Publ. RIMS, Kyoto Univ.
42 (2006), 1–8

On the Dimension Formula for the Hyperfunction Solutions of Some Holonomic D-modules

By

Jörg Schürmann*

Abstract

In this short note we improve a recent dimension formula of Takeuchi for the dimension of the hyperfunction solutions of some holonomic D-modules. Besides the constructibility result and the local index formula of Kashiwara for the holomorphic solution complex, we only use a vanishing theorem of Lebeau together with a simple calculation in terms of constructible functions.

§1. Introduction

One of the basic results about holonomic D-modules is the *constructibility result* of Kashiwara [6], that the holomorphic solution complex

$$\mathcal{S}ol(\mathcal{M}) := Rhom_{D_X}(\mathcal{M}, \mathcal{O}_X)$$

of a holonomic D-module $\mathcal{M}$ on the complex manifold X is a bounded *complex analytically constructible* complex of sheaves of complex vector spaces with *finite dimensional* stalks (compare also with [7, chapter 5], [10, thm.4.5.8, p.458] and [14, chap.III]). In particular, the function

$$\chi(\mathcal{M}) : X \to \mathbb{Z} \; ; \; x \mapsto \chi\big(\, \mathcal{S}ol(\mathcal{M})_x\big) \tag{1}$$

is well defined and *complex analytically constructible*. Here χ is the usual *Euler characteristic*. Moreover, one has by Kashiwara [5] the following beautiful

Communicated by M. Kashiwara. Received July 15, 2003.
2000 Mathematics Subject Classification(s): 32C38, 32S40, 35A27.
*Westf. Wilhelms-Universität, SFB 478 "Geometrische Strukturen in der Mathematik", Hittorfstr.27, 48149 Münster, Germany.
e-mail: `jschuerm@math.uni-muenster.de`

description of this local index in terms of the characteristic cycle of the holonomic D-module $\mathcal{M}$ (see [7, thm.6.3.1, p.127, cor.6.3.4, p.128] and [2, thm.2, p.574]):

Theorem 1.1 (local index formula). *Let Y_j be finitely many distinct irreducible closed complex analytic subsets of X such that the characteristic variety $char(\mathcal{M})$ of $\mathcal{M}$ can be estimated by*

$$char(\mathcal{M}) \subset \bigcup_j T^*_{Y_j}X\,, \quad with \quad T^*_Y X := cl(T^*_{Y_{reg}}X) \tag{2}$$

the closure of the conormal bundle to the regular part Y_{reg} of the irreducible complex analytic subset $Y \subset X$. Then

$$\chi(\mathcal{M})(x) = \sum_j (-1)^{d_j} \cdot m_j \cdot Eu_{Y_j}(x)\,, \tag{3}$$

with d_j the complex codimension of Y_j, Eu_{Y_j} the famous Euler obstruction of Y_j as defined by MacPherson (cf. [13]) and m_j the (generic) multiplicity of $\mathcal{M}$ along $T^*_{Y_j}X$.

Note that $Eu_Y = 1_Y$, if $Y \subset X$ is a *closed* complex analytic submanifold. So a very special case of the local index formula is given as in [7, ex. on p.129] by the

Example 1. Suppose all Y_j in the estimate (2) are *closed* connected complex analytic submanifolds of X. Then

$$\chi(\mathcal{M})(x) = \sum_j (-1)^{d_j} \cdot m_j \cdot 1_{Y_j}(x)\,, \tag{4}$$

with d_j the complex codimension of Y_j and m_j the (generic) multiplicity of $\mathcal{M}$ along $T^*_{Y_j}X$.

Let us now consider the case that X is the *complexification* of the real analytic manifold M, with $i : M \hookrightarrow X$ the closed inclusion. Assume M is purely n-dimensional. Then the sheaf complex

$$R\Gamma_M(\mathcal{O}_X)[n] \simeq Ri_*i^!(\mathcal{O}_X)[n]$$

is concentrated in degree zero, with

$$\mathcal{B}_M := h^0\big(\, i^!(\mathcal{O}_X)[n]\,\big) \otimes or_M$$

the sheaf of Sato's *hyperfunctions* on M, and or_M the *orientation sheaf* of M.

Then the hyperfunction solution complex

$$Rhom_{i^*D_X}(i^*\mathcal{M}, \mathcal{B}_M) \simeq i^!\mathcal{S}ol(\mathcal{M}) \otimes or_M[n] \tag{5}$$

of a holonomic D-module $\mathcal{M}$ on X is *subanalytically constructible* on M with *finite dimensional* stalks (compare [7, thm.5.1.7, p.115]). So it is natural to ask for a corresponding index formula like (3) or (4).

Remark. The same constructibility result is true for the solutions

$$Rhom_{i^*D_X}(i^*\mathcal{M}, \mathcal{A}_M) \simeq i^*\mathcal{S}ol(\mathcal{M})$$

in the sheaf $\mathcal{A}_M = i^*\mathcal{O}_X$ of real analytic functions on M. If $\mathcal{M}$ is a *regular holonomic* D-module on X, then one also has isomorphisms (see [8, cor.8.3, cor.8.5, p.360] or [1, p.326]):

$$Rhom_{i^*D_X}(i^*\mathcal{M}, \mathcal{B}_M) \simeq Rhom_{i^*D_X}(i^*\mathcal{M}, \mathcal{D}b_M)$$

and

$$Rhom_{i^*D_X}(i^*\mathcal{M}, \mathcal{A}_M) \simeq Rhom_{i^*D_X}(i^*\mathcal{M}, \mathcal{C}^\infty_M)\,,$$

with $\mathcal{D}b_M$ (or $\mathcal{C}^\infty_M$) the sheaf of distributions (or smooth functions) on M.

The following counterpart of (4) is the main result of this note:

Theorem 1.2 (local dimension formula). *Let M_j be finitely many distinct closed real analytic submanifolds of M such that the characteristic variety $char(\mathcal{M})$ of the holonomic D-module $\mathcal{M}$ on X can be estimated by*

$$char(\mathcal{M}) \subset \bigcup_j T^*_{Y_j}X\,, \tag{6}$$

with $Y_j \subset X$ the complexification of M_j. Assume the Y_j are irreducible (i.e. connected), with $Y_j \cap M = M_j$. Then one has for $x \in M$:

$$dim_C\big(hom_{i^*D_X}(i^*\mathcal{M}, \mathcal{B}_M)_x\big) = \sum_j m_j \cdot 1_{M_j}(x)\,, \tag{7}$$

with m_j the (generic) multiplicity of $\mathcal{M}$ along $T^*_{Y_j}X$.

This is indeed a counterpart of (4). The estimate (6) implies by a theorem of Lebeau [12] (compare also with [3, thm.2.1, rem., p.531] and [3, ex.(1), p.533]) the *vanishing result*

$$\mathcal{E}xt^k_{i^*D_X}(i^*\mathcal{M}, \mathcal{B}_M)_x = 0 \quad \text{for all } k \geq 1 \tag{8}$$

so that

$$(9) \qquad dim_C\big(hom_{i^*D_X}(i^*\mathcal{M}, \mathcal{B}_M)_x\big) = \chi\big(Rhom_{i^*D_X}(i^*\mathcal{M}, \mathcal{B}_M)_x\big) \,.$$

Theorem 1.2 answers affirmatively a question asked (or better, discussed) in [17, rem.3.5] at the end of a recent paper of Takeuchi [17], where he proves the dimension formula (7) under the special assumption, that in suitable local coordinates $(M,x) \simeq (\mathbb{R}^n, 0)$ the M_j are *linear subspaces* (passing through $x = 0$).

Note that this special case already covers (locally) the one-dimensional case $(X, M, x) \simeq (\mathbb{C}, \mathbb{R}, 0)$, with $\mathcal{M}$ a holonomic D-module such that

$$char(\mathcal{M}) \subset T^*_{\{x\}}X \cup T^*_X X \,.$$

In this case one gets back a classical result of Kashiwara [4, thm.4.2.7, p.69] (cf. [7, cor.3.2.36(b), p.88-89]) and Komatsu [11]:

$$dim_C\big(hom_{i^*D_X}(i^*\mathcal{M}, \mathcal{B}_M)_x\big) = d + d' \,,$$

with d or d' the multiplicity of $\mathcal{M}$ along $T^*_{\{x\}}X$ or $T^*_X X$.

Let $j : X\backslash M \to M$ be the open inclusion of the complement of M in X. Then the proof given in [17, sec.3] is based on the distinguished triangle

$$(10) \quad Ri_* i^! \mathcal{S}ol(\mathcal{M})[n] \longrightarrow \mathcal{S}ol(\mathcal{M})[n] \longrightarrow Rj_* j^* \mathcal{S}ol(\mathcal{M})[n] \xrightarrow{[1]} \,.$$

Moreover, he uses the *micro-local theory* of the characteristic cycles $CC(\cdot)$ for subanalytically constructible complexes of sheaves (as in [9, 10, 15, 16]), in particular a deep result of Schmid-Vilonen [15] about a description of

$$CC(Rj_* j^* \mathcal{S}ol(\mathcal{M})) \quad \text{in terms of} \quad CC(\mathcal{S}ol(\mathcal{M})) \,.$$

In the next section we explain our simple proof of Theorem 1.2, which doesn't make use of this sophisticated micro-local theory of characteristic cycles. Instead of this, we use the observation that the calculation of

$$\chi((i^!\mathcal{F})_x) \quad \text{for} \quad \mathcal{F} = \mathcal{S}ol(\mathcal{M})$$

can be done in terms of *subanalytically constructible functions*, i.e. the functor $i^!$ induces a corresponding (unique) $\mathbb{Z}$-linear transformation for the abelian groups $CF(\cdot)$ of subanalytically constructible functions such that the following diagram commutes (compare [10, sec.9.7] and [16, sec.2.3]):

$$\begin{array}{ccc} K_0(X) & \xrightarrow{i^!} & K_0(M) \\ \chi_X \big\downarrow & & \big\downarrow \chi_M \\ CF(X) & \xrightarrow{i^!} & CF(M)\,. \end{array} \tag{11}$$

Here $K_0(\cdot)$ is the *Grothendieck group* of subanalytically constructible (complexes of) sheaves with finite dimensional stalks, with $\chi_?$ induced by taking stalkwise the Euler characteristic.

Then the calculation of

$$\chi((i^! \mathcal{S}ol(\mathcal{M}) \otimes or_M[n])_x) = (-1)^n \cdot i^! \big(\chi_X(\mathcal{S}ol(\mathcal{M}))\big)(x) \tag{12}$$

becomes an easy exercise by the local index theorem and example 1, since the M_j and therefore also the Y_j are closed submanifolds!

If we allow in the estimate (6) also *singular subspaces*, then we get at least the following weak parity version of the local index theorem:

Theorem 1.3 (local index formula for hyperfunctions). *Let M_j be finitely many distinct real analytic subspaces of M such that the characteristic variety char$(\mathcal{M})$ of the holonomic D-module $\mathcal{M}$ on X can be estimated as in* (6), *with $Y_j \subset X$ the complexification of M_j. Assume the Y_j are irreducible, with $Y_j \cap M = M_j$. Then one has for $x \in M$:*

$$\chi\big(Rhom_{\,i^*D_X}(i^*\mathcal{M}, \mathcal{B}_M)_{\,x}\big) \equiv \sum_j m_j \cdot Eu_{Y_j}(x) \quad mod\ 2\,, \tag{13}$$

with Eu_{Y_j} the Euler obstruction of Y_j and m_j the (generic) multiplicity of $\mathcal{M}$ along $T^*_{Y_j}X$.

§2. Constructible Functions

In this final section we give the proof of Theorem 1.2 and 1.3 in terms of constructible functions. Let us start with the proof of Theorem 1.2.

By the estimate (6) and Example 1 we get

$$\chi_X(\mathcal{S}ol(\mathcal{M})) = \sum_j (-1)^{d_j} \cdot m_j \cdot 1_{Y_j}\ ,$$

with d_j the complex codimension of Y_j and m_j the (generic) multiplicity of $\mathcal{M}$ along $T^*_{Y_j}X$. By linearity of $i^!$ on the level of constructible functions one also has

$$i^!\big(\chi_X(\mathcal{S}ol(\mathcal{M}))\big) = \sum_j (-1)^{d_j} \cdot m_j \cdot i^!(1_{Y_j}) \,.$$

Then the dimension formula (7) follows from (5), (9), (12) and the simple formula

$$i^!(1_{Y_j}) = (-1)^{(n-d_j)} \cdot 1_{M_j} \,. \tag{14}$$

The formula (14) corresponds by the commutative diagram (11) to the *base change formula*

$$i^!(Rk_*\mathbb{C}_{Y_j}) \simeq Rk'_* i'^! \mathbb{C}_{Y_j}$$

for the cartesian diagram of inclusions

$$\begin{array}{ccc} M & \xrightarrow{\;i\;} & X \\ {\scriptstyle k'}\uparrow & & \uparrow{\scriptstyle k} \\ M_j & \xrightarrow{\;i'\;} & Y_j \,. \end{array}$$

Note that $i'^!\mathbb{C}_{Y_j} \simeq \mathbb{C}_{M_j}[-(n-d_j)]$ *locally* on M_j, since M_j is a closed submanifold of Y_j of real codimension equal to the complex dimension $n-d_j$ of Y_j.

For the proof of the parity formula (13) in Theorem 1.3 it is enough to show

$$\chi\big(\, i^*Rj_*j^*\mathcal{S}ol(\mathcal{M})_x \,\big) \equiv 0 \mod 2 \text{ for all } x \in M. \tag{15}$$

Use the local index formula (3) and the distinguished triangle (10). But this follows from the fact that the constructible function

$$\chi_X(\mathcal{S}ol(\mathcal{M}))\; mod\; 2$$

is invariant under the complex conjugation acting on the complexification X of M (with fixed point set M).

More precisely, by [16, lem.1.1.1, p.27] one gets the description:

$$\chi\big(i^*Rj_*j^*\mathcal{S}ol(\mathcal{M})_x\big) = \chi\big(R\Gamma(M_{f,x}, \mathcal{S}ol(\mathcal{M}))\big) \,,$$

with

$$M_{f,x} := \{||x|| \leq \delta,\ f = w\} \quad \text{for} \quad 0 < w \ll \delta \ll 1$$

(i.e. for w, δ small, with w also small compared to δ) a *local right Milnor fiber* of the function f at x, defined in local coordinates

$$(X, M, x) \simeq (\mathbb{C}^n, \mathbb{R}^n, 0) \quad \text{by} \quad z = (z_1, \ldots, z_n) \mapsto f(z) := \sum_{k=1}^{n} im(z_k)^2 \,.$$

Here $im(\cdot)$ is the imaginary part, with the complex conjugation acting on $(\mathbb{C}^n, \mathbb{R}^n, 0)$ in the usual way. This conjugation leaves the compact semi-analytic set $M_{f,x}$ invariant without any fixed point! But the Euler characteristic

$$\chi\big(R\Gamma(M_{f,x}, \mathcal{S}ol(\mathcal{M}))\big) \mod 2$$

can be calculated in terms of $\mathbb{Z}_2$-valued constructible functions:

$$\chi\big(R\Gamma(M_{f,x}, \mathcal{S}ol(\mathcal{M}))\big) \equiv (c \circ \pi)_* \alpha \mod 2,$$

with

$$\alpha := \chi_{M_{f,x}}\big(\mathcal{S}ol(\mathcal{M})|M_{f,x}\big) \mod 2 \quad \in CF(M_{f,x}, \mathbb{Z}_2)\,,$$

$$\pi : M_{f,x} \to M_{f,x}/conj. \quad \text{the quotient and} \quad c : M_{f,x}/conj. \to \{pt\}$$

a constant map. Here $(c \circ \pi)_*$ is induced by $R(c \circ \pi)_*$ similarly as in (11) by the commutative diagram (compare [10, sec.9.7] and [16, sec.2.3]):

$$\begin{array}{ccc} K_0(M_{f,x}) & \xrightarrow{R(c\circ\pi)_*} & K_0(\{pt\}) \simeq \mathbb{Z} \\ {\scriptstyle \chi_{M_{f,x}}}\downarrow & & \downarrow{\scriptstyle \chi_{\{pt\}}} \\ CF(M_{f,x}) & \xrightarrow{(c\circ\pi)_*} & CF(\{pt\}) \simeq \mathbb{Z} \\ {\scriptstyle mod\ 2}\downarrow & & \downarrow{\scriptstyle mod\ 2} \\ CF(M_{f,x}, \mathbb{Z}_2) & \xrightarrow{(c\circ\pi)_*} & CF(\{pt\}, \mathbb{Z}_2) \simeq \mathbb{Z}_2\,, \end{array} \tag{16}$$

with $CF(\cdot, \mathbb{Z}_2)$ the corresponding abelian group of $\mathbb{Z}_2$-valued subanalytically constructible functions.

Then $(c \circ \pi)_* = c_* \circ \pi_*$ by functoriality. But $\pi_*(\alpha) \equiv 0$, since α is invariant under the conjugation *conj.*, with $\pi : M_{f,x} \to M_{f,x}/conj.$ an unramified covering of degree two. Of course, here it is important to work with $\mathbb{Z}_2$-valued constructible functions.

References

[1] Björk, J. E., *Analytic D-modules and applications*, Kluwer, 1993.
[2] Brylinski, J. L., Dubson, A. and Kashiwara, M., Formule d'indice pour les modules holonomes et obstruction d'Euler locale, *C. R. Acad. Sci. Paris*, **293** (1981), 573-576.
[3] Honda, N. and Schapira, P., A vanishing theorem for holonomic modules with positive characteristic varieties, *Publ. RIMS, Kyoto Univ.*, **26** (1990), 529-534.
[4] Kashiwara, M., *Algebraic study of systems of linear differential equations*, Master's Thesis, University of Tokyo 1970, English translation in: *Mémoires de la SMF*, **63**, 1996.
[5] ———, Index theorem for maximally overdetermined systems of linear differential equations, *Proc. Japan Acad.*, **49** (1973), 803-804.
[6] ———, On the maximally overdetermined systems of linear differential equations I, *Publ. RIMS, Kyoto Univ.*, **10** (1975), 563-579.
[7] ———, *Systems of microdifferential equations*, *Progr. Math.*, **34**, Birkhäuser, Basel, 1983.
[8] ———, The Riemann-Hilbert problem for holonomic systems, *Publ. RIMS, Kyoto Univ.*, **20** (1984), 319-365.
[9] ———, Index theorem for constructible sheaves, *Astérisque*, **130** (1985), 193-209.
[10] Kashiwara, M. and Schapira, P., *Sheaves on Manifolds*, Springer, Berlin Heidelberg, 1990.
[11] Komatsu, H., On the index of ordinary differential operators, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, **18** (1971), 379-398.
[12] Lebeau, G., Annulation de la cohomologie hyperfonction de certains modules holonomes, *C. R. Acad. Sci. Paris Ser. A-B*, **290** (1980), A313-A316.
[13] MacPherson, R., Chern classes for singular varieties, *Ann. of Math.*, **100** (1974), 423-432.
[14] Mebkhout, Z. and Narnáez-Macarro, L., Le Théorème de Constructibilité de Kashiwara, In: *Images directes et constructibilité*, ed. Maisonobe, M. and Sabbah, C., *Travaux en cours*, Hermann, Paris, **46** (1997), 47-98.
[15] Schmid, W. and Vilonen, K., Characteristic cycles for constructible sheaves, *Inv. Math.*, **124** (1996), 451-502.
[16] Schürmann, J., *Topology of singular spaces and constructible sheaves*, *Monografie Matematyczne* (*New Series*), **63**, Birkhäuser, Basel, 2003.
[17] Takeuchi, K., Dimension formulas for the hyperfunction solutions to holonomic D-modules, *Adv. Math.*, **180** (2003), 134-145.