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Weak Solution of a Singular Semilinear Elliptic Equation in a Bounded Domain

By

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Abstract

We study the singular semilinear elliptic equation $\Delta u + f(.,u) = 0$ in $\mathcal{D}'(\Omega)$, where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain of class $C^{1,1}$. $f : \Omega \times (0,\infty) \to [0,\infty)$ is such that $f(.,u) \in L^1(\Omega)$ for $u > 0$ and $u \to f(x,u)$ is continuous and nonincreasing for a.e. x in Ω. We assume that there exists a subset $\Omega' \subset \Omega$ with positive measure such that $f(x,u) > 0$ for $x \in \Omega'$ and $u > 0$ and that $\int_\Omega f(x, cd(x,\partial\Omega))\, dx < \infty$ for all $c > 0$. Then we show that there exists a unique solution u in $W_0^{1,1}(\Omega)$ such that $\Delta u \in L^1(\Omega)$, $u > 0$ a.e. in Ω.

§1. Introduction

Let Ω be a sufficiently smooth (e.g. of class $C^{1,1}$) bounded domain in $\mathbb{R}^n$ $(n \geq 1)$. We consider the singular boundary value problem

$$\Delta u + f(.,u) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega)\,, \tag{1.1}$$

$$u \in W_0^{1,1}(\Omega)\,,\ f(.,u(.)) \in L^1(\Omega)\,, \tag{1.2}$$

where f satisfies the following conditions:

(H1) $f : \Omega \times (0,\infty) \to [0,\infty)$. For all $u > 0$, $x \to f(x,u)$ is in $L^1(\Omega)$, and $u \to f(x,u)$ is continuous and nonincreasing for a.e. x in Ω;

(H2) There exists $\Omega' \subset \Omega$ with positive measure such that $f(x,u) > 0$ for $x \in \Omega'$ and $u > 0$;

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(H3) For all $c > 0$

$$\int_\Omega f(x, cd(x, \partial\Omega))\, dx \;<\; +\infty\,.$$

This kind of singularity has been considered by several authors, particularly the case where

$$f(x, u) = p(x)u^{-\lambda}\,,$$

$\lambda > 0$ [4, 5, 6, 9, 10, 11, and their references].

Lazer and McKenna [10] for instance established the existence and uniqueness of a positive $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying

$$\Delta u + p(x)u^{-\lambda} = 0 \quad \text{in } \Omega\,, \tag{1.3}$$

$$u = 0 \quad \text{on } \partial\Omega\,, \tag{1.4}$$

when p is Hölder-continuous and strictly positive in $\overline{\Omega}$.

Del Pino [6] proved that if p is a bounded, nonnegative measurable function which is positive on a set of positive measure, then (1.3)–(1.4) has a unique positive weak solution in the sense that $u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ satisfies (1.4), $u > 0$ in Ω and

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega p u^{-\lambda} \varphi$$

for all $\varphi \in C_c^\infty(\Omega)$.

Lair and Shaker [9] considered the case

$$f(x, u) = p(x)g(u)\,,$$

under the following assumptions:

(A0) $p \in L^2(\Omega)$ is nontrivial and nonnegative;

(A1) $g'(s) \le 0$;

(A2) $g(s) > 0$ if $s > 0$;

(A3) $\int_0^\varepsilon g(s)\, ds < \infty$ for some $\varepsilon > 0$.

They established the existence of a unique weak solution in the sense that $u \in H_0^1(\Omega)$ satisfies

$$\int_\Omega (\nabla u \nabla v - p(x)g(u)v)\, dx = 0 \quad \forall\ v\ \in H_0^1(\Omega)\,.$$

Notice that, when $g(u) = u^{-\lambda}$, conditions (A1) and (A3) imply that $0 \le \lambda < 1$.

Finally Mâagli and Zribi [11] treated the general case $f(x,u)$. However their assumptions are different from ours and they lead them to the existence of a weak solution in $C(\overline{\Omega})$.

Our purpose is to give a general existence and uniqueness result under sufficiently weak conditions. We shall prove the following theorem.

Theorem 1. *Let* $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ *be a bounded domain of class* $C^{1,1}$ *and let* $f : \Omega \times (0,\infty) \to [0,\infty)$ *satisfy* (H1)–(H3). *Then problem* (1.1)–(1.2) *has a unique solution.*

§2. Proof of Theorem 1

1) *Uniqueness of the solution.* We shall need the following lemma ([7, Lemma 3]).

Lemma 1. *Let* $p \in C^1(\mathbb{R},\mathbb{R}) \cap L^\infty(\mathbb{R})$ *be a nondecreasing function satisfying* $p(0) = 0$. *For* $u \in W_0^{1,1}(\Omega)$ *such that* $\Delta u \in L^1(\Omega)$ *we have*

$$\int \Delta u . p(u) \ \leq 0 \, .$$

Let u_1, u_2 be two solutions of problem (1.1)–(1.2). Let $u = u_1 - u_2$. By (H1) we have $u\Delta u \geq 0$ a.e. in Ω. Now let $p \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a strictly increasing function satisfying $p(0) = 0$. Then $p(u)\Delta u \geq 0$ a.e. in Ω. Using Lemma 1 we deduce that $p(u)\Delta u = 0$ a.e. in Ω and therefore $\Delta u = 0$ a.e. in Ω. Since $u \in W_0^{1,1}(\Omega)$, this implies that $u = 0$ a.e. in Ω.

2) *Existence of a solution.* We first recall the following result ([1, Lemme 2.8]).

Lemma 2. *Let* $u \in W_0^{1,1}(\Omega)$ *be such that* $\Delta u \geq 0$ *in* $\mathcal{D}'(\Omega)$. *Then* $u \leq 0$ *a.e. in* Ω.

In the sequel $\mathbb{N}^\star$ denotes the set of positive integers.

Lemma 3. *Let* $j \in \mathbb{N}^\star$. *There exists a unique* $u_j \in W_0^{1,1}(\Omega)$ *such that* $f(.,u_j + \frac{1}{j}) \in L^1(\Omega)$, $u_j \geq 0$ *a.e. in* Ω *and* $\Delta u_j + f(.,u_j + \frac{1}{j}) = 0$ *in* $\mathcal{D}'(\Omega)$.

Proof. Define

$$\beta_j(x,u) = f\left(x, \frac{1}{j}\right) - f\left(x, u + \frac{1}{j}\right) , \quad x \in \Omega \, , \, u \geq 0 \, .$$

and

$$\beta_j(x,u) = 0\,, \quad x \in \Omega\,,\, u \leq 0\,.$$

□

Then we have:

- For all $u \in \mathbb{R}$, $x \to \beta_j(x,u)$ is in $L^1(\Omega)$;
- $\mathbb{R} \ni u \to \beta_j(x,u)$ is continuous and nondecreasing for a.e. x in Ω;
- $\beta_j(x,0) = 0$ for a.e. x in Ω.

Since $f(.,\frac{1}{j}) \in L^1(\Omega)$ Theorem 3 in [7] implies the existence of a unique $u_j \in W_0^{1,1}(\Omega)$ satisfying $\beta_j(.,u_j) \in L^1(\Omega)$ and

$$-\Delta u_j + \beta_j(.,u_j) \;=\; f\left(.,\frac{1}{j}\right) \quad \text{in} \quad \mathcal{D}'(\Omega)\,.$$

Since $\Delta u_j \leq 0$ in $\mathcal{D}'(\Omega)$, Lemma 2 implies that $u_j \geq 0$ a.e. in Ω. Therefore we have $f(.,u_j+\frac{1}{j}) \in L^1(\Omega)$ and

$$\Delta u_j + f\left(.,u_j+\frac{1}{j}\right) \;=\; 0 \quad \text{in} \quad \mathcal{D}'(\Omega)\,,$$

and the lemma is proved.

Lemma 4. *For every $j \in \mathbb{N}^\star$ there exists $a_j > 0$ such that*

$$u_j(x) \geq a_j d(x,\partial\Omega)\,, \quad \textit{for a.e. } x \in \Omega\,.$$

Proof. For $\varepsilon > 0$ we set $\Omega_\varepsilon = \{x \in \Omega;\, d(x,\partial\Omega) > \varepsilon\}$. Clearly (H2) implies that, for every $j \in \mathbb{N}^\star$, there exist $\varepsilon_j > 0$ and $M_j > 0$ such that the function $\tilde{f}_j$ defined by

$$\tilde{f}_j(x) \;=\; \min\left(f\left(x,u_j(x)+\frac{1}{j}\right),M_j\right)\mathbf{1}_{\Omega_{\varepsilon_j}}(x), \quad x \in \Omega\,,$$

satisfies $\tilde{f}_j \not\equiv 0$. Let v_j be the solution of the following boundary value problem

$$\Delta v_j + \tilde{f}_j = 0 \quad \text{in} \quad \Omega\,,$$
$$v_j = 0 \quad \text{on} \quad \partial\Omega\,.$$

It is well-known (see [8]) that, for $1 < p < \infty$, $v_j \in C^1(\overline{\Omega}) \cap W^{2,p}(\Omega)$. We have $\Delta(u_j - v_j) \leq 0$ in $\mathcal{D}'(\Omega)$, hence by Lemma 2 $u_j \geq v_j$ a.e. in Ω. Now the boundary point version of the Strong Maximum Principle for weak solutions ([12], Theorem 2) implies that there exists $a_j > 0$ such that $v_j(x) \geq a_j d(x,\partial\Omega)$ for $x \in \Omega$ and the lemma follows. □

Lemma 5. *For every $j \in \mathbb{N}^\star$ we have $u_j + \frac{1}{j} \geq u_{j+1} + \frac{1}{j+1}$ a.e. in Ω.*

Proof. Let $u = (u_{j+1}+\frac{1}{j+1})-(u_j+\frac{1}{j})$. Using a variant of Kato's inequality (see [3] Lemma A1 in the Appendix) we deduce that

$$\Delta u^+ \geq 0 \quad \text{in } \mathcal{D}'(\Omega)\,.$$

$u^+ \in W_0^{1,1}(\Omega)$ (see [1, Lemme 2.7]). Therefore Lemma 2 implies that $u \leq 0$ a.e. in Ω and the lemma is proved. □

Lemma 6. *For every $j \in \mathbb{N}^\star$ we have $u_j \leq u_{j+1}$ a.e. in Ω.*

Proof. Using (H1) and Lemma 5 we get

$$\Delta(u_{j+1} - u_j) = f\left(., u_j + \frac{1}{j}\right) - f\left(., u_{j+1} + \frac{1}{j+1}\right) \leq 0 \quad \text{a.e. in } \Omega\,,$$

and we conclude with the help of Lemma 2. □

Now we define

$$c_j \;= \int_\Omega f\left(x, u_j(x) + \frac{1}{j}\right) dx\,, \quad j \in \mathbb{N}^\star.$$

Using Lemma 4 and Lemma 6 we can write

$$c_j \;\leq \int_\Omega f(x, a_1 d(x, \partial\Omega))\, dx\,, \quad \forall\, j \in \mathbb{N}^\star.$$

Therefore

$$\sup_{j \in \mathbb{N}^\star} c_j \; < \infty\,. \tag{3.1}$$

Now we can prove the existence. By (H1) and Lemma 5 $j \to f(., u_j + \frac{1}{j})$ is nondecreasing. (3.1) and the Beppo Levi theorem for monotonic sequences imply that there exists $g \in L^1(\Omega)$ such that

$$f\left(., u_j + \frac{1}{j}\right) \to g \quad \text{in } L^1(\Omega) \quad \text{as } j \to \infty\,.$$

We have the following estimate [2, Theorem 8]: for $1 \leq q < N/(N-1)$ there exists $M_q > 0$ such that

$$||u_j||_{W^{1,q}(\Omega)} \;\leq\; M_q ||\Delta u_j||_{L^1(\Omega)} \quad \forall\; j \;\in \mathbb{N}^\star\,.$$

Therefore there exists $u \in W_0^{1,1}(\Omega)$ such that $u_j \to u$ in $W_0^{1,1}(\Omega)$. By Lemma 6 and the Fischer-Riesz theorem $u_j \to u$ a.e. in Ω. Lemma 4 and Lemma 6 imply that $u > 0$ a.e. in Ω. Clearly we have $g = f(.,u)$ and $\Delta u + f(.,u) = 0$ in $\mathcal{D}'(\Omega)$. The proof is complete.

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