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The Harmonic Volumes of Hyperelliptic Curves

By

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Abstract

We determine the harmonic volumes for all the hyperelliptic curves. This gives a geometric interpretation of a theorem established by A. Tanaka [10].

Contents

§1. Introduction

Let X be a compact Riemann surface of genus $g \geq 3$. A harmonic volume I of X was introduced by B. Harris [5], using Chen's iterated integrals [3]. The aim of this paper is to determine the harmonic volumes of all the hyperelliptic

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curves, which are 2-fold branched coverings of $\mathbb{C}P^1$. As was already pointed out by Harris, some important algebraic cycles in the Jacobian variety $J(X)$ are related to $2I$, which vanishes for all the hyperelliptic curves. The harmonic volumes of hyperelliptic curves, however, have been still unknown. First of all, we give the statement of the main theorem of this paper. We denote by H the first integral homology group of X. Harris defined the harmonic volume I as a homomorphism $(H^{\otimes 3})' \to \mathbb{R}/\mathbb{Z}$. Here $(H^{\otimes 3})'$ is a certain subgroup of $H^{\otimes 3}$. See Section 2 for the definition of $(H^{\otimes 3})'$. We denote by C a hyperelliptic curve.

Theorem 4.1. *For any hyperelliptic curve C, let $\{x_i, y_i\}_{i=1,2,\ldots,g}$ be a symplectic basis of $H = H_1(C;\mathbb{Z})$ in Figure 1, where ι is the hyperelliptic involution. We denote by z_i either x_i or y_i. Then,*

$$\begin{aligned} &I(z_i \otimes z_j \otimes z_k) = 0 \ \text{ for } i \neq j \neq k \neq i, \\ &I(x_i \otimes y_i \otimes z_k - x_{k+1} \otimes y_{k+1} \otimes z_k) \\ &\qquad = \begin{cases} 1/2 & \text{for } i < k, k = 2, 3, \ldots, g-1 \text{ and } z_k = y_k, \\ 0 & \text{for } i \geq k+2, k = 1, k = g \text{ or } z_k = x_k. \end{cases} \end{aligned}$$

The elements $z_i \otimes z_j \otimes z_k$ and $x_i \otimes y_i \otimes z_k - x_{k+1} \otimes y_{k+1} \otimes z_k$ are the parts of a basis of $(H^{\otimes 3})'$ whose harmonic volumes depend on the complex structure of Riemann surfaces.

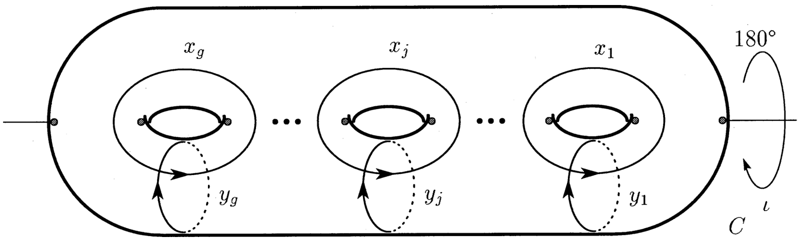

Figure 1.

By using the harmonic volume of the compact Riemann surface X whose coefficients are extended over $\mathbb{C}$, Harris [7] studied the problem of characterizing the condition when the cycles W_1 and W_1^- are algebraically equivalent to each other. Here W_1 is the image of the Abel-Jacobi map $X \to J(X)$ and W_1^- is the image of W_1 under the involution (-1) of $J(X)$. Harmonic volumes or extended ones tell us the non-triviality of $W_1 - W_1^-$ in $J(X)$ as follows. If $W_1 - W_1^-$ is trivial as an algebraic cycle, then $2I \equiv 0$ modulo $\mathbb{Z}$. As is well known, if X is hyperelliptic, then $W_1 - W_1^-$ is trivial. It is known that $I \equiv 0$ or $I \equiv 1/2$ modulo $\mathbb{Z}$ for any hyperelliptic curve C by the hyperelliptic involution. It has been still unknown which elements in $(H^{\otimes 3})'$ have nontrivial I or not. Our main theorem gives the complete answer for this problem.

We have two ways to compute the harmonic volumes of all the hyperelliptic curves in Theorem 4.1. One is an analytic way and the other is a topological. In the first way, the computation of the harmonic volumes of all the hyperelliptic curves can be reduced to that of a single hyperelliptic curve C_0, which is considered as a point of the moduli space of hyperelliptic curves, denoted by $\mathcal{H}_g$. The harmonic volume I varies continuously on the whole Torelli space $\mathcal{I}_g$, which is the space consisting of all the compact Riemann surfaces with a fixed symplectic basis of H. Gunning [4] obtained quadratic periods of hyperelliptic curves. The periods are defined by iterated integrals of holomorphic 1-forms along loops. In general, iterated integrals are not homotopy invariant with fixed endpoints. When we add some correction terms, they are homotopy invariant. Because of the correction terms, the computation of harmonic volumes is more difficult than that of quadratic periods. In the second way, we use basic results on the cohomology of the hyperelliptic mapping class group. It is denoted by Δ_g. The following theorem is obtained in the second topological way.

Theorem 5.9. *We have*

$$\mathrm{Hom}_{\Delta_g}((H^{\otimes 3})', \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

This theorem gives a geometric interpretation of a theorem established by Tanaka [10]. It is concerning about the first homology group of Δ_g with coefficients in H.

Theorem 5.10 (Tanaka [10], Theorem 1.1). *If $g \geq 2$, then*

$$H_1(\Delta_g; H) = \mathbb{Z}/2\mathbb{Z}.$$

We denote by δ a connected homomorphism $H^0(\Delta_g; ((H^{\otimes 3})')^*) \to H^1(\Delta_g; H^*)^{\oplus 3}$ defined in Section 5. We may regard the restriction of $\delta I|_H$ as the generator of $H_1(\Delta_g; H)$.

§2. Preliminaries

In this section, we define a harmonic volume of a compact Riemann surface X of genus $g \geq 3$. We begin with recalling the definition of an iterated integral on X. Let $\gamma : [0,1] \to X$ be a path in X, and $A^1(X)$ the 1-forms on X. The iterated integral of 1-forms $\omega_1, \omega_2, \dots, \omega_k \in A^1(X)$ along γ is defined by

$$\int_\gamma \omega_1\omega_2\cdots\omega_k = \int_{0\leq t_1\leq t_2\leq\cdots\leq t_k\leq 1} f_1(t_1)f_2(t_2)\cdots f_k(t_k)dt_1dt_2\cdots dt_k,$$

where $\gamma^*(\omega_i) = f_i(t)dt$ in terms of the coordinate t on the interval $[0,1]$. The integral is not invariant under homotopy with fixed endpoints. But, the following lemma is well known. See Chen [3] for details.

Lemma 2.1. *Let $\omega_{1,i}, \omega_{2,i}, i = 1, 2, \dots, m$ be closed 1-forms on X and γ a path in X. Suppose that $\sum_{i=1}^{m} \int_X \omega_{1,i} \wedge \omega_{2,i} = 0$. Take a 1-form η on X satisfying $d\eta = \sum_{i=1}^{m} \omega_{1,i} \wedge \omega_{2,i}$. Then the integral*

$$\sum_{i=1}^{m} \int_\gamma \omega_{1,i}\omega_{2,i} - \int_\gamma \eta$$

is invariant under homotopy with fixed endpoints.

Using iterated integrals, Harris [5] defined the harmonic volume in the following way. In order to define it, we have to define a pointed harmonic volume for (X, x_0), where x_0 is a point on X. We identify $H_1(X;\mathbb{Z})$ with $H^1(X;\mathbb{Z})$ by Poincaré duality and call them H. Let K be the kernel of $(\ ,\ ) : H \otimes H \to \mathbb{Z}$ induced by the intersection pairing. On the compact Riemann surface X, the Hodge star operator $* : A^1(X) \to A^1(X)$ is locally given by $*(f_1(z)dz + f_2(z)d\bar{z}) = -\sqrt{-1}f_1(z)dz + \sqrt{-1}f_2(z)d\bar{z}$ in a local coordinate z and depends only on the complex structure and not the choice of Hermitian metric. Let $\mathscr{H}_{\mathbb{Z}}$ denote the free abelian group of rank $2g$ spanned by all the real harmonic 1-forms on X with integral periods. We identify H with $\mathscr{H}_{\mathbb{Z}}$ by the Hodge theorem.

Definition 2.2 (The pointed harmonic volume [8]). The pointed harmonic volume for (X, x_0) is a linear form on $K \otimes H$ with values in $\mathbb{R}/\mathbb{Z}$ defined by

$$I_{x_0}\left(\sum_{k=1}^{m}\left(\sum_{i=1}^{n_k} a_{i,k}\otimes b_{i,k}\right)\otimes c_k\right) = \sum_{k=1}^{m}\left(\sum_{i=1}^{n_k}\int_{\gamma_k} a_{i,k}b_{i,k} - \int_{\gamma_k}\eta_k\right) \mod \mathbb{Z},$$

where γ_k is a loop in X with the base point x_0, whose homology class is Poincaré dual of the cohomology class of c_k and η_k is a 1-form on X, which satisfies $d\eta_k = \sum_{i=1}^{n_k} a_{i,k} \wedge b_{i,k}$ and $\int_X \eta_k \wedge *\alpha = 0$ for any closed 1-form α on X.

The harmonic volume is given as a restriction of the pointed harmonic volume. A natural homomorphism $p : H^{\otimes 3} \to H^{\oplus 3}$ is defined by $p(a \otimes b \otimes c) = ((a,b)c, (b,c)a, (c,a)b)$. We denote by $(H^{\otimes 3})'$ the kernel of p. It is a free $\mathbb{Z}$ module and satisfies the following short exact sequence

$$0 \longrightarrow (H^{\otimes 3})' \longrightarrow H^{\otimes 3} \xrightarrow{\ p\ } H^{\oplus 3} \longrightarrow 0\ .$$

The rank of $(H^{\otimes 3})'$ is $(2g)^3 - 6g$ and $(H^{\otimes 3})' \subset K \otimes H$. Harris [5] proved that the restriction of the pointed harmonic volume on $K \otimes H$ to $(H^{\otimes 3})'$ is independent of the choice of the base point.

Definition 2.3 (The harmonic volume [5]). The harmonic volume I for X is a linear form on $(H^{\otimes 3})'$ with values in $\mathbb{R}/\mathbb{Z}$ defined by

$$I\Big(\sum_i a_i \otimes b_i \otimes c_i\Big) = I_{x_0}\Big(\sum_i a_i \otimes b_i \otimes c_i\Big) \mod \mathbb{Z}.$$

The map I is a well-defined homomorphism $(H^{\otimes 3})' \to \mathbb{R}/\mathbb{Z}$. We have $I(\sum_i h_{\sigma(1),i} \otimes h_{\sigma(2),i} \otimes h_{\sigma(3),i}) = \mathrm{sgn}(\sigma) I(\sum_i h_{1,i} \otimes h_{2,i} \otimes h_{3,i})$, where $\sum_i h_{1,i} \otimes h_{2,i} \otimes h_{3,i} \in (H^{\otimes 3})'$ and σ is an element of the third symmetric group S_3. See Harris (Lemma 2.7 in [5]) and Pulte [8] for details. In the sequel, we regard $(H^{\otimes 3})'$ as an S_3-module by this action. We choose a symplectic basis $\{x_i, y_i\}_{i=1,2,\dots,g}$ of H so that $(x_i, x_j) = (y_i, y_j) = 0$ and $(x_i, y_j) = \delta_{ij} = -(y_j, x_i)$, where δ_{ij} is Kronecker's delta. Let z_i denote x_i or y_i. We define the subset $\mathfrak{A} \subset (H^{\otimes 3})'$ consisting of the following elements,

(1)	$z_i \otimes z_j \otimes z_k$	$(i \neq j \neq k \neq i)$
(2)	$x_i \otimes y_i \otimes z_k - x_{k+1} \otimes y_{k+1} \otimes z_k$	$(i \neq k$ and $i \neq k+1)$
(3a)	$x_i \otimes x_i \otimes z_k$	$(i \neq k)$
(3b)	$y_i \otimes y_i \otimes z_k$	$(i \neq k)$
(4a)	$x_i \otimes x_i \otimes x_i$	
(4b)	$y_i \otimes y_i \otimes y_i$	
(5a)	$x_{i+1} \otimes x_i \otimes y_{i+1} + y_{i+1} \otimes x_i \otimes x_{i+1}$	
(5b)	$y_{i+1} \otimes y_i \otimes x_{i+1} + x_{i+1} \otimes y_i \otimes y_{i+1}$	
(6a)	$x_i \otimes x_i \otimes y_i - x_i \otimes x_{i+1} \otimes y_{i+1} - x_{i+1} \otimes x_i \otimes y_{i+1}$	
(6b)	$y_i \otimes y_i \otimes x_i - y_i \otimes y_{i+1} \otimes x_{i+1} - y_{i+1} \otimes y_i \otimes x_{i+1}$.	

Here $i, j, k \in \{1, 2, \ldots, g\}$ and all subscripts are read modulo g. Then $\mathfrak{B} = \{\sigma(a); a \in \mathfrak{A}, \sigma \in S_3\}$ is a basis of $(H^{\otimes 3})'$.

By the definition of the harmonic volume, it is obvious that $I = 0 \bmod \mathbb{Z}$ for the type $(3), (4)$ and (5). Furthermore, $I = 1/2 \bmod \mathbb{Z}$ for the type (6). So it is enough to consider the type (1) and (2).

§3. The Periods and Iterated Integrals of a Hyperelliptic Curve

In this section, we compute the periods and iterated integrals of a hyperelliptic curve of genus $g \geq 3$. First of all, we take a symplectic basis of H.

§3.1. A homology basis of hyperelliptic curves

We define a hyperelliptic curve C as follows. Let $p_0, p_1, \ldots, p_{2g+1}$ be distinct points on $\mathbb{C}$. It is the compactification of the plane curve in the (z, w) plane $\mathbb{C}^2$

$$w^2 = \prod_{i=0}^{2g+1} (z - p_i),$$

and admits the hyperelliptic involution given by $\iota : (z, w) \mapsto (z, -w)$. Let π be the 2-sheeted covering $C \to \mathbb{C}P^1, (z, w) \mapsto z$, branched over $2g + 2$ branch points $\{p_i\}_{i=0,1,\ldots,2g+1}$ and $P_i \in C$ a ramification point so that $\pi(P_i) = p_i$. On the curve C, we choose endpoints $Q_0, Q_1 (= \iota(Q_0))$ as in Figure 2. We define by Ω the simply-connected domain $\mathbb{C}P^1 \setminus \bigcup_{j=0}^{g} p_{2j}p_{2j+1}$, where $p_{2j}p_{2j+1}$ is a simple arc connecting p_{2j} and p_{2j+1}. Then $\pi^{-1}(\Omega)$ consists of two connected components. We denote by Ω_0, Ω_1 the connected components of $\pi^{-1}(\Omega)$ which contain Q_0, Q_1 respectively. Let $e_j, j = 0, 1, \ldots, 2g + 1$, be a path in C which is to be followed from Q_0 to P_j and go to Q_1 along the arcs Q_0P_j and P_jQ_1. See Figure 2. We write simply $\overline{e}_j$ for $\pi(e_j)$. It is a loop in $\mathbb{C}P^1$ with the base point $\pi(Q_0)$.

It is obvious that $e_{j_1} \cdot \iota(e_{j_2})$ is a loop in C with the base point Q_0, where the product $e_{j_1} \cdot \iota(e_{j_2})$ indicates that we traverse e_{j_1} first, then $\iota(e_{j_2})$. So we have the homotopy equivalences relative to the base point Q_0

$$e_j \cdot \iota(e_j) \sim 1, \quad j = 0, 1, \ldots, 2g + 1,$$

and

$$e_0 \cdot \iota(e_1) \cdot \cdots \cdot e_{2g} \cdot \iota(e_{2g+1}) \sim 1.$$

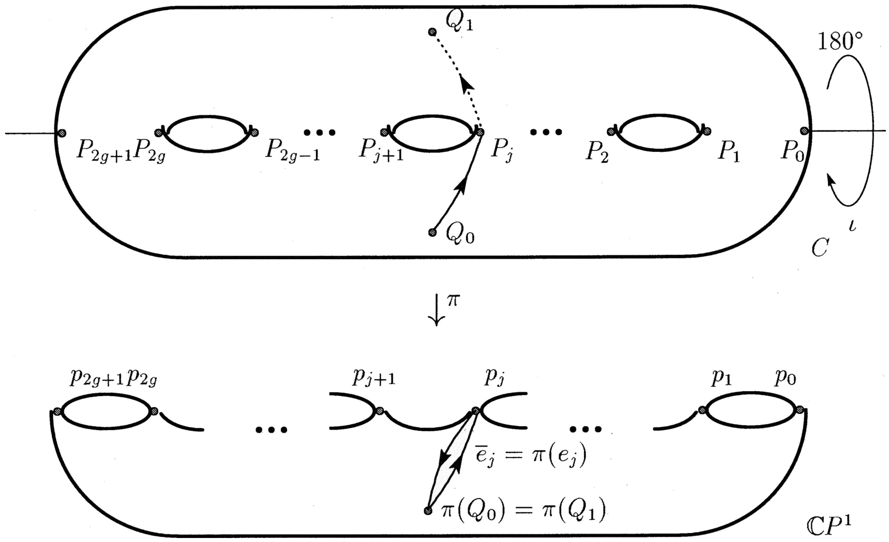

Figure 2.

We define the loops $a_i, b_i, i = 1, 2, \ldots, g$, in C with the base point Q_0 by

$$a_i = e_{2i-1} \cdot \iota(e_{2i}),$$
$$b_i = e_{2i-1} \cdot \iota(e_{2i-2}) \cdot \cdots \cdot e_1 \cdot \iota(e_0).$$

So a symplectic basis of $H_1(C;\mathbb{Z})$ can be given by $\{[a_i],[b_i]\}_{i=1,2,\ldots,g}$, where $[a_i]$ and $[b_i]$ are the homology classes of a_i and b_i respectively. In fact, we have $([a_i],[b_j]) = \delta_{ij} = -([b_j],[a_i])$ and $([a_i],[a_j]) = ([b_i],[b_j]) = 0$. It is clear that $[a_i]$ and $[b_i]$ are equal to x_i and y_i in Figure 1 respectively.

§3.2. The hyperelliptic curve C_0

A hyperelliptic curve C_0 is defined by the equation $w^2 = z^{2g+2} - 1$. We take $Q_i = (0, (-1)^i\sqrt{-1}), i = 0, 1$, and $P_j = (\zeta^j, 0), j = 0, 1, \ldots, 2g+1$, where $\zeta = e^{2\pi\sqrt{-1}/(2g+2)}$. We define a path $e_j : [0,1] \to C_0, j = 0, 1, \ldots, 2g+1$, by

$$\begin{cases} \left(2t\zeta^j, \sqrt{-1}\sqrt{1-(2t)^{2g+2}}\right) & \text{for } 0 \le t \le 1/2, \\ \left((2-2t)\zeta^j, -\sqrt{-1}\sqrt{1-(2-2t)^{2g+2}}\right) & \text{for } 1/2 \le t \le 1. \end{cases}$$

We denote $\omega_i = z^{i-1}dz/w, i = 1, 2, \ldots, g$, which are holomorphic 1-forms on C_0. Then $\{\omega_i\}_{i=1,2,\ldots,g}$ is a basis of the space of holomorphic 1-forms on

C_0. Let $B(u,v)$ denote the beta function $\int_0^1 x^{u-1}(1-x)^{v-1}dx$ for $u,v>0$. It is easy to show.

Lemma 3.1. *We have*

$$\int_{e_j}\omega_i = -2\sqrt{-1}\zeta^{ij}B(i/(2g+2),1/2)/(2g+2) = -\int_{\iota(e_j)}\omega_i.$$

We denote by ω_i' the holomorphic 1-form $\dfrac{(2g+2)\sqrt{-1}}{2B(i/(2g+2),1/2)}\omega_i$. The periods of C_0 are obtained by Lemma 3.1.

Lemma 3.2. *We have*

$$\int_{a_j}\omega_i' = \zeta^{i(2j-1)}(1-\zeta^i),$$
$$\int_{b_j}\omega_i' = \frac{\zeta^{2ij}-1}{\zeta^i+1},$$

where $i,j\in\{1,2,\dots,g\}$.

Remark 3.3. Since ω_i' is a closed 1-form, the integral $\int_\gamma \omega_i'$ depends only on the homology classes of γ.

In order to prove Lemma 3.5, we start with the following well known lemma.

Lemma 3.4. *Let* ω_1,ω_2 *be* 1*-forms on* X *and* $\gamma_1,\gamma_2,\dots,\gamma_m$ *paths in* X *so that* $\gamma_1\gamma_2\cdots\gamma_m$ *is a path. Then, we have*

$$\int_{\gamma_1\gamma_2\cdots\gamma_m}\omega_1\omega_2 = \sum_{i=1}^m\int_{\gamma_i}\omega_1\omega_2 + \sum_{i<j}\int_{\gamma_i}\omega_1\int_{\gamma_j}\omega_2.$$

Since ι is a diffeomorphism of C_0 and $\iota(e_k)=e_k^{-1}$, we have

$$\int_{e_k}\omega_i'\omega_j' = \int_{\iota(e_k)}\omega_i'\omega_j' = \int_{e_k^{-1}}\omega_i'\omega_j' = -\int_{e_k}\omega_i'\omega_j' - \int_{e_k}\omega_i'\int_{e_k^{-1}}\omega_j'.$$

Then $\displaystyle\int_{e_k}\omega_i'\omega_j' = \frac{1}{2}\int_{e_k}\omega_i'\int_{e_k}\omega_j'$. This formula, Lemma 3.2 and Lemma 3.4 give us iterated integrals of ω_i' along a_k and b_k.

Lemma 3.5. *We have*

$$\int_{a_k} \omega_i'\omega_j' = \frac{1}{2}\zeta^{(i+j)(2k-1)}(1-2\zeta^j+\zeta^{i+j}),$$

$$\int_{b_k} \omega_i'\omega_j' = \sum_{l=1}^{k}\frac{1}{2}\zeta^{(i+j)(2l-2)}(1-2\zeta^i+\zeta^{i+j}) + \sum_{1\le l<m\le k}(\zeta^i-1)(\zeta^j-1)\zeta^{i(2m-2)+j(2l-2)},$$

where $i,j\in\{1,2,\dots,g\}$.

For the rest of this section, we compute the iterated integrals of real harmonic 1-forms of C_0 with integral periods. Let Ω_a and Ω_b be the non-singular matrices

$$\begin{pmatrix} \int_{a_1}\omega_1' \cdots \int_{a_g}\omega_1' \\ \vdots \qquad\qquad \vdots \\ \int_{a_1}\omega_g' \cdots \int_{a_g}\omega_g' \end{pmatrix} \text{ and } \begin{pmatrix} \int_{b_1}\omega_1' \cdots \int_{b_g}\omega_1' \\ \vdots \qquad\qquad \vdots \\ \int_{b_1}\omega_g' \cdots \int_{b_g}\omega_g' \end{pmatrix},$$

respectively. It is clear that (ij)-entries of $(\Omega_a)^{-1}$ and $(\Omega_b)^{-1}$ are given by $\dfrac{1}{g+1}\dfrac{\zeta^j(-1+\zeta^{-2ij})}{1-\zeta^j}$ and $\dfrac{1}{g+1}\zeta^{-2ij}(1+\zeta^j)$ respectively. Then we obtain the period matrix $(\Omega_a)^{-1}\Omega_b$ denoted by Z. In general, it is well known that $Z\in GL(g,\mathbb{C})$ is symmetric and its imaginary part $\Im Z$ is positive definite. In particular, Schindler [9] proved the theorem below. We deduce it directly from Lemma 3.2.

Theorem 3.6 (Schindler [9], Theorem 2)**.** *Let Z be the period matrix on the curve C_0 as above. Then its (ij)-entry is given by*

$$\frac{1}{g+1}\sum_{k=1}^{g}\frac{\zeta^k(\zeta^{-2ik}-1)(\zeta^{2kj}-1)}{1-\zeta^{2k}}.$$

Furthermore, all the entries are pure imaginary.

Remark 3.7. We need some steps for another presentation of Z by

Schindler as follows.

$$\begin{aligned}\sum_{k=1}^{g}\frac{\zeta^k(\zeta^{-2ik}-1)(\zeta^{2kj}-1)}{1-\zeta^{2k}} &= \sum_{k=1}^{g}\zeta^k(\zeta^{2kj}-1)\zeta^{-2k}\frac{1-\left(\zeta^{-2k}\right)^i}{1-\zeta^{-2k}}\\ &= \sum_{k=1}^{g}\zeta^k(\zeta^{2kj}-1)\sum_{\nu=1}^{i}\zeta^{-2k\nu}\\ &= \sum_{\nu=1}^{i}\sum_{k=1}^{g}\left(\left(\zeta^{1-2\nu+2j}\right)^k-\left(\zeta^{1-2\nu}\right)^k\right)\\ &= \sum_{\nu=1}^{i}\left(\frac{1+\zeta^{1-2\nu+2j}}{1-\zeta^{1-2\nu+2j}}-\frac{1+\zeta^{1-2\nu}}{1-\zeta^{1-2\nu}}\right).\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{1}{g+1}\sum_{k=1}^{g}\frac{\zeta^k(\zeta^{-2ik}-1)(\zeta^{2kj}-1)}{1-\zeta^{2k}}\\ &\qquad=\frac{\sqrt{-1}}{g+1}\left(\sum_{\nu=1}^{i}\frac{1+\cos\frac{2\nu-1}{g+1}\pi}{\sin\frac{2\nu-1}{g+1}\pi}+\frac{1+\cos\frac{2(j-\nu)+1}{g+1}\pi}{\sin\frac{2(j-\nu)+1}{g+1}\pi}\right).\end{aligned}$$

We define real harmonic 1-forms $\alpha_i, \beta_i, i=1,2,\ldots,g$, by

$$\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_g\end{pmatrix}=(\Im Z)^{-1}\Im\left((\Omega_a)^{-1}\begin{pmatrix}\omega'_1\\ \vdots\\ \omega'_g\end{pmatrix}\right)\quad\text{and}\quad\begin{pmatrix}\beta_1\\ \vdots\\ \beta_g\end{pmatrix}=-\Re\left((\Omega_a)^{-1}\begin{pmatrix}\omega'_1\\ \vdots\\ \omega'_g\end{pmatrix}\right).$$

Using Theorem 3.6, we have $\Im Z=-\sqrt{-1}(\Omega_a)^{-1}\Omega_b$. Then

$$\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_g\end{pmatrix}=\Re\left((\Omega_b)^{-1}\begin{pmatrix}\omega'_1\\ \vdots\\ \omega'_g\end{pmatrix}\right).$$

It is clear that $\int_{a_j}\alpha_i=\int_{b_j}\beta_i=0$ and $\int_{b_j}\alpha_i=\delta_{ij}=-\int_{a_j}\beta_i$ by Lemma 3.2. Let PD denote the Poincaré dual $H_1(C_0;\mathbb{Z})\to H^1(C_0;\mathbb{Z})$. We have $\mathrm{PD}([a_i])=\alpha_i$ and $\mathrm{PD}([b_i])=\beta_i$ for $i=1,2,\ldots,g$. Hence, $\{\alpha_i,\beta_i\}_{i=1,2,\ldots,g}\subset H^1(C_0;\mathbb{Z})$ is a symplectic basis.

Let t_u be a complex number $\displaystyle\sum_{p=1}^{g}\zeta^{up}$ for any integer u. It is obvious that

$$t_u=\begin{cases}g & \text{for } u\in(2g+2)\mathbb{Z},\\ -1 & \text{for } u\notin(2g+2)\mathbb{Z} \text{ and } u:\text{even},\\ \dfrac{1+\zeta^u}{1-\zeta^u} & \text{for } u:\text{odd}.\end{cases}$$

Moreover, t_u is pure imaginary and $t_{-u} = -t_u$ when u is odd.

Using Lemma 3.5, we can calculate iterated integrals by means of t_u as follows.

Lemma 3.8. *On the curve C_0, we have the equations*

$$(1)\quad \int_{a_k} \beta_i\beta_j = \frac{1}{2(g+1)^2}\Big\{(t_{2k-2j} - t_{2k})\sum_{u=1}^{i} t_{2k-2u} + (t_{2k} - t_{2k-2i})\sum_{u=1}^{j} t_{2k-2u+2}\Big\},$$

$$(2)\quad \int_{b_k} \beta_i\beta_j = 0,$$

$$(3)\quad \int_{a_k} \alpha_i\alpha_j = 0,$$

$$(4)\quad \int_{b_k} \alpha_i\alpha_j = \frac{1}{2(g+1)^2}\Big\{\sum_{u=1}^{k}(t_{2u-2j}t_{2u-2i} - 2t_{2u-2j-2}t_{2u-2i} + t_{2u-2j-2}t_{2u-2i-2}) + \sum_{v=2}^{k} 2(t_{2v-2i} - t_{2v-2i-2})(t_{2v-2j-2} - t_{(-2j)})\Big\}.$$

Here $i, j, k \in \{1, 2, \ldots, g\}$.

Remark 3.9. For $k = 1$, $\sum_{v=2}^{k} 2(t_{2v-2i} - t_{2v-2i-2})(t_{2v-2j-2} - t_{(-2j)}) = 0$.

Proof. We compute $\int_{a_k} \beta_i\beta_j$ in the following way. Let $A_{i,j}$ be the (i,j)-entry of $(\Omega_a)^{-1}$. By the definition of β_i, we have $\beta_i = -\Re\Big(\sum_{l=1}^{g} A_{i,l}\omega'_l\Big)$. Using this, $\int_{a_k} \beta_i\beta_j$ can be given by

$$\begin{aligned}
&\int_{a_k} \Re\Big(\sum_{l=1}^{g} A_{i,l}\omega'_l\Big)\Re\Big(\sum_{m=1}^{g} A_{j,m}\omega'_m\Big) \\
&= \frac{1}{4}\int_{a_k} \sum_{l,m=1}^{g} \big(A_{i,l}A_{j,m}\omega'_l\omega'_m + A_{i,l}\overline{A}_{j,m}\omega'_l\overline{\omega}'_m + \overline{A}_{i,l}A_{j,m}\overline{\omega}'_l\omega'_m + \overline{A}_{i,l}\overline{A}_{j,m}\overline{\omega}'_l\overline{\omega}'_m\big) \\
&= \frac{1}{2}\Re\Big\{\sum_{l,m=1}^{g}\Big(A_{i,l}A_{j,m}\int_{a_k}\omega'_l\omega'_m + A_{i,l}\overline{A}_{j,m}\int_{a_k}\omega'_l\overline{\omega}'_m\Big)\Big\}.
\end{aligned}$$

Lemma 3.5 gives us

$$\begin{aligned}
&(g+1)^2 \sum_{l,m=1}^{g} A_{i,l}A_{j,m} \int_{a_k} \omega'_l\omega'_m \\
&= \sum_{l,m=1}^{g} \frac{\zeta^l(-1+\zeta^{-2il})}{1-\zeta^l}\frac{\zeta^m(-1+\zeta^{-2jm})}{1-\zeta^m}\frac{1}{2}\zeta^{(l+m)(2k-1)}(1-2\zeta^m+\zeta^{l+m}) \\
&= \frac{1}{2}\sum_{m=1}^{g}\frac{1-\zeta^{2jm}}{1-\zeta^m}\zeta^{m(2k-2j)}\sum_{l=1}^{g}\frac{1-\zeta^{2il}}{1-\zeta^l}\zeta^{l(2k-2i)}(1-2\zeta^m+\zeta^{l+m}) \\
&= \frac{1}{2}\sum_{m=1}^{g}\sum_{v=2k-2j}^{2k-1}\zeta^{mv}\sum_{l=1}^{g}\sum_{u=2k-2i}^{2k-1}\zeta^{lu}(1-2\zeta^m+\zeta^{l+m}) \\
&= \frac{1}{2}\sum_{m=1}^{g}\sum_{v=2k-2j}^{2k-1}\zeta^{mv}\Big\{\sum_{u=2k-2i}^{2k-1}\Big(t_u(1-2\zeta^m)+t_{u+1}\zeta^m\Big)\Big\} \\
&= \frac{1}{2}\sum_{m=1}^{g}\sum_{v=2k-2j}^{2k-1}\zeta^{mv}\Big\{\sum_{u=2k-2i}^{2k-1}t_u(1-\zeta^m)+(t_{2k}-t_{2k-2i})\zeta^m\Big\} \\
&= \frac{1}{2}\sum_{v=2k-2j}^{2k-1}\sum_{m=1}^{g}\left\{\sum_{u=2k-2i}^{2k-1}t_u\zeta^{mv}(1-\zeta^m)+(t_{2k}-t_{2k-2i})\zeta^{m(v+1)}\right\} \\
&= \frac{1}{2}\sum_{v=2k-2j}^{2k-1}\left\{\sum_{u=2k-2i}^{2k-1}t_u(t_v-t_{v+1})+(t_{2k}-t_{2k-2i})t_{v+1}\right\}.
\end{aligned}$$

So we obtain

$$\begin{aligned}
&(g+1)^2 \sum_{l,m=1}^{g} A_{i,l}A_{j,m} \int_{a_k} \omega'_l\omega'_m \\
&\quad = \frac{1}{2}\Big\{(t_{2k-2j}-t_{2k})\sum_{u=2k-2i}^{2k-1}t_u+(t_{2k}-t_{2k-2i})\sum_{v=2k-2j+1}^{2k}t_v\Big\}.
\end{aligned}$$

Using $\int_{a_k}\omega'_i\overline{\omega}'_j = \frac{1}{2}\zeta^{(i-j)(2k-1)}(1-2\zeta^{-j}+\zeta^{i-j})$, the value $(g+1)^2\sum_{l,m=1}^{g} A_{i,l}\overline{A}_{j,m}\int_{a_k}\omega'_l\overline{\omega}'_m$ can be computed by

$$\frac{1}{2}\Big\{\big(t_{-(2k-2j)}-t_{-(2k)}\big)\sum_{u=2k-2i}^{2k-1}t_u+(t_{2k}-t_{2k-2i})\sum_{v=2k-2j+1}^{2k}t_{-v}\Big\}.$$

Since $t_u = t_{-u}$ for $u \in 2\mathbb{Z}$ and t_u is pure imaginary for $u \in 2\mathbb{Z}+1$, we have (1). The values $\int_{b_k}\beta_i\beta_j$, $\int_{a_k}\alpha_i\alpha_j$ and $\int_{b_k}\alpha_i\alpha_j$ are calculated similarly. $\square$

§4. The Harmonic Volumes of Hyperelliptic Curves

In this section, we consider the harmonic volumes of hyperelliptic curves. They can be reduced to the computation for the hyperelliptic curve C_0.

Theorem 4.1. *For any hyperelliptic curve C, let $\{x_i, y_i\}_{i=1,2,\dots,g}$ be a symplectic basis of $H = H_1(C;\mathbb{Z})$ in Figure 1, where ι is the hyperelliptic involution. We denote by z_i either x_i or y_i. Then,*

$$\begin{aligned} I(z_i \otimes z_j \otimes z_k) =& \ 0 \ \text{ for } i \neq j \neq k \neq i, \\ I(x_i \otimes y_i \otimes z_k &- x_{k+1} \otimes y_{k+1} \otimes z_k) \\ &= \begin{cases} 1/2 & \text{for } i < k, k = 2, 3, \dots, g-1 \text{ and } z_k = y_k, \\ 0 & \text{for } i \geq k, k = 1, k = g \text{ or } z_k = x_k. \end{cases} \end{aligned}$$

In order to prove Theorem 4.1, we need the following two lemmas. Let $\mathscr{H}_{\mathbb{Z}}$ be all the real harmonic 1-forms on C_0 with integral periods.

Lemma 4.2. *On the curve C_0, let η be a 1-form on C_0 satisfying the conditions*

$$\begin{cases} d\eta = \displaystyle\sum_k h_{1,k} \wedge h_{2,k}, \\ \displaystyle\int_X \eta \wedge *\alpha = 0 \text{ for any closed 1-form } \alpha \text{ on } X, \\ \iota^*\eta = \eta, \end{cases}$$

where ι is the hyperelliptic involution of C_0 and $h_{1,k}, h_{2,k} \in \mathscr{H}_{\mathbb{Z}}$ such that $\sum_k (h_{1,k}, h_{2,k}) = \sum_k \int_{C_0} h_{1,k} \wedge h_{2,k} = 0$.
Then for any j

$$\int_{e_j} \eta = 0.$$

Proof. We will have η explicitly. For any $\displaystyle\sum_k h_{1,k} \wedge h_{2,k}$, there exist $a^1_{i,j}, a^2_{i,j} \in \mathbb{C}$ such that $\displaystyle\sum_k h_{1,k} \wedge h_{2,k} = \sum_{i,j} a^1_{i,j}\omega_i \wedge \overline{\omega_j} + a^2_{i,j}\overline{\omega_i} \wedge \omega_j$, where $i, j \in \{1, 2, \dots, g\}$. The $(1,1)$-form $\omega_i \wedge \overline{\omega}_j$ is $\dfrac{\lambda^{i-1}\overline{\lambda}^{j-1}}{\mu\overline{\mu}} d\lambda \wedge d\overline{\lambda}$ in a coordinate λ satisfying $\mu^2 = \lambda^{2g+2} - 1$. Take a polynomial $f(\lambda, \overline{\lambda})$ of degree at most $2g-2$ which belongs to $\mathbb{C}[\lambda, \overline{\lambda}]$ so that $\dfrac{f(\lambda)}{\mu\overline{\mu}} d\lambda \wedge d\overline{\lambda} = \displaystyle\sum_k h_{1,k} \wedge h_{2,k}$. It is clear that $\dfrac{f(\lambda)}{\mu\overline{\mu}} d\lambda \wedge d\overline{\lambda}$ is invariant under the action of the hyperelliptic involution

$\iota : (\lambda, \mu) \mapsto (\lambda, -\mu)$, since $\mu\overline{\mu} = |\mu^2| = |\lambda^{2g+2} - 1| = |(-\mu)^2|$. So we regard $\dfrac{f(\lambda)}{\mu\overline{\mu}} d\lambda \wedge d\overline{\lambda}$ as a 1-form on $\mathbb{C}P^1$. On the curve C_0, Harris ([5] in Section 5, 6 and [6]) gave η in the following explicit forms.

$$\begin{aligned}\eta &= \frac{-1}{2\pi}\int_{\lambda\in\mathbb{C}P^1} \Im\left(\frac{dz}{z-\lambda}\right)\frac{f(\lambda)}{|\lambda^{2g+2}-1|}d\lambda\wedge d\overline{\lambda}\\ &= \frac{-1}{2\pi}\frac{1}{2\sqrt{-1}}\left(dz\int\frac{1}{z-\lambda}\frac{f(\lambda)}{|\lambda^{2g+2}-1|}d\lambda\wedge d\overline{\lambda}\right.\\ &\qquad\qquad\left. -d\overline{z}\int\overline{\left(\frac{1}{z-\lambda}\right)}\frac{f(\lambda)}{|\lambda^{2g+2}-1|}d\lambda\wedge d\overline{\lambda}\right),\end{aligned}$$

in a coordinate z satisfying $w^2 = z^{2g+2} - 1$. It satisfies

$$\iota^*\eta = \eta.$$

This equation allows us to have

$$\int_{e_k}\eta = \int_{\iota(e_k)^{-1}}\eta = -\int_{\iota(e_k)}\eta = -\int_{e_k}\iota^*\eta = -\int_{e_k}\eta.$$

Then we obtain $\int_{e_j}\eta = 0$. □

Lemma 4.3. *On the curve C_0,*

$$\begin{aligned}&I(z_i\otimes z_j\otimes z_k) = \ 0 \ \textit{ for } i\neq j\neq k\neq i,\\ &I(x_i\otimes y_i\otimes z_k - x_{k+1}\otimes y_{k+1}\otimes z_k)\\ &\qquad = \begin{cases} 1/2 & \textit{for } i<k, k=2,3,\ldots,g-1 \textit{ and } z_k = y_k,\\ 0 & \textit{for } i\geq k+2, k=1, k=g \textit{ or } z_k = x_k.\end{cases}\end{aligned}$$

Proof. It is enough to consider the iterated integral part of the harmonic volume by Lemma 4.2.

Type (1)
Lemma 3.8 gives us $I(z_i\otimes z_j\otimes z_k)\equiv 0$ for $i\neq j\neq k\neq i$.

Type (2)
We compute $I(x_i\otimes y_i\otimes z_k - x_{k+1}\otimes y_{k+1}\otimes z_k)$ for $i\neq k$ and $i\neq k+1$. When

$i < k$, $k = 2, 3, \ldots, g-1$ and $z_k = y_k$,

$$
\begin{aligned}
& I\ (x_i \otimes y_i \otimes y_k - x_{k+1} \otimes y_{k+1} \otimes y_k) \\
&= \int_{a_i} \beta_i \beta_k - \int_{a_{k+1}} \beta_{k+1} \beta_k \\
&= \frac{1}{2(g+1)^2} \left\{ -(g+1) \sum_{u=1}^{k} t_{2i-2u+2} \right\} - \frac{1}{2(g+1)^2} \left\{ -(g+1) \sum_{u=1}^{k} t_{2(k+1)-2u+2} \right\} \\
&= \frac{1}{2(g+1)^2} \{ -(g+1)(g-k+1) \} - \frac{1}{2(g+1)^2} \{ -(g+1)(-k) \} \\
&= -1/2 \\
&= 1/2 \bmod \mathbb{Z}.
\end{aligned}
$$

It is similarly shown that $I(x_i \otimes y_i \otimes z_k - x_{k+1} \otimes y_{k+1} \otimes z_k) = 0$ for $i \geq k+2$, $k = 1$, $k = g$ or $z_k = x_k$. □

Before the proof of Theorem 4.1, we recall some results about the moduli space of compact Riemann surfaces. Let Σ_g be a closed oriented surface of genus g. Its mapping class group, denoted here by Γ_g, is the group of isotopy classes of orientation preserving diffeomorphisms of Σ_g. This group acts on the Teichmüller space $\mathcal{T}_g$ of Σ_g and the quotient space $\mathcal{M}_g$ is the moduli space of Riemann surfaces of genus g. The group Γ_g acts naturally on the first homology group $H_1(\Sigma_g; \mathbb{Z})$ of Σ_g. Let $\mathscr{I}_g$ be the subgroup of Γ_g, which acts trivially on $H_1(\Sigma_g; \mathbb{Z})$ and we call it the Torelli group. Its action on $\mathcal{T}_g$ is free and the quotient $\mathcal{I}_g = \mathscr{I}_g \backslash \mathcal{T}_g$, called the Torelli space, is the moduli space of compact Riemann surfaces with a fixed symplectic basis of $H_1(\Sigma_g; \mathbb{Z})$. There is a natural projection $p_T : \mathcal{I}_g \to \mathcal{M}_g$.

Let $\mathcal{H}_g \subset \mathcal{M}_g$ be the moduli space of hyperelliptic curves of genus g. The hyperelliptic mapping class group Δ_g is the subgroup of Γ_g defined by

$$\{\varphi \in \Gamma_g; \varphi\iota = \iota\varphi\},$$

where ι is the hyperelliptic involution of Σ_g. We choose $\widetilde{\mathcal{H}}_g$ a connected component of $p_T^{-1}(\mathcal{H}_g)$ with the symplectic basis in Figure 1. $\widetilde{\mathcal{H}}_g$ is a complex submanifold of dimension $2g-1$ of $\mathcal{I}_g$. Let $\mathscr{I}_g^H$ denote the group $\mathscr{I}_g \cap \Delta_g$. The moduli space $\mathcal{H}_g$ is known to be connected and has a natural structure of a quasi-projective orbifold. Hence we have $\mathcal{H}_g = p_T(\widetilde{\mathcal{H}}_g)$. The group Δ_g can be considered as its orbifold fundamental group and $\mathscr{I}_g^H$ is the fundamental group of $\widetilde{\mathcal{H}}_g$.

Proof. (Theorem 4.1) One of the key points of this proof is that the harmonic volume of C belongs to $\mathrm{Hom}_{\Delta_g}((H^{\otimes 3})', \mathbb{Z}/2\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}((H^{\otimes 3})',$

$\mathbb{Z}/2\mathbb{Z})^{\Delta_g}$. Let $E \to \mathcal{H}_g$ be a flat vector bundle with a fiber $\mathrm{Hom}_{\mathbb{Z}}((H^{\otimes 3})', \mathbb{Z}/2\mathbb{Z})$ and $(p_T|_{\widetilde{\mathcal{H}}_g})^* E$ the pullback of the flat vector bundle E. Harris [5] proved that I varies in $\mathcal{I}_g$ continuously. For any hyperelliptic curves, $I \equiv 0$ or $I \equiv 1/2$ modulo $\mathbb{Z}$. Hence the flat vector bundle $(p_T|_{\widetilde{\mathcal{H}}_g})^* E$ has a locally constant section $\widetilde{I}$ associated to I. Moreover, $\widetilde{\mathcal{H}}_g$ is arcwise connected and the monodromy representation $\mathscr{I}_g^H \to \mathrm{Aut}(\mathrm{Hom}_{\mathbb{Z}}((H^{\otimes 3})', \mathbb{Z}/2\mathbb{Z}))$ is trivial. Therefore $\widetilde{I}$ is constant on $\widetilde{\mathcal{H}}_g$. Since $\mathcal{H}_g = p_T(\widetilde{\mathcal{H}}_g)$, the harmonic volumes of hyperelliptic curves can be reduced to the calculation of C_0. The result follows from Lemma 4.3. □

§5. The Harmonic Volumes of Hyperelliptic Curves from a Topological Viewpoint

In this section, we study $\mathrm{Hom}_{\Delta_g}((H^{\otimes 3})', \mathbb{Z}_2)$ which contains the harmonic volume I. Let $\mathbb{Z}_2$ denote the field $\mathbb{Z}/2\mathbb{Z}$.

Birman and Hilden proved the following theorem.

Theorem 5.1 ([2], Theorem 8)**.** *The hyperelliptic mapping class group Δ_g admits the following presentation*;

- *generators*: $\sigma_1, \sigma_2, \dots, \sigma_{2g+1}$
- *relations*:

(1) $\sigma_n\sigma_m = \sigma_m\sigma_n, |n-m| \geq 2$,

(2) $\sigma_n\sigma_{n+1}\sigma_n = \sigma_{n+1}\sigma_n\sigma_{n+1}, 1 \leq n \leq 2g$,

(3) $\theta^{2g+2} = 1$,

(4) $(\theta\kappa)^2 = 1$,

(5) $\sigma_1(\theta\kappa) = (\theta\kappa)\sigma_1$,

where $\theta = \sigma_1\sigma_2\cdots\sigma_{2g+1}$ *and* $\kappa = \sigma_{2g+1}\sigma_{2g}\cdots\sigma_1$.

Remark 5.2. The generator σ_i, $1 \leq i \leq 2g+1$, is equal to the Dehn twist along the simple closed curve l_i in C in Figure 3.

Let $H_{\mathbb{Z}_2}$ denote $H_1(C;\mathbb{Z}_2)$. A homomorphism $\rho : \Delta_g \to \mathrm{Sp}(2g;\mathbb{Z}_2)$ is given by the action on the homology group $H_{\mathbb{Z}_2}$. So $H_{\mathbb{Z}_2}$ is a $\mathbb{Z}_2\Delta_g$-module, where $\mathbb{Z}_2\Delta_g$ is the group ring of Δ_g. We consider e_i, a_j and b_j for $0 \leq i \leq 2g+1$ and $1 \leq j \leq g$ in Section 3.1. The first homology classes of a_j and b_j are denoted by x_j and $y_j \in H_{\mathbb{Z}_2}$ respectively. Let B denote the branch locus

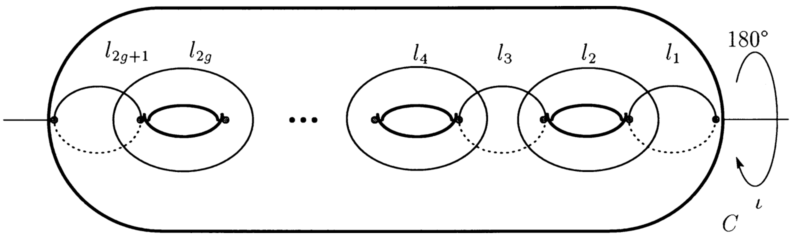

Figure 3.

$\{p_i\}_{i=0,1,\dots,2g+1}$. We deform e_i, denoted by e'_i, to avoid P_i in a sufficiently small neighborhood of P_i so that $\pi(e'_i)$ surrounds p_i and $\{\pi(e'_i)\}_{i=0,1,\dots,2g+1}$ is a generator of $H_1(\mathbb{C}P^1 - B;\mathbb{Z}_2)$. Since the coefficients are in $\mathbb{Z}_2$, the homology class of e'_i is independent of the choice of e'_i. See Figure 4.

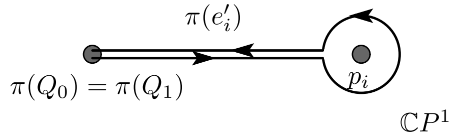

Figure 4.

Arnol'd [1] proved the following. A linear map $\nu : H_{\mathbb{Z}_2} \to H_1(\mathbb{C}P^1 - B;\mathbb{Z}_2)$ defined by $\nu(x_i) = \pi(e'_{2i-1}) + \pi(e'_{2i}), \nu(y_i) = \pi(e'_0) + \pi(e'_1) + \cdots + \pi(e'_{2i-1})$ is injective. This map gives the short exact sequence

$$0 \longrightarrow H_{\mathbb{Z}_2} \xrightarrow{\nu} H_1(\mathbb{C}P^1 - B;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Here the map $H_1(\mathbb{C}P^1 - B;\mathbb{Z}_2) \to \mathbb{Z}_2$ is the augmentation map $\pi(e'_i) \mapsto 1$. Let f_i denote $\pi(e'_0) + \pi(e'_i)$ for $i = 1, 2, \dots, 2g+1$. Using ν, we identify $H_{\mathbb{Z}_2}$ with the subgroup of $H_1(\mathbb{C}P^1 - B;\mathbb{Z}_2)$ generated by $f_1, f_2, \dots, f_{2g+1}$. It is clear that $f_1 + f_2 + \cdots + f_{2g+1} = 0$. A surjective homomorphism $\mu : \Delta_g \to S_{2g+2}$ is defined by $\mu(\sigma_j) = (j-1, j)$. Let $\rho' : S_{2g+2} \to \mathrm{Sp}(2g;\mathbb{Z}_2)$ be the homomorphism induced by the action on $H_1(\mathbb{C}P^1 - B;\mathbb{Z}_2)$ given by the permuting $\pi(e'_0), \pi(e'_1), \dots, \pi(e'_{2g+1})$. Arnol'd [1] obtained the commutative diagram

$$\begin{array}{ccc} H_{\mathbb{Z}_2} & \xrightarrow{\rho(\sigma_j)} & H_{\mathbb{Z}_2} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} \\ H_1(\mathbb{C}P^1 - B;\mathbb{Z}_2) & \xrightarrow{\rho'(j-1,j)} & H_1(\mathbb{C}P^1 - B;\mathbb{Z}_2). \end{array}$$

We identify the actions of $\sigma_1, \sigma_2, \ldots, \sigma_{2g}$ and σ_{2g+1} on $H_{\mathbb{Z}_2}$ with those of the transpositions $(0,1), (1,2), \ldots, (2g-1, 2g)$ and $(2g, 2g+1)$ on $H_1(\mathbb{C}P^1 - B; \mathbb{Z}_2)$ respectively.

We denote by $\Delta_g' = \{\sigma \in \Delta_g;\ \sigma(P_0) = P_0\}$ and $\Delta_g'' = \{\sigma \in \Delta_g;\ \sigma(P_0) = P_0$ and $\sigma(P_1) = P_1\}$. We have $\mu(\Delta_g') = S_{2g+1}$ and $\mu(\Delta_g'') = S_{2g}$, where $S_{2g+1} = \{\sigma \in S_{2g+2}; \sigma(\pi(e_0')) = \pi(e_0')\}$ and $S_{2g} = \{\sigma \in S_{2g+2}; \sigma(\pi(e_0')) = \pi(e_0')$ and $\sigma(\pi(e_1')) = \pi(e_1')\}$. As in the proof of Theorem 4.1, the pointed harmonic volume I_{P_0} is an element of $\mathrm{Hom}_{\Delta_g'}(K \otimes H, \mathbb{Z}_2)$. For a $\mathbb{Z}\Delta_g'$-module M, we denote $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}_2)$, which is naturally regarded as a $\mathbb{Z}_2\Delta_g'$-module. Clearly we have $H^* = H_{\mathbb{Z}_2}^*$.

The homomorphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K \otimes H & \longrightarrow & H^{\otimes 3} & \xrightarrow{(\ ,\ )\otimes \mathrm{id}} & H & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (H^{\otimes 3})' & \longrightarrow & H^{\otimes 3} & \xrightarrow{p} & H^{\oplus 3} & \longrightarrow & 0,
\end{array}$$

induces the homomorphism of long exact sequences,

(5.1)

$$\begin{array}{ccccccc}
H^0(S_{2g+1}; H^*) & \longrightarrow & H^0(S_{2g+1}; (H^{\otimes 3})^*) & \rightarrow & H^0(S_{2g+1}; (K \otimes H)^*) & \longrightarrow & H^1(S_{2g+1}; H^*) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^0(S_{2g+1}; (H^{\oplus 3})^*) & \rightarrow & H^0(S_{2g+1}; (H^{\otimes 3})^*) & \rightarrow & H^0(S_{2g+1}; ((H^{\otimes 3})')^*) & \rightarrow & H^1(S_{2g+1}; (H^{\oplus 3})^*).
\end{array}$$

Lemma 5.3. *We have*

$$H^0(S_{2g+1}; H^*) = 0.$$

Proof. We take $\varphi \in H^0(S_{2g+1}; H^*)$. Since φ is S_{2g+1}-equivariant, $\varphi(f_1) = \varphi(f_2) = \cdots = \varphi(f_{2g+1})$. Using $f_1 + f_2 + \cdots + f_{2g+1} = 0$, we have $0 = \varphi(f_1 + f_2 + \cdots + f_{2g+1}) = (2g+1)\varphi(f_1) = \varphi(f_1)$. From $\varphi(f_i) = 0$, $1 \leq i \leq 2g+1$, $H^0(S_{2g+1}; H^*) = 0$ follows. $\square$

We recall the notion of induced and co-induced modules. Let $\mathrm{Ind}_{S_{2g}}^{S_{2g+1}} \mathbb{Z}_2$ denote the induced module $\mathbb{Z}_2 S_{2g+1} \otimes_{\mathbb{Z}_2 S_{2g}} \mathbb{Z}_2$ and $\mathrm{Coind}_{S_{2g}}^{S_{2g+1}} \mathbb{Z}_2$ the co-induced module $\mathrm{Hom}_{S_{2g}}(\mathbb{Z}_2 S_{2g+1}, \mathbb{Z}_2)$. They are $(2g+1)$-dimensional vector spaces over $\mathbb{Z}_2$. We denote by $r_i = (i, 1) \otimes 1 \in \mathrm{Ind}_{S_{2g}}^{S_{2g+1}} \mathbb{Z}_2$ for $i = 1, 2, \ldots, 2g+1$. Then $\{r_i\}_{i=1,2,\ldots,2g+1}$ is a basis of $\mathrm{Ind}_{S_{2g}}^{S_{2g+1}} \mathbb{Z}_2$. Since $[S_{2g+1} : S_{2g}] < \infty$, we have a natural isomorphism $\lambda : \mathrm{Coind}_{S_{2g}}^{S_{2g+1}} \mathbb{Z}_2 \to \mathrm{Ind}_{S_{2g}}^{S_{2g+1}} \mathbb{Z}_2$ given by $\lambda(s) = \sum_{i=1}^{i=2g+1} (i, 1) \otimes s((i, 1))$ for $s \in \mathrm{Coind}_{S_{2g}}^{S_{2g+1}} \mathbb{Z}_2$. Let s_i be the element of

$\mathrm{Coind}_{S_{2g}}^{S_{2g+1}}\mathbb{Z}_2$ such that $\lambda(s_i) = r_i$. We have a natural exact sequence

$$0 \longrightarrow H_{\mathbb{Z}_2} \xrightarrow{\phi} \mathrm{Coind}_{S_{2g}}^{S_{2g+1}}\mathbb{Z}_2 \xrightarrow{\chi} \mathbb{Z}_2 \longrightarrow 0\ , \tag{5.2}$$

where $\phi(n_2 f_2 + n_3 f_3 + \cdots + n_{2g+1} f_{2g+1}) = n_2 s_1 + n_3 s_2 + \cdots + n_{2g+1} s_{2g} + (n_2 + n_3 + \cdots + n_{2g+1}) s_{2g+1}$ and χ is the augmentation map.

A transfer map is defined as follows. The canonical surjection τ of S_{2g+1}-modules $\mathrm{Ind}_{S_{2g}}^{S_{2g+1}}\mathbb{Z}_2 \to \mathbb{Z}_2$ is defined by $\tau(\sigma \otimes a) = \sigma a = a$. By Shapiro's lemma, we obtain $H^i(S_{2g+1}; \mathrm{Coind}_{S_{2g}}^{S_{2g+1}}\mathbb{Z}_2) = H^i(S_{2g}; \mathbb{Z}_2)$ for any i. A transfer map $\mathrm{cor}_{S_{2g}}^{S_{2g+1}} : H^i(S_{2g}; \mathbb{Z}_2) \to H^i(S_{2g+1}; \mathbb{Z}_2)$ is induced by Shapiro's lemma and the following composite mapping

$$\mathrm{Coind}_{S_{2g}}^{S_{2g+1}}\mathbb{Z}_2 \xrightarrow{\lambda} \mathrm{Ind}_{S_{2g}}^{S_{2g+1}}\mathbb{Z}_2 \xrightarrow{\tau} \mathbb{Z}_2.$$

It immediately follows that χ is equal to $\tau \circ \lambda$.

Lemma 5.4. *We have*

$$H^1(S_{2g+1}; H^*) = 0.$$

Proof. The exact sequence (5.2) induces the exact sequence

$$0 \longrightarrow H^0(S_{2g+1}; H_{\mathbb{Z}_2}) \xrightarrow{\phi^*} H^0(S_{2g+1}; \mathrm{Coind}_{S_{2g}}^{S_{2g+1}}\mathbb{Z}_2) \xrightarrow{\chi^*} H^0(S_{2g+1}; \mathbb{Z}_2)$$

$$\longrightarrow H^1(S_{2g+1}; H_{\mathbb{Z}_2}) \xrightarrow{\phi^*} H^1(S_{2g+1}; \mathrm{Coind}_{S_{2g}}^{S_{2g+1}}\mathbb{Z}_2) \xrightarrow{\chi^*} H^1(S_{2g+1}; \mathbb{Z}_2).$$

By Shapiro's lemma, we obtain $H^i(S_{2g+1}; \mathrm{Coind}_{S_{2g}}^{S_{2g+1}}\mathbb{Z}_2) = H^i(S_{2g}; \mathbb{Z}_2)$ for $i = 0, 1$. We have $H^0(S_{2g+1}; \mathbb{Z}_2) = \mathbb{Z}_2$ and $H^0(S_{2g}; \mathbb{Z}_2) = \mathbb{Z}_2$, since the actions of S_{2g+1} and S_{2g} on $\mathbb{Z}_2$ are trivial. Let sign_i be the signature map $S_i \to \mathbb{Z}_2$ for $i = 2g, 2g+1$. Since $2g, 2g+1 \geq 6 > 5$, we obtain $H^1(S_i; \mathbb{Z}_2) = \mathbb{Z}_2$ and sign_i generates $H^1(S_i; \mathbb{Z}_2)$ for $i = 2g, 2g+1$. In order to prove $H^1(S_{2g+1}; H^*) = H^1(S_{2g+1}; H_{\mathbb{Z}_2}) = 0$, it is enough to prove that χ^* is an isomorphism. Let 1_i denote the nontrivial element of $H^0(S_i; \mathbb{Z}_2)$ for $i = 2g, 2g+1$. Since $\chi = \tau \circ \lambda$, we have $\chi^* = \mathrm{cor}_{S_{2g}}^{S_{2g+1}} : H^0(S_{2g}; \mathbb{Z}_2) \to H^0(S_{2g+1}; \mathbb{Z}_2)$. Lemma 5.3 gives $H^0(S_{2g+1}; H_{\mathbb{Z}_2}) = H^0(S_{2g+1}; H^*) = 0$. Then we obtain $\mathrm{cor}_{S_{2g}}^{S_{2g+1}}(1_{2g}) = 1_{2g+1}$ and the isomorphism $\chi^* : H^0(S_{2g}; \mathbb{Z}_2) \to H^0(S_{2g+1}; \mathbb{Z}_2)$. We apply the transfer formula

$$\mathrm{cor}_{S_{2g}}^{S_{2g+1}}(\mathrm{res}_{S_{2g}}^{S_{2g+1}}(\mathrm{sign}_{2g}) \cup 1_{2g}) = \mathrm{sign}_{2g} \cup \mathrm{cor}_{S_{2g}}^{S_{2g+1}}(1_{2g})$$

to sign_{2g} and 1_{2g}. So $\mathrm{cor}_{S_{2g}}^{S_{2g+1}}\mathrm{res}_{S_{2g}}^{S_{2g+1}}(\mathrm{sign}_{2g}) = \mathrm{sign}_{2g}$. Since χ^* is surjective, we have the isomorphism $\chi^* = \mathrm{cor}_{S_{2g}}^{S_{2g+1}} : H^1(S_{2g}; \mathbb{Z}_2) \to H^1(S_{2g+1}; \mathbb{Z}_2)$. Then $H^1(S_{2g+1}; H^*) = H^1(S_{2g+1}; H_{\mathbb{Z}_2}) = 0$. □

Using the diagram (5.1), Lemma 5.3 and Lemma 5.4, we get the homomorphism of the commutative diagram

$$\begin{array}{ccc} H^0(S_{2g+1}; (H^{\otimes 3})^*) & \longrightarrow & H^0(S_{2g+1}; (K \otimes H)^*) \\ \uparrow & & \uparrow \\ H^0(S_{2g+1}; (H^{\otimes 3})^*) & \longrightarrow & H^0(S_{2g+1}; ((H^{\otimes 3})')^*)). \end{array}$$

The two horizontal and one left-hand vertical homomorphisms are isomorphisms. Then the other right-hand vertical homomorphism is an isomorphism. We have $H^0(S_{2g+1}; (H^{\otimes 3})^*) = H^0(S_{2g+1}; ((H^{\otimes 3})')^*)) = H^0(S_{2g+1}; (K \otimes H)^*)$.

Lemma 5.5.

$$H^0(S_{2g+1}; (H^{\otimes 3})^*) = \mathbb{Z}_2.$$

Moreover, the unique nontrivial element $\psi \in H^0(S_{2g+1}; (H^{\otimes 3})^*)$ *is an* S_{2g+1}-*homomorphism* $H^{\otimes 3} \to \mathbb{Z}_2$ *defined by*

$$\psi(f_i \otimes f_j \otimes f_k) = \begin{cases} 0 & for\ i \neq j \neq k \neq i, \\ 0 & for\ i = j = k, \\ 1 & otherwise. \end{cases}$$

Proof. Let ψ be an element of $H^0(S_{2g+1}; (H^{\otimes 3})^*)$. Since ψ is S_{2g+1}-equivariant, there exist a, b_1, b_2, b_3 and $c \in \mathbb{Z}_2$ such that $\psi(f_i \otimes f_i \otimes f_i) = a$, $\psi(f_j \otimes f_i \otimes f_i) = b_1, \psi(f_i \otimes f_j \otimes f_i) = b_2, \psi(f_i \otimes f_i \otimes f_j) = b_3$ for $i \neq j$ and $\psi(f_i \otimes f_j \otimes f_k) = c$ for $i \neq j \neq k \neq i$. The dimension of $\mathbb{Z}_2$-vector space $H^0(S_{2g+1}; (H^{\otimes 3})^*)$ is not greater than 1. Since $I(x_i \otimes x_i \otimes y_i - x_i \otimes x_{i+1} \otimes y_{i+1} - x_{i+1} \otimes x_i \otimes y_{i+1}) \equiv 1/2$, $I_{P_0} \in H^0(\Delta_g'; ((H^{\otimes 3})')^*)) = H^0(S_{2g+1}; ((H^{\otimes 3})^*)$ is not 0. We obtain $H^0(S_{2g+1}; ((H^{\otimes 3})')^*)) \neq 0$. Hence we have $H^0(S_{2g+1}; (H^{\otimes 3})^*) = \mathbb{Z}_2$. It is clear that the generator of $H^0(S_{2g+1}; (H^{\otimes 3})^*)$ is ψ as above. □

Corollary 5.6.

$$H^0(S_{2g+2}; (H^{\otimes 3})^*) = 0.$$

Proof. Take $\psi \in H^0(S_{2g+1}; (H^{\otimes 3})^*)$ in the proof of Lemma 5.5. Let b denote $b_1 = b_2 = b_3$. Using $\rho(\sigma_1)(f_i) = f_1 + f_i$ for $i = 2, 3, \ldots, 2g+1$, we have $0 = \psi(f_2 \otimes f_3 \otimes f_4) = \psi(\rho(\sigma_1)(f_2 \otimes f_3 \otimes f_4)) = 3b = b$. The equation $a = b_1 = b_2 = b_3 = c = 0$ gives $H^0(S_{2g+2}; (H^{\otimes 3})^*) = 0$. □

Using the diagram (5.1), Lemma 5.3, Lemma 5.4 and Lemma 5.5, we have

Proposition 5.7.

$$H^0(\Delta'_g; (K \otimes H)^*) = H^0(\Delta'_g; ((H^{\otimes 3})')^*) = \mathbb{Z}_2.$$

This gives us the following theorem.

Theorem 5.8.

$$H^0(\Delta_g; ((H^{\otimes 3})')^*) = \mathbb{Z}_2.$$

Proof. We have a natural injection $H^0(\Delta_g; ((H^{\otimes 3})')^*) \hookrightarrow H^0(\Delta'_g; ((H^{\otimes 3})')^*)$. Using Proposition 5.7, the dimension of $\mathbb{Z}_2$-vector space $H^0(\Delta_g; ((H^{\otimes 3})')^*)$ is not greater than 1. As in the proof of Lemma 5.5, the harmonic volume $I \in H^0(\Delta_g; ((H^{\otimes 3})')^*)$ is not 0. Hence $H^0(\Delta_g; ((H^{\otimes 3})')^*) = \mathbb{Z}_2$. □

Proof (The second proof of Theorem 4.1). Using Theorem 5.8, Proposition 5.7 and Lemma 5.5, we identify $H^0(S_{2g+1}; (H^{\otimes 3})^*)$ with $H^0(\Delta_g; ((H^{\otimes 3})')^*)$, whose generator is regarded as ψ in Lemma 5.5. We substitute

$$\begin{cases} x_i = f_{2i-1} + f_{2i}, \\ y_i = f_1 + f_2 + \cdots + f_{2i-1}, \end{cases}$$

for elements of the type (1) and (2) in Section 2. Then the direct computation of ψ gives us Theorem 4.1. □

The harmonic volume I gives a geometric interpretation of a theorem established by Tanaka.

Theorem 5.9 (Tanaka [10], Theorem 1.1). *If* $g \geq 2$, *then*

$$H_1(\Delta_g; H) = \mathbb{Z}_2.$$

Tanaka obtained the generator of $H_1(\Delta_g; H)$, using the relations of Δ_g in Theorem 5.1. Since Δ_g acts transitively on H, $H_0(\Delta_g, H) = 0$. By the universal coefficient theorem,

$$H^1(\Delta_g; H^*) = \mathrm{Hom}_{\mathbb{Z}}(H_1(\Delta_g; H); \mathbb{Z}_2) = \mathbb{Z}_2.$$

We have $H^1(\Delta_g; H^*) = \mathbb{Z}_2$.

By Corollary 5.6, it is clear that $H^0(\Delta_g; (H^{\otimes 3})^*) = 0$. The short exact sequence

$$0 \longrightarrow (H^{\otimes 3})' \longrightarrow H^{\otimes 3} \xrightarrow{p} H^{\oplus 3} \longrightarrow 0$$

gives us a connected homomorphism $\delta : H^0(\Delta_g; ((H^{\otimes 3})')^*) \to H^1(\Delta_g; (H^{\oplus 3})^*)$ and it is injective. Since I is S_3-invariant, we may consider $\delta I = (\delta I|_H, \delta I|_H, \delta I|_H) \in H^1(\Delta_g; H^*)^{\oplus 3}$. Here $\delta I|_H$ is the restriction $H^0(\Delta_g; ((H^{\otimes 3})')^*) \to H^1(\Delta_g; H^*)$.

Proposition 5.10. *The generator of $H^1(\Delta_g; H^*)$ is $\delta I|_H$.*

Proof. If $\delta I|_H$ is *not* the generator of $H^1(\Delta_g; H^*)$, we have $\delta I = 0 \in H^1(\Delta_g; (H^{\oplus 3})^*)$. This contradicts that δ is injective. □

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