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Hopf Bifurcation and Stability of a Competition Diffusion System with Distributed Delay

By

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Abstract

In this paper we investigate the effects of time delay and diffusion rate on the stability of the positive steady state for a competition diffusion system with distributed delay. We obtain the condition of instability of the positive uniform steady state. Regarding the time delay as bifurcation parameter, we reduce the original system on the center manifold and get the stability of the Hopf bifurcation periodic solutions. Finally, regarding the diffusion rate as a bifurcation parameter, we show that the one-dimensional competition diffusion system with infinite delay and Dirichlet boundary condition exhibits the spatiotemporal structures near the steady state of the system.

§1. Introduction

Yamada and Niikura [14] considered the following semilinear parabolic system

$$(1.1)\quad \begin{aligned} \frac{\partial u}{\partial t} &= D(\alpha)\Delta u + C(\alpha)u + \int_{-\infty}^{t} k(t-s,\alpha)u(x,s)ds + f(u_t,\alpha)(x), \\ \frac{\partial u}{\partial n} &= 0, t \in R, x \in \partial\Omega, \end{aligned}$$

where $u \in R^n, x \in \Omega$, a bounded domain in R^N with smooth boundary, α is a parameter in an open interval $I \subset R, u_t(x,s) = u(x,t+s)$ for $s \leq 0, \frac{\partial u}{\partial n}$ is the outward normal derivative on $\partial\Omega$. They established the existence conditions of

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nontrivial spatially homogeneous periodic solutions. J. H. Wu [13] illustrated the above result with the following semilinear Lotka-Volterra diffusion equation, modelling the population density of a species which lives in Ω undergoing diffusion and memory effects.

$$
\begin{aligned}
(1.2)\qquad & \frac{\partial u}{\partial t} = d\Delta u + u[a - bu - \int_{-\infty}^{t} k(t-s)u(x,s)ds], (x,t) \in \Omega \times R, \\
& \frac{\partial u}{\partial n} = 0, (x,t) \in \partial\Omega \times R,
\end{aligned}
$$

where $a > 0, b \geq 0, d > 0$ are some positive constants and $k : [0, +\infty) \to R$ is given so that $\int_0^{+\infty}(|k(t)| + t|k(t)|)dt < \infty$, for example, $k(t) = \frac{\alpha t}{\rho^2}e^{-\frac{t}{\rho}}$ with $\alpha > 0,\ \rho > 0$. In this case, the Laplace transformation $\widehat{k}(\lambda) = \int_0^{+\infty} e^{-\lambda t}k(t)dt = \frac{\alpha}{(1+\lambda\rho)^2}$. Consider α as a parameter, the nontrivial periodic solutions of (1.2) bifurcate from the positive equilibrium $u^* = \frac{a}{b+\int_0^{+\infty} k(t)dt}$ as α passing through the critical value $\alpha_0 = \frac{2(1+\rho b u^*)^2}{u^*\rho}$.

Delays are often incorporated into population models for resource regeneration times, maturing times or gestation periods [6]. Gourley et al. [1, 7] considered reaction diffusion systems with delay terms. The system represents two interacting species

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} &= F_1(u_1, u_2, G_{11} ** u_1, G_{12} ** u_2) + D_1 \frac{\partial^2 u_1}{\partial x^2}, \\
\frac{\partial u_2}{\partial t} &= F_2(u_1, u_2, G_{21} ** u_1, G_{22} ** u_2) + D_2 \frac{\partial^2 u_2}{\partial x^2}
\end{aligned}
$$

for $x \in (-\infty, +\infty), t \geq 0$, and the double convolution is defined by

$$
(G_{ij} ** u_j)(x,t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{t} G_{ij}(x-y, t-s)u_j(y,s)dsdy, i,j = 1,2,
$$

and each of the kernels G_{ij} satisfies that

(A) $G_{ij}(x,t) \geq 0, (x,t) \in (-\infty, +\infty) \times [0, +\infty)$,

(B) For each $t \geq 0$, $G_{ij}(x,t)$ is an even function of x,

(C) $\int_{-\infty}^{+\infty} \int_0^{+\infty} G_{ij}(x,t)dtdx = 1, i,j = 1,2.$

Gourley et al. [1, 7] presented a computational method for determining regions in parameter space corresponding to linear instability of a spatially uniform steady state solution. Similarly, for the variety of kernels, Beretta et al. [2], Hasting [10], Yamada et al. [14] discussed the corresponding models.

In this paper we consider a competition diffusion system with distributed delay

(1.3)

$$\frac{\partial U}{\partial t} = d\Delta U + r_1 U \left[1 - \int_{-\infty}^{t} f(t-\tau)U(\tau,x)d\tau - \mu_1 \int_{-\infty}^{t} f(t-\tau)V(\tau,x)d\tau\right],$$

$$\frac{\partial V}{\partial t} = d\Delta V + r_2 V \left[1 - \mu_2 \int_{-\infty}^{t} f(t-\tau)U(\tau,x)d\tau - \int_{-\infty}^{t} f(t-\tau)V(\tau,x)d\tau\right],$$

where the positive parameters μ_1, μ_2, r_1, r_2 satisfy that

$$\mu_1 < 1 < 1/\mu_2. \tag{1.4}$$

The system has a unique positive spatially uniform steady state (u^*, v^*), that is,

$$(u^*, v^*) = ((1-\mu_1)/(1-\mu_1\mu_2), (1-\mu_2)/(1-\mu_1\mu_2)). \tag{1.5}$$

The distributed time delay should be viewed as the effects of past history [2, 10]. $f(t)$ is the delay kernel or memory function such that $\int_0^{+\infty} f(t)dt = 1$. For example, $f(t) = \alpha e^{-\alpha t}$ and $f(t) = \alpha^2 t e^{-\alpha t}$ (which are called the weak and strong generic kernels, respectively) have been used in the context of various spatially homogeneous predator-prey models with delay [1–11]. The weak generic kernel and the strong generic kernel have quite different behavior in terms of modelling the memory or delay effects. While the former decays monotonically for $t > 0$ and represents a memory that progressively becomes weaker as one goes further into the past, the latter increases for a range $t < 1/\alpha$. Thus, for the strong kernel, the most important moment in the past is $1/\alpha$ units before the present time t, and the effect of the past decreases as one goes even further into the past [4]. The occurrence of spatial form and pattern evolving from a spatially homogeneous initial state is a fundamental problem in developmental biology. For the memory function $f(t) = \delta(t)$ with δ being the Dirac delta function, one gets the case of the so-called zero delay, (u^*, v^*) is globally asymptotically stable for all positive initial values [16], for $f(t) = \delta(t-r)$, the case of discrete delay, there exists a positive critical value r_0 such that (u^*, v^*) is locally asymptotically stable for $0 \le r < r_0$ and the Hopf bifurcation periodic solutions bifurcate from (u^*, v^*) as r passing through r_0 [15, 17, 18].

In this paper we consider the effects of time delay and diffusion rate on Hopf bifurcations in competition diffusion system (1.3) with the strong generic kernel. In section 2 we discuss the local stability of the positive equilibrium and describe the condition for the Hopf bifurcation, regarding α as a bifurcation parameter. In section 3 we study the stability of the bifurcation periodic solutions near the bifurcation value by applying the method published by Hassard et al. [9] for unbounded delay functional differential equations. We prove that a Hopf bifurcation occurs at α_0 for increasing values of α, toward stable periodic solutions. In section 4 we discuss the effects of diffusion rate on stability of spatially uniform steady state (u^*, v^*) of the one-dimensional competition diffusion system with infinite delay. Regarding the diffusion rate d as a bifurcation parameter, we prove that a Hopf bifurcation occurs at d_0 for decreasing values of d, and show that the system exhibits spatiotemporal structures near the steady state of the system.

§2. Local Stability and Hopf Bifurcation

We now consider the modification of competition diffusion system with distributed delay and Neumann boundary condition

(2.1)

$$\begin{aligned}
&\frac{\partial U}{\partial t} = d\Delta U + r_1 U\left[1 - \int_{-\infty}^{t} f(t-\tau)U(\tau,x)d\tau - \mu_1 \int_{-\infty}^{t} f(t-\tau)V(\tau,x)d\tau\right],\\
&\frac{\partial V}{\partial t} = d\Delta V + r_2 V\left[1 - \mu_2 \int_{-\infty}^{t} f(t-\tau)U(\tau,x)d\tau - \int_{-\infty}^{t} f(t-\tau)V(\tau,x)d\tau\right],\\
&\frac{\partial U}{\partial n}(t,x) = \frac{\partial V}{\partial n}(t,x) = 0, \quad t \in R, x \in \partial\Omega,\\
&(U,V) = (\widetilde{\varphi}_1(t,x), \widetilde{\varphi}_2(t,x)), \quad (t,x) \in (-\infty, 0] \times \overline{\Omega},
\end{aligned}$$

where Ω is a bounded domain in R^N, for simplicity, we choose $d = 1$ and assume that the positive parameters μ_1, μ_2, r_1, r_2 satisfy that (1.4). Following the usual linearization procedure, set

$$u(t,x) = U(t,x) - u^*, v(t,x) = V(t,x) - v^*,$$

and substitute them into (2.1), we have

$$\begin{aligned}
(2.2)\quad &\frac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = \Delta \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} - B\int_{-\infty}^{0} f_\alpha(-s)\begin{pmatrix} u(t+s,x) \\ v(t+s,x) \end{pmatrix} ds + F,\\
&\frac{\partial u}{\partial n}(t,x) = \frac{\partial v}{\partial n}(t,x) = 0, \qquad t \in R, x \in \partial\Omega,\\
&(u,v) = (\varphi_1(t,x), \varphi_2(t,x)), \qquad (t,x) \in (-\infty, 0] \times \overline{\Omega},
\end{aligned}$$

where $f_\alpha(s) = \frac{s}{\alpha^2}e^{-\frac{s}{\alpha}}$, $s \in [0, +\infty)$, $(\varphi_1, \varphi_2) \in \mathcal{C}((-\infty, 0], X \times X)$, $X = L^2(\overline{\Omega})$, and

$$B = \begin{pmatrix} r_1 u^* & r_1\mu_1 u^* \\ r_2\mu_2 v^* & r_2 v^* \end{pmatrix},$$

$$F = -\begin{pmatrix} r_1 u(t,x) & r_1\mu_1 u(t,x) \\ r_2\mu_2 v(t,x) & r_2 v(t,x) \end{pmatrix} \int_{-\infty}^{0} f_\alpha(-s) \begin{pmatrix} u(t+s,x) \\ v(t+s,x) \end{pmatrix} ds.$$

The associated linearized problem is

$$\begin{aligned} &\frac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = \Delta \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} - B\int_{-\infty}^{0} f_\alpha(-s)\begin{pmatrix} u(t+s,x) \\ v(t+s,x) \end{pmatrix} ds, \\ &\frac{\partial u}{\partial n}(t,x) = \frac{\partial v}{\partial n}(t,x) = 0, \qquad t \in R, x \in \partial\Omega, \\ &(u,v) = (\varphi_1(t,x), \varphi_2(t,x)), \qquad (t,x) \in (-\infty, 0] \times \overline{\Omega}. \end{aligned} \tag{2.3}$$

In terms of the eigenvalues $0 = \nu_0 < \nu_1 \leq \nu_2 \leq \cdots$ for $-\Delta$ subject to the Neumann boundary condition and their corresponding eigenfunctions $e_0(x) \equiv 1, e_1(x), e_2(x), \cdots$, we introduce the operator $\mathcal{A} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} : \mathcal{D}(\mathcal{A}) \to X \times X$, with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in (H^2(\Omega) \times H_0^1(\overline{\Omega}))^2 \right\}.$$

The operator $\mathcal{A}$ is an infinitesimal generator of a compact C_0−semigroup [12]. The study of the stability of (u^*, v^*) leads to the investigation of the eigenvalue problem

$$\Delta(\alpha,\lambda)\begin{pmatrix} y \\ z \end{pmatrix} \equiv \left(\mathcal{A} - B\int_{-\infty}^{0} f_\alpha(-s)e^{\lambda s} ds - \lambda I\right)\begin{pmatrix} y \\ z \end{pmatrix} = 0, 0 \neq \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(\mathcal{A}). \tag{2.4}$$

Or the study of the point spectrum $P_\sigma(\mathcal{A}_1(\alpha))$, where $\mathcal{A}_1(\alpha)$ is the infinitesimal generator of the semigroup induced by solutions of equation (2.3)([8, 13]) with

$$\begin{aligned} &\mathcal{A}_1(\alpha)\varphi(\theta) = \frac{d\varphi}{d\theta}, \quad -\infty < \theta \leq 0, \\ &\mathcal{D}(\mathcal{A}_1(\alpha)) = \left\{ \varphi \in \mathcal{C}^1, \varphi(0) \in \mathcal{D}(\mathcal{A}), \dot{\varphi}(0) = \mathcal{A}\varphi(0) - B\int_{-\infty}^{0} f_\alpha(-s)\varphi(s) ds \right\} \end{aligned} \tag{2.5}$$

where $\mathcal{C}^1 = \mathcal{C}((-\infty, 0]) \bigcap \mathcal{C}^1((-\infty, 0))$. The set of point spectrum of $\mathcal{A}$ consists of a sequence of real numbers $\{-\nu_n | 0 = \nu_0 < \nu_1 \le \nu_2 \le \dots\}$ with the corresponding eigenspaces

$$\mathrm{ES}_0 = \mathrm{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}, \mathrm{ES}_n = \mathrm{Span}\left\{\begin{pmatrix}e_n(x)\\0\end{pmatrix}, \begin{pmatrix}0\\e_n(x)\end{pmatrix}\right\}, n = 1, 2, \dots.$$

Therefore, (2.4) is equivalent to a sequence of eigenvalue problems

$$\text{(2.6)} \quad \Delta(n, \alpha, \lambda)\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix} \equiv \left(-\nu_n I - B\int_{-\infty}^{0} f_\alpha(-s)e^{\lambda s}ds - \lambda I\right)\begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix} = 0,$$
$$\begin{pmatrix}y\\ z\end{pmatrix}_n = \begin{pmatrix}\alpha_n\\ \beta_n\end{pmatrix} e_n(x), n = 0, 1, 2, \dots,$$

then we obtain a sequence of characteristic equations

$$\text{(2.7)} \qquad det\left[(\lambda + \nu_n)I + B\int_{-\infty}^{0} f_\alpha(-s)e^{\lambda s}ds\right] = 0, n = 0, 1, 2, \dots.$$

For the strong generic kernel $f_\alpha(s) = \frac{s}{\alpha^2}e^{-\frac{s}{\alpha}}$, $\int_{-\infty}^{0} f_\alpha(-s)e^{\lambda s}ds = \frac{1}{(1+\lambda\alpha)^2}$, then we have

(2.8)
$$[(\lambda + \nu_n)(\alpha^2\lambda^2 + 2\lambda\alpha + 1) + M_1][(\lambda + \nu_n)(\alpha^2\lambda^2 + 2\lambda\alpha + 1) + M_2] = 0,$$

$n = 0, 1, 2, \dots$, which are equivalent to (2.9) and/or (2.10)

$$\text{(2.9)} \qquad \lambda^3 + p_1\lambda^2 + p_2\lambda + p_{31} = 0, \quad n = 0, 1, 2, \dots.$$
$$\text{(2.10)} \qquad \lambda^3 + p_1\lambda^2 + p_2\lambda + p_{32} = 0, \quad n = 0, 1, 2, \dots.$$

where

$$p_1 = \nu_n + \frac{2}{\alpha}, \quad p_2 = \frac{2\nu_n}{\alpha} + \frac{1}{\alpha^2}, \quad p_{31} = \frac{\nu_n + M_1}{\alpha^2}, \quad p_{32} = \frac{\nu_n + M_2}{\alpha^2}$$
$$M_{1,2} = \frac{1}{2}\left[r_1u^* + r_2v^* \mp \sqrt{(r_1u^* + r_2v^*)^2 - 4r_1r_2(1 - \mu_1\mu_2)u^*v^*}\right].$$

The condition (1.4) implies that $0 < M_1 < M_2$. By the Routh-Hurwitz criterion, the necessary and sufficient conditions for the characteristic roots of (2.9) and (2.10) to have negative parts are (i) $p_1 > 0$, (ii) $p_{31} > 0$ or $p_{32} > 0$, (iii) $p_1p_2 > p_{31}$ or $p_1p_2 > p_{32}$. Obviously, (i) and (ii) are satisfied. Since

$$p_1p_2 - p_{31} = \frac{1}{\alpha^3}(2\alpha^2\nu_n^2 + 4\alpha\nu_n + 2 - \alpha M_1),$$
$$p_1p_2 - p_{32} = \frac{1}{\alpha^3}(2\alpha^2\nu_n^2 + 4\alpha\nu_n + 2 - \alpha M_2),$$

if $0 \le \alpha < \alpha_0 \equiv \frac{2}{M_2}$, then for any $n \ge 0$, $p_1p_2 - p_{31} > p_1p_2 - p_{32} > 0$. According to the Routh-Hurwitz criterion, all roots of (2.9) and (2.10) have negative real parts, this implies that the positive steady state (u^*, v^*) is locally asymptotically stable.

If $\alpha = \alpha_0$, then we have

$$p_1p_2 - p_{31} > 0, \quad n = 0, 1, 2, \ldots;$$
$$p_1p_2 - p_{32} > 0, \quad n = 1, 2, \ldots; \qquad p_1p_2 - p_{32} = 0, \quad n = 0.$$

The Routh-Hurwitz criterion implies that all roots of (2.8) have negative real parts for $n = 1, 2, \ldots$. For $n = 0$, all roots of (2.9) have negative real parts and (2.10) has three roots, a pair of complex, simple eigenvalues $\lambda(\alpha_0) = \frac{i}{\alpha_0}$ and $\overline{\lambda}(\alpha_0) = -\frac{i}{\alpha_0}$ and one negative real root $\lambda_3(\alpha_0) = -\frac{2}{\alpha_0}$.

Therefore, we know that $\alpha = \alpha_0$ is a critical value and some consequences as follows.

(1) All eigenvalues of (2.8) have negative real parts for $\alpha < \alpha_0$;

(2) If $\alpha = \alpha_0$, (2.8) has a pair of pure imaginary eigenvalues $\lambda(\alpha_0) = \frac{i}{\alpha_0}$ and $\overline{\lambda}(\alpha_0)$ for $n = 0$, and other eigenvalues have negative real parts for $n \ge 0$.

Hence we have the following result about the Hopf bifurcation.

Theorem 2.1. *A Hopf bifurcation of the steady state (u^*, v^*) occurs as α increasingly passes through α_0.*

Proof. According to the Hopf bifurcation Theorem, it suffices to verify the transversality condition. For $n = 0$, (2.10) can be rewritten as

$$\lambda^3 + \frac{2}{\alpha}\lambda^2 + \frac{1}{\alpha^2}\lambda + \frac{M_2}{\alpha^2} = 0.$$

Find the derivative on both sides with respect to α, we have

$$\frac{d\lambda}{d\alpha} = \frac{2(\alpha\lambda^2 + \lambda + M_2)}{\alpha(3\alpha^2\lambda^2 + 4\alpha\lambda + 1)}$$

Since $\lambda(\alpha_0) = \frac{i}{\alpha_0}$, we have

$$\frac{d\lambda}{d\alpha}|_{\alpha=\alpha_0} = \frac{1-3i}{5\alpha_0^2}, \quad \frac{d\mathrm{Re}\lambda}{d\alpha}|_{\alpha=\alpha_0} = \frac{1}{5\alpha_0^2} > 0.$$

This implies that the transversality condition holds true. Therefore, a Hopf bifurcation of (u^*, v^*) occurs as α increasingly passes through α_0. This completes the proof. □

Remark. The above result shows that for system (2.1), if the time delay α is less than a critical value α_0, there is no spatial, temporal or spatiotemporal structure and the effect of delay is to make the species distributed uniform over the region Ω as $t \to +\infty$. When α passes through the critical value, the Hopf bifurcation occurs and demonstrates that the system exhibits certain temporal structures.

In the problem (2.1), we use the strong generic kernel $f_\alpha(s) = \frac{s}{\alpha^2}e^{-\frac{s}{\alpha}}$, $s \in [0, +\infty)$. For the weak generic kernel $f_\alpha(s) = \frac{1}{\alpha}e^{-\frac{s}{\alpha}}$, $s \in [0, +\infty)$, we have $\int_{-\infty}^{0} f_\alpha(-s)e^{\lambda s}ds = \frac{1}{\lambda\alpha+1}$, then (2.7) becomes

$$\left[\lambda^2 + \left(\frac{1}{\alpha} + \nu_n\right)\lambda + \frac{\nu_n^2 + M_1}{\alpha}\right]\left[\lambda^2 + \left(\frac{1}{\alpha} + \nu_n\right)\lambda + \frac{\nu_n^2 + M_2}{\alpha}\right] = 0, \tag{2.11}$$

where $n = 0, 1, 2, \ldots, \alpha > 0, \nu_n \geq 0, 0 < M_1 < M_2$. Using the Routh-Hurwitz criterion, we obtain that for every $n \geq 0$ the roots of (2.11) all lie in the left half complex plane, that is, the steady state (u^*, v^*) is linearly stable. This illustrates that the weak generic kernel and the strong generic kernel exhibit quite different behavior in terms of modelling the memory (or delay) effects [3, 4, 5].

§3. Stability of the Hopf Bifurcation Solutions

We now investigate the stability of the Hopf bifurcation periodic solutions. Rewrite (2.1) into the operator differential equation

$$\dot{x}_t = \mathcal{A}_1(\alpha)x_t + \mathcal{X}_0 x_t \tag{3.1}$$

where $x = (u, v)^T, x_t(\theta) = x(t + \theta), \theta \in (-\infty, 0]$. The linear operator $\mathcal{A}_1(\alpha)$ is defined in (2.4) and the nonlinear one $\mathcal{X}_0$ is defined as follows.

$$\mathcal{X}_0\varphi(\theta) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \theta \in (-\infty, 0), \\ -\begin{pmatrix} r_1\varphi_1(0) & r_1\mu_1\varphi_1(0) \\ r_2\mu_2\varphi_2(0) & r_2\varphi_2(0) \end{pmatrix} \int_{-\infty}^{0} f_\alpha(-s)\varphi(s)ds, & \theta = 0. \end{cases} \tag{3.2}$$

The adjoint operator $\mathcal{A}_1^*(\alpha_0)$ is defined as follows.

$$\begin{aligned} \mathcal{A}_1^*(\alpha_0)\psi(\delta) &= -\frac{d\psi}{d\delta}, \quad 0 \leq \delta < +\infty, \\ \mathcal{D}(\mathcal{A}_1^*(\alpha_0)) &= \Big\{\psi \in \mathcal{C}^1, \psi(0) \in \mathcal{D}(\mathcal{A}), -\dot{\psi}(0) \\ &\quad = \mathcal{A}\psi(0) - B^T \int_{-\infty}^{0} f_{\alpha_0}(-s)\psi(-s)ds\Big\}. \end{aligned} \tag{3.3}$$

here $\mathcal{C}^1 = \mathcal{C}[0,+\infty)\bigcap\mathcal{C}^1(0,+\infty)$. In order to determine the Poincaré normal form of operator $\mathcal{A}_1(\alpha_0)$, we need to compute the eigenvector $q(\theta)$ of the operator $\mathcal{A}_1(\alpha_0)$ belonging to the eigenvalue $\frac{i}{\alpha_0}$, and the eigenvector $q^*(\theta)$ of the operator $\mathcal{A}_1^*(\alpha_0)$ belonging to the eigenvalue $-\frac{i}{\alpha_0}$. We know that the eigenvalue problem $\mathcal{A}_1(\alpha_0)\varphi = \pm\frac{i}{\alpha_0}\varphi$ has the eigenfunctions

$$\Phi(\theta) = (q(\theta), \overline{q}(\theta)), \qquad q(\theta) = \begin{pmatrix} 1 \\ \beta_0 \end{pmatrix} e^{\frac{\theta}{\alpha_0}i}, \quad -\infty < \theta \le 0, \tag{3.4}$$

where the vector $(1,\beta_0)^T$ satisfies that

$$\Delta\left(0, \alpha_0, \frac{i}{\alpha_0}\right)\begin{pmatrix} 1 \\ \beta_0 \end{pmatrix} = 0,$$

then we have $\beta_0 = \frac{M_2 - r_1 u^*}{r_1\mu_1 u^*}$.

Similarly, the eigenvalue problem $\mathcal{A}_1^*(\alpha_0)\psi = \mp\frac{i}{\alpha_0}\psi$ has the eigenfunctions

$$\Psi(\theta) = (q^*(\theta), \overline{q}^*(\theta)), \qquad q^*(\theta) = D\begin{pmatrix} 1 \\ \beta^* \end{pmatrix} e^{\frac{\theta}{\alpha_0}i}, \quad 0 \le \theta < +\infty, \tag{3.5}$$

where the vector $(1,\beta^*)^T$ satisfies that

$$(1,\beta^*)\Delta\left(0, \alpha_0, -\frac{i}{\alpha_0}\right) = 0,$$

then we have $\beta^* = \frac{M_2 - r_1 u^*}{r_2\mu_2 v^*}$, D is a free constant which can be determined by the condition $\langle q^*, q\rangle = 1$, where $\langle\cdot,\cdot\rangle$ is the scalar product which is defined by

$$\langle\psi,\phi\rangle = \overline{\psi}^T(0)\cdot\phi(0) - \int_{-\infty}^{0}\left(\int_{s}^{0}\overline{\psi}^T(t-s)Bf_{\alpha_0}(-s)\varphi(t)dt\right)ds. \tag{3.6}$$

Here $\overline{\psi}^T$ denotes the transpose of the complex conjugate of ψ. Hence $\langle q^*, q\rangle = 1$ implies that

$$\overline{D}\left[1 + \beta_0\beta^* - (1,\beta^*)B\begin{pmatrix} 1 \\ \beta_0 \end{pmatrix}\frac{2\alpha_0}{(1+i)^3}\right] = 1.$$

Since $(1,\beta^*)\frac{2}{\alpha_0} = (1,\beta^*)B$, we have

$$\overline{D} = \frac{2-i}{5(1+\beta_0\beta^*)}. \tag{3.7}$$

Now we reduce the vector-valued function $x_t \in (\mathcal{C}(-\infty,0]\bigcap\mathcal{C}^1(-\infty,0))^2$ to the two dimensional case. It is obvious that $\langle q^*, \overline{q}\rangle = 0$, so we consider the transformation

$$z = \langle q^*, x_t\rangle, \quad w = x_t - zq - \overline{zq}, \tag{3.8}$$

so that z and $\overline{z}$ are the local coordinates for the center manifold $\mathcal{C}_0$ in the directions of q and $\overline{q}$. In the variables z and w, the operator equation (3.1) becomes

$$\begin{aligned} \dot{z} &= \frac{i}{\alpha_0} z + \langle q^*(\theta), \mathcal{X}_0(w + 2\mathrm{Re}\{zq(\theta)\})\rangle, \\ \dot{w} &= \mathcal{A}_1(\alpha_0)w - 2\mathrm{Re}\{\langle q^*(\theta), \mathcal{X}_0(w + 2\mathrm{Re}\{zq(\theta)\})\rangle q(\theta)\} \\ &\quad + \mathcal{X}_0(w + 2\mathrm{Re}\{zq(\theta)\}). \end{aligned} \tag{3.9}$$

On the manifold $\mathcal{C}_0, w(t,\theta) = w(z(t), \overline{z}(t), \theta)$, where

$$w(z, \overline{z}, \theta) = w_{20}\frac{z^2}{2} + w_{11}z\overline{z} + w_{02}\frac{\overline{z}^2}{2} + \cdots . \tag{3.10}$$

According to the definition of the nonlinear operator $\mathcal{X}_0$, we have

$$\langle q^*(\theta), \mathcal{X}_0(w + 2\mathrm{Re}\{zq(\theta)\})\rangle = \overline{q}^{*T}(0) \cdot \mathcal{X}_0(w(z, \overline{z}, 0) + 2\mathrm{Re}\{zq(0)\}) =: g(z, \overline{z}),$$

denote $H(z, \overline{z}, \theta) = \mathcal{X}_0(w + 2\mathrm{Re}\{zq(\theta)\}) - 2\mathrm{Re}\{g(z, \overline{z})q(\theta)\}$, then

$$\begin{aligned} \dot{z} &= \frac{i}{\alpha_0} z + g(z, \overline{z}), \\ \dot{w} &= \mathcal{A}_1(\alpha_0)w + H(z, \overline{z}, \theta) \end{aligned} \tag{3.11}$$

Now we expand the functions $g(z, \overline{z})$ and $H(z, \overline{z}, \theta)$ in powers of $z, \overline{z}$ on the center manifold $\mathcal{C}_0$:

$$g(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots , \tag{3.12}$$

$$H(z, \overline{z}, \theta) = H_{20}\frac{z^2}{2} + H_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots , \tag{3.13}$$

then, according to the definition of $g(z, \overline{z})$ and $H(z, \overline{z}, \theta)$, we can determine the coefficients of the expansions (3.12)(3.13).

For simplicity, we just calculate the Hopf bifurcation direction with the coefficients of (2.1) in a special case: $r_1 = r_2, \mu_1 = \mu_2$. On the case, we have

$$u^* = v^* = \frac{1}{1 + \mu_1}, M_2 = r_1, \beta_0 = \beta^* = 1. \tag{3.14}$$

Since

$$\mathcal{X}_0(w + 2\mathrm{Re}\{zq(\theta)\}) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \theta \in (-\infty, 0), \\ \begin{pmatrix} f_0^1 \\ f_0^2 \end{pmatrix}, & \theta = 0. \end{cases}$$

$$\begin{pmatrix} f_0^1 \\ f_0^2 \end{pmatrix} = -\begin{pmatrix} (w^{(1)}(0) + z + \overline{z})[r_1(\widetilde{w}^{(1)} - \frac{z}{2}i + \frac{\overline{z}}{2}i) + r_1\mu_1(\widetilde{w}^{(2)} - \frac{z}{2}i + \frac{\overline{z}}{2}i)] \\ (w^{(2)}(0) + z + \overline{z})[r_1\mu_1(\widetilde{w}^{(1)} - \frac{z}{2}i + \frac{\overline{z}}{2}i) + r_1(\widetilde{w}^{(2)} - \frac{z}{2}i + \frac{\overline{z}}{2}i)] \end{pmatrix},$$

where $\widetilde{w} = (\widetilde{w}^{(1)}, \widetilde{w}^{(2)})^T = \int_{-\infty}^{0} f_{\alpha_0}(-s)w(s)ds$. Use the result $\beta_0 = \beta^* = 1$, we have

$$g(z,\overline{z}) = \overline{D}(1,\beta^*)\cdot\begin{pmatrix} f_0^1 \\ f_0^2 \end{pmatrix} = \overline{D}(f_0^1 + \beta^* f_0^2) = \overline{D}(f_0^1 + f_0^2),$$

$$H(z,\overline{z},\theta) = -2\mathrm{Re}\{\overline{D}(f_0^1 + f_0^2)q(\theta)\} + \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & \theta \in (-\infty, 0), \\ \begin{pmatrix} f_0^1 \\ f_0^2 \end{pmatrix}, & \theta = 0. \end{cases}$$

Now we calculate $H_{20}(\theta)$. Since

$$\frac{\partial^2 f_0^j}{\partial z^2}|_{z=\overline{z}=0} = \frac{\partial^2 \overline{f}_0^j}{\partial z^2}|_{z=\overline{z}=0} = r_1(1+\mu_1)i, \quad j = 1,2,$$

we have

$$H_{20}(\theta) = \frac{\partial^2 H}{\partial z^2}|_{z=\overline{z}=0} = \begin{cases} 2\Gamma\overline{D}q(\theta) + 2\Gamma D\overline{q}(\theta), & \theta \in (-\infty,0), \\ -\frac{1}{5}\Gamma\begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \theta = 0. \end{cases}$$

where $\Gamma = -r_1(1+\mu_1)i$. Similarly, we have $H_{02}(\theta) = -H_{20}(\theta)$ and $H_{11}(\theta) = 0$. In fact

$$\frac{\partial^2 f_0^j}{\partial z \partial \overline{z}}|_{z=\overline{z}=0} = \frac{\partial^2 \overline{f}_0^j}{\partial z \partial \overline{z}}|_{z=\overline{z}=0} = 0, \quad j = 1,2.$$

On the other hand, near the origin, we use the chain rule and get

$$\begin{aligned} \dot{w} &= w_z(z,\overline{z})\dot{z} + w_{\overline{z}}(z,\overline{z})\dot{\overline{z}} \\ &= (w_{20}(\theta)z + w_{11}(\theta)\overline{z} + \cdots)\left(\frac{i}{\alpha_0}z + g(z,\overline{z})\right) \\ &\quad + (w_{11}(\theta)z + w_{02}(\theta)\overline{z} + \cdots)\left(-\frac{i}{\alpha_0}\overline{z} + \overline{g(z,\overline{z})}\right) \\ &= \mathcal{A}_1(\alpha_0)\left[w_{20}\frac{z^2}{2} + w_{11}z\overline{z} + w_{02}\frac{\overline{z}^2}{2} + \cdots\right] + H_{20}\frac{z^2}{2} + H_{11}z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots. \end{aligned}$$

Equating the coefficients on both sides, we have

$$\left(\frac{2i}{\alpha_0}I - \mathcal{A}_1(\alpha_0)\right)w_{20}(\theta) = H_{20}(\theta), \tag{3.15}$$

$$-\mathcal{A}_1(\alpha_0))w_{11}(\theta) = H_{11}(\theta), \tag{3.16}$$

$$\left(-\frac{2i}{\alpha_0}I - \mathcal{A}_1(\alpha_0)\right)w_{02}(\theta) = H_{02}(\theta). \tag{3.17}$$

Since $H_{11}(\theta) = 0$, according to the definition of $\mathcal{A}_1(\alpha_0)$, we get $w_{11}(\theta) = 0$. It is easy to check that $w_{02}(\theta) = \overline{w}_{02}(\theta)$. Now we solve the first equation. Explicitly writing the equation (3.15), we have

$$\left(\frac{2i}{\alpha_0} - \frac{d}{d\theta}\right)\begin{pmatrix} w_{20}^{(1)}(\theta) \\ w_{20}^{(2)}(\theta)\end{pmatrix} = \left(2\Gamma\overline{D}e^{\frac{\theta i}{\alpha_0}} + 2\Gamma D e^{-\frac{\theta i}{\alpha_0}}\right)\begin{pmatrix}1\\1\end{pmatrix}, \theta \in (-\infty, 0), \tag{3.18}$$

with the initial condition at $\theta = 0$

(3.19)
$$\frac{2i}{\alpha_0}\begin{pmatrix} w_{20}^{(1)}(0) \\ w_{20}^{(2)}(0)\end{pmatrix} + \int_{-\infty}^{0}\frac{r_1 f_{\alpha_0}(-s)}{1+\mu_1}\begin{pmatrix} w_{20}^{(1)}(s) + \mu_1 w_{20}^{(2)}(s) \\ \mu_1 w_{20}^{(1)}(s) + w_{20}^{(2)}(s)\end{pmatrix} ds = -\frac{1}{5}\Gamma\begin{pmatrix}1\\1\end{pmatrix}.$$

The general solution of (3.18) is given by

$$\begin{pmatrix} w_{20}^{(1)}(\theta) \\ w_{20}^{(2)}(\theta)\end{pmatrix} = \begin{pmatrix} T_1 e^{\frac{\theta i}{\alpha_0}} + T_2 e^{-\frac{\theta i}{\alpha_0}} + T_3 e^{\frac{2\theta i}{\alpha_0}} \\ S_1 e^{\frac{\theta i}{\alpha_0}} + S_2 e^{-\frac{\theta i}{\alpha_0}} + S_3 e^{\frac{2\theta i}{\alpha_0}}\end{pmatrix}, \tag{3.20}$$

where

$$T_1 = S_1 = -2\Gamma\alpha_0\overline{D}i, \quad T_2 = S_2 = -\frac{2}{3}\Gamma\alpha_0 D i. \tag{3.21}$$

In the problem we must require that $w(\theta)$ is continuous in $\theta \in (-\infty, 0]$, that is, we have $\lim_{\theta\to 0-} w_{20}(\theta) = w_{20}(0)$. Therefore

$$T_3 = w_{20}^{(1)}(0) - (T_1 + T_2), \quad S_3 = w_{20}^{(2)}(0) - (S_1 + S_2). \tag{3.22}$$

Now we put (3.20)–(3.22) into (3.19) to get

$$\begin{pmatrix} \frac{2i}{\alpha_0} + r_1 A & r_1\mu_1 A \\ r_1\mu_1 A & \frac{2i}{\alpha_0} + r_1 A\end{pmatrix}\begin{pmatrix} w_{20}^{(1)}(0) \\ w_{20}^{(2)}(0)\end{pmatrix} = \begin{pmatrix} C_{20}^{(1)} \\ C_{20}^{(2)}\end{pmatrix}, \tag{3.23}$$

where $A = -\frac{3+4i}{25(1+\mu_1)}$ and

$$\begin{pmatrix} C_{20}^{(1)} \\ C_{20}^{(2)}\end{pmatrix} = \frac{3r_1(3i-4)(1+\mu_1)}{125}\begin{pmatrix}1\\1\end{pmatrix}.$$

Denote $\Delta = (\frac{2i}{\alpha_0} + r_1 A)^2 - (r_1\mu_1 A)^2$, the solution of (3.23) is

$$\begin{pmatrix} w_{20}^{(1)}(0) \\ w_{20}^{(2)}(0)\end{pmatrix} = \frac{(1+i)(1+\mu_1)}{10}\begin{pmatrix}1\\1\end{pmatrix}. \tag{3.24}$$

Since

$$\begin{pmatrix}\widetilde{w}_{20}^{(1)}(0)\\ \widetilde{w}_{20}^{(2)}(0)\end{pmatrix} = \int_{-\infty}^{0} f_{\alpha_0}(-s)\begin{pmatrix}w_{20}^{(1)}(s)\\ w_{20}^{(2)}(s)\end{pmatrix} ds = \frac{(1+i)(1+\mu_1)}{10}\begin{pmatrix}1\\ 1\end{pmatrix},$$

and $g(z,\overline{z}) = \overline{D}(f_0^1 + f_0^2)$, we obtain

$$\begin{aligned} g(z,\overline{z}) = &-\overline{D}r_1\left(w_{20}^{(1)}(0)\frac{z^2}{2}+w_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+z+\overline{z}\right)\left(\widetilde{w}_{20}^{(1)}\frac{z^2}{2}+\widetilde{w}_{02}^{(1)}\frac{\overline{z}^2}{2}-\frac{zi}{2}+\frac{\overline{z}i}{2}\right)\\ &-\overline{D}r_1\mu_1\left(w_{20}^{(1)}(0)\frac{z^2}{2}+w_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+z+\overline{z}\right)\left(\widetilde{w}_{20}^{(2)}\frac{z^2}{2}+\widetilde{w}_{02}^{(2)}\frac{\overline{z}^2}{2}-\frac{zi}{2}+\frac{\overline{z}i}{2}\right)\\ &-\overline{D}r_1\mu_1\left(w_{20}^{(2)}(0)\frac{z^2}{2}+w_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+z+\overline{z}\right)\left(\widetilde{w}_{20}^{(1)}\frac{z^2}{2}+\widetilde{w}_{02}^{(1)}\frac{\overline{z}^2}{2}-\frac{zi}{2}+\frac{\overline{z}i}{2}\right)\\ &-\overline{D}r_1\left(w_{20}^{(2)}(0)\frac{z^2}{2}+w_{02}^{(2)}(0)\frac{\overline{z}^2}{2}+z+\overline{z}\right)\left(\widetilde{w}_{20}^{(2)}\frac{z^2}{2}+\widetilde{w}_{02}^{(2)}\frac{\overline{z}^2}{2}-\frac{zi}{2}+\frac{\overline{z}i}{2}\right)\end{aligned}$$

Equating the coefficients of both sides, we have

$$\begin{aligned} &g_{20} = 2\overline{D}r_1(1+\mu_1)i, \quad g_{02} = -2\overline{D}r_1(1+\mu_1)i,\\ &g_{11} = 0, \quad g_{21} = -\frac{r_1(1+\mu_1)^2}{20}(1+i).\end{aligned}$$

For determining the Floquet index, we transform equation (3.11) into the Poincaré normal form [13, 9, 8]

$$\dot{\xi} = \frac{i}{\alpha_0}\xi + C_1(0)\xi|\xi|^2 + o(|\xi|^3),$$

where

$$\begin{aligned} &C_1(0) = \frac{\alpha_0 i}{2}\left[g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right] + \frac{g_{21}}{2},\\ &\mathrm{Re}C_1(0) = \frac{1}{2}\mathrm{Re}g_{21} = -\frac{r_1(1+\mu_1)^2}{40} < 0,\\ &\alpha - \alpha_0 = \mu_2\epsilon^2 + o(\epsilon^2),\ \mu_2 = -\frac{\mathrm{Re}C_1(0)}{\mathrm{Re}\lambda'(\alpha_0)} > 0,\ \beta_2 = 2\mathrm{Re}C_1(0) < 0,\end{aligned}$$

where β_2 is the coefficient of the main part of the Floquet index and μ_2 determines the bifurcation direction, the bifurcating periodic solutions are orbitally asymptotically stable with asymptotic phase, and bifurcate from the steady state (u^*, v^*) for $\alpha > \alpha_0$. It should be observed that these bifurcating periodic solutions are spatially homogeneous:

$$\begin{pmatrix}U\\ V\end{pmatrix} = \begin{pmatrix}u^*\\ v^*\end{pmatrix} + \sqrt{\frac{\alpha-\alpha_0}{\mu_2}}\begin{pmatrix}1\\ \beta_0\end{pmatrix}\cos\frac{t}{\alpha_0} + o(\sqrt{\alpha-\alpha_0}) \quad (\alpha \to \alpha_0 + 0),$$

the period is $T(\epsilon) \approx 2\pi\alpha_0$.

Summarizing the above analysis, we have the following theorem.

Theorem 3.1. *A Hopf bifurcation occurs from the steady state (u^*, v^*) as α increasingly passes through α_0. Moreover, the bifurcating periodic solutions are orbitally asymptotically stable with asymptotic phase.*

§4. Hopf Bifurcation and Spatiotemporal Structure

We now consider the one dimensional competition diffusion system with distributed delay and Dirichlet boundary conditions

(4.1)
$$\begin{aligned}
&\frac{\partial U}{\partial t} = d\Delta U + r_1 U\left[1 - \int_{-\infty}^{t} f(t-\tau)U(\tau,x)d\tau - \mu_1\int_{-\infty}^{t} f(t-\tau)V(\tau,x)d\tau\right],\\
&\frac{\partial V}{\partial t} = d\Delta V + r_2 V\left[1 - \mu_2\int_{-\infty}^{t} f(t-\tau)U(\tau,x)d\tau - \int_{-\infty}^{t} f(t-\tau)V(\tau,x)d\tau\right],\\
&U(t,x) = u^*, V(t,x) = v^*, \qquad t\in R, x = 0,\pi,\\
&(U,V) = (\widetilde{\varphi}_1(t,x), \widetilde{\varphi}_2(t,x)), \qquad (t,x)\in(-\infty,0]\times[0,\pi],
\end{aligned}$$

where $f(s) = \alpha^2 s e^{-\alpha s}$, $s \in [0, +\infty)$. Here we use Dirichlet boundary conditions at the boundaries of the domain. On the boundaries of the process, both populations are maintained at the density level corresponding to the stationary state (u^*, v^*). Following the usual linearization procedure, set

$$u(t,x) = U(t,x) - u^*, v(t,x) = V(t,x) - v^*,$$

and substitute them into (4.1), we have

$$\begin{aligned}
(4.2)\quad &\frac{\partial}{\partial t}\begin{pmatrix} u(t,x)\\ v(t,x)\end{pmatrix} = d\Delta\begin{pmatrix} u(t,x)\\ v(t,x)\end{pmatrix} - B\int_{-\infty}^{0} f(-s)\begin{pmatrix} u(t+s,x)\\ v(t+s,x)\end{pmatrix} ds + F,\\
&u(t,x) = v(t,x) = 0, \qquad t\in R, x = 0,\pi,\\
&(u,v) = (\varphi_1(t,x), \varphi_2(t,x)), \qquad (t,x)\in(-\infty,0]\times[0,\pi],
\end{aligned}$$

where $(\varphi_1, \varphi_2) \in \mathcal{C}((-\infty,0], X\times X)$,$X = L^2([0,\pi])$, and

$$\begin{aligned}
&B = \begin{pmatrix} r_1u^* & r_1\mu_1u^*\\ r_2\mu_2v^* & r_2v^*\end{pmatrix},\\
&F = -\begin{pmatrix} r_1u(t,x) & r_1\mu_1u(t,x)\\ r_2\mu_2v(t,x) & r_2v(t,x)\end{pmatrix}\int_{-\infty}^{0} f(-s)\begin{pmatrix} u(t+s,x)\\ v(t+s,x)\end{pmatrix} ds.
\end{aligned}$$

The associated linearized problem is

$$(4.3)\qquad \begin{aligned}&\frac{\partial}{\partial t}\begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix} = d\Delta\begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix} - B\int_{-\infty}^{0} f(-s)\begin{pmatrix}u(t+s,x)\\ v(t+s,x)\end{pmatrix} ds,\\ &u(t,x)=v(t,x)=0, \qquad\qquad t\in R, x=0,\pi,\\ &(u,v)=(\varphi_1(t,x),\varphi_2(t,x)), \qquad\qquad (t,x)\in(-\infty,0]\times[0,\pi].\end{aligned}$$

We introduce an operator $\mathcal{A} = \begin{pmatrix}\partial^2/\partial x^2 & 0\\ 0 & \partial^2/\partial x^2\end{pmatrix} : \mathcal{D}(\mathcal{A}) \to X\times X$, with domain

$$\mathcal{D}(\mathcal{A}) = \left\{\begin{pmatrix}y\\ z\end{pmatrix} \in H_0^2([0,\pi])\times H_0^2([0,\pi])\right\}.$$

In terms of the eigenvalues and eigenfunctions of the operator $\partial^2/\partial x^2$ subject to the Dirichlet boundary conditions, the operator $\mathcal{A}$ has the eigenvalues $-k^2, k = 1,2,\cdots$ and corresponding eigenspaces

$$\text{ES}_k = \text{Span}\left\{\begin{pmatrix}\sin kx\\ 0\end{pmatrix}, \begin{pmatrix}0\\ \sin kx\end{pmatrix}\right\}, k=1,2,\cdots .$$

The operator $\mathcal{A}$ is an infinitesimal generator of a compact C_0-semigroup [12]. The study of the stability of (u^*,v^*) leads to the investigation of the eigenvalue problem

(4.4)

$$\Delta(d,\lambda)\begin{pmatrix}y\\ z\end{pmatrix} \equiv \left(d\mathcal{A} - B\int_{-\infty}^{0} f(-s)e^{\lambda s}ds - \lambda I\right)\begin{pmatrix}y\\ z\end{pmatrix} = 0, 0\neq\begin{pmatrix}y\\ z\end{pmatrix}\in\mathcal{D}(\mathcal{A}).$$

Let

$$\begin{pmatrix}y\\ z\end{pmatrix} = \sum_{k=1}^{\infty}\begin{pmatrix}\alpha_k\\ \beta_k\end{pmatrix}\sin kx,$$

then (4.4) is equivalent to a sequence of eigenvalue problems

$$(4.5)\ \Delta(k,d,\lambda)\begin{pmatrix}\alpha_k\\ \beta_k\end{pmatrix} \equiv \left(-dk^2 I - B\int_{-\infty}^{0} f(-s)e^{\lambda s}ds - \lambda I\right)\begin{pmatrix}\alpha_k\\ \beta_k\end{pmatrix} = 0,$$

then we obtain a sequence of characteristic equations

$$(4.6)\qquad det\left[(\lambda + dk^2)I + B\int_{-\infty}^{0} f(-s)e^{\lambda s}ds\right] = 0, n = 1,2,\ldots .$$

For the strong generic kernel $f(s) = \alpha^2 s e^{-\alpha s}$, $\int_{-\infty}^{0} f(-s)e^{\lambda s} ds = \frac{\alpha^2}{(\lambda+\alpha)^2}$, then we have

$$[(\lambda + dk^2)(\lambda + \alpha)^2 + M_1\alpha^2][(\lambda + dk^2)(\lambda + \alpha)^2 + M_2\alpha^2] = 0, \tag{4.7}$$

$k = 1, 2, \ldots$, which are equivalent to (4.8) and/or (4.9)

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_{31} = 0, \quad k = 1, 2, \ldots. \tag{4.8}$$

$$\lambda^3 + p_1\lambda^2 + p_2\lambda + p_{32} = 0, \quad k = 1, 2, \ldots. \tag{4.9}$$

where

$$\begin{aligned}
&p_1 = 2\alpha + dk^2, \quad p_2 = \alpha^2 + 2\alpha dk^2, \quad p_{31} = \alpha^2 dk^2 + M_1\alpha^2,\\
&p_{32} = \alpha^2 dk^2 + M_2\alpha^2,\\
&M_{1,2} = \frac{1}{2}\left[r_1u^* + r_2v^* \mp \sqrt{(r_1u^* + r_2v^*)^2 - 4r_1r_2(1 - \mu_1\mu_2)u^*v^*}\right].
\end{aligned}$$

The condition (1.4) implies that $0 < M_1 < M_2$. By the Routh-Hurwitz criterion, the necessary and sufficient conditions for the characteristic roots of (4.8) and (4.9) to have negative parts are (i) $p_1 > 0$, (ii) $p_{31} > 0$ or $p_{32} > 0$, (iii) $p_1p_2 > p_{31}$ or $p_1p_2 > p_{32}$. Obviously, (i) and (ii) are satisfied. Since

$$\begin{aligned}
p_1p_2 - p_{31} &= \alpha(2k^4d^2 + 4\alpha k^2 d + 2\alpha^2 - \alpha M_1),\\
p_1p_2 - p_{32} &= \alpha(2k^4d^2 + 4\alpha k^2 d + 2\alpha^2 - \alpha M_2).
\end{aligned}$$

If $\alpha > \frac{M_2}{2}$, then for any $d > 0, k \geq 1, p_1p_2 - p_{31} > p_1p_2 - p_{32} > 0$. According to the Routh-Hurwitz criterion, all roots of (4.8) and (4.9) have negative real parts. This implies that the positive steady state (u^*, v^*) is locally asymptotically stable. Analysis of these shows that no diffusive instability occurs.

If $\frac{M_2}{2} > \alpha > \frac{M_1}{2}$, then for any $d > 0, k \geq 1, p_1p_2 - p_{31} > 0$. Denote $d_0 = \sqrt{\frac{\alpha M_2}{2}} - \alpha$, we have the following results.

	$k = 1$	$k > 1$
$d = 0$	$p_1p_2 - p_{31} > 0, p_1p_2 - p_{32} < 0$	$p_1p_2 - p_{31} > 0, p_1p_2 - p_{32} < 0$
$d > d_0$	$p_1p_2 - p_{31} > 0, p_1p_2 - p_{32} > 0$	$p_1p_2 - p_{31} > 0, p_1p_2 - p_{32} > 0$
$d = d_0$	$p_1p_2 - p_{31} > 0, p_1p_2 - p_{32} = 0$	$p_1p_2 - p_{31} > 0, p_1p_2 - p_{32} > 0$

where the critical value

$$\begin{aligned}
d_0 &= \max\{d > 0 | p_1p_2 - p_{32} = 0, k \geq 1\}\\
&= \max\left\{\left(\sqrt{\frac{\alpha M_2}{2}} - \alpha\right)/k^2 | k \geq 1\right\} = \sqrt{\frac{\alpha M_2}{2}} - \alpha.
\end{aligned}$$

If $\alpha < \frac{M_1}{2}$, then we can also get the above table, where

$$\begin{aligned} d_0 &= \max\{d > 0 | p_1p_2 - p_{31} = 0 \quad or \quad p_1p_2 - p_{32} = 0, k \geq 1\} \\ &= \max\left\{\left(\sqrt{\frac{\alpha M_2}{2}} - \alpha\right)/k^2, \left(\sqrt{\frac{\alpha M_1}{2}} - \alpha\right)/k^2 | k \geq 1\right\} = \sqrt{\frac{\alpha M_2}{2}} - \alpha. \end{aligned}$$

Therefore, for $d = d_0$, the above table shows that all roots of (4.8) for $k \geq 1$ and (4.9) for $k \geq 2$ have negative real parts, and for $k = 1$, (4.9) has three roots, a pair of complex, simple eigenvalues $\lambda(d_0) = \omega_0 i, \overline{\lambda}(d_0) = -\omega_0 i$, $\omega_0 = \sqrt{\alpha^2 + 2\alpha d_0}$ and one negative real root $\lambda_3(d_0) = -(2\alpha + d_0)$. Thus, for $\alpha < \frac{M_2}{2}$, we know that $d = d_0$ is a critical value and some consequences as follows.

(1) All eigenvalues of (4.7) have negative real parts for $d > d_0$;

(2) If $d = d_0$, (4.7) has a pair of pure imaginary eigenvalues $\lambda(d_0) = \omega_0 i$ and $\overline{\lambda}(d_0)$ for $k = 1$, and other eigenvalues have negative real parts for $k \geq 1$.

Hence we have the following result about the Hopf bifurcation.

Theorem 4.1. *If $\alpha < \frac{M_2}{2}$, then a Hopf bifurcation of the steady state (u^*, v^*) of system* (4.1) *occurs as the diffusion rate d decreasingly passes through d_0.*

Proof. According to the Hopf bifurcation Theorem, it suffices to verify the transversality condition. For $k = 1$, find the derivative on both sides of (4.9) with respect to d, we have

$$\lambda'(d) = \frac{-(\lambda^2 + 2\alpha\lambda + \alpha^2)}{3\lambda^2 + 2(2\alpha + d)\lambda + \alpha^2 + 2\alpha d},$$

Since $\lambda(d_0) = \omega_0 i, \omega_0 = \sqrt{\alpha^2 + 2\alpha d_0}$, we have

$$\text{Re}\lambda'(d)|_{d=d_0} = \frac{-2\alpha(\alpha + d_0)}{\omega_0^2 + (2\alpha + d_0)^2} < 0.$$

This implies that the transversality condition holds true. Therefore, a Hopf bifurcation from (u^*, v^*) occurs as d decreasingly passes through d_0. This completes the proof. □

Remark. For $d = d_n, n = 2, 3, \cdots$, the instability of the Hopf bifurcation periodic solutions is due to the fact that the center manifold for $d \approx d_n, n = 2, 3, \cdots$ is unstable.

The above-mentioned result shows that the system (4.1) exhibits spatiotemporal structure when the diffusion rate d passes through the critical value

d_0. That is, the Hopf bifurcation occurs as d decreasingly passes through the critical value d_0 and has the asymptotical expressions

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} u^* \\ v^* \end{pmatrix} + \sqrt{\left|\frac{d-d_0}{\mu_2}\right|} \begin{pmatrix} 1 \\ \beta_0 \end{pmatrix} \cos(\omega_0 t)\sin x + o(\sqrt{|d-d_0|}),$$
$$(d \to d_0 - 0).$$

It should be observed that these bifurcating periodic solutions are spatially inhomogeneous. We omit the detailed proof for the stability of the bifurcation periodic solutions and their asymptotic expressions as the calculation is tedious. Readers interested in the details of these arguments are referred to [8, 9, 13, 18], where a detailed discussion of these stability results is given.

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