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Nilpotent Orbits of $\mathbb{Z}_4$-Graded Lie Algebra and Geometry of Moment Maps Associated to the Dual Pair $(U(p,q),\, U(r,s))$

Dedicated to Professor Ryoshi Hotta on his 60th birthday

By

Takuya Ohta*

Abstract

Let $\mathfrak{s}^1 \leftarrow L_+ \rightarrow \mathfrak{s}^2$ be the $K_{\mathbb{C}}$-versions of the moment maps associated to the dual pair $(U(p,q), U(r,s))$ and $\mathcal{N}(\mathfrak{s}^1) \leftarrow \mathcal{N}(L_+) \rightarrow \mathcal{N}(\mathfrak{s}^2)$ their restrictions to the nilpotent varieties. In this paper, we first describe the nilpotent orbit correspondence via the moment maps explicitly. Second, under the condition $\min\{p,q\} \geq \max\{r,s\}$, we show that there are open subvariety L'_+ (resp. $(\mathfrak{s}^2)'$) of L_+ (resp. $\mathfrak{s}^2$) and locally closed subvariety $(\mathfrak{s}^1)'$ of $\mathfrak{s}^1$ such that the restrictions of the moment maps $\mathcal{N}((\mathfrak{s}^1)') \leftarrow \mathcal{N}(L'_+) \rightarrow \mathcal{N}((\mathfrak{s}^2)')$ give bijections of nilpotent orbits. Furthermore, we show that the bijections preserve the closure relation and the equivalence class of singularities.

§0. Introduction

In [KrP1], H. Kraft and C. Procesi made a comparison of singularities between closures of nilpotent orbits in $\mathfrak{gl}(n,\mathbb{C})$ and those in $\mathfrak{gl}(m,\mathbb{C})$ $(n-m > 0)$, that is:

Theorem ([KrP1, Proposition 3.1]). *Let η and σ be Young diagrams with n boxes which have (non-empty) $n-m$ rows. Let η' and σ' be the Young diagrams with m boxes which we obtain from η and σ by erasing the coincident*

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*Department of Mathematics Tokyo Denki University, Kanda-nisiki-cho, Chiyoda-ku, Tokyo 101-8457, Japan.
e-mail: ohta@cck.dendai.ac.jp

first column respectively. We write C_η *and* C_σ *(resp.* $C_{\eta'}$ *and* $C_{\sigma'}$*) the nilpotent orbits in* $\mathfrak{gl}(n,\mathbb{C})$ *(resp.* $\mathfrak{gl}(m,\mathbb{C})$*) corresponding to* η *and* σ *(resp.* η' *and* σ'*) respectively. Suppose that* $\overline{C_\eta} \supset C_\sigma$. *Then* $\overline{C_{\eta'}} \supset C_{\sigma'}$ *and we have*

$$Sing(\overline{C_\eta}, C_\sigma) = Sing(\overline{C_{\eta'}}, C_{\sigma'})$$

(for the definition of smooth equivalence class $Sing(\ ,\ )$*, see Definition* 2.14*).*

On the singularities of nilpotent orbits, they proved a similar correspondence between $\mathfrak{o}(n,\mathbb{C})$ and $\mathfrak{sp}(m,\mathbb{C})$ in [KrP2].

On the other hand, in [O1] and [O2], the author showed that the similar correspondence of singularities between closures of nilpotent orbits in the following pairs of complex symmetric pairs:

$((\mathfrak{gl}(n,\mathbb{C}), \mathfrak{o}(n,\mathbb{C}))\,, (\mathfrak{gl}(m,\mathbb{C}), \mathfrak{o}(m,\mathbb{C})))$ [O1],

$((\mathfrak{gl}(2n,\mathbb{C}), \mathfrak{sp}(2n,\mathbb{C}))\,, (\mathfrak{gl}(2m,\mathbb{C}), \mathfrak{sp}(2m,\mathbb{C})))$ [O1],

$((\mathfrak{gl}(p+q,\mathbb{C}), \mathfrak{gl}(p,\mathbb{C}) + \mathfrak{gl}(q,\mathbb{C})))\,, ((\mathfrak{gl}(r+s,\mathbb{C}), \mathfrak{gl}(r,\mathbb{C}) + \mathfrak{gl}(s,\mathbb{C})))$ [O2],

$((\mathfrak{o}(p+q,\mathbb{C}), \mathfrak{o}(p,\mathbb{C}) + \mathfrak{o}(q,\mathbb{C}))\,, (\mathfrak{sp}(2n,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C})))$ [O2],

$((\mathfrak{sp}(p+q,\mathbb{C}), \mathfrak{sp}(p,\mathbb{C}) + \mathfrak{sp}(q,\mathbb{C}))\,, (\mathfrak{o}(2n,\mathbb{C}), \mathfrak{gl}(n,\mathbb{C})))$ [O2].

Recently, we have come to understand that the quotient maps which give these correspondences, are the moment maps associated to the dual pairs corresponding to the pairs of complex Lie algebras (cases of complex dual pairs) and those of symmetric pairs (cases of real dual pairs).

For the moment maps $\mathfrak{g}^1 \xleftarrow{\rho} L \xrightarrow{\pi} \mathfrak{g}^2$ associated to the complex dual pairs $(G^1, G^2) \hookrightarrow Sp(L)$ $((G^1, G^2) = (GL(n,\mathbb{C}), GL(m,\mathbb{C})), (O(n,\mathbb{C}), Sp(m,\mathbb{C})), (Sp(n,\mathbb{C}), O(m,\mathbb{C})))$, by using the construction of [KrP1] and [KrP2], A. Daszkiewicz, W. Kraśkiewicz and T. Przebinda ([DKP]) showed that for any nilpotent G^2-orbit $\mathcal{O}_2$ in $\mathfrak{g}^2 = Lie(G^2)$, $\rho(\pi^{-1}(\overline{\mathcal{O}}_2))$ is a closure of a single nilpotent G^1-orbit $\mathcal{O}_1$.

For certain real dual pairs $(G^1_{\mathbb{R}}, G^2_{\mathbb{R}})$ in the stable range with $G^2_{\mathbb{R}}$ the smaller member, K. Nishiyama noticed that an analogue of the above correspondence $\mathcal{O}_2 \mapsto \mathcal{O}_1$ is injective (he call this a θ-lifting of nilpotent orbits) and studied the relation of the structure of the ring of regular functions on $\overline{\mathcal{O}_1}$ and that on $\overline{\mathcal{O}_2}$ via the moment maps ([N1], [N2]).

It is known that, for some representations of $G^2_{\mathbb{R}}$ corresponding to small nilpotent orbits, Howe's correspondence of representations and θ-lifting of nilpotent orbits are compatible via taking associated varieties(cf., [N3], [NOT], [NZ], [Y]). The relationship of the restriction of a representation to a reductive subgroup and the projection of the associated variety to the Lie subalgebra, was studied earlier by T. Kobayashi in [Ko1] and [Ko2], and the similar results also had been obtained as a consequence.

Let $(\ , \ )_{L_{\mathbb{R}}}$ be a non-degenerate symplectic form on a real vector space $L_{\mathbb{R}}$ and $(G^1_{\mathbb{R}}, G^2_{\mathbb{R}}) = (U(p,q), U(r,s))$ $(\dim_{\mathbb{R}} L_{\mathbb{R}} = 2(p+q)(r+s))$ be a dual pair contained in the real symplectic group $Sp(L_{\mathbb{R}})$ defined by $(\ , \ )_{L_{\mathbb{R}}}$. Let $(G^j, K^j)(j = 1,2)$ be the complex symmetric pair corresponding to the real group $G^j_{\mathbb{R}}$ and $Lie(G^j) = \mathfrak{g}^j = \mathfrak{k}^j + \mathfrak{s}^j$ a complexfied Cartan decomposition corresponding to $G^j_{\mathbb{R}}$. For $z \in L_{\mathbb{R}}$, we define a linear form $\mu_z \in \mathfrak{sp}(L_{\mathbb{R}})^*$ by

$$\mu_z(x) = \frac{1}{2}(xz, z)_{L_{\mathbb{R}}} \ (x \in \mathfrak{sp}(L_{\mathbb{R}})).$$

By restricting to $\mathfrak{g}^j_{\mathbb{R}}$, we obtain maps

$$L_{\mathbb{R}} \to (\mathfrak{g}^j_{\mathbb{R}})^*, \ z \mapsto \mu_z|_{\mathfrak{g}^j_{\mathbb{R}}} \ (j = 1,2).$$

Via the usual identification $(\mathfrak{g}^j_{\mathbb{R}})^* \simeq \mathfrak{g}^j_{\mathbb{R}}$, we obtain maps

$$\mathfrak{g}^1_{\mathbb{R}} \xleftarrow{\rho} L_{\mathbb{R}} \xrightarrow{\pi} \mathfrak{g}^2_{\mathbb{R}},$$

which we call the moment maps associated to the dual pair $(G^1_{\mathbb{R}}, G^2_{\mathbb{R}}) \hookrightarrow Sp(L_{\mathbb{R}})$. By the complexification, we obtain complex moment maps

$$\mathfrak{g}^1 \xleftarrow{\rho} L \xrightarrow{\pi} \mathfrak{g}^2.$$

By restricting to a suitable maximally totally isotropic subspace L_+, we obtain $K^1 \times K^2$-equivariant maps

$$\mathfrak{s}^1 \xleftarrow{\rho} L_+ \xrightarrow{\pi} \mathfrak{s}^2.$$

For the simplicity, we also call these restrictions "moment maps" associated to the dual pair $(G^1_{\mathbb{R}}, G^2_{\mathbb{R}}) \hookrightarrow Sp(L_{\mathbb{R}})$. In this paper, we show that these moment maps are obtained by a $\mathbb{Z}_4$-gradation of $\mathfrak{gl}(p+q+r+s, \mathbb{C})$, and we consider the nilpotent orbits correspondence among $\mathfrak{s}^1$, L_+, $\mathfrak{s}^2$ via these maps and genalalization of the θ-lifting of nilpotent orbits.

In §1, we describe the classification of nilpotent $K^1 \times K^2$-orbits in L_+ and their closure relation due to Kempken [Ke].

In §2, we first give the explicit description of the nilpotent orbit correspondence

$$\mathcal{N}(\mathfrak{s}^1)/K^1 \leftarrow \mathcal{N}(L_+)/K^1 \times K^2 \rightarrow \mathcal{N}(\mathfrak{s}^2)/K^2$$

induced by ρ and π. The main theorems of this paper are the following:

Theorem 2.9. *Suppose that* $min\{p,q\} \geq max\{r,s\}$ *and,* $p-r>0$ *or* $q-s>0$. *There exists an open subvariety* L'_+ *with the following properties*: $(\mathfrak{s}^1)' := \rho(L'_+)$ *is a locally closed subvariety of* $\mathfrak{s}^1$ *and* $(\mathfrak{s}^2)' := \pi(L'_+)$ *is an open subvariety of* $\mathfrak{s}^2$. *Then we have the following*:
(i) $\rho|_{\mathcal{N}(L'_+)} : \mathcal{N}(L'_+) \to \mathcal{N}((\mathfrak{s}^1)')$ *is locally trivial in the classical topology with typical fibre isomorphic to* K^2.
(ii) $\pi|_{\mathcal{N}(L'_+)} : \mathcal{N}(L'_+) \to \mathcal{N}((\mathfrak{s}^2)')$ *is smooth and each fibre of* $\pi|_{\mathcal{N}(L'_+)}$ *is a single* K^1*-orbit.*
(iii) *The induced maps*

$$\mathcal{N}((\mathfrak{s}^1)')/K^1 \leftarrow \mathcal{N}(L'_+)/K^1 \times K^2 \to \mathcal{N}((\mathfrak{s}^2)')/K^2$$

are bijections.
(iv) *The bijections in* (iii) *preserve the closure relation. That is, for* $\mathcal{O}_j \in \mathcal{N}(L'_+)/K^1 \times K^2$ $(j=1,2)$ *and the corresponding orbits* $\mathcal{O}^1_j = \rho(\mathcal{O}_j) \in \mathcal{N}((\mathfrak{s}^1)')/K^1$, $\mathcal{O}^2_j = \pi(\mathcal{O}_j) \in \mathcal{N}((\mathfrak{s}^2)')/K^2$, *we have*

$$\overline{\mathcal{O}^1_1} \supset \mathcal{O}^1_2 \Longleftrightarrow \overline{\mathcal{O}_1} \supset \mathcal{O}_2 \Longleftrightarrow \overline{\mathcal{O}^2_1} \supset \mathcal{O}^2_2.$$

Theorem 2.14. *Under the assumption of Theorem* 2.9, (iv), *suppose* $\overline{\mathcal{O}_1} \supset \mathcal{O}_2$. *Then we have*

$$Sing(\overline{\mathcal{O}^1_1}, \mathcal{O}^1_2) = Sing(\overline{\mathcal{O}_1}, \mathcal{O}_2) = Sing(\overline{\mathcal{O}^2_1}, \mathcal{O}^2_2).$$

Thus the correspondence

$$\mathcal{N}((\mathfrak{s}^1)')/K^1 \simeq \mathcal{N}((\mathfrak{s}^2)')/K^2$$

obtained by the moment maps is considered as a good duality, which gives the correspondence of nilpotent orbits of Kraft-Procesi type simultaneously.

If $(G^1_{\mathbb{R}}, G^2_{\mathbb{R}}) = (U(p,q), U(r,s))$ is in the stable range (i.e. $\min\{p,q\} \geq r+s$), we see $\mathcal{N}((\mathfrak{s}^2)') = \mathcal{N}(\mathfrak{s}^2)$. Hence, via the bijection of Theorem 2.9, (iii), each nilpotent orbit in $\mathfrak{s}^2$ corresponds to some nilpotent orbit in $\mathcal{N}((\mathfrak{s}^1)')$ which coincides with Nishiyama's θ-lifting. Thus, in our general setting, the bijection $\mathcal{N}((\mathfrak{s}^2)')/K^2 \simeq \mathcal{N}((\mathfrak{s}^1)')/K^1$ given by Theorem 2.9, (iii), is considered as a generalization of Nishiyama's θ-lifting.

On the other hand, if $C_2 \in [\mathcal{N}(\mathfrak{s}_2) \backslash \mathcal{N}(\mathfrak{s}'_2)]/K^2$, $\rho(\pi^{-1}(\overline{C_2}))$ is not a closure of a single K^1-orbit in general (cf. Remark 2.15, (iii)) and hence the analogue of the main result of [DKP] does not holds in our case. $\mathcal{N}(\mathfrak{s}'_2)/K^2$ is considered as a domain on which a "good" correspondence

$$\mathcal{N}((\mathfrak{s}^2)')/K^2 \simeq \mathcal{N}((\mathfrak{s}^1)')/K^1, \quad \mathcal{O}_2 \mapsto \mathcal{O}_1 \ (\rho(\pi^{-1}(\overline{\mathcal{O}_2})) = \overline{\mathcal{O}_1})$$
$$\text{(generalization of } \theta - \text{lifting)}$$

is defined.

In §3, we explain the reason why the maps $\mathfrak{s}^1 \overset{\rho}{\leftarrow} L_+ \overset{\pi}{\rightarrow} \mathfrak{s}^2$ constructed in §2 can be interpreted as the $K_{\mathbb{C}}$-version of the original real moment maps $\mathfrak{g}^1_{\mathbb{R}} \overset{\rho}{\leftarrow} L_{\mathbb{R}} \overset{\pi}{\rightarrow} \mathfrak{g}^2_{\mathbb{R}}$.

Finally we mention the generalization of the correspondences

$$\mathcal{N}((\mathfrak{s}^1)')/K^1 \leftarrow \mathcal{N}(L'_+)/K^1 \times K^2 \rightarrow \mathcal{N}((\mathfrak{s}^2)')/K^2.$$

These correspondences can be extended to the general orbit correspondences

$$(\mathfrak{s}^1)'/K^1 \leftarrow L'_+/K^1 \times K^2 \rightarrow (\mathfrak{s}^2)'/K^2$$

and the analogue of Theorem 2.9 and Theorem 2.14 also hold for these generalizations. Furthermore these results also hold for all reductive dual pairs in the real symplectic groups. These will be given in forthcoming paper ([O3]).

§1. Nilpotent Orbits of $\mathbb{Z}_m$-Graded Lie Algebras

To understand the nilpotent orbits correspondence via the moment maps, we give a combinatorial description of the classification of nilpotent orbits of Θ-representations in the spirit of *ab*-diagrams in [O1, O2]. With this combinatorial description, we review the results by [Ke] on the closure relation of nilpotent orbits in §1.

In §2, we shall use these results with $m = 4$(the order of Θ).

§1.1. $\mathbb{Z}_m$-graded Lie algebras

Let G be a complex reductive algebraic group with Lie algebra $\mathfrak{g}$ and m a positive integer. Let $\Theta : G \to G$ be an automorphism of G such that $\Theta^m = id$ and $\Theta^j \neq id$ $(1 \leq j < m)$. We write $\Theta : \mathfrak{g} \to \mathfrak{g}$ for the induced automorphism. We put $\zeta := e^{2\pi i/m}$,

$$G_1 = \{g \in G; \Theta(g) = g\} \ \text{ and } \ \mathfrak{g}_\delta := \{X \in \mathfrak{g}; \Theta(X) = \delta X\} \ (\delta \in \langle\zeta\rangle),$$

where $\langle\zeta\rangle$ denotes the multiplicative group generated by ζ. Then $\mathfrak{g}$ is decomposed as

$$\mathfrak{g} = \oplus_{\delta \in \langle\zeta\rangle} \mathfrak{g}_\delta$$

and we obtain a $\mathbb{Z}_m$-graded Lie algebra. For each $\delta \in \langle\zeta\rangle$, the isotropy group G_1 acts on $\mathfrak{g}_\delta$ by the adjoint action. In this paper, we call the group G_1 a Θ-group and the representation $(G_1\ ,\ \mathfrak{g}_\zeta)$ of G_1 on $\mathfrak{g}_\zeta$ a Θ-representation.

§1.2. Classification of nilpotent orbits of the Θ-representation defined by an automorphism of a vector space

Let V be a finite dimensional complex vector space and $S : V \to V$ an automorphism of V such that $S^m = id$ and $S^j \neq id$ $(1 \leq j < m)$, where m is a positive integer. Put $G = GL(V)$ and $\mathfrak{g} = \mathfrak{gl}(V)$. Then S defines an automorphism $\Theta : G \to G, \Theta(g) = SgS^{-1}$ $(g \in G)$. As before we write $\zeta := e^{2\pi i/m}$. Then we obtain a Θ-representation $(G_1\ ,\ \mathfrak{g}_\zeta)$. For $\delta \in \langle\zeta\rangle$, we write $V_\delta := \{v \in V; Sv = \delta v\}$. Then V decomposed as

$$V = V_1 \oplus V_\zeta \oplus V_{\zeta^2} \oplus \cdots \oplus V_{\zeta^{m-1}}$$

and $\mathfrak{g}_\zeta$ can be written as

$$\mathfrak{g}_\zeta = \{X \in \mathfrak{g}; XV_\delta \subset V_{\zeta\delta}\ (\delta \in \langle\zeta\rangle)\}.$$

We write $\mathcal{N}(\mathfrak{g}_\zeta)$ the set of nilpotent elements of $\mathfrak{g}$ contained in $\mathfrak{g}_\zeta$. To describe the G_1-orbits in $\mathcal{N}(\mathfrak{g}_\zeta)$, we introduce the following notion.

Definition 1.1. (i) For a Young diagram η for which an element of $\langle\zeta\rangle$ is placed in each box, we say η a $\langle\zeta\rangle$-signed diagram (called "word" in [Ke], a generalization of "ab-diagram" in [O1, O2]) if, for each box placed $\delta \in \langle\zeta\rangle$, the right adjacent box is placed $\zeta\delta$. e.g.

$$\eta = \begin{array}{llllll} i & i^2 & i^3 & 1 & i & i^2 \\ 1 & i & i^2 & i^3 & 1 & i \\ i^3 & 1 & i & i^2 & & \end{array} \quad \text{in case } m = 4.$$

(ii) For a $\langle\zeta\rangle$-signed diagram η and $\delta \in \langle\zeta\rangle$, we denote by $n_\delta(\eta)$ the number of δ's which occur in η. We write $D(n_0, n_1, n_2, \dots n_{m-1})$ the set of $\langle\zeta\rangle$-signed diagrams η such that $n_{\zeta^j}(\eta) = n_j$ $(0 \leq j \leq m-1)$.

(iii) For a $\langle\zeta\rangle$-signed diagram η, we write η' the $\langle\zeta\rangle$-signed diagram which we obtain from η by erasing the first column. We define $\eta^{(j)}$ by $\eta^{(j)} = (\eta^{(j-1)})'$. e.g. for the $\langle i\rangle$-signed diagram η of (i),

$$\eta' = \begin{array}{lllll} i^2 & i^3 & 1 & i & i^2 \\ i & i^2 & i^3 & 1 & i \\ 1 & i & i^2 & & \end{array} \quad \text{and} \quad \eta^{(2)} = \begin{array}{llll} i^3 & 1 & i & i^2 \\ i^2 & i^3 & 1 & i \\ i & i^2 & & \end{array}.$$

Write $n_j := \dim V_{\zeta^j}$ $(0 \leq j \leq m-1)$. Then the G_1-orbits in $\mathcal{N}(\mathfrak{g}_\zeta)$ are classified by $D(n_0, n_1, n_2, \dots, n_{m-1})$ as follows:

Proposition 1.2 ([Ke]). (i) *For any $x \in \mathcal{N}(\mathfrak{g}_\zeta)$, there exists a basis $\{v_j^k; 1 \le k \le p, 0 \le j \le r_k\}$ of V contained in $V_1 \cup V_\zeta \cup V_{\zeta^2} \cup \ldots \cup V_{\zeta^{m-1}}$ such that*

$$v_0^k \xrightarrow{x} v_1^k \xrightarrow{x} v_2^k \xrightarrow{x} \ldots \xrightarrow{x} v_{r_k}^k \xrightarrow{x} 0,$$

i.e., $xv_j^k = v_{j+1}^k$ $(0 \le j \le r_k - 1)$ and $xv_{r_k}^k = 0$.
(ii) *For $1 \le k \le p$, if $v_0^k \in V_{\delta_k}$ $(\delta_k \in \langle\zeta\rangle)$, we write*

$$\eta_k := \delta_k \, \zeta\delta_k \, \zeta^2\delta_k \ldots \zeta^{r_k}\delta_k .$$

Thus we obtain a $\langle\zeta\rangle$-signed diagram $\eta \in D(n_0, n_1, n_2, \ldots, n_{m-1})$ with p rows, whose rows are $\eta_1, \eta_2, \ldots, \eta_p$.

$$\eta = \eta_1 + \eta_2 + \cdots + \eta_p = \begin{matrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_p \end{matrix} .$$

Then η is independent of choice of the basis $\{v_j^k\}$. We write $\eta = \eta_x$ and call η_x the $\langle\zeta\rangle$-signed diagram of x.
(iii) *The correspondence*

$$\mathcal{N}(\mathfrak{g}_\zeta) \to D(n_0, n_1, n_2, \ldots, n_{m-1}), \quad x \mapsto \eta_x$$

of (ii) *defines a bijection*

$$\mathcal{N}(\mathfrak{g}_\zeta)/G_1 \simeq D(n_0, n_1, n_2, \ldots n_{m-1}).$$

For the reader's convenience, we give a proof (which parallel to [O2]).

Proof of Proposition 1.2. (i) For $x \in \mathcal{N}(\mathfrak{g}_\zeta) \setminus \{0\}$, as in the proof of [[KrP2], Lemma 7.3], we can take $h \in \mathfrak{g}_1$, $y \in \mathfrak{g}_{\zeta^{-1}}$ such that (h, x, y) is an S-triple;

$$[h, x] = 2x, \; [h, y] = -2y, \; [x, y] = h.$$

Since $Sy = \zeta^{-1}yS$, $K := \ker(y : V \to V)$ is decomposed as

$$K = \oplus_{\delta \in \langle\zeta\rangle}(K \cap V_\delta).$$

Since each $K \cap V_\delta$ is h-stable, we can take a basis $\{v_0^k; 1 \le k \le p\}$ of K consisting of h-weight vectors. Define r_k by $x^{r_k}v_0^k \neq 0$ and $x^{r_k+1}v_0^k = 0$ and write $v_j^k := x^j v_0^k$ $(0 \le j \le r_k)$. We obtain a basis $\{v_j^k; 1 \le k \le p, 0 \le j \le r_k\}$ of (i).

(ii) Since $x^q V = \mathbb{C}\{v_j^k; 1 \le k \le p, q \le j\}$ $(q \ge 0)$, we have

$$n_\delta(\eta^{(q)}) = {}^\sharp\{v_j^k; 1 \le k \le p,\ q \le j, v_j^k \in V_\delta\} = \dim(x^q V \cap V_\delta)$$

for $\delta \in \langle\zeta\rangle$ and $q \ge 0$. Hence η is uniquely determined by x.
(iii) Suppose $\{v_j^k\}$ is a basis of V corresponding to $x \in \mathcal{N}(\mathfrak{g}_\zeta)$. We put $x' = \mathrm{Ad}(g)x$ $(g \in G_1)$. Then clearly $\{gv_j^k\}$ is a basis of V corresponding to x' and hence $\eta_x = \eta_{x'}$. Therefore the map $\mathcal{N}(\mathfrak{g}_\zeta)/G_1 \to D(n_0, n_1, n_2, \ldots, n_{m-1})$ is defined.

For $x, x' \in \mathcal{N}(\mathfrak{g}_\zeta)$ such that $\eta_x = \eta_{x'}$, take a basis $\{v_j^k\}$ (resp. $\{u_j^k\}$) of V corresponding to x (resp. x') by (i). Here we can assume that v_0^k and u_0^k contained in the same V_δ for each k. Defined $g \in GL(V)$ by $gv_j^k = u_j^k$. Since $gV_\delta = V_\delta$ for each $\delta \in \langle\zeta\rangle$, we have $g \in G_1$. We easily see that $x' = \mathrm{Ad}(g)x$ and hence the map $\mathcal{N}(\mathfrak{g}_\zeta)/G_1 \to D(n_0, n_1, n_2, \ldots, n_{m-1})$ is injective. The surjectivity of this map is easily shown. □

§1.3. On the closure relation

Let us define an ordering of $\langle\zeta\rangle$-signed diagrams as follows.

Definition 1.3. For $\langle\zeta\rangle$-signed diagrams $\eta, \mu \in D(n_0, n_1, n_2, \ldots, n_{m-1})$, we write $\eta \ge \mu$ if $n_\delta(\eta^{(j)}) \ge n_\delta(\mu^{(j)})$ for all $\delta \in \langle\zeta\rangle$ and $j \ge 0$.

For the closure relation, we refer to [Ke] for the proof.

Theorem 1.4. *For two nilpotent orbits $\mathcal{O}_j \in \mathcal{N}(\mathfrak{g}_\zeta)/G_1$ $(j = 1, 2)$, we denote by $\eta_j \in D(n_0, n_1, n_2, \ldots, n_{m-1})$ the $\langle\zeta\rangle$-signed diagrams corresponding to $\mathcal{O}_j$. Then $\mathcal{O}_2$ is contained in the Zariski closure $\overline{\mathcal{O}_1}$ of $\mathcal{O}_1$ if and only if $\eta_1 \ge \eta_2$:*

$$\overline{\mathcal{O}_1} \supset \mathcal{O}_2 \Leftrightarrow \eta_1 \ge \eta_2$$

§2. Geometry of the Moment Maps Associated to the Dual Pairs $(U(p,q), U(r,s))$

§2.1. The moment maps

Let V be a finite dimensional complex vector space and $s_V : V \to V$ a linear involution. We call such a pair (V, s_V) a vector space with involution.

Define an involution θ_V of $GL(V)$ by $\theta_V(g) = s_V g s_V$ $(g \in GL(V))$ and put

$$\begin{aligned}
V_a &:= \{v \in V; s_V v = v\}, \quad V_b := \{v \in V; s_V v = -v\},\\
n_a &:= \dim V_a, \quad n_b := \dim V_b,\\
K(V) &:= GL(V)_1 = \{g \in GL(V); \theta_V(g) = g\} \simeq GL(V_a) \times GL(V_b),\\
\mathfrak{k}(V) &:= \mathfrak{gl}(V)_1 = \{X \in \mathfrak{gl}(V); \theta_V(X) = X\}\\
\mathfrak{s}(V) &:= \mathfrak{gl}(V)_{-1} = \{X \in \mathfrak{gl}(V); \theta_V(X) = -X\}.
\end{aligned}$$

Thus we obtain a symmetric pair $(GL(V), K(V))$ which corresponds to the real group $U(n_a, n_b)$.

By (1.2), nilpotent $K(V)$-orbits in $\mathfrak{s}(V)$ are classified by $\langle -1\rangle$-signed diagrams:

$$\mathcal{N}(\mathfrak{s}(V))/K(V) \simeq D(n_a, n_b).$$

Via the identification $a = 1$ and $b = -1$, we consider $D(n_a, n_b)$ as the set of ab-diagrams with n_a a's and n_b b's.

Let (U, s_U) be another vector space with an involution s_U. Define θ_U, U_a, U_b, $K(U)$, $\mathfrak{k}(U)$ and $\mathfrak{s}(U)$ as above and put $m_a = \dim U_a$, $m_b = \dim U_b$. Then $(GL(U), K(U))$ is the symmetric pair corresponding to the real group $U(m_a, m_b)$.

For (V, s_V) and (U, s_U), we consider the vector space

$$L := \mathrm{Hom}_{\mathbb{C}}(U, V) \oplus \mathrm{Hom}_{\mathbb{C}}(V, U)$$

on which $GL(V) \times GL(U)$ acts by

$$(g, h)(P, Q) = (gPh^{-1}, hQg^{-1}) \ ((g, h) \in GL(V) \times GL(U), (P, Q) \in L).$$

We also consider a subspace

$$L_+ := \{(P, Q) \in L; s_V P s_U = P, s_U Q s_V = -Q\}$$

on which $K(V) \times K(U)$ acts by the above action. We define $GL(V) \times GL(U)$-equivariant morphisms

$$\mathfrak{gl}(V) \stackrel{\rho}{\leftarrow} L \stackrel{\pi}{\rightarrow} \mathfrak{gl}(U), \quad \rho(P, Q) = PQ, \quad \pi(P, Q) = QP \ ((P, Q) \in L).$$

Then the restrictions of ρ and π to L_+ defines $K(V) \times K(U)$-equivariant morphisms

$$\mathfrak{s}(V) \stackrel{\rho}{\leftarrow} L_+ \stackrel{\pi}{\rightarrow} \mathfrak{s}(U).$$

These morphisms were treated in [[O2], §3] and certain duality between nilpotent orbits in $\mathfrak{s}(V)$ and $\mathfrak{s}(U)$ was shown there. In §3, we explain that these maps can be interpreted as the moment maps.

In §3, we will construct the following:

(a) A non-degenerate symplectic form $(\ , \ )_L$ on L.

(b) A real vector subspace $L_{\mathbb{R}}$ of L such that $\dim_{\mathbb{R}} L_{\mathbb{R}} = \dim L$ and $(\ , \ )_L | L_{\mathbb{R}}$ is real valued and non-degenerate.

(c) A real form $GL(V)_{\mathbb{R}} \simeq U(n_a, n_b)$ (resp. $GL(U)_{\mathbb{R}} \simeq U(m_a, m_b)$) of $GL(V)$ (resp. $GL(U)$) with Cartan involution $\theta_V|_{GL(V)_{\mathbb{R}}}$ (resp. $\theta_U|_{GL(U)_{\mathbb{R}}}$).

We will show the following:

Proposition 2.1. (i) *The commuting actions of $GL(V)_{\mathbb{R}}$ and $GL(U)_{\mathbb{R}}$ on L stabilize $L_{\mathbb{R}}$ and preserve the symplectic form $(\ , \ )_L$;*

$$(GL(V)_{\mathbb{R}}, GL(U)_{\mathbb{R}}) \hookrightarrow Sp(L_{\mathbb{R}}).$$

(ii) $-i\rho(L_{\mathbb{R}}) \subset \mathfrak{gl}(V)_{\mathbb{R}} = Lie(GL(V)_{\mathbb{R}})$ *and* $i\pi(L_{\mathbb{R}}) \subset \mathfrak{gl}(U)_{\mathbb{R}} = Lie(GL(U)_{\mathbb{R}})$.

(iii) *By the identification* $\mathfrak{gl}(V)_{\mathbb{R}} \simeq \mathfrak{gl}(V)^*_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{gl}(V)_{\mathbb{R}}, \mathbb{R})$ (*resp.* $\mathfrak{gl}(U)_{\mathbb{R}} \simeq \mathfrak{gl}(U)^*_{\mathbb{R}}$) *via the trace form on V* (*resp.* U), $-i\rho|_{L_{\mathbb{R}}} : L_{\mathbb{R}} \to \mathfrak{gl}(V)^*_{\mathbb{R}}$ (*resp.* $i\pi|_{L_{\mathbb{R}}} : L_{\mathbb{R}} \to \mathfrak{gl}(U)^*_{\mathbb{R}}$) *coincides with the moment map with respect to the action of $GL(V)_{\mathbb{R}}$* (*resp.* $GL(U)_{\mathbb{R}}$) *on the symplectic manifold* $(L_{\mathbb{R}}, (\ , \ )_L|_{L_{\mathbb{R}}})$.

(iv) L_+ *is a maximally totally isotropic subspace of* $(L, (\ , \ )_L)$.

Then

$$\mathfrak{gl}(V)_{\mathbb{R}} \overset{-i\rho|_{L_{\mathbb{R}}}}{\longleftarrow} L_{\mathbb{R}} \overset{i\pi|_{L_{\mathbb{R}}}}{\longrightarrow} \mathfrak{gl}(U)_{\mathbb{R}}$$

are moment maps and

$$\mathfrak{gl}(V) \overset{-i\rho}{\longleftarrow} L \overset{i\pi}{\longrightarrow} \mathfrak{gl}(U)$$

are the complexification. Since

$$\mathfrak{s}(V) \overset{-i\rho|_{L_+}}{\longleftarrow} L_+ \overset{i\pi|_{L_+}}{\longrightarrow} \mathfrak{s}(U)$$

are the restrictions to the maximally totally isotropic subspace L_+ of the complexified moment maps, we may call $\rho|_{L_+}$ and $\pi|_{L_+}$ the moment maps.

§2.2. Geometry of moment maps

Let (V, s_V) and (U, s_U) be as in (2.1). We put $W := V \oplus U$, $G := GL(W)$, $\mathfrak{g} = \mathfrak{gl}(W)$ and define a linear automorphism $S : W \to W$ by

$$S = \begin{pmatrix} s_V & 0 \\ 0 & -is_U \end{pmatrix}.$$

S defines an automorphism

$$\Theta : G \to G, \quad \Theta(g) = SgS^{-1} \ (g \in G)$$

of order 4 and we obtain a Θ-representation $(G_1, \mathfrak{g}_i)$. Clearly we have

$$G_1 = \left\{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} ; g \in K(V),\ h \in K(U) \right\} \simeq K(V) \times K(U).$$

Since

$$\Theta \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} s_V A s_V & i s_V B s_U \\ -i s_U C s_V & s_U D s_U \end{pmatrix},$$

we have

$$\mathfrak{g}_i = \left\{ \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} ; P \in \mathrm{Hom}_{\mathbb{C}}(U, V),\ Q \in \mathrm{Hom}_{\mathbb{C}}(V, U), \right.$$
$$\left. s_V P s_U = P,\ s_U Q s_V = -Q \right\} \simeq L_+.$$

It is easily verified that the isomorphism

$$L_+ \simeq \mathfrak{g}_i, \quad (P, Q) \mapsto \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$$

is $G_1 = K(V) \times K(U)$-equivariant.

Remark 2.2. (i) Since $V_a = W_1$, $U_b = W_i$, $V_b = W_{-1}$, $U_a = W_{-i}$ and

$$\mathfrak{g}_i = \{X \in \mathrm{End}(W); XW_\delta \subset W_{i\delta}\ (\delta \in \langle i \rangle)\},$$

we can see $\mathfrak{g}_i$ as the set of quadruples of linear maps $W_\delta \to W_{i\delta}$ $(\delta \in \langle i \rangle)$;

$$\mathfrak{g}_i = \left\{ \begin{array}{ccccc} & & Q_a & & \\ & V_a & \to & U_b & \\ P_a & \uparrow & & \downarrow & P_b \\ & U_a & \leftarrow & V_b & \\ & & Q_b & & \end{array} \right\}.$$

(ii) For $X = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \in \mathfrak{g}_i$, since

$$X^2 = \begin{pmatrix} PQ & 0 \\ 0 & QP \end{pmatrix} = \begin{pmatrix} \rho(X) & 0 \\ 0 & \pi(X) \end{pmatrix},$$

we can see that

$$\rho(X) = X^2|_V \quad \text{and} \quad \pi(X) = X^2|_U.$$

By (1.2), we have the bijections

$$\begin{aligned}
&\mathcal{N}(\mathfrak{s}(V))/K(V) \simeq D(n_a, n_b),\ C_\eta \leftrightarrow \eta \\
&\mathcal{N}(\mathfrak{s}(U))/K(U) \simeq D(m_a, m_b),\ C_\sigma \leftrightarrow \sigma \\
&\mathcal{N}(\mathfrak{g}_i)/G_1 = \mathcal{N}(\mathfrak{g}_i)/K(V) \times K(U) \simeq D(n_a, m_b, n_b, m_a), \quad \mathcal{O}_\mu \leftrightarrow \mu,
\end{aligned}$$

where we consider $D(n_a, n_b)$ and $D(m_a, m_b)$ as the sets of ab-diagrams by the identification $a = 1$ and $b = -1$.

It is easy to see that the image $\rho(\mathcal{O}_\mu)$ (resp. $\pi(\mathcal{O}_\mu)$) of $\mathcal{O}_\mu \in \mathcal{N}(\mathfrak{g}_i)/K(V) \times K(U)$ is a nilpotent $K(V)$-orbit (resp. $K(U)$-orbit) in $\mathfrak{s}(V)$ (resp. $\mathfrak{s}(U)$). We define ab-diagrams $\rho(\mu) \in D(n_a, n_b)$ and $\pi(\mu) \in D(m_a, m_b)$ by

$$\rho(\mathcal{O}_\mu) = C_{\rho(\mu)} \quad \text{and} \quad \pi(\mathcal{O}_\mu) = C_{\pi(\mu)}.$$

Then $\rho(\mu)$ and $\pi(\mu)$ are given as follows:

Proposition 2.3. *For a $\langle i \rangle$-signed diagram $\mu \in D(n_a, m_b, n_b, m_a)$, $\rho(\mu)$ is the ab-diagram which we obtain from μ by erasing $\pm i$ and replacing 1 and -1 by a and b respectively. On the other hand, $\pi(\mu)$ is the ab-diagram which we obtain from μ by erasing ± 1 and replacing $-i$ and i by a and b respectively.*

Example. For

$$\mu = \begin{array}{llllll} i & -1 & -i & 1 & i & -1 \\ 1 & i & -1 & -i & 1 & i \\ -i & 1 & i & -1 & & \end{array} \in D(4,5,4,3),$$

$$\rho(\mu) = \begin{array}{lll} b & a & b \\ a & b & a \\ a & b & \end{array} \quad \text{and} \quad \pi(\mu) = \begin{array}{lll} b & a & b \\ b & a & b \\ a & b & \end{array}.$$

Now we write $d_a := n_a - m_a$ and $d_b := n_b - m_b$. To obtain a good duality between nilpotent orbits in $\mathfrak{s}(V)$ and those of $\mathfrak{s}(U)$ via the moment maps $\mathfrak{s}(V) \xleftarrow{\rho} \mathfrak{g}_i \xrightarrow{\pi} \mathfrak{s}(U)$, from now on, we assume the following:

Assumption 2.4. (i) $\min\{n_a, n_b\} \geq \max\{m_a, m_b\}$, *and*
(ii) $d_a > 0$ *or* $d_b > 0$.

Then we have the following:

Proposition 2.5 ([[O2], Proposition 3]). (i) $\pi : \mathfrak{g}_i \to \mathfrak{s}(U)$ *is surjective and*

$$\rho(\mathfrak{g}_i) = \{X \in \mathfrak{s}(V); \mathrm{rk}(X|_{V_a} : V_a \to V_b) \leq m_b,\ \mathrm{rk}(X|_{V_b} : V_b \to V_a) \leq m_a\}.$$

(ii) $\pi : \mathfrak{g}_i \to \mathfrak{s}(U)$ *and* $\rho : \mathfrak{g}_i \to \mathfrak{s}(V)$ *are quotient maps under* $K(V)$ *and* $K(U)$ *respectively*, *that is*

$$\pi^*(\mathbb{C}[\mathfrak{s}(U)]) = \mathbb{C}[\mathfrak{g}_i]^{K(V)} \ \text{ and } \ \rho^*(\mathbb{C}[\mathfrak{s}(V)]) = \mathbb{C}[\mathfrak{g}_i]^{K(U)}.$$

Let us consider the following subsets $\mathfrak{g}_i'$, $\mathfrak{s}(V)'$, $\mathfrak{s}(U)'$ of $\mathfrak{g}_i$, $\mathfrak{s}(V)$, $\mathfrak{s}(U)$ respectively:

$$\mathfrak{g}_i' := \left\{ \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}; \mathrm{rk}(P) \text{ and } \mathrm{rk}(Q) \text{ attain their maximum} \right\}$$

$$= \left\{ \begin{array}{ccc} & Q_a & \\ V_a & \rightarrow & U_b \\ P_a \ \uparrow & & \downarrow \ P_b \\ U_a & \leftarrow & V_b \\ & Q_b & \end{array} ; Q_a,\ Q_b \text{ are surjective and } P_a, P_b \text{ are injective} \right\},$$

$$\mathfrak{s}(V)' := \{X \in \mathfrak{s}(V); \mathrm{rk}(X|_{V_a} : V_a \to V_b) = m_b,\ \mathrm{rk}(X|_{V_b} : V_b \to V_a) = m_a\},$$
$$\mathfrak{s}(U)' := \{Y \in \mathfrak{s}(U); \mathrm{rk}(Y|_{U_a} : U_a \to U_b) \geq m_b - d_a,\ \mathrm{rk}(Y|_{U_b} : U_b \to U_a) \geq m_a - d_b\}.$$

Then $\mathfrak{g}_i'$ (resp. $\mathfrak{s}(U)'$) is an open subvariety of $\mathfrak{g}_i$ (resp. $\mathfrak{s}(U)$) and $\mathfrak{s}(V)'$ is a locally closed subvariety of $\mathfrak{s}(V)$ which is open in $\rho(\mathfrak{g}_i)$. We have the following:

Proposition 2.6 (cf. [[O2], Lemma 9]). (i) $\pi(\mathfrak{g}_i') = \mathfrak{s}(U)'$ *and* $\rho(\mathfrak{g}_i') = \mathfrak{s}(V)'$.

(ii) *The restriction* $\rho|_{\mathfrak{g}_i'} : \mathfrak{g}_i' \to \mathfrak{s}(V)'$ *is locally trivial in the classical topology with typical fibre isomorphic to* $K(U)$.

(iii) $\pi|_{\mathfrak{g}_i'} : \mathfrak{g}_i' \to \mathfrak{s}(U)'$ *is smooth.*

Proof. (i) follows from elementary computation of linear algebra. The proofs of (ii) and the smoothness of (iii) are similar to that of [[KrP1], Lemma 5.2]. □

Remark 2.7. Let $f : X \to Y$ be a smooth morphism of complex varieties of relative dimension r and $f(x) = y$ $(x \in X)$. Then some neighbourhoods (in the classical topology) of $x \in X$ and $(y, 0) \in Y \times \mathbb{C}^r$ are analytically isomorphic (cf. [[KrP2], 12.2]).

Let us consider an ab-diagram

$$\mathbf{d} = \begin{matrix} a & \uparrow \\ \vdots & d_a \\ a & \downarrow \\ b & \uparrow \\ \vdots & d_b \\ b & \downarrow \end{matrix}$$

with a single column, and subsets of signed-diagrams:

$$\begin{aligned} D(n_a, m_b, n_b, m_a)' &:= \{\mu \in D(n_a, m_b, n_b, m_a); \text{each row of} \\ &\qquad \mu \text{ starts with} \pm 1 \text{ and ends with} \pm 1\} \\ D(n_a, n_b)' &:= \{\eta \in D(n_a, n_b);\ \text{first column of } \eta \text{ coincides with } \mathbf{d}\} \\ D(m_a, m_b)' &:= \{\sigma \in D(m_a, m_b); n_a(\sigma_1) \le d_b,\ n_b(\sigma_1) \le d_a\}, \end{aligned}$$

where σ_1 denotes the first column of σ. For $\sigma \in D(m_a, m_b)$, we easily see that $\sigma \in D(m_a, m_b)'$ if and only if there exists $\eta \in D(n_a, n_b)$ such that $\eta' = \sigma$ and the first column of η coincides with $\mathbf{d}$. We write $\mathcal{N}(\mathfrak{g}_i')$ (resp. $\mathcal{N}(\mathfrak{s}(V)')$, $\mathcal{N}(\mathfrak{s}(U)')$) the set of nilpotent elements in $\mathfrak{g}_i'$ (resp. $\mathfrak{s}(V)'$, $\mathfrak{s}(U)'$). Then we have the following

Lemma 2.8. (i) *For a nilpotent orbit* $\mathcal{O}_\mu \in \mathcal{N}(\mathfrak{g}_i)/K(V) \times K(U)$ $(\mu \in D(n_a, m_b, n_b, m_a))$, $\mathcal{O}_\mu \subset \mathfrak{g}_i'$ *if and only if* $\mu \in D(n_a, m_b, n_b, m_a)'$;

$$\mathcal{N}(\mathfrak{g}_i')/K(V) \times K(U) \simeq D(n_a, m_b, n_b, m_a)'.$$

(ii) *For* $C_\eta \in \mathcal{N}(\mathfrak{s}(V))/K(V)$ $(\eta \in D(n_a, n_b))$, $C_\eta \subset \mathfrak{s}(V)'$ *if and only if* $\eta \in D(n_a, n_b)'$;

$$\mathcal{N}(\mathfrak{s}(V)')/K(V) \simeq D(n_a, n_b)'.$$

(iii) *For* $C_\sigma \in \mathcal{N}(\mathfrak{s}(U))/K(U)$ $(\sigma \in D(m_a, m_b))$, $C_\sigma \subset \mathfrak{s}(U)'$ *if and only if* $\sigma \in D(m_a, m_b)'$;

$$\mathcal{N}(\mathfrak{s}(U)')/K(U) \simeq D(m_a, m_b)'.$$

Proof. For

$$(P,Q)=\left(\begin{array}{ccccc} & & Q_a & & \\ & V_a & \rightarrow & U_b & \\ P_a & \uparrow & & \downarrow & P_b \\ & U_a & \leftarrow & V_b & \\ & & Q_b & & \end{array}\right)\in \mathcal{O}_\mu,$$

we see

$$Q \text{ is surjective } \Leftrightarrow \text{each row of } \mu \text{ starts with } \pm 1$$

and

$$P \text{ is injective } \Leftrightarrow \text{ each row of } \mu \text{ ends with } \pm 1.$$

Hence (i) follows.

For $\eta \in D(n_a, n_b)$, we write η_1 the first column of η. Then for $X \in C_\eta$, since $\mathrm{rk}(X|_{V_a} : V_a \to V_b) = n_b(\eta')$ and $\mathrm{rk}(X|_{V_b} : V_b \to V_a) = n_a(\eta')$, we have

$$C_\eta \subset \mathfrak{s}(V)'$$

$$\Leftrightarrow n_b(\eta') = m_b,\, n_a(\eta') = m_a$$

$$\Leftrightarrow n_a(\eta_1) = n_a - m_a = d_a,\, n_b(\eta_1) = n_b - m_b = d_b \Leftrightarrow \eta_1 = \mathbf{d}.$$

Hence (ii) follows.

For $Y \in C_\sigma$, since $\mathrm{rk}(Y|_{U_a} : U_a \to U_b) = n_b(\sigma')$ and $\mathrm{rk}(Y|_{U_b} : U_b \to U_a) = n_a(\sigma')$, we have $n_a(\sigma_1) = m_a - n_a(\sigma')$ and $n_b(\sigma_1) = m_b - n_b(\sigma')$. Then

$$C_\sigma \subset \mathfrak{s}(U)'$$

$$\Leftrightarrow n_b(\sigma') \geq m_b - d_a \text{ and } n_a(\sigma') \geq m_a - d_b$$

$$\Leftrightarrow n_a(\sigma_1) \leq d_b,\, n_b(\sigma_1) \leq d_a$$

Hence (iii) follows. □

Theorem 2.9. (i) $\rho|_{\mathcal{N}(\mathfrak{g}_i')} : \mathcal{N}(\mathfrak{g}_i') \to \mathcal{N}(\mathfrak{s}(V)')$ *is locally trivial in the classical topology with typical fibre isomorphic to* $K(U)$.

(ii) $\pi|_{\mathcal{N}(\mathfrak{g}_i')} : \mathcal{N}(\mathfrak{g}_i') \to \mathcal{N}(\mathfrak{s}(U)')$ *is smooth.*

(iii) *There exists bijections*

$$\begin{array}{ccccc} \mathcal{N}(\mathfrak{s}(V)')/K(V) & \overset{\rho}{\xleftarrow{\sim}} & \mathcal{N}(\mathfrak{g}_i')/K(V)\times K(U) & \overset{\pi}{\xrightarrow{\sim}} & \mathcal{N}(\mathfrak{s}(U)')/K(U) \\ \downarrow \wr & \rho & \downarrow \wr & \pi & \downarrow \wr \\ D(n_a,n_b)' & \xleftarrow{\sim} & D(n_a,m_b,n_b,m_a)' & \xrightarrow{\sim} & D(m_a,m_b)' \end{array}.$$

(iv) *The bijections in the first row of* (iii) *preserve the closure relation. That is, for* $\mathcal{O}_{\mu_j} \in \mathcal{N}(\mathfrak{g}_i')/K(V)\times K(U)$ $(j=1,2)$ *and the corresponding orbits* $C_{\rho(\mu_j)} = \rho(\mathcal{O}_{\mu_j}) \in \mathcal{N}(\mathfrak{s}(V)')/K(V)$, $C_{\pi(\mu_j)} = \pi(\mathcal{O}_{\mu_j}) \in \mathcal{N}(\mathfrak{s}(U)')/K(U)$ *respectively, we have*

$$\overline{C_{\rho(\mu_1)}} \supset C_{\rho(\mu_2)} \Longleftrightarrow \overline{\mathcal{O}_{\mu_1}} \supset \mathcal{O}_{\mu_2} \Longleftrightarrow \overline{C_{\pi(\mu_1)}} \supset C_{\pi(\mu_2)}.$$

Proof. (i) Since $\rho|_{\mathfrak{g}_i'} : \mathfrak{g}_i' \to \mathfrak{s}(V)'$ is locally trivial and $(\rho|_{\mathfrak{g}_i'})^{-1}(\mathcal{N}(\mathfrak{s}(V)') = \mathcal{N}(\mathfrak{g}_i')$, (i) follows.

(ii) Since $(\pi|_{\mathfrak{g}_i'})^{-1}(\mathcal{N}(\mathfrak{s}(U)')) = \mathcal{N}(\mathfrak{g}_i')$,

$$\begin{array}{ccc} \mathcal{N}(\mathfrak{g}_i') & \hookrightarrow & \mathfrak{g}_i' \\ \pi|_{\mathcal{N}(\mathfrak{g}_i')} \downarrow & & \downarrow \pi|_{\mathfrak{g}_i'} \\ \mathcal{N}(\mathfrak{s}(U)') & \hookrightarrow & \mathfrak{s}(U)' \end{array}$$

is a fibre product. Since $\pi|_{\mathfrak{g}_i'} : \mathfrak{g}_i' \to \mathfrak{s}(U)'$ is smooth, so is $\pi|_{\mathcal{N}(\mathfrak{g}_i')} : \mathcal{N}(\mathfrak{g}_i') \to \mathcal{N}(\mathfrak{s}(U)')$.

(iii) The subjectivities of

$$D(n_a, n_b)' \xleftarrow{\rho} D(n_a, m_b, n_b, m_a)' \xrightarrow{\pi} D(m_a, m_b)'$$

follow from Proposition 2.6, (i).

For $\eta \in D(n_a, n_b)'$, since η has $d_a + d_b$ rows, we write η as a sum of rows;

$$\eta = \eta_1 + \eta_2 + \cdots + \eta_{d_a} + \eta_{d_a+1} + \cdots + \eta_{d_a+d_b},$$

where each η_j $(1 \le j \le d_a)$ starts with a and each η_j $(d_a + 1 \le j \le d_a + d_b)$ starts with b. For each η_j, we define an $\langle i\rangle$-signed diagram $\tilde{\eta}_j$ with a single row as follows:

$$\begin{aligned}
\eta_j &= \overbrace{ab\cdots ab}^{2k} \to \tilde{\eta}_j = \overbrace{1\, i\, {-1} \cdots 1\, i\, {-1}}^{4k-1}, \\
\eta_j &= \overbrace{ab\cdots ba}^{2k+1} \to \tilde{\eta}_j = \overbrace{1\, i\, {-1} \cdots {-1}\, {-i}\, 1}^{4k+1}, \\
\eta_j &= \overbrace{ba\cdots ba}^{2k} \to \tilde{\eta}_j = \overbrace{{-1}\, {-i}\, 1 \cdots {-1}\, {-i}\, 1}^{4k-1}, \\
\eta_j &= \overbrace{ba\cdots ab}^{2k+1} \to \tilde{\eta}_j = \overbrace{{-1}\, {-i}\, 1 \cdots 1\, i\, {-1}}^{4k+1}.
\end{aligned}$$

As the sum of $\tilde{\eta}_j$ $(1 \le j \le d_a + d_b)$, we obtain an $\langle i\rangle$-signed diagram

$$\tilde{\eta} = \tilde{\eta}_1 + \tilde{\eta}_2 + \cdots + \tilde{\eta}_{d_a} + \tilde{\eta}_{d_a+1} + \cdots + \tilde{\eta}_{d_a+d_b}.$$

Then it is easy to see that

$$\begin{aligned} n_i(\tilde{\eta}_j) &= n_b(\eta_j),\ n_{-i}(\tilde{\eta}_j) = n_a(\eta_j) - 1 \quad (1 \le j \le d_a), \\ n_{-i}(\tilde{\eta}_j) &= n_a(\eta_j),\ n_i(\tilde{\eta}_j) = n_b(\eta_j) - 1 \quad (d_a + 1 \le j \le d_a + d_b). \end{aligned}$$

Thus we have

$$\begin{aligned} n_i(\tilde{\eta}) &= \sum_{j=1}^{d_a} n_i(\tilde{\eta}_j) + \sum_{j=d_a+1}^{d_a+d_b} n_i(\tilde{\eta}_j) \\ &= \sum_{j=1}^{d_a} n_b(\eta_j) + \sum_{j=d_a+1}^{d_a+d_b} \{n_b(\eta_j) - 1\} = n_b - d_b = m_b. \end{aligned}$$

Similarly we have $n_{-i}(\tilde{\eta}) = m_a$, and hence $\tilde{\eta} \in D(n_a, m_b, n_b, m_a)'$. It is clear that $\rho(\tilde{\eta}) = \eta$. If $\rho(\mu) = \eta$ for $\mu \in D(n_a, m_b, n_b, m_a)$, μ must contain $\tilde{\eta}$ in part and hence $\mu = \tilde{\eta}$. Therefore $\rho : D(n_a, m_b, n_b, m_a)' \to D(n_a, n_b)'$ is injective.

For $\sigma \in D(m_a, m_b)'$, as before we write

$$\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_{d_a} + \sigma_{d_a+1} + \cdots + \sigma_{d_a+d_b}$$

as a sum of rows such that each σ_j $(1 \le j \le d_a)$ is empty or starts with b and each σ_j $(d_a + 1 \le j \le d_a + d_b)$ is empty or starts with a. For each σ_j, we define an $\langle i \rangle$-signed diagram $\tilde{\sigma}_j$ with a single row as follows:

$$\begin{aligned} \sigma_j &= \overbrace{ba \cdots ba}^{2k} \ (1 \le j \le d_a,\ k \ge 0) \ \to \tilde{\sigma}_j = \overbrace{1\, i\, {-1}\, {-i} \cdots i\, {-1}\, {-i}\, 1}^{4k+1}, \\ \sigma_j &= \overbrace{ba \cdots bab}^{2k+1} \ (1 \le j \le d_a,\ k \ge 0) \ \to \tilde{\sigma}_j = \overbrace{1\, i\, {-1}\, {-i} \cdots {-i}\, 1\, i\, {-1}}^{4k+3}, \\ \sigma_j &= \overbrace{ab \cdots ab}^{2k} \ (d_a + 1 \le j \le d_a + d_b,\ k \ge 0) \ \to \tilde{\sigma}_j = \overbrace{-1\, {-i}\, 1\, i \cdots {-i}\, 1\, i\, {-1}}^{4k+1}, \\ \sigma_j &= \overbrace{ab \cdots ba}^{2k+1} \ (d_a + 1 \le j \le d_a + d_b,\ k \ge 0) \ \to \tilde{\sigma}_j = \overbrace{-1\, {-i}\, 1\, i \cdots i\, {-1}\, {-i}\, 1}^{4k+3}. \end{aligned}$$

As the sum of $\tilde{\sigma}_j$ $(1 \le j \le d_a + d_b)$, we obtain an $\langle i \rangle$-signed diagram

$$\tilde{\sigma} = \tilde{\sigma}_1 + \tilde{\sigma}_2 + \cdots + \tilde{\sigma}_{d_a} + \tilde{\sigma}_{d_a+1} + \cdots + \tilde{\sigma}_{d_a+d_b}.$$

Then it is easy to see that

$$\begin{aligned} n_1(\tilde{\sigma}_j) &= n_a(\sigma_j) + 1,\ n_{-1}(\tilde{\sigma}_j) = n_b(\sigma_j) \quad (1 \le j \le d_a), \\ n_{-1}(\tilde{\sigma}_j) &= n_b(\sigma_j) + 1,\ n_1(\tilde{\sigma}_j) = n_a(\sigma_j) \quad (d_a + 1 \le j \le d_a + d_b). \end{aligned}$$

Thus we have

$$\begin{aligned} n_1(\tilde{\sigma}) &= \sum_{j=1}^{d_a}\{n_a(\sigma_j)+1\} + \sum_{j=d_a+1}^{d_a+d_b} n_a(\sigma_j) \\ &= \sum_{j=1}^{d_a+d_b} n_a(\sigma_j) + d_a = m_a + d_a = n_a, \\ n_{-1}(\tilde{\sigma}) &= \sum_{j=1}^{d_a} n_b(\sigma_j) + \sum_{j=d_a+1}^{d_a+d_b}\{n_b(\sigma_j)+1\} \\ &= \sum_{j=1}^{d_a+d_b} n_b(\sigma_j) + d_b = m_b + d_b = n_b. \end{aligned}$$

Hence $\tilde{\sigma} \in D(n_a, m_b, n_b, m_a)'$ and $\pi(\tilde{\sigma}) = \sigma$. It is easy to see that $\tilde{\sigma}$ is the unique element of $D(n_a, m_b, n_b, m_a)'$ which maps onto σ via π. Therefore $\pi : D(n_a, m_b, n_b, m_a)' \to D(m_a, m_b)'$ is injective.

(iv) Since ρ is quotient map under $K(U)$, $\rho(\overline{\mathcal{O}_{\mu_1}})$ is closed ([MF, Chap. 1, §2]) and hence

$$\rho(\overline{\mathcal{O}_{\mu_1}}) = \overline{\rho(\mathcal{O}_{\mu_1})} = \overline{C_{\rho(\mu_1)}}.$$

Therefore, if $\overline{\mathcal{O}_{\mu_1}} \supset \mathcal{O}_{\mu_2}$, we have

$$\overline{C_{\rho(\mu_1)}} = \rho(\overline{\mathcal{O}_{\mu_1}}) \supset \rho(\mathcal{O}_{\mu_2}) = C_{\rho(\mu_2)}.$$

Conversely, suppose that $\overline{C_{\rho(\mu_1)}} \supset C_{\rho(\mu_2)}$. Take $z \in \mathcal{O}_{\mu_2}$ and put $z_1 := \rho(z) \in C_{\rho(\mu_2)}$. Since $\rho|_{\mathcal{N}(\mathfrak{g}_i')} : \mathcal{N}(\mathfrak{g}_i') \to \mathcal{N}(\mathfrak{s}(V)')$ is smooth of relative dimension $r := \dim K(U)$, there exist neighborhoods (in the classical topology) N_z of z in $\mathcal{N}(\mathfrak{g}_i')$, N_{z_1} of z_1 in $\mathcal{N}(\mathfrak{s}(V)')$, N_O of O in $\mathbb{C}^r$ and an analytic isomorphism $\iota : N_z \to N_{z_1} \times N_O$ such that the diagram

$$\begin{array}{ccc} N_z & \xrightarrow{\iota} & N_{z_1} \times N_O \\ \rho \downarrow & \swarrow & p_1 \\ N_{z_1} & & \end{array}$$

is commutative, where p_1 is the projection to the first factor (cf. Remark 2.7). Then we easily see the implication $z_1 \in \overline{C_{\rho(\mu_1)}} \Rightarrow z \in \overline{\mathcal{O}_{\mu_1}}$. Therefore we obtain

$$\overline{C_{\rho(\mu_1)}} \supset C_{\rho(\mu_2)} \Leftrightarrow \overline{\mathcal{O}_{\mu_1}} \supset \mathcal{O}_{\mu_2}.$$

Similarly we have

$$\overline{\mathcal{O}_{\mu_1}} \supset \mathcal{O}_{\mu_2} \Leftrightarrow \overline{C_{\pi(\mu_1)}} \supset C_{\pi(\mu_2)}.$$

□

Remark 2.10. By the definitions of $D(n_a,n_b)'$ and $D(m_a,m_b)'$, the correspondence $\eta \mapsto \eta'$ defines a bijection

$$D(n_a,n_b)' \tilde{\to} D(m_a,m_b)'.$$

By the proof of Theorem 2.9, (iii), the bijection $D(n_a,n_b)' \tilde{\to} D(m_a,m_b)'$ defined in the second row of (iii) coincides with the above bijection:

$$\pi((\rho|_{D(n_a,m_b,n_b,m_a)'})^{-1}(\eta)) = \eta' \ (\eta \in D(n_a,n_b)').$$

Proposition 2.11. *For an ab-diagram $\eta \in D(n_a,n_b)'$, denote by $\tilde{\eta} \in D(n_a,m_b,n_b,m_a)'$ and $\eta' \in D(m_a,m_b)'$ the diagrams which correspond to η by the bijections of Theorem* 2.9, (iii) *respectively. We write $C_\eta \in \mathcal{N}(\mathfrak{s}(V)')/K(V)$, $\mathcal{O}_{\tilde{\eta}} \in \mathcal{N}(\mathfrak{g}_i')/K(V)\times K(U)$ and $C_{\eta'} \in \mathcal{N}(\mathfrak{s}(U)')/K(U)$ the corresponding nilpotent orbits respectively. Then we have*

(i) $\rho^{-1}(C_\eta) = \mathcal{O}_{\tilde{\eta}}$

(ii) $\pi(\rho^{-1}(C_\eta)) = C_{\eta'}$

(iii) $\rho(\pi^{-1}(\overline{C_{\eta'}})) = \overline{C_\eta}$

(iv) $\pi^{-1}(\overline{C_{\eta'}}) = \overline{\mathcal{O}_{\tilde{\eta}}}$. *In particular, $\pi^{-1}(\overline{C_{\eta'}})$ is irreducible.*

Proof. The proofs of (i), (ii) and (iii) are essentially the same as those of [O2, Lemma 10], hence we omit them.

(iv) Let $\mathcal{O} \subset \overline{\mathcal{O}_{\tilde{\eta}}}$ be a $K(V)\times K(U)$ orbit. Since π is a quotient map and $\overline{\mathcal{O}_{\tilde{\eta}}}$ is a $K(V)$-invariant closed subset, we have

$$\pi(\mathcal{O}) \subset \pi(\overline{\mathcal{O}_{\tilde{\eta}}}) = \overline{\pi(\mathcal{O}_{\tilde{\eta}})} = \overline{C_{\pi(\tilde{\eta})}} = \overline{C_{\eta'}}.$$

Hence $\mathcal{O} \subset \pi^{-1}(\overline{C_{\eta'}})$ and we have $\overline{\mathcal{O}_{\tilde{\eta}}} \subset \pi^{-1}(\overline{C_{\eta'}})$.

Next suppose that $\mathcal{O}_\mu \subset \pi^{-1}(\overline{C_{\eta'}})$ is a $K(V)\times K(U)$ orbit corresponding to a diagram $\mu \in D(n_a,m_b,n_b,m_a)$. Put $\sigma := \pi(\mu)$. Then we have $\sigma = \pi(\mu) \leq \pi(\tilde{\eta}) = \eta'$. Take $x \in \mathcal{O}_{\tilde{\eta}}$ and $y \in \mathcal{O}_\mu$. Write $X := \pi(x) = x^2|_U \in C_{\eta'}$ and $Y := \pi(y) = y^2|_U \in C_\sigma$. To show that $\mu \leq \tilde{\eta}$, it is sufficient to show that

$$n_\delta(\tilde{\eta}^{(k)}) = \mathrm{rk}(W_{i^{-k}\delta} \xrightarrow{x^k} W_\delta) \geq n_\delta(\mu^{(k)}) = \mathrm{rk}(W_{i^{-k}\delta} \xrightarrow{y^k} W_\delta)$$

for any $\delta \in \langle i \rangle$ and $k \geq 1$.

If $k = 2\ell$ is even, we have

$$\begin{aligned} n_i(\tilde{\eta}^{(2\ell)}) &= \mathrm{rk}(W_{i^{-2\ell}i} \xrightarrow{x^{2\ell}} W_i) = \mathrm{rk}(W_{(-1)^\ell i} \xrightarrow{x^{2\ell}} U_b) \\ &= \mathrm{rk}(W_{(-1)^\ell i} \xrightarrow{X^\ell} U_b) = n_b((\eta')^{(\ell)}). \\ n_i(\mu^{(2\ell)}) &= \mathrm{rk}(W_{i^{-2\ell}i} \xrightarrow{y^{2\ell}} W_i) = \mathrm{rk}(W_{(-1)^\ell i} \xrightarrow{y^{2\ell}} U_b) \\ &= \mathrm{rk}(W_{(-1)^\ell i} \xrightarrow{Y^\ell} U_b) = n_b(\sigma^{(\ell)}). \end{aligned}$$

Since $\eta' \geq \sigma$, we have

$$n_i(\tilde{\eta}^{(2\ell)}) = n_b((\eta')^{(\ell)}) \geq n_b(\sigma^{(\ell)}) = n_i(\mu^{(2\ell)}).$$

Similarly we have $n_{-i}(\tilde{\eta}^{(2\ell)}) \geq n_{-i}(\mu^{(2\ell)})$.

We also have

$$\begin{aligned} n_1(\tilde{\eta}^{(2\ell)}) &= \mathrm{rk}(W_{i^{-2\ell}} \xrightarrow{x^{2\ell}} W_1) = \mathrm{rk}(W_{(-1)^\ell} \xrightarrow{x} W_{(-1)^\ell i} \xrightarrow{x^{2(\ell-1)}} U_a \xhookrightarrow{x} V_a) \\ &= \mathrm{rk}(W_{(-1)^\ell i} \xrightarrow{x^{2(\ell-1)}} U_a) = \mathrm{rk}(W_{(-1)^\ell i} \xrightarrow{X^{\ell-1}} U_a) = n_a((\eta')^{(\ell-1)}), \\ n_1(\mu^{(2\ell)}) &= \mathrm{rk}(W_{(-1)^\ell} \xrightarrow{y} W_{(-1)^\ell i} \xrightarrow{y^{2(\ell-1)}} U_a \xrightarrow{y} V_a) \\ &\leq \mathrm{rk}(W_{(-1)^\ell i} \xrightarrow{Y^{\ell-1}} U_a) = n_a(\sigma^{(\ell-1)}). \end{aligned}$$

Since $\eta' \geq \sigma$, we have

$$n_1(\tilde{\eta}^{(2\ell)}) = n_a((\eta')^{(\ell-1)}) \geq n_a(\sigma^{(\ell-1)}) \geq n_1(\mu^{(2\ell)}).$$

Similarly we have $n_{-1}(\tilde{\eta}^{(2\ell)}) \geq n_{-1}(\mu^{(2\ell)})$.

Suppose that $k = 2\ell+1$ is odd. Then by the similar computation as above, we have

$$\begin{aligned} n_1(\tilde{\eta}^{(2\ell+1)}) &= n_a((\eta')^{(\ell)}) \geq n_a(\sigma^{(\ell)}) \geq n_1(\mu^{(2\ell+1)}), \\ n_{-1}(\tilde{\eta}^{(2\ell+1)}) &= n_b((\eta')^{(\ell)}) \geq n_b(\sigma^{(\ell)}) \geq n_{-1}(\mu^{(2\ell+1)}), \\ n_i(\tilde{\eta}^{(2\ell+1)}) &= n_b((\eta')^{(\ell)}) \geq n_b(\sigma^{(\ell)}) \geq n_i(\mu^{(2\ell+1)}), \\ n_{-i}(\tilde{\eta}^{(2\ell+1)}) &= n_a((\eta')^{(\ell)}) \geq n_a(\sigma^{(\ell)}) \geq n_{-i}(\mu^{(2\ell+1)}). \end{aligned}$$

Therefore we have $\tilde{\eta} \geq \mu$ and hence $\mathcal{O}_\mu \subset \overline{\mathcal{O}_{\tilde{\eta}}}$ by Theorem 1.4. Thus we have $\pi^{-1}(\overline{C_{\eta'}}) \subset \overline{\mathcal{O}_{\tilde{\eta}}}$. □

Remark 2.12. In the setting of Proposition 2.11,

(i) $\pi^{-1}(C_{\eta'})$ is not a single $K(V) \times K(U)$-orbit in general.

(ii) It holds $\pi(\rho^{-1}(\overline{C_\eta})) \supset \overline{C_{\eta'}}$ but the equality does not holds in general.

Example. Let us consider the case when $n_a = 5$, $n_b = 3$, $m_a = m_b = 2$. Thus $d_a = 5 - 2 = 3$ and $d_b = 3 - 2 = 1$. For

$$\eta = \begin{array}{l} a\ b\ a \\ a\ b \\ b\ a \\ a \end{array} \in D(5,3),$$

since the number of a's (resp. b's) in the first column of η is $3 = d_a$ (resp. $1 = d_b$), $\eta \in D(5,3)'$. Then

$$\tilde{\eta} = \begin{array}{lllll} 1 & i & -1 & -i & 1 \\ 1 & i & -1 & & \\ -1 & -i & 1 & & \\ 1 & & & & \end{array} \in D(5,2,3,2)'$$

is the unique element $\mu \in D(5,2,3,2)$ such that $\rho(\mu) = \eta$. Hence $\rho^{-1}(C_\eta) = \mathcal{O}_{\tilde{\eta}}$. Clearly

$$\pi(\tilde{\eta}) = \eta' = \begin{array}{ll} b & a \\ b & \\ a & \end{array} \in D(2,2)'$$

is the ab-diagram which we obtain from η by erasing first column. We see that

$$\tilde{\eta}, \quad \begin{array}{lll} i & -1 & -i \\ 1 & i & -1 \\ -1 & -i & 1 \\ 1 & & \\ 1 & & \\ 1 & & \end{array} \in \pi^{-1}(\eta')$$

and hence $\pi^{-1}(C_{\eta'})$ is not a single $K(V) \times K(U)$-orbit. Take

$$\sigma = \begin{array}{lll} a & b & a \\ a & & \\ a & & \\ a & & \\ b & & \\ b & & \end{array} \in D(5,3) \setminus D(5,3)'.$$

We easily see $\sigma \leq \eta$ ($\Leftrightarrow C_\sigma \subset \overline{C_\eta}$). Since

$$\mu := \begin{array}{lllllll} -i & 1 & i & -1 & -i & 1 & i \\ 1 & & & & & & \\ 1 & & & & & & \\ 1 & & & & & & \\ -1 & & & & & & \\ -1 & & & & & & \end{array} \in \rho^{-1}(\sigma)$$

and $\pi(\mu) = abab \in D(2,2)$, we have

$$\pi(\rho^{-1}(\overline{C_\eta})) \supset C_{\pi(\mu)} \not\subset \overline{C_{\eta'}}.$$

Thus $\pi(\rho^{-1}(\overline{C_\eta})) \neq \overline{C_{\eta'}}$.

Definition 2.13 ([KrP2]). Consider two varieties X, Y and $x \in X$, $y \in Y$. The singularity of X at x is said to be smoothly equivalent to that of Y at y if there exists a variety Z, a point $z \in Z$ and two morphisms $Y \overset{\psi}{\leftarrow} Z \overset{\varphi}{\rightarrow} X$ such that $\varphi(z) = x$, $\psi(z) = y$ and φ, ψ are smooth at z. This clearly defines an equivalence relation among pointed varieties (X, x). We denote by $Sing(X, x)$ the equivalence class to which (X, x) belongs.

Suppose that an algebraic group G acts on a variety X. For a G-orbit $\mathcal{O}$ of X, the equivalence class $Sing(X, x)$ is independent of the choice of $x \in \mathcal{O}$. We denote this equivalence class by $Sing(X, \mathcal{O})$.

Theorem 2.14. *For two $\langle i\rangle$-signed diagrams $\tilde{\eta}, \tilde{\sigma} \in D(n_a, m_b, n_b, m_a)'$, let $\eta = \rho(\tilde{\eta}), \sigma = \rho(\tilde{\sigma}) \in D(n_a, n_b)'$ and $\eta' = \pi(\tilde{\eta}), \sigma' = \pi(\tilde{\sigma}) \in D(m_a, m_b)'$ the corresponding ab-diagrams by the bijections of Theorem* 2.9, (iii)*. Suppose that $\mathcal{O}_{\tilde{\sigma}} \subset \overline{\mathcal{O}_{\tilde{\eta}}}$. Hence $C_\sigma \subset \overline{C_\eta}$ and $C_{\sigma'} \subset \overline{C_{\eta'}}$ by Theorem* 2.9*. Then we have*

$$Sing(\overline{C_\eta}, C_\sigma) = Sing(\overline{\mathcal{O}_{\tilde{\eta}}}, \mathcal{O}_{\tilde{\sigma}}) = Sing(\overline{C_{\eta'}}, C_{\sigma'}).$$

Proof. Since π and ρ are quotient maps and $\overline{\mathcal{O}_{\tilde{\eta}}}$ is a $K(V) \times K(U)$-invariant closed subset of $\mathfrak{g}_i$, we have

$$\pi(\overline{\mathcal{O}_{\tilde{\eta}}}) = \overline{\pi(\mathcal{O}_{\tilde{\eta}})} = \overline{C_{\eta'}},\ \rho(\overline{\mathcal{O}_{\tilde{\eta}}}) = \overline{\rho(\mathcal{O}_{\tilde{\eta}})} = \overline{C_\eta}$$

and we obtain morphisms

$$\overline{C_\eta} \overset{\rho}{\leftarrow} \overline{\mathcal{O}_{\tilde{\eta}}} \overset{\pi}{\rightarrow} \overline{C_{\eta'}}.$$

Since $\rho(\mathcal{O}_{\tilde{\sigma}}) = C_\sigma$ and $\pi(\mathcal{O}_{\tilde{\sigma}}) = C_{\sigma'}$, it is sufficient to show that $\rho|_{\overline{\mathcal{O}_{\tilde{\eta}}}}$ and $\pi|_{\overline{\mathcal{O}_{\tilde{\eta}}}}$ are smooth at a point $Y \in C_{\tilde{\sigma}}$. Since $\pi|_{\mathfrak{g}_i'} : \mathfrak{g}_i' \to \mathfrak{s}(U)'$ is smooth and

$$\begin{array}{ccc} \mathfrak{g}_i' \cap \pi^{-1}(\overline{C_{\eta'}} \cap \mathfrak{s}(U)') & \hookrightarrow & \mathfrak{g}_i' \\ \pi \downarrow & & \downarrow \pi \\ \overline{C_{\eta'}} \cap \mathfrak{s}(U)' & \hookrightarrow & \mathfrak{s}(U)' \end{array}$$

is a fibre product,

$$\mathfrak{g}_i' \cap \pi^{-1}(\overline{C_{\eta'}} \cap \mathfrak{s}(U)') \overset{\pi}{\rightarrow} \overline{C_{\eta'}} \cap \mathfrak{s}(U)'$$

is also smooth. By Theorem 2.9, (iii) and (iv), we have

$$\mathfrak{g}_i' \cap \pi^{-1}(\overline{C_{\eta'}} \cap \mathfrak{s}(U)') = \overline{\mathcal{O}_{\tilde{\eta}}} \cap \mathfrak{g}_i',$$

and hence

$$\overline{\mathcal{O}_{\tilde{\eta}}} \cap \mathfrak{g}_i' \xrightarrow{\pi} \overline{C_{\eta'}} \cap \mathfrak{s}(U)'$$

is smooth. Since $Y \in \mathcal{O}_{\tilde{\sigma}} \subset \overline{\mathcal{O}_{\tilde{\eta}}} \cap \mathfrak{g}_i'$, $\pi|_{\overline{\mathcal{O}_{\tilde{\eta}}}}$ is smooth at Y. Similarly, we can show that $\rho|_{\overline{\mathcal{O}_{\tilde{\eta}}}}$ is smooth at $Y \in \mathcal{O}_{\tilde{\sigma}}$. □

Remark 2.15. (i) Let us consider the condition $\mathfrak{s}(U) = \mathfrak{s}(U)'$, that is, $\mathfrak{s}(U)$ coincides with the image of the smooth morphism $\pi|_{\mathfrak{g}_i'} : \mathfrak{g}_i' \to \mathfrak{s}(U)'$. Then we have

$\mathfrak{s}(U) = \mathfrak{s}(U)'$
$\Leftrightarrow m_b - d_a \leq 0,\ m_a - d_b \leq 0$
$\Leftrightarrow m_a + m_b \leq \min\{n_a, n_b\}$
$\Leftrightarrow$ The dual pair $(U(n_a, n_b), U(m_a, m_b))$ is in the stable range.

(ii) When the dual pair $(U(n_a, n_b), U(m_a, m_b))$ is in the stable range, K. Nishiyama showed that for $C_2 \in \mathcal{N}(\mathfrak{s}(U))/K(U)$ ($\mathfrak{s}(U) = \mathfrak{s}(U)'$), there exists $C_1 \in \mathcal{N}(\mathfrak{s}(V))/K(V)$ such that

$$\rho(\pi^{-1}(\overline{C_2})) = \overline{C_1}$$

and called the correspondence $C_2 \mapsto C_1$ the θ-lifting. By Proposition 2.11, (iii), this correspondence coincides with the bijection

$$\mathcal{N}(\mathfrak{s}(U)')/K(U) \xrightarrow{\sim} \mathcal{N}(\mathfrak{s}(V)')/K(V)$$

given by Theorem 2.9, (iii) (the inverse of the map of Remark 2.10). Thus, in our general setting (Assumption 2.4), the above bijection is considered as a generalization of Nishiyama's θ-lifting.

(iii) Under Assumption 2.4, for any $C_2 \in \mathcal{N}(\mathfrak{s}(U)')/K(U)$, $\pi^{-1}(\overline{C_2})$ is a closure of a single $K(V)\times K(U)$-orbit in $\mathcal{N}(\mathfrak{g}_i')$ (Proposition 2.11, (iv)) and $\rho(\pi^{-1}(\overline{C_2}))$ is also a closure of a single $K(V)$-orbit C_1 in $\mathcal{N}(\mathfrak{s}(V))$. But if $C_2 \in [\mathcal{N}(\mathfrak{s}(U)) \setminus \mathcal{N}(\mathfrak{s}(U)')]/K(U)$, $\pi^{-1}(\overline{C_2})$ is not a closure of a single $K(V) \times K(U)$-orbit and $\rho(\pi^{-1}(\overline{C_2}))$ is not a closure of a single $K(V)$-orbit in general (see the following example) and hence the analogue of the main result of [DKP] does not hold in our case. Thus $\mathcal{N}(\mathfrak{s}(U)')/K(U)$ is considered as a domain on which a "good" correspondence

$$C_2 \mapsto C_1 \quad (\text{generalization of } \theta - \text{lifting})$$

is defined.

Example. Let us consider the case when $n_a = 2$, $n_b = 1$, $m_a = m_b = 1$. For a diagram η in $D(2,1)$, $D(2,1,1,1)$ or $D(1,1)$, let us denote also by η the corresponding orbit in $\mathcal{N}(\mathfrak{s}(V))/K(V)$, $\mathcal{N}(\mathfrak{g}_i)/K(V) \times K(U)$ or $\mathcal{N}(\mathfrak{s}(U))/K(U)$. We have the following:

(1) $\mathcal{N}(\mathfrak{s}(V)')/K(V) = \{a\ \ b\ \ a\}$, $\mathcal{N}(\mathfrak{g}_i')/K(V) \times K(U) = \{1\ \ i\ \ -1\ \ -i\ \ 1\}$, $\mathcal{N}(\mathfrak{s}(U)')/K(U) = \{b\ \ a\}$.
The correspondence of Theorem 2.9 is given by

$$a\ \ b\ \ a \leftarrow 1\ \ i\ \ -1\ \ -i\ \ 1 \rightarrow b\ \ a.$$

By Proposition 2.11, we have

$$\pi^{-1}\left(\overline{b\ \ a}\right) = \overline{1\ \ i\ \ -1\ \ -i\ \ 1},\ \ \rho(\pi^{-1}\left(\overline{b\ \ a}\right)) = \overline{a\ \ b\ \ a}.$$

(2) $[\mathcal{N}(\mathfrak{s}(U)) \setminus \mathcal{N}(\mathfrak{s}(U)')]/K(U) = \{a\ \ b, \begin{smallmatrix} a \\ b \end{smallmatrix}\}$.
By Proposition 2.3 and Theorem 1.4, we have

$$\pi^{-1}\left(\overline{a\ \ b}\right) = \overline{\begin{matrix} -1 & -i & 1 & i \\ 1 & & & \end{matrix}} \cup \overline{\begin{matrix} -i & 1 & i & -1 \\ 1 & & & \end{matrix}}, \rho\left(\pi^{-1}\left(\overline{a\ \ b}\right)\right) = \overline{\begin{matrix} b & a \\ a & \end{matrix}} \cup \overline{\begin{matrix} a & b \\ a & \end{matrix}},$$

$$\pi^{-1}\left(\overline{\begin{matrix} a \\ b \end{matrix}}\right) = \overline{\begin{matrix} -1 & -i & 1 \\ 1 & i & \end{matrix}} \cup \overline{\begin{matrix} 1 & i & -1 \\ -i & 1 & \end{matrix}}, \rho\left(\pi^{-1}\left(\overline{\begin{matrix} a \\ b \end{matrix}}\right)\right) = \overline{\begin{matrix} b & a \\ a & \end{matrix}} \cup \overline{\begin{matrix} a & b \\ a & \end{matrix}}.$$

§3. Relation Between $\mathfrak{s}(V) \overset{\rho|_{L_+}}{\longleftarrow} L_+ \overset{\pi|_{L_+}}{\longrightarrow} \mathfrak{s}(U)$ and the Moment Maps of the Dual Pair $(U(n_a, n_b), U(m_a, m_b))$

In this section, we give the reason why the maps $\mathfrak{s}(V) \overset{\rho|_{L_+}}{\longleftarrow} L_+ \overset{\pi|_{L_+}}{\longrightarrow} \mathfrak{s}(U)$ constructed in §2 can be interpreted as the $K_{\mathbb{C}}$-versions of the original real moment maps $\mathfrak{u}(n_a, n_b) \leftarrow L_{\mathbb{R}} \rightarrow \mathfrak{u}(m_a, m_b)$.

§3.1 Let V be a finite dimensional vector space and $(\ ,\ )_V$ a non-degenerate hermitian form on V;

$$(u, \alpha v)_V = \alpha(u, v)_V,\ (u, v)_V = \overline{(v, u)_V}\ (u, v \in V, \alpha \in \mathbb{C}).$$

Then we can take complex vector subspaces V_a and V_b of V such that

(a) $V = V_a \oplus V_b$

(b) $(V_a, V_b)_V = \{0\}$

(c) $(\ ,\ )_V|_{V_a}$ is positive definite and $(\ ,\ )_V|_{V_b}$ is negative definite.

We define a linear involution s_V of V by $s_V|_{V_a} = id_{V_a}$ and $s_V|_{V_b} = -id_{V_b}$. For $A \in \mathrm{End}V$, we define the adjoint $A^* \in \mathrm{End}V$ of A by

$$(Av_1, v_2)_V = (v_1, A^*v_2)_V \ (v_1, v_2 \in V).$$

Then we easily see the following:

Remark 3.1. (1) $(s_V v_1, v_2)_V = (v_1, s_V v_2)_V \ (v_1, v_2 \in V)$.
(2) $(s_V A s_V)^* = s_V A^* s_V \ (A \in \mathrm{End}V)$.

For the vector space (V, s_V) with involution, we use the notations $K(V)$, $\mathfrak{k}(V)$, $\mathfrak{s}(V)$ of (2.1). We define a real group $GL(V)_{\mathbb{R}}$ and its Lie algebra $\mathfrak{gl}(V)_{\mathbb{R}}$ by

$$GL(V)_{\mathbb{R}} = \{g \in GL(V); g^* = g^{-1}\},\ \mathfrak{gl}(V)_{\mathbb{R}} = \{X \in \mathfrak{gl}(V); X^* = -X\}.$$

Then $GL(V)_{\mathbb{R}} \simeq U(\dim V_a, \dim V_b)$ the indefinite unitary group. Clearly the restriction $\theta_V|_{GL(V)_{\mathbb{R}}}$ of $\theta_V : GL(V) \to GL(V)$ $(\theta_V(g) = s_V g s_V)$ to $GL(V)_{\mathbb{R}}$ is a Cartan involution of $GL(V)_{\mathbb{R}}$.

§3.2 Let V and U be two vector spaces with non-degenerate hermitian forms $(\ ,\ )_V$ and $(\ ,\ )_U$ respectively. Then

$$\left(\begin{pmatrix} v_1 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ u_2 \end{pmatrix} \right)_{V \oplus U} := (v_1, v_2)_V + (u_1, u_2)_U \quad (v_j \in V,\ u_j \in U)$$

is also a hermitian form on $V \oplus U$. We put $n_a = \dim V_a$, $n_b = \dim V_b$, $m_a = \dim U_a$, $m_b = \dim U_b$.

For $A \in \mathrm{Hom}(U, V)$ (resp. $A \in \mathrm{Hom}(V, U)$), we define the adjoint $A^* \in \mathrm{Hom}(V, U)$ (resp.$A^* \in \mathrm{Hom}(U, V)$) by

$(Au, v)_V = (u, A^*v)_U$ (resp. $(Av, u)_U = (v, A^*u)_V$) for $u \in U$ and $v \in V$.

Remark 3.2. For $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{End}(V \oplus U)$ ($A \in \mathrm{End}(V)$, $B \in \mathrm{Hom}(U, V)$, $C \in \mathrm{Hom}(V, U)$, $D \in \mathrm{End}(U)$), the adjoint X^* of X with respect to the hermitian form $(\ ,\ )_{V \oplus U}$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}.$$

We define a complex conjugation $\tau : GL(V\oplus U) \to GL(V\oplus U)$ by $\tau(g) = (g^*)^{-1}$ ($g \in GL(V \oplus U)$). We also denote by $\tau : \mathfrak{gl}(V \oplus U) \to \mathfrak{gl}(V \oplus U)$ the differential of τ. Then τ defines a real form

$$GL(V \oplus U)_{\mathbb{R}} = \{g \in GL(V \oplus U)_{\mathbb{R}}; \tau(g) = g\} \simeq U(n_a + m_a, n_b + m_b)$$

and its Lie algebra

$$\mathfrak{gl}(V \oplus U)_{\mathbb{R}} = \{X \in \mathfrak{gl}(V \oplus U); \tau(X) = X\}.$$

As in (2.2), let us consider a linear automorphism $S : V \oplus U \to V \oplus U$ by $S = \begin{pmatrix} s_V & 0 \\ 0 & -is_U \end{pmatrix}$. Now we define a bilinear from $(\ ,\ )_L$ on

$$\begin{aligned} L &:= \mathrm{Hom}_{\mathbb{C}}(U, V) \oplus \mathrm{Hom}_{\mathbb{C}}(V, U) \\ &= \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}; A \in \mathrm{Hom}_{\mathbb{C}}(U, V), B \in \mathrm{Hom}_{\mathbb{C}}(V, U) \right\} \\ &= \{X \in \mathfrak{gl}(V \oplus U); \mathrm{Ad}(S^2)X = -X\} \end{aligned}$$

by

$$\begin{aligned} \left(\begin{pmatrix} 0 & A_1 \\ B_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & A_2 \\ B_2 & 0 \end{pmatrix} \right)_L &:= i\{\mathrm{tr}_U(B_1A_2) - \mathrm{tr}_V(A_1B_2)\} \\ &= i\{\mathrm{tr}_V(A_2B_1) - \mathrm{tr}_V(A_1B_2)\}. \end{aligned}$$

Since τ stabilizes L, we can consider the real subspace

$$L_{\mathbb{R}} := \{X \in L; \tau(X) = X\} = \left\{ \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix}; A \in \mathrm{Hom}_{\mathbb{C}}(U, V) \right\}$$

of L whose dimension is $\dim_{\mathbb{R}} L_{\mathbb{R}} = \dim_{\mathbb{C}} L$. Then we have the following:

Lemma 3.3. (i) *For $z \in L$, $\tau(\mathrm{Ad}(S)z) = \mathrm{Ad}(S)(\tau(z))$.*

(ii) *$(\ ,\ )_L$ is a $GL(V) \times GL(U)$-invariant symplectic form on L.*

(iii) *$(\tau(z_1), \tau(z_2))_L = \overline{(z_1, z_2)_L}$ $(z_1, z_2 \in L)$.*
In particular, $(\ ,\ )_L$ is real valued on $L_{\mathbb{R}}$.

(iv) *$(\mathrm{Ad}(S)z_1, \mathrm{Ad}(S)z_2)_L = (z_1, z_2)_L$ $(z_1, z_2 \in L)$.*

(v) *For $z \in L_{\mathbb{R}}$, we have*

$$(\mathrm{Ad}(S)z, z)_L \le 0$$

and it holds $(\mathrm{Ad}(S)z, z)_L = 0$ if and only if $z = 0$. In particular, $(\ ,\ )_L|_{L_{\mathbb{R}}}$ is non-degenerate and so is $(\ ,\ )_L$.

Proof. (i) For $z = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$,

$$\begin{aligned}\tau(\mathrm{Ad}(S)z) &= \tau \begin{pmatrix} 0 & is_V A s_U \\ -is_U B s_V & 0 \end{pmatrix} = - \begin{pmatrix} 0 & (-is_U B s_V)^* \\ (is_V A s_U)^* & 0 \end{pmatrix} \\ &= - \begin{pmatrix} 0 & is_V B^* s_U \\ -is_U A s_V & 0 \end{pmatrix} = -\mathrm{Ad}(S) \begin{pmatrix} 0 & B^* \\ A^* & 0 \end{pmatrix} = \mathrm{Ad}(S)(\tau(z)).\end{aligned}$$

(ii) For $z_j = \begin{pmatrix} 0 & A_j \\ B_j & 0 \end{pmatrix} \in L$ $(j = 1, 2)$, since

$$\left(\begin{pmatrix} 0 & A_1 \\ B_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & A_2 \\ B_2 & 0 \end{pmatrix} \right)_L = i\{\mathrm{tr}_V(A_2 B_1) - \mathrm{tr}_V(A_1 B_2)\},$$

$(\ ,\ )_L$ is symplectic. It is clearly $GL(V) \times GL(U)$-invariant.
(iii) For the above z_j, we have

$$\begin{aligned}(\tau(z_1), \tau(z_2))_L &= \left(- \begin{pmatrix} 0 & B_1^* \\ A_1^* & 0 \end{pmatrix}, - \begin{pmatrix} 0 & B_2^* \\ A_2^* & 0 \end{pmatrix} \right)_L \\ &= i\{\mathrm{tr}_V(B_2^* A_1^*) - \mathrm{tr}_V(B_1^* A_2^*)\} \\ &= i\{\mathrm{tr}_V((A_1 B_2)^*) - \mathrm{tr}_V((A_2 B_1)^*)\} \\ &= i\{\overline{\mathrm{tr}_V(A_1 B_2)} - \overline{\mathrm{tr}_V(A_2 B_1)}\} \\ &= \overline{-i\{\mathrm{tr}_V(A_1 B_2) - \mathrm{tr}_V(A_2 B_1)\}} \\ &= \overline{(z_1, z_2)_L}\end{aligned}$$

(iv)
$$\begin{aligned}&(\mathrm{Ad}(S)z_1, \mathrm{Ad}(S)z_2)_L \\ &= \left(\begin{pmatrix} 0 & is_V A_1 s_U \\ -is_U B_1 s_V & 0 \end{pmatrix}, \begin{pmatrix} 0 & is_V A_2 s_U \\ -is_U B_2 s_V & 0 \end{pmatrix} \right)_L \\ &= i\{\mathrm{tr}_V((is_V A_2 s_U)(-is_U B_1 s_V)) - \mathrm{tr}_V((is_V A_1 s_U)(-is_U B_2 s_V))\} \\ &= i\{\mathrm{tr}_V(A_2 B_1) - \mathrm{tr}_V(A_1 B_2)\} = (z_1, z_2)_L.\end{aligned}$$

(v) Let v_j $(1 \le j \le n_a + n_b)$ (resp. u_j $(1 \le j \le m_a + m_b)$) be an orthogonal basis of V (resp. U) such that

$$(v_j, v_j) = \begin{cases} 1 & (0 \le j \le n_a) \\ -1 & (n_a + 1 \le j \le n_a + n_b) \end{cases}$$
$$\left(\text{resp. } (u_j, u_j) = \begin{cases} 1 & (0 \le j \le m_a) \\ -1 & (m_a + 1 \le j \le m_a + m_b) \end{cases} \right).$$

Then, by the obvious identification $V = \mathbb{C}^{n_a+n_b}$ (resp. $U = \mathbb{C}^{m_a+m_b}$) via this basis, we see

$$(v, v') = {}^t\bar{v} \begin{pmatrix} 1_{n_a} & 0 \\ 0 & -1_{n_b} \end{pmatrix} v' \ (v, v' \in V = \mathbb{C}^{n_a+n_b}) \text{ and } s_V = \begin{pmatrix} 1_{n_a} & 0 \\ 0 & -1_{n_b} \end{pmatrix}$$

$$\left(\text{resp. } (u, u') = {}^t\bar{u} \begin{pmatrix} 1_{m_a} & 0 \\ 0 & -1_{m_b} \end{pmatrix} u'(u, u' \in U = \mathbb{C}^{m_a+m_b}) \right.$$
$$\left. \text{and } s_U = \begin{pmatrix} 1_{m_a} & 0 \\ 0 & -1_{m_b} \end{pmatrix}\right),$$

where $\bar{v}$ denotes the ordinary complex conjugation of v. The adjoint $A^* \in \mathrm{Hom}(V, U)$ of $A \in \mathrm{Hom}(U, V)$ can be written as

$$A^* = \begin{pmatrix} 1_{m_a} & 0 \\ 0 & -1_{m_b} \end{pmatrix} {}^t\bar{A} \begin{pmatrix} 1_{n_a} & 0 \\ 0 & -1_{n_b} \end{pmatrix}.$$

Then for $z = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in L_{\mathbb{R}}$ $(B = -A^*)$, we compute $(\mathrm{Ad}(S)z, z)_L$ as follows:

$$\begin{aligned}
(\mathrm{Ad}(S)z, z)_L &= \left(\begin{pmatrix} s_V & 0 \\ 0 & -is_U \end{pmatrix} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \begin{pmatrix} s_V & 0 \\ 0 & is_U \end{pmatrix}, \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \right)_L \\
&= \left(\begin{pmatrix} 0 & is_V A s_U \\ -is_U B s_V & 0 \end{pmatrix}, \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \right)_L \\
&= i\{-i\mathrm{tr}_U(s_U B s_V A) - i\mathrm{tr}_V(s_V A s_U B)\} \\
&= 2\mathrm{tr}_U(s_U B s_V A) = -2\mathrm{tr}_U(s_U A^* s_V A) \\
&= -2\mathrm{tr}_U(s_U(s_U {}^t\bar{A} s_V) s_V A) \\
&= -2\mathrm{tr}_U({}^t\bar{A}A).
\end{aligned}$$

Thus (v) easily follows from this. □

Since $(\ ,\ )_L$ is a non-degenerate symplectic form on L and $\tau : L \to L$ is an antilinear involution such that $(\tau(z_1), \tau(z_2))_L = \overline{(z_1, z_2)_L}$ $(z_1, z_2 \in L)$, $(\ ,\ )_L$ and τ define the symplectic group

$$Sp(L) = \{g \in GL(L); (gz_1, gz_2)_L = (z_1, z_2)_L \ (z_1, z_2 \in L)\}$$

and a real form

$$\begin{aligned}
Sp(L_{\mathbb{R}}) &= \{g \in GL(L_{\mathbb{R}}); (gz_1, gz_2)_L = (z_1, z_2)_L \ (z_1, z_2 \in L_{\mathbb{R}})\} \\
&= \{g \in Sp(L); \tau \circ g \circ \tau^{-1} = g\} \\
&\simeq Sp(\dim_{\mathbb{R}} L_{\mathbb{R}}, \mathbb{R}) = Sp(2(n_a + n_b)(m_a + m_b), \mathbb{R}).
\end{aligned}$$

As in (2.1), $GL(V) \times GL(U)$ acts on L.

Lemma 3.4. *The real subspace $L_{\mathbb{R}}$ of L is stable under the action of $GL(V)_{\mathbb{R}} \times GL(U)_{\mathbb{R}}$.*

Proof. Let us consider the involution

$$\sigma := \mathrm{Ad}(S^2) : GL(V \oplus U) \to GL(V \oplus U).$$

We easily see that $\sigma \circ \tau = \tau \circ \sigma$ and hence $(GL(V \oplus U)_{\mathbb{R}}, \sigma)$ is a real symmetric pair. Furthermore, we see

$$GL(V \oplus U)^{\sigma}_{\mathbb{R}} = \{g \in GL(V \oplus U)_{\mathbb{R}}; \sigma(g) = g\} = GL(V)_{\mathbb{R}} \times GL(U)_{\mathbb{R}}$$

and

$$\mathfrak{gl}(V \oplus U)^{-\sigma}_{\mathbb{R}} = \{X \in \mathfrak{gl}(V \oplus U)_{\mathbb{R}}; \sigma(X) = -X\} = L_{\mathbb{R}}.$$

Thus $GL(V)_{\mathbb{R}} \times GL(U)_{\mathbb{R}}$ stabilizes $L_{\mathbb{R}}$. $\square$

Since the symplectic form $(\ ,\ )_L$ is $GL(V) \times GL(U)$-invariant and the actions of $GL(V)$ and $GL(U)$ on L are clearly faithful, we obtain embeddings $GL(V) \hookrightarrow Sp(L)$ and $GL(U) \hookrightarrow Sp(L)$. Clearly the actions of $GL(V)$ and $GL(U)$ on L commute. Since $L_{\mathbb{R}}$ is stable under the action of $GL(V)_{\mathbb{R}} \times GL(U)_{\mathbb{R}}$ and

$$\begin{aligned}(U(n_a, n_b), U(m_a, m_b)) &\simeq (GL(V)_{\mathbb{R}}, GL(U)_{\mathbb{R}}) \hookrightarrow Sp(L_{\mathbb{R}}) \\ &\simeq Sp(2(n_a + n_b)(m_a + m_b), \mathbb{R}),\end{aligned}$$

we obtain a dual pair

$$(GL(V)_{\mathbb{R}}, GL(U)_{\mathbb{R}}) \hookrightarrow Sp(L_{\mathbb{R}}).$$

As in §2, we consider the automorphism $\Theta : \mathfrak{gl}(V \oplus U) \to \mathfrak{gl}(V \oplus U)$, $\Theta(X) = \mathrm{Ad}(S)X$. Then $s_L := -i\mathrm{Ad}(S)|L$ defines a linear involution $s_L : L \to L$ and we have

$$\mathfrak{g}_i = \{z \in L; s_L(z) = z\} \ \text{ and } \ \mathfrak{g}_{-i} = \{z \in L; s_L(z) = -z\},$$

so that $L = \mathfrak{g}_i \oplus \mathfrak{g}_{-i}$. Later we will show that $\mathfrak{g}_{\pm i}$ are maximally totally isotropic subspaces of L and hence $L = \mathfrak{g}_i \oplus \mathfrak{g}_{-i}$ is a polar decomposition of L.

Clearly we have the following:

Lemma 3.5. (i) $\tau \circ s_L(z) = -s_L \circ \tau(z)$ $(z \in L)$. *In particular $\tau(\mathfrak{g}_{\pm i}) = \mathfrak{g}_{\mp i}$.*

(ii) $(s_L z_1, s_L z_2)_L = -(z_1, z_2)_L$ $(z_1, z_2 \in L)$.

For $g \in Sp(L)$, we see $\theta_L(g) := s_L g s_L \in Sp(L)$ by Lemma 3.5, (ii) and hence obtain an involution

$$\theta_L : Sp(L) \to Sp(L).$$

By Lemma 3.5, (i), θ_L commutes with $\tau : Sp(L) \to Sp(L)$. Furthermore, we can verify that $\theta_L | Sp(L_{\mathbb{R}})$ is a Cartan involution of $Sp(L_{\mathbb{R}})$. By the emmbedding $(GL(V), GL(U)) \hookrightarrow Sp(L)$, we see

$$\begin{aligned}(K(V), K(U)) \hookrightarrow Sp(L)^{\theta_L} &:= \{g \in Sp(L); \theta_L(g) = g\},\\ (\mathfrak{s}(V), \mathfrak{s}(U)) \hookrightarrow \mathfrak{sp}(L)^{-\theta_L} &:= \{X \in \mathfrak{sp}(L); \theta_L(X) = -X\}.\end{aligned}$$

§3.3 Recall the $GL(V) \times GL(U)$-equivariant morphisms

$$\mathfrak{gl}(V) \xleftarrow{\rho} L \xrightarrow{\pi} \mathfrak{gl}(U), \quad \rho \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = AB, \quad \pi \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = BA$$

of (2.1). Let us consider the $GL(V) \times GL(U)$-equivariant morphisms $\mathfrak{gl}(V) \xleftarrow{\rho'} L \xrightarrow{\pi'} \mathfrak{gl}(U)$,

$$\rho' \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = -i\rho \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = -iAB, \quad \pi' \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = i\pi \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = iBA$$

instead of ρ and π. Then we have the following:

Lemma 3.6. *We have $\rho'(L_{\mathbb{R}}) \subset \mathfrak{gl}(V)_{\mathbb{R}}$ and $\pi'(L_{\mathbb{R}}) \subset \mathfrak{gl}(U)_{\mathbb{R}}$. Thus we obtain $GL(V)_{\mathbb{R}} \times GL(U)_{\mathbb{R}}$-equivariant maps*

$$\mathfrak{gl}(V)_{\mathbb{R}} \xleftarrow{\rho'|_{L_{\mathbb{R}}}} L_{\mathbb{R}} \xrightarrow{\pi'|_{L_{\mathbb{R}}}} \mathfrak{gl}(U)_{\mathbb{R}}.$$

Proof. For $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in L_{\mathbb{R}}$, since

$$\tau(X) = -X^* = -\begin{pmatrix} 0 & B^* \\ A^* & 0 \end{pmatrix} = X,$$

we have $B = -A^*$. Then $\rho'(X) = iAA^*$ and

$$\rho'(X)^* = (iAA^*)^* = -i(A^*)^* A^* = -iAA^* = -\rho'(X).$$

Hence $\rho'(X) \in \mathfrak{gl}(V)_{\mathbb{R}}$.

Similarly we have $\pi'(X) \in \mathfrak{gl}(U)_{\mathbb{R}}$. $\square$

For $z \in L$, we define a linear form $\mu_z \in \mathfrak{sp}(L)^*$ by

$$\mu_z(x) = \frac{1}{2}(xz, z)_L \ (x \in \mathfrak{sp}(L)).$$

Then we obtain a map

$$\mu : L \to \mathfrak{sp}(L)^*, \quad z \mapsto \mu_z.$$

Since $(\ , \ )_L$ is real valued on $L_{\mathbb{R}}$, we see $\mu_z \in \mathfrak{sp}(L_{\mathbb{R}})^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{sp}(L_{\mathbb{R}}), \mathbb{R})$ for $z \in L_{\mathbb{R}}$. Hence we obtain a map

$$\mu|_{L_{\mathbb{R}}} : L_{\mathbb{R}} \to \mathfrak{sp}(L_{\mathbb{R}})^*, \quad z \mapsto \mu_z.$$

It is known that $\mu|_{L_{\mathbb{R}}}$ is the moment map with respect to the action of $Sp(L_{\mathbb{R}})$ on the symplectic manifold $(L_{\mathbb{R}}, (\ , \ )_L|_{L_{\mathbb{R}}})$ (see for example [[CG], Proposition 1.4.6]).

Via the embeddings

$$\mathfrak{gl}(V) \hookrightarrow \mathfrak{sp}(L), \quad \mathfrak{gl}(U) \hookrightarrow \mathfrak{sp}(L),$$

we can define linear forms $\rho_z^* \in \mathfrak{gl}(V)^*$, $\pi_z^* \in \mathfrak{gl}(U)^*$ by

$$\rho_z^* := \mu_z|_{\mathfrak{gl}(V)}, \quad \pi_z^* := \mu_z|_{\mathfrak{gl}(U)}$$

and we obtain

$$\rho^* : L \to \mathfrak{gl}(V)^* \ (z \mapsto \rho_z^*), \quad \pi^* : L \to \mathfrak{gl}(U)^* \ (z \mapsto \pi_z^*).$$

If $z \in L_{\mathbb{R}}$, we easily see $\rho_z^* \in \mathfrak{gl}(V)_{\mathbb{R}}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{gl}(V)_{\mathbb{R}}, \mathbb{R})$ and $\pi_z^* \in \mathfrak{gl}(U)_{\mathbb{R}}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{gl}(U)_{\mathbb{R}}, \mathbb{R})$. Thus

$$\rho^*|_{L_{\mathbb{R}}} : L_{\mathbb{R}} \to \mathfrak{gl}(V)_{\mathbb{R}}^* \ \ (\text{resp. } \pi^*|_{L_{\mathbb{R}}} : L_{\mathbb{R}} \to \mathfrak{gl}(U)_{\mathbb{R}}^*)$$

is the moment map with respect to the action of $GL(V)_{\mathbb{R}}$ (resp. $GL(U)_{\mathbb{R}}$) on the symplectic manifold $(L_{\mathbb{R}}, (\ , \ )_L|_{L_{\mathbb{R}}})$.

Now let us show that $\rho' : L \to \mathfrak{gl}(V)$ (resp. $\pi' : L \to \mathfrak{gl}(U)$) coincides with $\rho^* : L \to \mathfrak{gl}(V)^*$ (resp. $\pi^* : L \to \mathfrak{gl}(U)^*$) via the trace form on V (resp. U).

Proposition 3.7. *For $X \in L$, we have*

$$\mathrm{tr}_V(\rho'(X)x) = \rho_X^*(x) \ (x \in \mathfrak{gl}(V)) \ \textit{and} \ \ \mathrm{tr}_U(\pi'(X)y) = \pi_X^*(y) \ (y \in \mathfrak{gl}(U)).$$

Proof. For $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in L$ ($A \in \mathrm{Hom}(U,V)$, $B \in \mathrm{Hom}(V,U)$) and $x = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}(V)$, we compute

$$\begin{aligned}
\rho_X^*(x) = \mu_X(x) &= \frac{1}{2}((\mathrm{ad}x)X, X)_L = \frac{1}{2}\left(\left[\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right)_L \\
&= \frac{1}{2}\left(\begin{pmatrix} 0 & xA \\ -Bx & 0 \end{pmatrix}, \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right)_L = \frac{i}{2}\{\mathrm{tr}_U(-BxA) - \mathrm{tr}_V(xAB)\} \\
&= -\frac{i}{2}\{\mathrm{tr}_V(xAB) + \mathrm{tr}_V(xAB)\} \\
&= -i\mathrm{tr}_V(xAB) = \mathrm{tr}_V((-iAB)x) = \mathrm{tr}_V(\rho'(X)x).
\end{aligned}$$

For $y = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \in \mathfrak{gl}(U)$, similarly we have

$$\begin{aligned}
\pi_X^*(y) = \mu_X(y) &= \frac{1}{2}((\mathrm{ad}y)X, X)_L = \frac{1}{2}\left(\left[\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right], \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right)_L \\
&= \frac{1}{2}\left(\begin{pmatrix} 0 & -Ay \\ yB & 0 \end{pmatrix}, \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right)_L = \frac{i}{2}\{\mathrm{tr}_U(yBA) - \mathrm{tr}_V((-Ay)B)\} \\
&= \frac{i}{2}\{\mathrm{tr}_U(yBA) + \mathrm{tr}_U(yBA)\} = \mathrm{tr}_U(y(iBA)) \\
&= \mathrm{tr}_U(y\pi'(X)) = \mathrm{tr}_U(\pi'(X)y).
\end{aligned}$$

□

By Proposition 3.7, $\rho'|_{L_\mathbb{R}} : L_\mathbb{R} \to \mathfrak{gl}(V)_\mathbb{R}$ (resp. $\pi'|_{L_\mathbb{R}} : L_\mathbb{R} \to \mathfrak{gl}(U)_\mathbb{R}$) coincides with the moment map $\rho^*|_{L_\mathbb{R}} : L_\mathbb{R} \to \mathfrak{gl}(V)^*_\mathbb{R}$ (resp. $\pi^*|_{L_\mathbb{R}} : L_\mathbb{R} \to \mathfrak{gl}(U)^*_\mathbb{R}$) via the above identification. Thus we can see that ρ' and π' are the complexification of the moment maps.

Finally we show that

$$L_+ = \mathfrak{g}_i = \{X \in \mathfrak{gl}(V \oplus U); \Theta(X) = iX\}$$

is a maximally totally isotropic subspace of $(L, (\ ,\ )_L)$.

Lemma 3.8. *L_+ is a maximally totally isotropic subspace of $(L, (\ ,\ )_L)$.*

Proof. If $z_1, z_2 \in L_+ = \mathfrak{g}_i$, we have

$$(z_1, z_2)_L = (\mathrm{Ad}(S)z_1, \mathrm{Ad}(S)z_2)_L = (iz_1, iz_2)_L = -(z_1, z_2)_L$$

by Lemma 3.3, (iv). Hence $(z_1, z_2)_L = 0$. Thus $\mathfrak{g}_i$ is totally isotropic in L. Similarly $\mathfrak{g}_{-i}$ is also totally isotropic in L. Notice that $L = \mathfrak{g}_i \oplus \mathfrak{g}_{-i}$ and $\tau(\mathfrak{g}_{\pm i}) = \mathfrak{g}_{\mp i}$. Therefore $\dim \mathfrak{g}_i = \dim \mathfrak{g}_{-i}$ and we have $\dim \mathfrak{g}_i = \dim L/2$. Therefore $\mathfrak{g}_i$ is maximally totally isotropic in L. □

In such a way, our maps

$$\mathfrak{s}(V) \overset{\rho'|_{L_+}}{\longleftarrow} L_+ \overset{\pi'|_{L_+}}{\longrightarrow} \mathfrak{s}(U)$$

are the restrictions to the maximally totally isotropic subspace L_+, of the complexified moment maps

$$\mathfrak{gl}(V) \overset{\rho'}{\leftarrow} L \overset{\pi'}{\rightarrow} \mathfrak{gl}(U).$$

Remark 3.9. For the map $\mu : L \to \mathfrak{sp}(L)^*$, if $z \in L_+ = \mathfrak{g}_i$ and $x \in \mathfrak{sp}(L)^{\theta_L}$, we have $\mu_z(x) = \dfrac{1}{2}(xz, z)_L = 0$, since $xz \in L_+$ and L_+ is totally isotropic. Thus the restriction of μ to L_+ defines a map

$$\mu|_{L_+} : L_+ \to (\mathfrak{sp}(L)^{-\theta_L})^*.$$

Then our maps

$$\mathfrak{s}(V) \overset{\rho'|_{L_+}}{\longleftarrow} L_+ \overset{\pi'|_{L_+}}{\longrightarrow} \mathfrak{s}(U)$$

are obtained by the restrictions of the above maps to $\mathfrak{s}(V)$ and $\mathfrak{s}(U)$ respectively via the embedding $(\mathfrak{s}(V), \mathfrak{s}(U)) \hookrightarrow \mathfrak{sp}(L)^{-\theta_L}$.

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