in

$$\Gamma_i = \{\alpha \in \Delta^+ \setminus \Delta_I \mid \alpha + \gamma_j \notin \Delta \text{ and } \alpha - \gamma_j \notin \Delta \cup \{0\} \text{ for all } j \leq i\}.$$

Let $r = r(\mathfrak{g}, i_0)$ be the index such that $\Gamma_{r-1} \neq \emptyset$ and $\Gamma_r = \emptyset$. Note that $(\gamma_i, \gamma_j) = 0$ for $i \neq j$. It is known that all γ_i have the same length (see Moore [4]). For $1 \leq i \leq r$ we set $\lambda_i = -(\gamma_1 + \cdots + \gamma_i)$. The following fact is known (see [2], [9], [11]).

Lemma 5.1. *As an* $\mathrm{ad}(\mathfrak{l}_I)$*-module,* $U(\mathfrak{n}_I^-)$ *is multiplicity free, and*

$$U(\mathfrak{n}_I^-) = \bigoplus_{\mu \in \sum_{i=1}^r \mathbb{Z}_{\geq 0}\lambda_i} I(\mu),$$

where $I(\mu)$ *is an irreducible* $\mathfrak{l}_I$*-submodule of* $U(\mathfrak{n}_I^-)$ *with highest weight* μ*.*

Let $f_i \in U(\mathfrak{n}_I^-)$ be the highest weight vector of $I(\lambda_i)$. Since $U(\mathfrak{n}_I^-)$ is naturally identified with the symmetric algebra $S(\mathfrak{n}_I^-)$, we can determine the degree of $f \in U(\mathfrak{n}_I^-)$. If $f \in U(\mathfrak{n}_I^-)$ is a weight vector with weight $\mu \in -d\alpha_{i_0} + \sum_{i \in I} \mathbb{Z}_{\leq 0}\alpha_i$, then f is homogeneous and $\deg f = d$. In particular $\deg f_i = i$.

Considering the $[\mathfrak{l}_I, \mathfrak{l}_I]$-module homomorphism $U(\mathfrak{n}_I^-) \to V(\varpi_{i_0})$ such that $u \mapsto uv_{\varpi_{i_0}}$, we have the following corollary.

Corollary 5.2. *There exists a finite subset* M *of* $\sum_{i=1}^r \mathbb{Z}_{\geq 0}\lambda_i$ *such that*

$$V(\varpi_{i_0}) = \bigoplus_{\mu \in M} I(\mu)v_{\varpi_{i_0}}.$$

We have the following facts on L_I-orbits in $\mathfrak{n}_I^+$ (see Tanisaki [12] §1 or Wachi [13] §12).

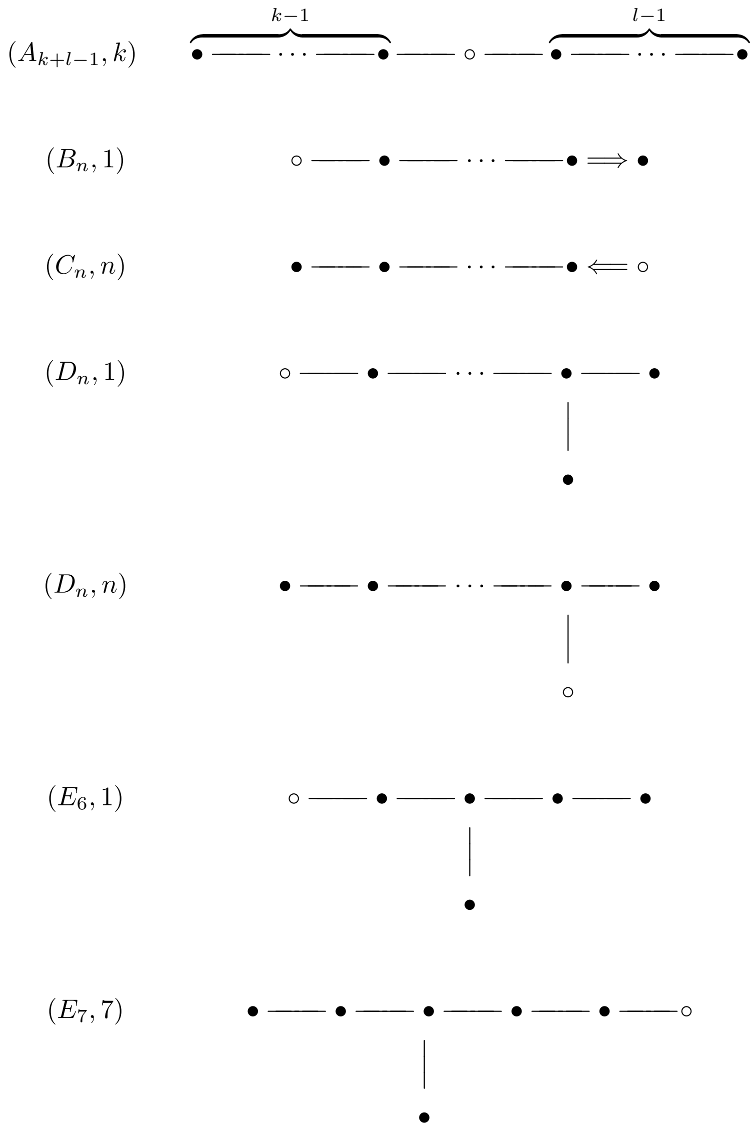

Figure 1. Commutative Parabolic Type

Proposition 5.3.

(i) $\mathfrak{n}_I^+$ *consists of* L_I*-orbits* $C_0, C_1, \dots, C_r$ *satisfying the closure relation*

$$\{0\} = C_0 \subset \overline{C_1} \subset \cdots \subset \overline{C_r} = \mathfrak{n}_I^+.$$

Here $\overline{C_i}$ *is the Zariski closure of* C_i*.*

(ii) *For* $1 \leq i \leq r$ *we set* $\mathcal{I}_i = \mathbb{C}[\mathfrak{n}_I^+]I(\lambda_i)$*. Then* $\mathcal{I}_i$ *is the defining ideal of* $\overline{C_{i-1}}$*.*

Remark 3. The orbit C_r is open. Set $S = \mathfrak{n}_I^+ \setminus C_r$. Then $S = \coprod_{i=0}^{r-1} C_i = \overline{C_{r-1}}$. By Lemma 2.1, the following are equivalent.

(i) $(L_I, \mathfrak{n}_I^+)$ is regular.

(ii) f_r is the unique basic relative invariant.

(iii) $\dim I(\lambda_r) = 1$.

Let h_{γ_i} be the coroot corresponding to γ_i. We set $\mathfrak{h}^- = \sum_{i=1}^r \mathbb{C}h_{\gamma_i}$. Then we have the following lemmas.

Lemma 5.4 (Moore [4]). *For* $\beta \in \Delta^+ \cap \Delta_I$ *there are three possible forms of the restriction* $\beta|_{\mathfrak{h}^-}$:

(i) $\beta|_{\mathfrak{h}^-} = 0$*. Then* $\beta \pm \gamma_i \notin \Delta$ *for all* i*.*

(ii) $\beta|_{\mathfrak{h}^-} = -\frac{\gamma_j}{2}|_{\mathfrak{h}^-}$*. Then* $\beta \pm \gamma_i \notin \Delta$ *for all* $i \neq j$*.*

(iii) $\beta|_{\mathfrak{h}^-} = \frac{\gamma_j - \gamma_k}{2}|_{\mathfrak{h}^-}$ $(j > k)$*. Then* $\beta \pm \gamma_i \notin \Delta$ *for all* $i \neq j, k$ *and* $\beta + \gamma_j \notin \Delta$*.*

Set $D = \{\alpha_i \mid i \in I\}$. For a subset Δ' of Δ, $\Delta'(\mathfrak{h}^-)$ is defined by

$$\Delta'(\mathfrak{h}^-) = \left\{ \beta \in \sum_{i=1}^r \mathbb{Q}\gamma_i \mid \beta|_{\mathfrak{h}^-} = \alpha|_{\mathfrak{h}^-} \ (\alpha \in \Delta') \right\}.$$

Lemma 5.5 (Moore [4]). *There are two possibilities as follows.*

Case (a):

$$D(\mathfrak{h}^-) = \left\{ \frac{1}{2}(\gamma_{i+1} - \gamma_i) \mid 1 \leq i \leq r-1 \right\} \cup \{0\},$$

$$(\Delta_I \cap \Delta^+)(\mathfrak{h}^-) = \left\{ \frac{1}{2}(\gamma_j - \gamma_i) \mid 1 \leq i \leq j \leq r \right\},$$

$$(\Delta^+ \setminus \Delta_I)(\mathfrak{h}^-) = \left\{ \frac{1}{2}(\gamma_j + \gamma_i) \mid 1 \leq i \leq j \leq r \right\}.$$

Case (b):

$$\begin{aligned} D(\mathfrak{h}^-) &= \left\{\frac{1}{2}(\gamma_{i+1}-\gamma_i) \mid 1 \le i \le r-1\right\} \cup \left\{-\frac{1}{2}\gamma_r\right\} \cup \{0\}, \\ (\Delta_I \cap \Delta^+)(\mathfrak{h}^-) &= \left\{\frac{1}{2}(\gamma_j-\gamma_i) \mid 1 \le i \le j \le r\right\} \cup \left\{-\frac{1}{2}\gamma_i \mid 1 \le i \le r\right\}, \\ (\Delta^+ \setminus \Delta_I)(\mathfrak{h}^-) &= \left\{\frac{1}{2}(\gamma_j+\gamma_i) \mid 1 \le i \le j \le r\right\} \cup \left\{\frac{1}{2}\gamma_i \mid 1 \le i \le r\right\}. \end{aligned}$$

Remark 4. By Weyl's dimension formula, there exists $\beta \in \Delta_I \cap \Delta^+$ such that $\beta|_{\mathfrak{h}^-} = -\frac{1}{2}\gamma_i|_{\mathfrak{h}^-}$ if and only if $\dim I(\lambda_r) \neq 1$. In other words the case where $(L_I, \mathfrak{n}^{\pm})$ are regular coincides with the case (a) in Lemma 5.5.

Lemma 5.6. *If $(L_I, \mathfrak{n}_I^+)$ is regular, then γ_r is the highest root θ. If $(L_I, \mathfrak{n}_I^+)$ is not regular, then $\theta|_{\mathfrak{h}^-} = \frac{\gamma_r}{2}|_{\mathfrak{h}^-}$.*

Proof. Assume that $(L_I, \mathfrak{n}_I^+)$ is regular. Let us show that $\gamma_r + \alpha_j \notin \Delta$ for any $j \in I_0$. Since $\gamma_r, \alpha_{i_0} \in \Delta^+ \setminus \Delta_I$, $\gamma_r + \alpha_{i_0}$ is not a root. If $j \in I$, then we have $\alpha_j = \frac{\gamma_k - \gamma_{k-1}}{2}$ or 0 on $\mathfrak{h}^-$ by Lemma 5.5 and Remark 4. So we have $\alpha_j + \gamma_r \notin \Delta$ by Lemma 5.4. Hence γ_r is the highest root.

Next we assume that $(L_I, \mathfrak{n}_I^+)$ is not regular. Then there exists a simple root α_{j_0} such that $\alpha_{j_0} = -\frac{\gamma_r}{2}$ on $\mathfrak{h}^-$. Since $(\gamma_r, \alpha_{j_0}) < 0$, we have $\gamma_r + \alpha_{j_0} \in \Delta$. In particular $\theta \in \gamma_r + \alpha_{j_0} + \sum_{j \neq i_0} \mathbb{Z}_{\ge 0}\alpha_j$. So we have on $\mathfrak{h}^-$

$$\begin{aligned} \theta &= \frac{\gamma_r}{2} + \sum_{i=1}^{r-1} a_i \frac{\gamma_{i+1}-\gamma_i}{2} - a_r\frac{\gamma_r}{2} \\ &= -a_1\frac{\gamma_1}{2} + \sum_{i=2}^{r-1}(a_{i-1}-a_i)\frac{\gamma_i}{2} + (1-a_r+a_{r-1})\frac{\gamma_r}{2}, \end{aligned}$$

where $a_i \in \mathbb{Z}_{\ge 0}$. By Lemma 5.5, we have $\theta|_{\mathfrak{h}^-} = \frac{\gamma_k+\gamma_l}{2}|_{\mathfrak{h}^-}$ or $\frac{\gamma_k}{2}|_{\mathfrak{h}^-}$. Therefore all a_i must be 0, and $\theta|_{\mathfrak{h}^-} = \frac{\gamma_r}{2}|_{\mathfrak{h}^-}$. □

Since $\mathfrak{n}_I^+$ is an irreducible $\mathfrak{l}_I$-module, we have $w_I\alpha_{i_0} = \theta$ and $w_I\varpi_{i_0} = \varpi_{i_0}$.

Lemma 5.7. $\lambda_r = w_I w_0 \varpi_{i_0} - \varpi_{i_0}$.

Proof. Let $v_{w_0\varpi_{i_0}}$ be the lowest weight vector of $V(\varpi_{i_0})$. Then $w_I w_0 \varpi_{i_0}$ is the highest weight of the irreducible $[\mathfrak{l}_I, \mathfrak{l}_I]$-submodule generated by $v_{w_0\varpi_{i_0}}$.

By Corollary 5.2 there exists $\mu = \sum_{i=1}^{r} m_i\lambda_i \in \sum_{i=1}^{r} \mathbb{Z}_{\geq 0}\lambda_i$ such that $I(\mu)v_{\varpi_{i_0}} = U([\mathfrak{l}_I, \mathfrak{l}_I])v_{w_0\varpi_{i_0}}$. In particular $\mu = w_I w_0\varpi_{i_0} - \varpi_{i_0}$. Now

$$(w_I w_0\varpi_{i_0} - \varpi_{i_0}, \alpha_{i_0}) = (w_0\varpi_{i_0}, \theta) - (\varpi_{i_0}, \alpha_{i_0}) = (\varpi_{i_0}, -\theta) - (\varpi_{i_0}, \alpha_{i_0})$$
$$= -2(\varpi_{i_0}, \alpha_{i_0}) = -(\alpha_{i_0}, \alpha_{i_0})$$

and

$$(\mu, \alpha_{i_0}) = \sum_{i=1}^{r} m_i(\lambda_i, \alpha_{i_0}) = -\sum_{i=1}^{r} m_i(\alpha_{i_0}, \alpha_{i_0}).$$

Therefore we have $\sum_{i=1}^{r} m_i = 1$, and $\mu = \lambda_k$ for an index k. Hence it is enough to show that $k = r$. We define the index $j \in I_0$ by $w_0\alpha_{i_0} = -\alpha_j$. Then we have $w_I(\lambda_k + \varpi_{i_0}) = w_0\varpi_{i_0} = -\varpi_j$. Since $w_I\alpha_{i_0} = \theta$, we have

$$(-\varpi_j, \alpha_{i_0}) = (w_I(\lambda_k + \varpi_{i_0}), \alpha_{i_0}) = (\lambda_k + \varpi_{i_0}, \theta).$$

Assume that $(L_I, \mathfrak{n}_I^+)$ is regular. Then we have $\theta = \gamma_r$ by Lemma 5.6. Hence $(\lambda_k + \varpi_{i_0}, \theta) = -\delta_{kr}(\alpha_{i_0}, \alpha_{i_0}) + \frac{(\alpha_{i_0}, \alpha_{i_0})}{2}$. In particular $(-\varpi_j, \alpha_{i_0}) \neq 0$. Hence $j = i_0$, and $(\lambda_k + \varpi_{i_0}, \theta) = -\frac{(\alpha_{i_0}, \alpha_{i_0})}{2}$. So we have $k = r$.

Next we assume that $(L_I, \mathfrak{n}_I^+)$ is not regular. By Lemma 5.6 we have $\theta|_{\mathfrak{h}^-} = \frac{\gamma_r}{2}|_{\mathfrak{h}^-}$, so

$$(\lambda_k + \varpi_{i_0}, \theta) = -\delta_{kr}\frac{(\alpha_{i_0}, \alpha_{i_0})}{2} + \frac{(\alpha_{i_0}, \alpha_{i_0})}{2} \geq 0.$$

Since $(-\varpi_j, \alpha_{i_0}) \leq 0$, we have $(\lambda_k + \varpi_{i_0}, \theta) = (-\varpi_j, \alpha_{i_0}) = 0$. Therefore $j \neq i_0$ and $k = r$. □

Remark 5. By the proof of Lemma 5.7 we see that the prehomogeneous vector space $(L_I, \mathfrak{n}_I^+)$ is regular if and only if $w_0\alpha_{i_0} = -\alpha_{i_0}$. Hence the pairs $(\mathfrak{g}, i_0)$ such that the corresponding prehomogeneous vector spaces are regular are as follows: (A_{2n-1}, n), $(B_n, 1)$, (C_n, n), $(D_n, 1)$, $(D_{2n}, 2n)$ and $(E_7, 7)$.

For $\mu \in \mathfrak{h}_{I,+}^* = \mathbb{Z}_{\geq 0}\varpi_{i_0}$ we take the lowest weight vector $v_{w_0\mu}$ of $V(\mu)$. Then the $U_{R_I}(\mathfrak{g})$-module $U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} V(\mu))$ is generated by $1 \otimes 1_c \otimes v_{w_0\mu}$. For the $U_{R_I}(\mathfrak{g})$-module homomorphism ψ_μ defined in Section 3, there exists a non-zero element $u_\mu \in U_{R_I}(\mathfrak{n}_I^-)$ such that $\psi_\mu(1 \otimes 1_c \otimes v_{w_0\mu}) = u_\mu \otimes 1_{c+\mu}$. Since $y(1 \otimes 1_c \otimes v_{w_0\mu}) = 0$ for any $y \in \mathfrak{l}_I \cap \mathfrak{n}^-$, $u_\mu \in U_{R_I}(\mathfrak{n}_I^-)$ is a lowest weight vector with weight $w_0\mu - \mu$ as an $\mathrm{ad}(\mathfrak{l}_I)$-module. By Lemma 5.1 such a lowest weight vector is unique up to constant multiple. Therefore $u_\mu = a_\mu u_\mu^0$ where $a_\mu \in R_I \setminus \{0\}$ and $u_\mu^0 \in U(\mathfrak{n}_I^-)$ is the unique lowest weight vector with weight $w_0\mu - \mu$. If $x(1 \otimes 1_c \otimes v_{w_0\mu}) = 0$ for $x \in U_{R_I}(\mathfrak{g})$, then we have $xu_\mu^0 \otimes 1_{c+\mu} = 0$

since $a_\mu \neq 0$. Hence we can define a $U_{R_I}(\mathfrak{g})$-module homomorphism ψ_μ^0 from $U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} V(\mu))$ to $M_{R_I}(c+\mu)$ by

$$\psi_\mu^0(x(1 \otimes 1_c \otimes v_{w_0\mu})) = xu_\mu^0 \otimes 1_{c+\mu}$$

for any $x \in U_{R_I}(\mathfrak{g})$. We set $\xi_\mu^0 = \psi_\mu^0 \iota_\mu \in \Xi_\mu$.

From the uniqueness of u_μ^0 we have

$$\psi(1 \otimes 1_c \otimes v_{w_0\mu}) = au_\mu^0 \otimes 1_{c+\mu} = a\psi_\mu^0(1 \otimes 1_c \otimes v_{w_0\mu}) \quad (a \in R_I)$$

for any $\psi \in \mathrm{Hom}_{U_{R_I}(\mathfrak{g})}\big(U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} V(\mu)),\ M_{R_I}(c+\mu)\big)$. Therefore we have the following.

Proposition 5.8. *Let $\mu \in \mathfrak{h}_{I,+}^*$. We have $\Xi_\mu = R_I \xi_\mu^0$.*

We call the above homomorphism ψ_μ^0 the minimal map in this paper.

Let $\tilde{f}_r \in U(\mathfrak{n}_I^-)$ be the lowest weight vector of the irreducible $\mathfrak{l}_I$-submodule $I(\lambda_r)$.

Proposition 5.9. *Let $\mu = m\varpi_{i_0} \in \mathfrak{h}_{I,+}^*$. Under the identification* $\exp : \mathfrak{n}_I^- \simeq N_I^-$ *we have*

$$(\tilde{P}_\mu(\psi_\mu^0)\varphi)|_{\mathfrak{n}_I^-} = \tilde{f}_r(\partial)^m(\varphi|_{\mathfrak{n}_I^-}).$$

Proof. Let $\{v_i\}_{0 \leq i \leq n}$ be a basis of $V(\mu)$ consisting of weight vectors such that v_n has the lowest weight $w_0\mu$. We denote the dual basis by $\{v_i^*\}$. We define elements $Y_i' \in U_{R_I}(\mathfrak{n}_I^-)$ by $\psi_\mu^0(1 \otimes 1_c \otimes v_i) = Y_i' \otimes 1_{c+\mu}$. Set $Y_i = \pi(Y_i')$ Then we have

$$(\tilde{P}_\mu(\psi_\mu^0)\varphi)(g) = \sum_{i=0}^n \langle gv_i^*, v_n\rangle (R(Y_i)\varphi)(g).$$

For $g \in N_I^-$ we have $\langle gv_i^*, v_n\rangle = \delta_{i,n}$. Therefore it is sufficient to show that

$$R(Y_n) = \tilde{f}_r^m(\partial) \tag{5.1}$$

By the definition of ψ_μ^0, Y_n is the lowest weight vector of $\mathrm{ad}(\mathfrak{l}_I)$-module $U(\mathfrak{n}_I^-)$ with weight $w_0\mu - \mu = m(w_0\varpi_{i_0} - \varpi_{i_0})$. By Lemma 5.7 the weight of $\tilde{f}_r$ is $w_0\varpi_{i_0} - \varpi_{i_0}$. Hence we have $Y_n = \tilde{f}_r^m$ up to constant multiple. Since $\mathfrak{n}_I^-$ is commutative, we have $R(y) = y(\partial)$ for any $y \in U(\mathfrak{n}_I^-)$. Hence the equation (5.1) holds. □

Finally we define subalgebras of $\mathfrak{g}$. For $1 \leq p \leq r = r(\mathfrak{g}, i_0)$ we set

$$\Delta_{(p)}^+ = \left\{\beta \in \Delta^+ \mid \beta|_{\mathfrak{h}^-} = \frac{\gamma_j + \gamma_k}{2}|_{\mathfrak{h}^-} \ \text{ for some } 1 \leq j \leq k \leq p\right\}.$$

By Lemma 5.5 we have $\Delta^+_{(p)} \subset \Delta^+ \setminus \Delta_I$. We define subspaces $\mathfrak{n}^{\pm}_{(p)}$ of $\mathfrak{g}$ by $\mathfrak{n}^{\pm}_{(p)} = \sum_{\beta \in \Delta^+_{(p)}} \mathfrak{g}_{\pm\beta}$. Set $\mathfrak{l}_{(p)} = [\mathfrak{n}^+_{(p)}, \mathfrak{n}^-_{(p)}]$ and $I_{(p)} = \{i \in I \mid \mathfrak{g}_{\alpha_i} \subset \mathfrak{l}_{(p)}\}$. Then we have the following.

Lemma 5.10 (see Wallach [14] and Wachi [13]). *We set $\mathfrak{g}_{(p)} = \mathfrak{n}^-_{(p)} \oplus \mathfrak{l}_{(p)} \oplus \mathfrak{n}^+_{(p)}$. Then $\mathfrak{g}_{(p)}$ is a simple subalgebra of $\mathfrak{g}$ with simple roots $\{\alpha_{i_0}\} \sqcup \{\alpha_i \mid i \in I_{(p)}\}$, and the pair $(\mathfrak{g}_{(p)}, i_0)$ is of regular commutative parabolic type. For any $1 \le j \le p$ we have $f_j \in U(\mathfrak{n}^-_{(p)})$, and f_p is a basic relative invariant of $(L_{(p)}, \mathfrak{n}^+_{(p)})$, where $L_{(p)}$ is the subgroup of L_I corresponding to $\mathfrak{l}_{(p)}$.*

Note that if $(L_I, \mathfrak{n}^+_I)$ is regular, then $\mathfrak{g}_{(r)} = \mathfrak{g}$, and that if not regular, then $\mathfrak{g}_{(r)} \subsetneq \mathfrak{g}$.

§6. Regular Type

In this section we assume that the prehomogeneous vector spaces $(L_I, \mathfrak{n}^{\pm}_I)$ are regular. By Remark 5 we have $w_0\varpi_{i_0} = -\varpi_{i_0}$. We take γ_i, λ_i and f_i $(1 \le i \le r = r(\mathfrak{g}, i_0))$ as in Section 5. Then the highest weight vector $f_r \in U(\mathfrak{n}^-_I) \simeq \mathbb{C}[\mathfrak{n}^+_I]$ is the unique basic relative invariant of $(L_I, \mathfrak{n}^+_I)$ with character $2\varpi_{i_0}$, and it is also the lowest weight vector of the irreducible $\mathfrak{l}_I$-module $I(\lambda_r)$.

Proposition 6.1. *Let $b(s)$ be the b-function of the basic relative invariant of $(L_I, \mathfrak{n}^-_I)$. Then for $m \in \mathbb{Z}_{>0}$ we have*

$$\xi^0_{m\varpi_{i_0}}(s\varpi_{i_0}) = b(s+m-1)b(s+m-2)\cdots b(s)$$

up to constant multiple.

Proof. For any $l \in L_I$ and $n \in \mathfrak{n}^-_I$ we have

$$\tilde{f}^{\varpi_{i_0}}(l\exp(n)l^{-1}) = (w_0\varpi_{i_0} - \varpi_{i_0})(l)\tilde{f}^{\varpi_{i_0}}(\exp(n)) = -2\varpi_{i_0}(l)\tilde{f}^{\varpi_{i_0}}(\exp(n)).$$

Thus $\tilde{f}^{\varpi_{i_0}}|_{\mathfrak{n}^-_I}$ is the basic relative invariant of $(L_I, \mathfrak{n}^-_I)$ under the identification $\mathfrak{n}^-_I \simeq N^-_I$. Hence we have

$$\begin{aligned} f_r(\partial)^m \tilde{f}^{(s+m)\varpi_{i_0}}|_{\mathfrak{n}^-_I} &= f_r(\partial)^m (\tilde{f}^{\varpi_{i_0}}|_{\mathfrak{n}^-_I})^{s+m} \\ &= b(s+m-1)b(s+m-2)\cdots b(s)\tilde{f}^{s\varpi_{i_0}}|_{\mathfrak{n}^-_I}. \end{aligned}$$

By Corollary 4.3 we have

$$\tilde{P}_{m\varpi_{i_0}}(\psi^0_{m\varpi_{i_0}})\tilde{f}^{(s+m)\varpi_{i_0}} = \xi^0_{m\varpi_{i_0}}(s\varpi_{i_0})\tilde{f}^{s\varpi_{i_0}}.$$

Therefore the statement holds by Proposition 5.9. □

In the rest of this section we shall show that $\xi_{\varpi_{i_0}} = \xi^0_{\varpi_{i_0}}$ up to constant multiple.

Lemma 6.2. *For any $1 \leq i \leq r$ we have $w_I\gamma_i = \gamma_{r-i+1}$.*

Proof. We show the statement by the induction on i. By Lemma 5.6 we have $w_I\gamma_1 = \theta = \gamma_r$. Assume that $i > 1$ and $w_I\gamma_j = \gamma_{r-j+1}$ for $1 \leq j \leq i-1$. Then we have $\gamma_{r-i+1} \pm w_I\gamma_j = \gamma_{r-i+1} \pm \gamma_{r-j+1} \notin \Delta \cup \{0\}$, and we have $w_I\gamma_{r-i+1} \pm \gamma_j \notin \Delta \cup \{0\}$. Hence $w_I\gamma_{r-i+1} \in \Gamma_{i-1}$. In particular $w_I\gamma_{r-i+1} - \gamma_i \in \sum_{k\in I} \mathbb{Z}_{\geq 0}\alpha_k$. By Lemma 5.5 there exist γ_k and γ_l such that $k \leq l$ and $w_I\gamma_i = \frac{\gamma_k+\gamma_l}{2}$ on $\mathfrak{h}^-$. For any $m \geq r-i+2$ we have $(w_I\gamma_i, \gamma_m) = (\gamma_i, w_I\gamma_m) = (\gamma_i, \gamma_{r-m+1}) = 0$. Since $(w_I\gamma_i, \gamma_l) > 0$, we have $l \leq r-i+1$ and $\gamma_l - w_I\gamma_i \in \Delta \cup \{0\}$. Now we have $\gamma_l - w_I\gamma_i = \frac{\gamma_l-\gamma_k}{2}$ on $\mathfrak{h}^-$. By Lemma 5.5 if $\gamma_l - w_I\gamma_i \neq 0$, it is a positive root. Therefore we have $\gamma_{r-i+1} - w_I\gamma_i = (\gamma_{r-i+1} - \gamma_l) + (\gamma_l - w_I\gamma_i) \in \sum_{k\in I} \mathbb{Z}_{\geq 0}\alpha_k$. Hence $w_I\gamma_{r-i+1} - \gamma_i \in \sum_{k\in I} \mathbb{Z}_{\leq 0}\alpha_k$, and we have $w_I\gamma_{r-i+1} = \gamma_i$. □

By Lemma 6.2 we can show the following easily.

Corollary 6.3. *The lowest weight $w_I\lambda_{r-1}$ of $I(\lambda_{r-1})$ is $\lambda_r + \alpha_{i_0}$.*

Lemma 6.4. *For any $1 \leq p \leq r = r(\mathfrak{g}, i_0)$ we have*

$$e_{i_0} f_p \otimes 1_{c+\mu} \in U_{R_I}(\mathfrak{l}_I \cap \mathfrak{n}^-)(f_{p-1} \otimes 1_{c+\mu}) \subset M_{R_I}(c+\mu),$$

where $e_{i_0} \in \mathfrak{g}_{\alpha_{i_0}} \setminus \{0\}$.

Proof. By Lemma 5.10 it is sufficient to show that the statement holds for $p = r$. We define $y \in U_{R_I}(\mathfrak{n}_I^-)$ by

$$e_{i_0}(f_r \otimes 1_{c+\mu}) = y \otimes 1_{c+\mu}.$$

Since $[e_{i_0}, \mathfrak{l}_I \cap \mathfrak{n}^-] = \{0\}$ and f_r is the lowest weight vector of the $\mathrm{ad}(\mathfrak{l}_I)$-module $U(\mathfrak{n}_I^-)$, y is the lowest weight vector as an $\mathrm{ad}(\mathfrak{l}_I)$-module. Moreover the weight of y is $\lambda_r + \alpha_{i_0} = w_I\lambda_{r-1}$, which is the lowest weight of the irreducible component $I(\lambda_{r-1}) = \mathrm{ad}(U(\mathfrak{l}_I))f_{r-1}$. Therefore we have

$$y \otimes 1_{c+\mu} \in U_{R_I}(\mathfrak{l}_I \cap \mathfrak{n}^-)(f_{r-1} \otimes 1_{c+\mu}).$$

□

Corollary 6.5. *Let $u \in U(\mathfrak{n}^+)$ with weight $k\alpha_{i_0} + \sum_{i \in I} m_i \alpha_i$. Then we have*

$$uf_r \otimes 1_{c+\mu} \in U_{R_I}(\mathfrak{l}_I \cap \mathfrak{n}^-)(f_{r-k} \otimes 1_{c+\mu}).$$

Proof. We shall show the statement by the induction on k. If $k = 0$, then the statement is clear. Assume that $k > 0$, and the statement holds for $k - 1$. We write $u = \sum_j u_j e_{i_0} u'_j$, where $u_j \in U(\mathfrak{l}_I \cap \mathfrak{n}^+)$ and $u'_j \in U(\mathfrak{n}^+)$. Then the weight of u'_j is in $(k-1)\alpha_{i_0} + \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$, and hence we have

$$\begin{aligned} uf_r \otimes 1_{c+\mu} &\in \sum_j u_j e_{i_0} U_{R_I}(\mathfrak{l}_I \cap \mathfrak{n}^-)(f_{r-k+1} \otimes 1_{c+\mu}) \\ &\subset U_{R_I}(\mathfrak{l}_I)(e_{i_0} f_{r-k+1} \otimes 1_{c+\mu}). \end{aligned}$$

Here note that $[e_{i_0}, U_{R_I}(\mathfrak{l}_I \cap \mathfrak{n}^-)] = 0$. By Lemma 6.4 we have

$$e_{i_0} f_{r-k+1} \otimes 1_{c+\mu} \in U_{R_I}(\mathfrak{l}_I \cap \mathfrak{n}^-)(f_{r-k} \otimes 1_{c+\mu}).$$

Therefore we obtain

$$uf_r \otimes 1_{c+\mu} \in U_{R_I}(\mathfrak{l}_I)(f_{r-k} \otimes 1_{c+\mu}) = U_{R_I}(\mathfrak{l}_I \cap \mathfrak{n}^-)(f_{r-k} \otimes 1_{c+\mu}).$$

□

Theorem 6.6. *We have $\xi_{\varpi_{i_0}} = \prod_{j=1}^{r} p_{\varpi_{i_0}, \lambda_j + \varpi_{i_0}} \in \mathbb{C}^\times \xi^0_{\varpi_{i_0}}$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.*

Proof. Let $v_{-\varpi_{i_0}}$ be the lowest weight vector of $V(\varpi_{i_0})$. Since f_r is the lowest weight vector of $U(\mathfrak{n}_I^-)$ with weight $-2\varpi_0$, we have $\psi^0_{\varpi_{i_0}}(1 \otimes 1_c \otimes v_{-\varpi_{i_0}}) = f_r \otimes 1_{c+\varpi_{i_0}}$. It is clear that

$$\varpi_{i_0} - w_0 \varpi_{i_0} = 2\varpi_{i_0} = -\lambda_r \in r\alpha_{i_0} + \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i.$$

Set $P(j) = \{\lambda \mid \varpi_{i_0} - \lambda \in j\alpha_{i_0} + \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i\}$. We define an $\mathfrak{l}_I$-submodule V_j of $V(\varpi_{i_0})$ by

$$V_j = \bigoplus_{\lambda \in P(j)} V(\varpi_{i_0})_\lambda$$

(cf. Section 3). Note that $V_j \neq 0$ for $0 \leq j \leq r$. We take the irreducible decomposition of V_j

$$V_j = \tilde{W}(\eta_{j,1}) \oplus \cdots \oplus \tilde{W}(\eta_{j,N_j}),$$

where $\tilde{W}(\eta)$ is an irreducible $\mathfrak{l}_I$-module with highest weight η. Let $v_{j,k}$ be the highest weight vector of $\tilde{W}(\eta_{j,k})$. There exists an element $u_{j,k} \in U(\mathfrak{n}^+)$ such that $u_{j,k} v_{-\varpi_{i_0}} = v_{j,k}$. Then the weight of $u_{j,k}$ is in $(r-j)\alpha_{i_0} + \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. By Corollary 6.5 we have

$$\begin{aligned}\psi^0_{\varpi_{i_0}}(1 \otimes 1_c \otimes v_{j,k}) &= u_{j,k} \psi^0_{\varpi_{i_0}}(1 \otimes 1_c \otimes v_{-\varpi_{i_0}}) \\ &= u_{j,k} f_r \otimes 1_{c+\varpi_{i_0}} \in U_{R_I}(\mathfrak{l}_I \cap \mathfrak{n}^-) f_j \otimes 1_{c+\varpi_{i_0}}.\end{aligned}$$

Since $v_{j,k}$ is the highest weight vector, we have

$$\psi^0_{\varpi_{i_0}}(1 \otimes 1_c \otimes v_{j,k}) \in R_I(f_j \otimes 1_{c+\varpi_{i_0}}).$$

In particular $\eta_{j,k} = \lambda_j + \varpi_{i_0}$ for $1 \leq k \leq N_j$, and the irreducible decomposition of $V(\varpi_{i_0})$ as an $\mathfrak{l}_I$-module is given by

$$V(\varpi_{i_0}) = \bigoplus_{j=0}^{r} \tilde{W}(\lambda_j + \varpi_{i_0})^{\oplus N_j},$$

where we set $\lambda_0 = 0$. (By Corollary 5.2 we have $N_j = 1$ for any j.) Therefore we have $\xi_{\varpi_{i_0}} = \prod_{j=1}^r p_{\varpi_{i_0}, \lambda_j + \varpi_{i_0}}$, which is regarded as a polynomial function on $\mathbb{C}\varpi_{i_0}$. Since $\deg p_{\varpi_{i_0}, \lambda_j + \varpi_{i_0}} = 1$ for $j \geq 1$, we have $\deg \xi_{\varpi_{i_0}} = r$. Now we have $\deg \xi^0_{\varpi_{i_0}} = \deg b(s) = \deg f_r = r$. From Proposition 5.8 we have $\xi_{\varpi_{i_0}} \in \mathbb{C}^\times \xi^0_{\varpi_{i_0}}$, hence the statement holds. □

For $1 \leq i < j \leq r$ we set $c_{i,j} = \sharp\{\alpha \in \Delta_I \cap \Delta^+ \mid \alpha|_{\mathfrak{h}^-} = \frac{\gamma_j - \gamma_i}{2}|_{\mathfrak{h}^-}\}$. It is known that $c_{i,j} = \sharp\{\alpha \in \Delta^+ \setminus \Delta_I \mid \alpha|_{\mathfrak{h}^-} = \frac{\gamma_j + \gamma_i}{2}|_{\mathfrak{h}^-}\}$ and this number is independent of i or j (see [15]). Set $c_0 = c_{i,j}$. Then we have $(2\rho, \gamma_j) = d_0(1 + c_0(j-1))$, where $d_0 = (\alpha_{i_0}, \alpha_{i_0})$. In particular $(2\rho, \lambda_j) = -j d_0 (1 + \frac{j-1}{2} c_0)$. Since $(\gamma_i, \gamma_j) = \delta_{i,j} d_0$, we have $(\varpi_{i_0}, \varpi_{i_0}) = (\lambda_j + \varpi_{i_0}, \lambda_j + \varpi_{i_0})$ for $1 \leq j \leq r$. Hence we have

$$p_{\varpi_{i_0}, \lambda_j + \varpi_{i_0}}(s\varpi_{i_0}) = -2(s\varpi_{i_0} + \rho, \lambda_j) = j d_0 \left(s + 1 + \frac{j-1}{2} c_0\right)$$

and the b-function is written by

$$b(s) = \prod_{j=1}^{r} \left(s + 1 + \frac{j-1}{2} c_0\right)$$

up to constant multiple (cf. Muller, Rubenthaler and Schiffmann [5]).

§7. Non-regular Type

Assume that the prehomogeneous vector space $(L_I, \mathfrak{n}_I^+)$ is not regular. We take γ_i, λ_i and f_i $(1 \le i \le r = r(\mathfrak{g}, i_0))$ as in Section 5. For $\mu = m\varpi_{i_0} \in \mathfrak{h}_{I,+}^*$ we denote by $\tilde{v}_\mu$ the highest weight vector of the irreducible $\mathfrak{l}_I$-submodule of $V(\mu)$ generated by the lowest weight vector of $V(\mu)$. The weight of $\tilde{v}_\mu$ is $w_I w_0 \mu$. We take $u \in U_{R_I}(\mathfrak{n}_I^-)$ as $\psi_\mu^0(1 \otimes 1_c \otimes \tilde{v}_\mu) = u \otimes 1_{c+\mu}$. By the definition of ψ_μ^0 we have $u \in U(\mathfrak{n}_I^-)$. Moreover u is the highest weight vector of $U(\mathfrak{n}_I^-)$ with weight $w_I w_0 \mu - \mu = w_I(w_0\mu - \mu)$. By Lemma 5.7 we have $w_I(w_0\mu - \mu) = m\lambda_r$. Therefore we have $u = f_r^m$. Set $\xi_\mu^0 = \psi_\mu^0 \iota_\mu \in R_I$.

We define subalgebras $\mathfrak{g}_{(r)}$, $\mathfrak{l}_{(r)}$ and $\mathfrak{n}_{(r)}^\pm$ of $\mathfrak{g}$ as in Lemma 5.10. We set $\tilde{\mathfrak{p}}^+ = \mathfrak{l}_{(r)} \oplus \mathfrak{n}_{(r)}^+$. We denote by $\tilde{V}(\mu)$ the irreducible $\mathfrak{g}_{(r)}$-module with highest weight μ. Let $\tilde{I}_0$ be an index set of simple roots of $\mathfrak{g}_{(r)}$, that is, $\tilde{I}_0 = I_{(r)} \sqcup \{i_0\}$ (see Lemma 5.10). We set $\tilde{I} = I_{(r)}$ and $\tilde{\mathfrak{g}} = \mathfrak{g}_{(r)}$ for simplicity. Let $\tilde{R}$ be an enveloping algebra of $\sum_{i \in \tilde{I}_0} \mathbb{C}h_i / \sum_{i \in \tilde{I}} \mathbb{C}h_i$. Since we have the canonical identification $R_I \simeq \tilde{R}$, a $U_{R_I}(\tilde{\mathfrak{g}})$-submodule

$$\tilde{M}(c+\mu) = U_{R_I}(\tilde{\mathfrak{g}}) \otimes_{U_{R_I}(\tilde{\mathfrak{p}}^+)} R_{I,c+\mu}$$

of $M_{R_I}(c+\mu)$ is a universal scalar generalized Verma module associated with $\tilde{\mathfrak{g}}$. We define an element $\tilde{\xi}_\mu^0$ of $\tilde{R} \simeq R_I$ by the multiplication map on $\tilde{M}(c+\mu)$ induced by the minimal map

$$\tilde{\psi}_\mu^0 : U_{R_I}(\tilde{\mathfrak{g}}) \otimes_{U_{R_I}(\tilde{\mathfrak{p}}^+)} (R_{I,c} \otimes_{\mathbb{C}} \tilde{V}(\mu)) \to \tilde{M}(c+\mu).$$

Then we have the following.

Proposition 7.1.

(i) *Under the identification* $\tilde{R} \simeq R_I$ *we have* $\xi_\mu^0 = \tilde{\xi}_\mu^0$ *for* $\mu \in \mathfrak{h}_{I,+}^*$.

(ii) $\xi_{\varpi_{i_0}} \in \mathbb{C}^\times \xi_{\varpi_{i_0}}^0$.

Proof. (i) We have $U(\tilde{\mathfrak{g}})v_\mu \simeq \tilde{V}(\mu)$, and $\tilde{v}_\mu$ is its lowest weight vector. The restriction ψ_μ^0 on $U_{R_I}(\tilde{\mathfrak{g}}) \otimes_{U_{R_I}(\tilde{\mathfrak{p}}^+)} (R_{I,c} \otimes_{\mathbb{C}} U(\tilde{\mathfrak{g}})v_\mu)$ is $\tilde{\psi}_\mu^0$ since $\tilde{\psi}_\mu^0(1 \otimes 1_c \otimes \tilde{v}_\mu) = f_r^m \otimes 1_{c+\mu}$. Hence we have $\xi_\mu^0 \otimes 1_{c+\mu} = \psi_\mu^0(1 \otimes 1_c \otimes v_\mu) = \tilde{\psi}_\mu^0(1 \otimes 1_c \otimes v_\mu) = \tilde{\xi}_\mu^0 \otimes 1_{c+\mu}$.

(ii) Since the pair $(\tilde{\mathfrak{g}}, i_0)$ is of regular type, we have $\deg \tilde{\xi}_{\varpi_{i_0}}^0 = r$ (see the proof of Theorem 6.6). Similarly to the proof of Theorem 6.6 we can show that $\deg \xi_{\varpi_{i_0}} = r$. By (i) we have $\deg \xi_{\varpi_{i_0}}^0 = \deg \xi_{\varpi_{i_0}}$. Since $\Xi_{\varpi_{i_0}} = R_I \xi_{\varpi_{i_0}}^0$ and $\xi_{\varpi_{i_0}} \in \Xi_{\varpi_{i_0}}$, we have $\xi_{\varpi_{i_0}} \in \mathbb{C}^\times \xi_{\varpi_{i_0}}^0$. □

As a result, we have the following.

Theorem 7.2. *For any pair* $(\mathfrak{g}, i_0)$ *of commutative parabolic type, the ideal* $\Xi_{\varpi_{i_0}}$ *is generated by* $\xi_{\varpi_{i_0}}$.

§8. Irreducibility of Generalized Verma Modules

Let $(L_I, \mathfrak{n}_I^-)$ be a prehomogeneous vector space of commutative parabolic type. Set $\{i_0\} = I_0 \setminus I$. In this section we give a new proof of the following well-known fact (Suga [10], Gyoja [1], Wachi [13]).

Theorem 8.1. *Let* $\lambda = s_0\varpi_{i_0} \in \mathfrak{h}_I^*$. $M_I(\lambda)$ *is irreducible if and only if* $\xi^0_{\varpi_{i_0}}(\lambda - m\varpi_{i_0}) \neq 0$ *for any* $m \in \mathbb{Z}_{>0}$.

We define $f_i \in U(\mathfrak{n}_I^-)$ and $\mathfrak{g}_{(i)}$ $(1 \le i \le r = r(\mathfrak{g}, i_0))$ as in Section 5.

Lemma 8.2. *For any* $m \in \mathbb{Z}_{>0}$, $I(m\lambda_r) \subset \mathbb{C}[\mathfrak{n}_I^+]I(m\lambda_{r-1})$.

Proof. Let v_m be the highest weight vector of the irreducible $\mathfrak{g}_{(r)}$-module $V_{(r),m}$ with highest weight $m\varpi_{i_0}$. Then any irreducible $[\mathfrak{l}_{(r)}, \mathfrak{l}_{(r)}]$-submodule of $V_{(r),m}$ is isomorphic to $I_{(r)}(\mu)$ for $\mu \in \sum_{i=1}^r \mathbb{Z}_{\ge 0}\lambda_i$ by Corollary 5.2. Here $I_{(r)}(\mu)$ is the irreducible $\mathfrak{l}_{(r)}$-submodule of $U(\mathfrak{n}_{(r)}^-)$ with highest weight μ. So it is enough to show that $I_{(r)}(m\lambda_r)v_m \subset \mathbb{C}[\mathfrak{n}_{(r)}^+]I_{(r)}(m\lambda_{r-1})v_m$. Now, $f_r^m v_m$ is the lowest weight vector of $V_{(r),m}$ and $f_{r-1}^m v_m$ is the lowest weight vector of the irreducible $\mathfrak{g}_{(r-1)}$-submodule $\mathfrak{g}_{(r-1)}v_m$ of $V_{(r),m}$. Hence there exists $y \in U(\mathfrak{g}_{(r)} \cap \mathfrak{n}^-)$ such that $yf_{r-1}^m v_m = f_r^m v_m$. Since $yf_{r-1}^m v_m \in \mathbb{C}[\mathfrak{n}_{(r)}^+]I_{(r)}(m\lambda_{r-1})v_m$, we have $f_r^m v_m \in \mathbb{C}[\mathfrak{n}_I^+]I_{(r)}(m\lambda_{r-1})v_m$. □

Corollary 8.3. *Let* $2 \le j \le r$. *For any* $m \in \mathbb{Z}_{>0}$ *and* $n \in \mathbb{Z}_{\ge 0}$ *there exists* $y \in U(\mathfrak{g}_{(j)} \cap \mathfrak{n}^-)$ *such that*

$$yf_{j-1}^m f_j^n \otimes 1_\lambda = f_j^{m+n} \otimes 1_\lambda \in M_I(\lambda).$$

Proof. It is enough to show the statement in the case where $j = r$. By the proof of Lemma 8.2 there exists $y_i \in U(\mathfrak{n}_{(r)}^-)$ and $y_i' \in U(\mathfrak{l}_{(r)} \cap \mathfrak{n}^-)$ such that

$$\sum_i y_i \mathrm{ad}(y_i')(f_{r-1}^m) = f_r^m.$$

Since $\mathrm{ad}(y_i')f_r = 0$, we have $\sum_i y_i y_i' f_{r-1}^m f_r^n \otimes 1_\lambda = f_r^{m+n} \otimes 1_\lambda$. □

Proposition 8.4. *Let* $K(\neq 0)$ *be a submodule of* $M_I(\lambda)$ *for* $\lambda \in \mathfrak{h}_I^*$. *We have* $f_r^n M_I(\lambda) \subset K$ *for* $n \gg 0$.

Proof. If $K = M_I(\lambda)$, then the statement is clear. Assume that $\{0\} \neq K \subsetneq M_I(\lambda)$. By Lemma 5.1 any highest weight vector of $M_I(\lambda)$ as an $\mathfrak{l}_I$-module is given by the following form:

$$f_1^{a_1} \cdots f_r^{a_r} \otimes 1_\lambda.$$

Since K has the highest weight vector as an $\mathfrak{l}_I$-module, there exists an element $f_1^{a_1} \cdots f_r^{a_r} \otimes 1_\lambda \in K$ such that $(a_1, \ldots, a_r) \neq 0$. By Corollary 8.3 there exists $u_1 = \sum_i u_{1,i} u'_{1,i} \in U(\mathfrak{n}^-_{(2)}) U(\mathfrak{l}_{(2)} \cap \mathfrak{n}^-)$ such that

$$u_1 f_1^{a_1} f_2^{a_2} \otimes 1_\lambda = f_2^{a_1+a_2} \otimes 1_\lambda.$$

Since $\mathrm{ad}(\mathfrak{l}_{(2)} \cap \mathfrak{n}^-) f_j = 0$ for $j \geq 2$, we have

$$f_2^{a_1+a_2} \cdots f_r^{a_r} \otimes 1_\lambda = u_1 f_1^{a_1} f_2^{a_2} \cdots f_r^{a_r} \otimes 1_\lambda \in K.$$

Similarly, there exist $u_1, u_2, \ldots, u_{r-1} \in U(\mathfrak{n}^-)$ such that $f_r^{a_1+a_2+\cdots a_r} \otimes 1_\lambda = u_{r-1} \cdots u_2 u_1 f_1^{a_1} f_2^{a_2} \cdots f_r^{a_r} \otimes 1_\lambda$, that is, we have $f_r^{a_1+a_2+\cdots a_r} \otimes 1_\lambda \in K$. Hence for any $y \in U(\mathfrak{n}_I^-)$ we have

$$f_r^{a_1+\cdots a_r}(y \otimes 1_\lambda) = y f_r^{a_1+\cdots a_r} \otimes 1_\lambda \in K,$$

and the statement holds. □

Set $\mu = \mu_m = m\varpi_{i_0}$ for any positive integer m, and let us prove Theorem 8.1 by using the commutative diagram

$$\begin{array}{ccc} M_{R_I}(c+\mu) & = & M_{R_I}(c+\mu) \\ {\scriptstyle \iota_\mu} \downarrow & & \downarrow {\scriptstyle \xi^0_\mu} \\ U_{R_I}(\mathfrak{g}) \otimes_{U_{R_I}(\mathfrak{p}_I^+)} (R_{I,c} \otimes_{\mathbb{C}} V(\mu)) & \xrightarrow[\psi^0_\mu]{} & M_{R_I}(c+\mu). \end{array} \tag{8.1}$$

Set $\lambda = s_0 \varpi_{i_0}$. We denote the highest weight vector of $V(\mu)$ by v_μ. Let $\tilde{v}_\mu$ be the highest weight vector of the irreducible $\mathfrak{l}_I$-module generated by the lowest weight vector of $\mathfrak{g}$-module $V(\mu)$. Considering the functor $\mathbb{C} \otimes_{R_I} (\ \cdot\ )$, where $\mathbb{C}$ has the R_I-module structure via $c(h_i)1 = (\lambda - \mu)(h_i)$, we obtain the following commutative diagram from (8.1):

$$\begin{array}{ccc} M_I(\lambda) & = & M_I(\lambda) \\ {\scriptstyle \iota_m} \downarrow & & \downarrow {\scriptstyle \xi^0_\mu(\lambda-\mu)} \\ U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I^+)} (\mathbb{C}_{I,\lambda-\mu} \otimes_{\mathbb{C}} V(\mu)) & \xrightarrow[\psi^0_m]{} & M_I(\lambda), \end{array}$$

where $\iota_m(1 \otimes 1_\lambda) = 1 \otimes 1_{\lambda-\mu} \otimes v_\mu$ and $\psi^0_m(1 \otimes 1_{\lambda-\mu} \otimes \tilde{v}_\mu) = f_r^m \otimes 1_\lambda$.

We assume that $M_I(\lambda)$ is irreducible. Since $\psi_m^0 \neq 0$, we have $\mathrm{Im}\psi_m^0 = M_I(\lambda)$. The weight space of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I^+)} (\mathbb{C}_{I,\lambda-\mu} \otimes_{\mathbb{C}} V(\mu))$ with weight λ is $\mathbb{C}(1 \otimes 1_{\lambda-\mu} \otimes v_\mu)$, hence there exists $a \in \mathbb{C} \setminus \{0\}$ such that

$$1 \otimes 1_\lambda = \psi_m^0(a \otimes 1_{\lambda-\mu} \otimes v_\mu) = a\psi_m^0 \iota_m(1 \otimes 1_\lambda) = a\xi_\mu^0(\lambda-\mu) \otimes 1_\lambda \neq 0.$$

By Propositions 6.1 and 7.1 we have

$$\xi_\mu^0(\lambda-\mu) = \xi_{\varpi_{i_0}}^0(\lambda-\varpi_{i_0})\xi_{\varpi_{i_0}}^0(\lambda-2\varpi_{i_0})\cdots\xi_{\varpi_{i_0}}^0(\lambda-m\varpi_{i_0}).$$

Therefore we have $\xi_{\varpi_{i_0}}^0(\lambda-m\varpi_{i_0}) \neq 0$ for any $m \in \mathbb{Z}_{>0}$.

Conversely, we assume that $\xi_{\varpi_{i_0}}^0(\lambda-m\varpi_{i_0}) \neq 0$ for any $m \in \mathbb{Z}_{>0}$. We set

$$N(m) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I^+)} (\mathbb{C}_{I,\lambda-\mu_m} \otimes_{\mathbb{C}} V(\mu_m)).$$

Since $\xi_{\mu_m}^0(\lambda-\mu_m) = \xi_{\varpi_{i_0}}^0(\lambda-\varpi_{i_0})\xi_{\varpi_{i_0}}^0(\lambda-2\varpi_{i_0})\cdots\xi_{\varpi_{i_0}}^0(\lambda-m\varpi_{i_0}) \neq 0$, we have

$$\begin{aligned}\psi_m^0(\xi_{\mu_m}^0(\lambda-\mu_m)^{-1} \otimes 1_{\lambda-\mu_m} \otimes v_{\mu_m}) &= \xi_{\mu_m}^0(\lambda-\mu_m)^{-1}\psi_m^0\iota_m(1 \otimes 1_\lambda) \\ &= 1 \otimes 1_\lambda.\end{aligned}$$

Hence ψ_m^0 is surjective, and we have an isomorphism

$$N(m)/\mathrm{Ker}\psi_m^0 \simeq M_I(\lambda) : \ \overline{1 \otimes 1_{\lambda-\mu_m} \otimes \tilde{v}_{\mu_m}} \mapsto f_r^m \otimes 1_\lambda$$

for any m. Under this identification we have

$$\overline{1 \otimes 1_{\lambda-\mu_{n+1}} \otimes \tilde{v}_{\mu_{n+1}}} = f_r^{n+1} \otimes 1_\lambda = f_r^n (f_r \otimes 1_\lambda) = f_r^n \overline{1 \otimes 1_{\lambda-\mu_1} \otimes \tilde{v}_{\mu_1}}.$$

Let $K \neq 0$ be a submodule of $M_I(\lambda)$. By Proposition 8.4 for $n \gg 0$ we have

$$\overline{1 \otimes 1_{\lambda-\mu_{n+1}} \otimes \tilde{v}_{\mu_{n+1}}} = f_r^n \overline{1 \otimes 1_{\lambda-\mu_1} \otimes \tilde{v}_{\mu_1}} \in K.$$

Hence we have

$$\begin{aligned}M_I(\lambda) &= N(n+1)/\mathrm{Ker}\psi_{n+1}^0 \\ &= U(\mathfrak{g})\overline{1 \otimes 1_{\lambda-\mu_{n+1}} \otimes \tilde{v}_{\mu_{n+1}}} \subset K.\end{aligned}$$

Therefore $K = M_I(\lambda)$, and $M_I(\lambda)$ is irreducible. We complete the proof of Theorem 8.1.

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