Obstructions to Deforming Space Curves and Non-reduced Components of the Hilbert Scheme

By

Hirokazu Nasu*

Abstract

Let $H_{\mathbb{P}^3}^S$ denote the Hilbert scheme of smooth connected curves in $\mathbb{P}^3$. We consider maximal irreducible closed subsets $W \subset H_{\mathbb{P}^3}^S$ whose general member C is contained in a smooth cubic surface and investigate the conditions for W to be a component of $(H_{\mathbb{P}^3}^S)_{\mathrm{red}}$. We especially study the case where the dimension of the tangent space of $H_{\mathbb{P}^3}^S$ at $[C]$ is greater than $\dim W$ ($\geq 4\deg(C)$) by one. We compute obstructions to deforming C in $\mathbb{P}^3$ and prove that for every W in this case, $H_{\mathbb{P}^3}^S$ is non-reduced along W and W is a component of $(H_{\mathbb{P}^3}^S)_{\mathrm{red}}$.

§1. Introduction

Mumford [10] showed that the Hilbert scheme $H_{\mathbb{P}^3}^S$ of smooth connected curves in $\mathbb{P}^3$ is non-reduced. $H_{\mathbb{P}^3}^S$ is the disjoint union of the open subscheme $H_{d,g}^S$ consisting of curves of degree d and genus g. He considered a 56-dimensional irreducible closed subset $W \subset H_{14,24}^S$ whose general member C is contained in a smooth cubic surface. He showed that the dimension of the tangent space of $H_{14,24}^S$ at $[C]$ is equal to 57. Moreover, he proved that W is maximal as a subvariety of $(H_{14,24}^S)_{\mathrm{red}}$, and hence $H_{14,24}^S$ is non-reduced.

We consider a generalization of Mumford's example. Let W be an irreducible closed subset of $H_{d,g}^S$ whose general member C is contained in a smooth cubic surface. Suppose that W is maximal among all such subsets. We ask the next question:

Question 1.1. *Is W an irreducible component of $(H_{d,g}^S)_{\mathrm{red}}$? If so, is $H_{d,g}^S$ non-reduced at the generic point of W?*

See §4 ((4.2) in particular) for more explicit description of W. First we observe that $\dim W = d + g + 18$ when $d > 9$, while every irreducible component of $H_{d,g}^S$ is of dimension at least $4d$. Hence we consider the natural range $\Omega := \{(d,g) \in \mathbb{Z}^2 | d > 9, g \geq 3d - 18\}$ of pairs (d,g), where the above question makes sense. Secondly we consider the cohomology group $H^1(\mathbb{P}^3, \mathcal{I}_C(3))$ for a general member C of W. The dimension $h^1(\mathcal{I}_C(3))$ as a vector space is the gap between $\dim W$ and the dimension of the tangent space of $H_{d,g}^S$ at $[C]$. This corresponds to the extra embedded first order infinitesimal deformations of $C \subset \mathbb{P}^3$ other than the ones coming from W. Thus if $h^1(\mathcal{I}_C(3)) = 0$, then W is an irreducible component of $H_{d,g}^S$ of $d + g + 18$, and moreover, $H_{d,g}^S$ is non-singular at the generic point of W.

In this paper, we concentrate on the case where $h^1(\mathcal{I}_C(3)) = 1$. This is the first non-vanishing case, which includes Mumford's example. In this case, there are only the two possibilities:

(A) $H_{d,g}^S$ is non-reduced along W. Moreover, W is an irreducible component of $(H_{d,g}^S)_{\mathrm{red}}$.

(B) There exists an irreducible component $V \supsetneq W$ of $H_{d,g}^S$ such that $\dim V = \dim W + 1$ and a general member is not contained in a cubic. Moreover, $H_{d,g}^S$ is generically smooth along W.

We show that the case (B) does not occur.

Theorem 1.2 (Main Theorem). *Let $(d,g) \in \Omega$ and let W be an irreducible closed subset of $H_{d,g}^S$ whose general member C is contained in a smooth cubic surface. Suppose that W is maximal among all such subsets. If $h^1(\mathcal{I}_C(3)) = 1$, then W is an irreducible component of $(H_{d,g}^S)_{\mathrm{red}}$ of dimension $d + g + 18$, and $H_{d,g}^S$ is non-reduced along W.*

For this kind of problem, two approaches are known. One is to show that (B) leads to a contradiction, using e.g. liaison. This was used by Mumford in [10]. It has been also used to show that $H_{16,30}^S$ is non-reduced in [11]. But it depends on case by case arguments. Hence we cannot apply it for our general case that $h^1(\mathcal{I}_C(3)) = 1$.

In the proof of Theorem 1.2, we use the other approach described as follows. Let C be a general member of W. If $H_{d,g}^S$ is non-singular at $[C]$, then every first order infinitesimal deformation φ (i.e. a deformation over $\operatorname{Spec} k[t]/t^2$) of

$C \subset \mathbb{P}^3$ can be lifted to a deformation over $\operatorname{Spec} k[t]/(t^{n+1})$ for any integer $n \geq 2$. We prove that there exists a first order infinitesimal deformation φ of $C \subset \mathbb{P}^3$ that cannot be lifted to any deformation over $\operatorname{Spec} k[t]/(t^3)$ (cf. Proposition 3.1). This implies that $H_{d,g}^S$ is singular along W, and hence we obtain (A). This approach was first used by Curtin in [1], who proved our result for the case of Mumford's example. We generalize a calculation method used in his proof. More precisely, we compute the obstruction map

$$\operatorname{Hom}(\mathcal{I}_C, \mathcal{O}_C) \to \operatorname{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$$
$$\cup \qquad\qquad \cup$$
$$\varphi \qquad \mapsto \quad \varphi \cup \mathbf{e} \cup \varphi,$$

where $\mathbf{e} \in \operatorname{Ext}^1(\mathcal{O}_C, \mathcal{I}_C)$ is the extension class of the basic exact sequence

$$(1.1) \qquad\qquad 0 \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

(cf. §2.1). We use linear systems on the cubic surface S containing C for the computation. Furthermore, we find an interesting relation between the obstruction map and some geometry arising from a conic pencil on the cubic S (cf. §3.3).

Generalizations of Mumford's example were also studied by Kleppe [6],[7] and Ellia [3]. They gave a conjecture concerning non-reduced components of the Hilbert scheme $H_{\mathbb{P}^3}^S$ with some results which partially prove it (see Conjecture 4.7). Our theorem differently partially proves the conjecture. See Remark 4.8 for the relation between their work and our theorem. Constructions of non-reduced components of $H_{\mathbb{P}^3}^S$ by liaison or Rao module have been developed by Martin-Deschamps and Perrin [9], and by Fløystad [4]. See [4] for another generalization of Mumford's example.

Acknowledgements

Notation and Conventions

We work in $\mathbb{P}^3$, the 3-dimensional projective space over an algebraically closed field k of characteristic 0. Given a closed subscheme V of $\mathbb{P}^3$, we denote by $\mathcal{I}_V$ the ideal sheaf of V in $\mathbb{P}^3$. If $X \subseteq V$ is a closed subscheme, we indicate the ideal sheaf of X in V by $\mathcal{I}_{X/V}$. $\mathcal{N}_V \cong \mathcal{H}om(\mathcal{I}_V, \mathcal{O}_V)$ and $\mathcal{N}_{X/V}$ denote the normal sheaf of V in $\mathbb{P}^3$ and the normal sheaf of X in V respectively. Given $\mathcal{O}_{\mathbb{P}^3}$-modules $\mathcal{F}$ and $\mathcal{G}$, $h^i(\mathcal{F})$, $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ and $\mathrm{Ext}^i(\mathcal{F},\mathcal{G})$ denote $\dim H^i(\mathbb{P}^3, \mathcal{F})$, $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{F},\mathcal{G})$ and $\mathrm{Ext}^i_{\mathcal{O}_{\mathbb{P}^3}}(\mathcal{F},\mathcal{G})$ respectively. We denote the p-th Čech cohomology group of $\mathcal{F}$ with respect to an open covering $\mathfrak{U}$ by $\check{H}^p(\mathfrak{U},\mathcal{F})$. If D is a Cartier divisor on a variety X, $\mathcal{O}_X(D)$ and $|D|$ respectively denote the invertible sheaf and the complete linear system associated to D. For a linear system Λ on X, we denote the fixed part of Λ by $\mathrm{Bs}\,\Lambda$. $\mathcal{O}_X(1)$ and $\mathbf{h}$ denote the restriction of the tautological line bundle $\mathcal{O}_{\mathbb{P}^3}(1)$ to X and the divisor class corresponding to $\mathcal{O}_X(1)$ respectively. We denote by $\mathrm{Rat}(\mathcal{L})$ the constant sheaf of global rational sections of a line bundle $\mathcal{L}$ on X. For a non-zero rational section s of $\mathcal{L}$, we denote the divisor $(s)_0 - (s)_\infty$ of zeros minus poles of s by $\mathrm{div}(s)$. $\mathcal{L}(D)$ denotes the subsheaf of $\mathrm{Rat}(\mathcal{L})$ which consists of rational sections s of $\mathcal{L}$ such that $\mathrm{div}(s) + D$ is effective. We have $\mathcal{L}(D) \cong \mathcal{L} \otimes \mathcal{O}_X(D)$ by the usual multiplication map.

§2.   Preliminaries

§2.1.

In this subsection, we recall some basic facts on the infinitesimal study of the Hilbert scheme of space curves. In what follows, we refer to [8, I.2] for the proofs, where there is a very thorough discussion of general embedded deformations.

Let C be a smooth connected curve in $\mathbb{P}^3$. Then an (*embedded*) *n-th order* (*infinitesimal*) *deformation* of $C \subset \mathbb{P}^3$ is a closed subscheme $\mathcal{C}_n$ of $\mathbb{P}^3 \times \mathrm{Spec}\,k[t]/(t^{n+1})$ which is flat over $k[t]/(t^{n+1})$ and $\mathcal{C}_n \otimes_{k[t]/(t^{n+1})} k = C$. The set of all first order deformations of $C \subset \mathbb{P}^3$ is the Zariski tangent space of $H^S_{\mathbb{P}^3}$ at the point $[C]$. Let $\mathcal{C}_1$ be a first order deformation of $C \subset \mathbb{P}^3$. If there exists no second order deformation $\mathcal{C}_2$ of $C \subset \mathbb{P}^3$ such that $\mathcal{C}_2 \otimes_{k[t]/(t^3)} k[t]/(t^2) = \mathcal{C}_1$, we say $\mathcal{C}_1$ is *obstructed at the second order*. The set of all first order deformations of $C \subset \mathbb{P}^3$ is parametrized by $\mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C)$. So we abusively identify them from now. The basic exact sequence (1.1) induces the isomorphism

$$\delta : \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C) \xrightarrow{\sim} \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C).$$

Let $\varphi \in \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C)$ be a first order deformation of $C \subset \mathbb{P}^3$. Then φ is obstructed at the second order if and only if the cup product $o(\varphi) := \delta(\varphi) \cup \varphi$ by

$$(2.1) \qquad \cup_1 : \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C) \times \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C) \xrightarrow{\cup} \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$$

is non-zero. $o(\varphi)$ is called the *obstruction* to extend φ to second order deformations. Since C and $\mathbb{P}^3$ are both non-singular, C is a local complete intersection in $\mathbb{P}^3$. Therefore the obstruction $o(\varphi)$ is contained in $H^1(\mathcal{N}_C)$, which is regarded as a subspace of $\mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C)$ by the exact sequence

$$0 \longrightarrow H^1(\mathcal{H}om(\mathcal{I}_C, \mathcal{O}_C)) \longrightarrow \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C) \longrightarrow H^0(\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_C)) \longrightarrow 0$$

obtained from local-global spectral sequence for Ext.

From now on, we treat the case where C is contained in a smooth cubic surface S. The natural sheaf inclusion $\mathcal{O}_{\mathbb{P}^3}(-3) \cong \mathcal{I}_S \xhookrightarrow{\iota} \mathcal{I}_C$ induces the homomorphisms

$$\psi : \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C) \longrightarrow \mathrm{Hom}(\mathcal{I}_S, \mathcal{O}_C) \cong H^0(\mathcal{O}_C(3)),$$
$$\psi' : \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C) \longrightarrow \mathrm{Ext}^1(\mathcal{I}_S, \mathcal{I}_C) \cong H^1(\mathcal{I}_C(3)), \qquad \text{and}$$
$$\psi'' : \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C) \longrightarrow \mathrm{Ext}^1(\mathcal{I}_S, \mathcal{O}_C) \cong H^1(\mathcal{O}_C(3)).$$

We denote by π the composite

$$(2.2) \qquad \psi'' \circ o = \psi'' \circ \cup_1 \circ (\delta \times \mathrm{id}) : \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C) \to H^1(\mathcal{O}_C(3)).$$

Then the following is obvious.

Proposition 2.1 ([4] Corollary 1.3). *Let φ be an embedded first order infinitesimal deformation of a curve $C \subset \mathbb{P}^3$ on a smooth cubic surface S. If $\pi(\varphi)$ is non-zero in $H^1(\mathcal{O}_C(3))$, then φ is obstructed at the second order.* $\qquad \square$

Let us give another expression of π. A natural cup product map

$$(2.3) \qquad \cup_2 : H^1(\mathcal{I}_C(3)) \times \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C) \xrightarrow{\cup} H^1(\mathcal{O}_C(3))$$

satisfies the commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C) \times \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C) & \xrightarrow{\cup_1} & \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_C) \\
\downarrow{\scriptstyle (\psi', \mathrm{id})} & & \downarrow{\scriptstyle \psi''} \\
H^1(\mathcal{I}_C(3)) \times \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C) & \xrightarrow{\cup_2} & H^1(\mathcal{O}_C(3)).
\end{array}$$

Moreover, ψ and ψ' naturally satisfy a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C) & \xrightarrow{\delta} & \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C) \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi'} \\
H^0(\mathcal{O}_C(3)) & \xrightarrow{\bar{\delta}} & H^1(\mathcal{I}_C(3)),
\end{array}$$

where $\bar{\delta}$ is the coboundary map of

$$(2.4) \quad 0 \longrightarrow H^0(\mathcal{I}_C(3)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \longrightarrow H^0(\mathcal{O}_C(3)) \overset{\bar{\delta}}{\longrightarrow} H^1(\mathcal{I}_C(3)) \longrightarrow 0$$

induced from $(1.1) \otimes \mathcal{O}_{\mathbb{P}^3}(3)$. Hence we have another expression of π as

$$(2.5) \qquad \pi = \psi'' \circ \cup_1 \circ (\delta \times \mathrm{id}) = \cup_2 \circ ((\psi' \circ \delta) \times \mathrm{id}) = \cup_2 \circ ((\bar{\delta} \circ \psi) \times \mathrm{id}).$$

By definition, ψ maps an element φ of $\mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C)$ to $u = \varphi_3(f)$, where f is the cubic polynomial which defines the isomorphism $\mathcal{O}_{\mathbb{P}^3}(-3) \cong \mathcal{I}_S$, and φ_3 is the homomorphism $H^0(\mathcal{I}_C(3)) \to H^0(\mathcal{O}_C(3))$ induced from φ. Moreover, ψ is surjective.

§2.2.

In this subsection, we recall some basic facts on linear systems on a smooth cubic surface. Let $\mathcal{L}$ be an invertible sheaf on a smooth cubic surface S. We may consider S to be a $\mathbb{P}^2$ blown up at 6 points in a general position and embedded by anti-canonical linear system $|-K_S|$ in $\mathbb{P}^3$. The classes of the pull back $\mathbf{l}$ of a line in $\mathbb{P}^2$ and six exceptional curves $\mathbf{e}_i$ $(1 \leq i \leq 6)$ form a $\mathbb{Z}$-free basis of the Picard group $\mathrm{Pic}\, S$ of S. Thus there is an isomorphism $\mathrm{Pic}\, S \cong \mathbb{Z}^{\oplus 7}$ sending the class $\mathcal{L} = a\mathbf{l} - \sum_{i=1}^{6} b_i \mathbf{e}_i$ to a 7-tuple $(a; b_1, \ldots, b_6)$ of integers. We denote the class $3\mathbf{l} - \sum_{i=1}^{6} \mathbf{e}_i$ of hyperplane sections by $\mathbf{h}$. Recall that the Weyl group $W(\mathbb{E}_6)$ acts on $\mathrm{Pic}\, S$. By virtue of this action, we can choose a suitable blow-up $S \to \mathbb{P}^2$ for $\mathcal{L}$ such that

$$(2.6) \qquad\qquad b_1 \geq b_2 \geq \cdots \geq b_6 \quad \text{and} \quad a \geq b_1 + b_2 + b_3$$

holds. When (2.6) holds, we say the basis $\{\mathbf{l}, \mathbf{e}_1, \ldots, \mathbf{e}_6\}$ is $\mathbb{E}$-*standard* for $\mathcal{L}$. The 7-tuple $(a; b_1, \ldots, b_6)$ is uniquely determined for each invertible sheaf $\mathcal{L}$ on S. We call it the $\mathbb{E}$-*multidegree* of $\mathcal{L}$. For a divisor D on S, we define the $\mathbb{E}$-multidegree of D as that of the associated invertible sheaf $\mathcal{O}_S(D)$.

$\mathbb{E}$-standard basis is useful for analyzing the linear system $|D|$ associated to a divisor D on a smooth cubic surface.

Lemma 2.2.   *Let D be a divisor of $\mathbb{E}$-multidegree $(a; b_1, \ldots, b_6)$ on a smooth cubic surface S.*

(i) *The following are equivalent:*

   (a) $D \geq 0$ *and* $|D|$ *is (base point) free;*

   (b) D *is nef (i.e.* $D \cdot C \geq 0$ *for any curve C on S);*

   (c) $b_6 \geq 0$.

(ii) *If $b_6 \geq 0$, then $D^2 \geq 0$. The equality holds if and only if $a = b_1$.*

(iii) *If $|D| \neq \emptyset$, then the fixed part of $|D|$ is*

$$F = \sum_{\substack{i=1 \\ b_i < 0}}^{6} (-b_i)\mathbf{e}_i$$

for an $\mathbb{E}$-standard basis for $\mathcal{O}_S(D)$.

Here we abusively identify the class $\mathbf{e}_i$ with the unique effective divisor in the class. We refer to Geramita [5] for the proof.

When C is a smooth connected curve on a smooth cubic surface, the $\mathbb{E}$-multidegree $(a; b_1, \ldots, b_6)$ of C satisfies $b_6 \geq 0$ if C is not a line, and $a > b_1$ if C is not a conic.

Let F be a "multiple line" or a "multiple conic" on a smooth cubic surface. We compute h^i $(i = 0, 1)$ of the structure sheaf of F:

Lemma 2.3. *Let $m > 0$ and let mE (resp. mD) be a member of the linear system $|m\mathbf{e}_1|$ (resp. $|m(\mathbf{l} - \mathbf{e}_1)|$) on a smooth cubic surface S. Then we have*

$$\dim H^0(\mathcal{O}_{mE}) = \frac{m(m+1)}{2}, \qquad H^1(\mathcal{O}_{mE}) = 0,$$
$$\dim H^0(\mathcal{O}_{mD}) = m, \qquad and \qquad H^1(\mathcal{O}_{mD}) = 0.$$

Proof. We prove the assertion for a multiple line mE by induction on $m \in \mathbb{N}$. It is clear for $m = 1$. There exists an exact sequence

$$0 \longrightarrow \ker q \longrightarrow \mathcal{O}_{mE} \overset{q}{\longrightarrow} \mathcal{O}_{(m-1)E} \longrightarrow 0.$$

Since the sheaf $\bar{\mathcal{I}}_{mE}$ of ideals defining mE in S is isomorphic to $\mathcal{O}_S(-mE)$, we have isomorphisms

$$\ker q \cong \bar{\mathcal{I}}_{mE}/\bar{\mathcal{I}}_{(m-1)E} \cong \mathcal{O}_S(-(m-1)E)\big|_E \cong \mathcal{O}_{\mathbb{P}^1}(m-1).$$

Therefore, by the inductive assumption, we get

$$h^0(\mathcal{O}_{mE}) = h^0(\ker q) + h^0(\mathcal{O}_{(m-1)E}) = m + m(m-1)/2 = m(m+1)/2$$

and $H^1(\mathcal{O}_{mE}) = 0$. The proof for a multiple conic mD is similar (use $D^2 = 0$). $\square$

We next characterize the freeness of $|D|$ for a divisor D by the vanishing of $H^1(S, -D)$. Let D be a non-zero effective divisor on a smooth cubic surface S. Then $|D|$ has the unique decomposition

$$|D| = |D'| + F,$$

where $F := \mathrm{Bs}\,|D|$ and $|D'|$ is free by Lemma 2.2 (i). (When $|D|$ is free, $D = D'$ and $F = 0$.)

Lemma 2.4.   *Let D, D', and F be as above. Then*

(1) *We have*
$$h^1(S, -D) = h^0(\mathcal{O}_{D'}) + h^0(\mathcal{O}_F) - 1.$$

  If $(D')^2 > 0$, then a general member of $|D'|$ is a smooth connected curve and hence $h^0(\mathcal{O}_{D'}) = 1$. If $(D')^2 = 0$, then a general member of $|D'|$ is a disjoint union of m conics for some $m \in \mathbb{Z}_{\geq 0}$ and hence $h^0(\mathcal{O}_{D'}) = m$.

(2) *Suppose that $D^2 > 0$. Then we have $h^1(S, -D) = h^0(\mathcal{O}_F)$. In particular, $|D|$ is free if and only if $H^1(S, -D) = 0$.*[*]

(3) *If $|D|$ is free, then $H^i(S, D) = 0$ for $i = 1, 2$.*

Proof.   (1) Let D, D', and F be as above. Since D is effective, we have $h^1(S, -D) = h^0(\mathcal{O}_D) - 1$ by an exact sequence $0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$. By Lemma 2.2 (iii), D' and F have disjoint supports. Therefore, we have $\mathcal{O}_D \cong \mathcal{O}_{D'} \oplus \mathcal{O}_F$ and $h^1(S, -D) = h^0(\mathcal{O}_{D'}) + h^0(\mathcal{O}_F) - 1$. When $(D')^2 > 0$, D' is ample or a pull-back of an ample divisor on a $\mathbb{P}^2$ blown-up at less than 6 points. Therefore, a general member of $|D'|$ is a smooth connected curve by Bertini's theorem. When $(D')^2 = 0$, D' is linearly equivalent to $m(\mathbf{l} - \mathbf{e}_1)$ for some $m \geq 0$ by Lemma 2.2 (ii), which is the class of m conics. Therefore, the case is also a consequence of Bertini's theorem together with Lemma 2.3.

(2) Let F be the fixed part of $|D|$. Then F is a disjoint sum of (multiple) lines or zero. Thus we have $F^2 \leq 0$. Since $D'(\sim D - F)$ and F are disjoint, we

---

[*]The only-if part is a particular consequence of Kawamata-Viehweg's vanishing theorem that $H^i(X, K_X + D) = 0$ for a nef (i.e. $D \cdot C \geq 0$ for any curve C) and big (i.e. $D^2 > 0$) Cartier divisor D on a smooth surface X and for $i > 0$. In what follows, we say "D is nef and big" to mean that $|D|$ is free and $D^2 > 0$.

get $D \cdot F = F^2$. Therefore $D^2 > 0$ implies $(D')^2 > 0$ by $(D')^2 = (D - F)^2 = D^2 - 2D \cdot F + F^2 = D^2 - F^2$. If $F \neq 0$, then we get $h^1(S, -D) = h^0(\mathcal{O}_F) \neq 0$ by (1). If $F = 0$ (i.e. $|D|$ is free), then we get $h^1(S, -D) = h^0(\mathcal{O}_D) - 1 = 0$.

(3) Let $\mathbf{h}$ be the class of hyperplane sections of S. Since $|D|$ is free, $D + \mathbf{h}$ is very ample. By the Serre duality and the Kodaira vanishing theorem, we have $H^i(S, D) \cong H^{2-i}(S, -(D + \mathbf{h})) = 0$ for $i = 1, 2$. $\qquad\square$

We use Lemma 2.4 to compute $h^1(\mathcal{I}_C(n))$ $(n \in \mathbb{Z})$ for a curve C on a smooth cubic surface S. Let $(a; b_1, \ldots, b_6)$ be the $\mathbb{E}$-multidegree of C. Given $n \in \mathbb{Z}_{\geq 0}$, we consider the linear system $\Lambda_n := |C - n\mathbf{h}|$ on S, where $\mathbf{h} = (3; 1, \ldots, 1)$ is the class of hyperplane sections. Suppose that $\Lambda_n \neq \emptyset$. Then by Lemma 2.2 (iii), the fixed part F of Λ_n is a disjoint sum of (multiple) lines as follows:

$$(2.7) \qquad F = \sum_{\substack{i=1 \\ b_i < n}}^{6} F_i, \quad F_i := (n - b_i)E_i$$

for an $\mathbb{E}$-standard basis for C. Here each E_i $(1 \leq i \leq 6)$ denotes the line corresponding to the class $\mathbf{e}_i$ of exceptional curve. Since all F_i's are disjoint, we have $\mathcal{O}_F \cong \bigoplus_{b_i < n} \mathcal{O}_{F_i}$. By Lemma 2.3, we get $h^0(\mathcal{O}_{F_i}) = (n + 1 - b_i)(n - b_i)/2$ for every i. The exact sequence $0 \to \mathcal{I}_S(n) \to \mathcal{I}_C(n) \to \mathcal{I}_{C/S}(n) \to 0$ induces an isomorphism

$$(2.8) \qquad H^1(\mathcal{I}_C(n)) \cong H^1(\mathcal{I}_{C/S}(n)) \cong H^1(S, -(C - n\mathbf{h})).$$

Thus we have the next corollary by applying Lemma 2.4 (2) to $D = C - n\mathbf{h}$.

Corollary 2.5. *Let C be a smooth connected curve of $\mathbb{E}$-multidegree $(a, b_1, \ldots, b_6)$ on a smooth cubic surface S. Assume that $\Lambda_n := |C - n\mathbf{h}| \neq \emptyset$ and $(C - n\mathbf{h})^2 > 0$ for $n \in \mathbb{Z}_{\geq 0}$. Then we have*

$$h^1(\mathcal{I}_C(n)) = h^0(\mathcal{O}_F) = \sum_{\substack{i=1 \\ b_i < n}}^{6} \frac{(n + 1 - b_i)(n - b_i)}{2},$$

where $F = \operatorname{Bs} \Lambda_n$. In particular, Λ_n is free if and only if $H^1(\mathcal{I}_C(n)) = 0$.

§2.3.

In this subsection, we define some restriction maps. In what follows, when X is a subscheme of $\mathbb{P}^3$ and F is a polynomial of degree d, we sometimes use the same symbol F to denote the element $F\big|_X$ of $H^0(\mathcal{O}_X(d))$ if there is no confusion.

Let S be a smooth cubic surface and let E be a line on S. Let x, y be two linear forms on $\mathbb{P}^3$ defining E. Then the cubic polynomial f defining S is

$$(2.9) \qquad f = Ax + By$$

for two quadratic polynomials A, B on $\mathbb{P}^3$. By definition, x, y form a basis of $H^0(\mathcal{O}_S(1)(-E))$. The corresponding linear system $\Lambda = |\mathbf{h} - E|$ defines the projection $p : S \to \mathbb{P}^1$ from E. By this map, S has a conic bundle structure. Let x', y' be the sections of $p^*\mathcal{O}_{\mathbb{P}^1}(1)$ corresponding to x, y. Then S is covered by two open subsets $D(x')$ and $D(y')$ of S. Let s be a rational section of $\mathcal{O}_S(1)$ defined by

$$(2.10) \qquad s = \begin{cases} \dfrac{-B}{x} & \text{on } D(x'), \\[2mm] \dfrac{A}{y} & \text{on } D(y'). \end{cases}$$

Then by construction, s is a global section of $\mathcal{O}_S(1)(E)$. Moreover, by the correspondence

$$m : \mathcal{O}_E \xrightarrow{\sim} \mathcal{O}_S(1)(E)\big|_E$$
$$\cup \qquad\qquad \cup$$
$$\mu \longleftrightarrow \mu(s\big|_E),$$

we get a trivialization of the line bundle $\mathcal{O}_S(1)(E)\big|_E \cong \mathcal{O}_{\mathbb{P}^1}$. Applying $\otimes \mathcal{O}_E(n-1)$ to m^{-1}, we have a natural isomorphism $m_n : \mathcal{O}_S(n)(E)\big|_E \xrightarrow{\sim} \mathcal{O}_E(n-1)$. We define a homomorphism r_E of $\mathcal{O}_S$-modules by the composite

$$r_E : \mathcal{O}_S(n)(E) \xrightarrow{res} \mathcal{O}_S(n)(E)\big|_E \xrightarrow{m_n} \mathcal{O}_E(n-1),$$

where res is the restriction map. Then we have an exact sequence

$$(2.11) \qquad 0 \longrightarrow \mathcal{O}_S(n) \longrightarrow \mathcal{O}_S(n)(E) \xrightarrow{r_E} \mathcal{O}_E(n-1) \longrightarrow 0.$$

We explicitly describe the restriction map $H^0(r_E)$ for any positive integer n. Let v be an element of $H^0(\mathcal{O}_S(n)(E))$. Then the multiplication map

$$H^0(\mathcal{O}_S(1)(-E)) \otimes H^0(\mathcal{O}_S(n)(E)) \longrightarrow H^0(\mathcal{O}_S(n+1))$$

gives two elements xv, yv in $H^0(\mathcal{O}_S(n+1))$. Since $H^0(\mathcal{O}_{\mathbb{P}^3}(n+1)) \to H^0(\mathcal{O}_S(n+1))$ is surjective, there exist two polynomials η_1, η_2 of degree $n+1$ such that their restrictions to S are xv, yv. Hence we have an equality

$$(2.12) \qquad v = \frac{\eta_1}{x} = \frac{\eta_2}{y} \quad \text{in} \quad \text{Rat}(\mathcal{O}_S(n)).$$

Since v is globally defined, there exists a polynomial ξ of degree $n-1$ such that

$$(2.13) \qquad x\eta_2 - y\eta_1 = \xi f.$$

Here we see that $\xi\big|_E$ does not depend on the choice of η_1, η_2. Here and later, for a polynomial F, we denote $F \pmod{\langle x, y \rangle}$ by $F\big|_E$. We show that $\xi\big|_E$ agrees with $r_E(v)$.

Claim 2.6. $\quad r_E(v) = \xi\big|_E.$

Proof. Since $f = Ax + By$, we have $x(\eta_2 - A\xi) = y(\eta_1 + B\xi)$ by (2.13). Since x and y are coprime, there exists a polynomial η' of degree n such that $\eta_1 = -B\xi + x\eta'$ and $\eta_2 = A\xi + y\eta'$. Therefore, we obtain $v = \xi s + \eta'$ from (2.12) and (2.10). We see $r_E(\eta') = 0$ because η' is a polynomial. Hence we get $r_E(v) = r_E(\xi s + \eta') = \xi\big|_E$ from $(\xi s)\big|_E = (\xi\big|_E)(s\big|_E)$ and the trivialization m. $\qquad \square$

Thus we get the description of $H^0(r_E)$.

Remark 2.7. Let Λ be the linear system $|\mathbf{h} - E|$ corresponding to $\mathcal{O}_S(1)(-E)$. Then the restriction $\Lambda\big|_E$ is a subpencil of $|\mathcal{O}_E(2)| \cong |\mathcal{O}_{\mathbb{P}^1}(2)|$ since $(\mathbf{h}-E)\cdot E = 2$. Writing the cubic equation f in the form $f = Ax+By$ is also useful to describe the restriction map in this case. We see that planes H through E are parametrized by $\mathbb{P}^1_{(t_0, t_1)}$ and $H = H_{(t_0, t_1)}$ defined by $t_0 x + t_1 y = 0$. A member of Λ is a conic defined by $t_0 x + t_1 y = t_0(-B) + t_1 A = 0$. Hence a member of $\Lambda\big|_E$ is a divisor of degree two on E, which is defined by $t_0(-B)\big|_E + t_1 A\big|_E = 0$.

By a similar argument, we have a natural isomorphism $\mathcal{O}_S(1)(-E)\big|_E \cong \mathcal{O}_E(2)$. The composition of the restriction map $\mathcal{O}_S(1)(-E) \overset{res}{\to} \mathcal{O}_S(1)(-E)\big|_E$ and the isomorphism induces

$$r_E : H^0(\mathcal{O}_S(1)(-E)) \longrightarrow H^0(\mathcal{O}_E(2)),$$

which sends $t_0 x + t_1 y$ to $t_0(-B)\big|_E + t_1 A\big|_E$. We can see the one-to-one correspondence between $|\operatorname{im} r_E|$ and $\Lambda\big|_E$ by taking the divisor of zeros.

§3.   Obstructed Deformation of Space Curves

We devote the whole section to the proof of the next proposition.

Proposition 3.1 (Core Proposition).   *Let S be a smooth cubic surface, let $\mathbf{h}$ be the class of hyperplane sections, and let $\mathbf{D}$ be a divisor class of S satisfying*

(i) *The fixed part of the linear system $|\mathbf{D} - 3\mathbf{h}|$ on S is exactly a line E,*

(ii) $|\mathbf{D} - 4\mathbf{h}| \neq \emptyset$.

Then any general member C of $|\mathbf{D}|$ has some embedded first order infinitesimal deformation which is obstructed at the second order.

First we observe $|\mathbf{D}| \neq \emptyset$ by (ii). Moreover, since both $|\mathbf{D} - 3\mathbf{h} - E|$ and $|3\mathbf{h} + E|$ are free by assumption and Lemma 2.2 (i), a general member C of $|\mathbf{D}|$ is a smooth connected curve by Bertini's theorem. Let S, $\mathbf{h}$, $\mathbf{D}$, E, and C be as in the statement. Let x, y, A, B and f be as in §2.3. We fix these notation throughout the proof. Now we start the proof.

Claim 3.2.   *Let $Z := C \cap E$, then Z is of length two.*

Proof. Let $(a; b_1, \ldots, b_6)$ be the multidegree of C on S and let $\{\mathbf{l}, \mathbf{e}_1, \ldots, \mathbf{e}_6\}$ be an $\mathbb{E}$-standard basis of $\mathrm{Pic}\,S$ for C. Then by Lemma 2.2 (iii), the fixed part $\mathrm{Bs}\,|C - 3\mathbf{h}|$ is a sum $\sum(3 - b_i)\mathbf{e}_i$ over all $b_i < 3$. On the other hand, we have $\mathrm{Bs}\,|C - 3\mathbf{h}| = E$ by assumption. Hence we have $E = \mathbf{e}_6$ and $b_6 = 2$. This implies $C \cdot E = b_6 = 2$.   □

Lemma 3.3.   *Let Λ be the conic pencil $|\mathbf{h} - E|$ on S and let $\Lambda\big|_E$ be its restriction to E. (We refer to Remark 2.7.) Then, Z is not a member of $\Lambda\big|_E$.*

Proof.   There exists an exact sequence

$$0 \longrightarrow \mathcal{O}_S(\mathbf{D} - E) \longrightarrow \mathcal{O}_S(\mathbf{D}) \longrightarrow \mathcal{O}_S(\mathbf{D})\big|_E \longrightarrow 0.$$

Then Lemma 2.4 (3) shows $H^1(S, \mathbf{D} - E) = 0$ because $|\mathbf{D} - E|$ is free. Hence the restriction map $H^0(\mathcal{O}_S(\mathbf{D})) \rightarrow H^0(\mathcal{O}_S(\mathbf{D})\big|_E)$ is surjective. We know $\dim |\mathcal{O}_S(\mathbf{D})\big|_E| = 2$ by Claim 3.2, while $\Lambda\big|_E$ is a pencil. Thus we have $\Lambda\big|_E \subsetneqq |\mathcal{O}_S(\mathbf{D})\big|_E|$. Therefore, any general member C of $|\mathbf{D}|$ meets E at $Z \notin \Lambda\big|_E$.   □

Claim 3.4. $C - 3\mathbf{h} - E$ *is nef and big.*

Proof. Put $D := C - 3\mathbf{h} - E$. Since D is clearly nef by assumption, it suffices to show $D^2 > 0$. Put $D_1 := C - 4\mathbf{h}$ and $D_2 := \mathbf{h} - E$. Then $D = D_1 + D_2$. We obtain

$$D^2 \geq D \cdot D_2 = (D_1 + D_2) \cdot D_2 = D_1 \cdot D_2,$$

because D is nef, D_1 is effective, and $(D_2)^2 = 0$. Since $D_1 \cdot D_2 = (C - 4\mathbf{h}) \cdot (\mathbf{h} - E) \geq (C - 4\mathbf{h}) \cdot (-E) = -2 + 4 = 2$, we have $D^2 > 0$. $\square$

Since $\mathrm{Bs}\,|C - 3\mathbf{h}| = E$ and $(C - 3\mathbf{h} - E)^2 > 0$, we have $h^1(S, -(C - 3\mathbf{h})) = 1$ by Lemma 2.4 (1). Hence we get $h^1(\mathcal{I}_C(3)) = 1$ by (2.8). Thus there exists an element u of $H^0(\mathcal{O}_C(3))$ which is not (the image of) a cubic polynomial, and an element φ of $\mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C)$ such that $\varphi(f) = u$ (cf. the last paragraph of §2.1). Let π be the map defined by (2.2). Then we have $\pi(\varphi) = (\bar{\delta}(\psi(\varphi))) \cup \varphi = (\bar{\delta}(\varphi(f))) \cup \varphi = \bar{\delta}(u) \cup \varphi$ by the alternative expression (2.5) of π. Thus it suffices to show the following: the cup product $\bar{\delta}(u) \cup \varphi$ by

$$\cup_2 : H^1(\mathcal{I}_C(3)) \times \mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C) \xrightarrow{\cup} H^1(\mathcal{O}_C(3))$$

is non-zero in $H^1(\mathcal{O}_C(3))$. (See §2.1 for $\bar{\delta}, \cup_2$ etc.) If it is proved, then by Proposition 2.1, φ is obstructed at the second order. Our procedure for this is as follows: we relate the above cup product map to familiar Serre duality pairing via several cup product maps, and eventually obtain the non-zero of the original product from the perfect pairing. First of all, since $\mathrm{Hom}(\mathcal{I}_C, \mathcal{O}_C) \cong H^0(\mathcal{H}om_{\mathbb{P}^3}(\mathcal{I}_C, \mathcal{O}_C)) \cong H^0(\mathcal{N}_C)$ is a cohomology group on C, the above $\cup_2$ is compatible with the cup product map

$$\cup_3 : H^1(\mathcal{N}_C{}^\vee(3)) \times H^0(\mathcal{N}_C) \xrightarrow{\cup} H^1(\mathcal{O}_C(3))$$

via natural maps. Here $\mathcal{N}_C{}^\vee$ is the conormal bundle $\mathcal{I}_C/\mathcal{I}_C{}^2$ of C. Moreover, since Z is an effective divisor on C, by tensoring $\mathcal{O}_C(2Z)$ with the first and the last sheaves of $\cup_3$, we get another cup product map

$$\cup_4 : H^1((\mathcal{N}_C{}^\vee(3)(2Z)) \times H^0(\mathcal{N}_C) \xrightarrow{\cup} H^1(\mathcal{O}_C(3)(2Z)),$$

which is also compatible with the previous ones $\cup_i$ ($i = 1, 2, 3$) via natural maps.

§3.1.

In this subsection, we compute the obstruction. Let u be as above. By the exact sequence (2.11) as $n = 3$, we have a commutative diagram of exact

sequences

$$
\begin{array}{ccccc}
& 0 & & 0 & & 0 \\
& \downarrow & & \downarrow & & \downarrow \\
0 \to \mathcal{O}_S(3)(-C) & \longrightarrow & \mathcal{O}_S(3)(E-C) & \xrightarrow{r_E} & \mathcal{O}_E(2)(-Z) \to 0 \\
& \downarrow & & \downarrow & & \downarrow \\
(3.1) \qquad 0 \to \quad \mathcal{O}_S(3) & \longrightarrow & \mathcal{O}_S(3)(E) & \xrightarrow{r_E} & \mathcal{O}_E(2) \quad \to 0 \\
& \downarrow {\scriptstyle res} & & \downarrow & & \downarrow \\
0 \to \quad \mathcal{O}_C(3) & \longrightarrow & \mathcal{O}_C(3)(Z) & \longrightarrow & \mathcal{O}_Z \quad \to 0 \\
& \downarrow & & \downarrow & & \downarrow \\
& 0 & & 0 & & 0.
\end{array}
$$

Since $C - 3\mathbf{h} - E$ is nef and big by Claim 3.4, we have

$$H^i(\mathcal{O}_S(3)(E-C)) = 0 \qquad (i = 0, 1)$$

by Lemma 2.4 (2). Hence the diagram induces an isomorphism $H^0(\mathcal{O}_S(3)(E)) \xrightarrow{\sim} H^0(\mathcal{O}_C(3)(Z))$. Thus there exists an element $\hat{u}$ of $H^0(\mathcal{O}_S(3)(E))$ such that $\hat{u}\big|_C = u$. In particular, as we saw in §2.3 (cf. (2.12) and (2.13)), there exist a quadratic polynomial ξ and two quartic polynomials η_1, η_2 such that

(3.2)
$$u = \frac{\eta_1}{x} = \frac{\eta_2}{y} \quad \text{in} \quad \mathrm{Rat}(\mathcal{O}_C(3)) \quad \text{and} \quad x\eta_2 - y\eta_1 = \xi f \quad \text{as a polynomial.}$$

Moreover, by the snake lemma, we have

$$H^1(\mathcal{O}_S(3)(-C)) = \mathrm{Coker}\, H^0(res) \xrightarrow{\sim} H^0(\mathcal{O}_E(2)(-Z)).$$

By the choice of u (not a cubic polynomial), we have the following:

(3.3) $r_E(\hat{u}) = \xi\big|_E \neq 0 \quad \text{in} \quad H^0(\mathcal{O}_E(2)), \quad \text{and}$

(3.4) $\mathrm{div}(\xi\big|_E) = Z.$

These respectively follow from the explicit description of r_E in Claim 2.6 and the direct diagram chasing.

Before we start the computation, we observe one sheaf inclusion $\mathcal{O}_C(2Z) \subset \mathcal{N}_C^\vee(3)(2Z)$. We get the inclusion by taking the dual of the exact sequence of normal bundles

(3.5) $$0 \longrightarrow \underbrace{\mathcal{N}_{C/S}}_{\cong\, \omega_C(1)} \longrightarrow \mathcal{N}_C \longrightarrow \underbrace{\mathcal{N}_S \otimes \mathcal{O}_C}_{\cong\, \mathcal{O}_C(3)} \longrightarrow 0$$

and then tensoring with $\mathcal{O}_C(3)(2Z)$. We see that the inclusion induces an injection between their H^1. For the injectivity, it is enough to show that $\mathcal{N}_{C/S}^\vee(3)(2Z) \cong \mathcal{O}_S(3\mathbf{h} + 2E - C)\big|_C$ does not have global sections. Indeed, we have

$$(3\mathbf{h} + 2E - C) \cdot C = -(C - 3\mathbf{h} - E)^2 - 3\mathbf{h} \cdot (C - 3\mathbf{h} - E) + 2 < 0,$$

since $C - 3\mathbf{h} - E$ is nef (hence effective) and big. Therefore we get the injection.

Lemma 3.5. *Let φ, u, and ξ be as above. Let $\mathbf{t}$ be the image of $\bar{\delta}(u)$ by the map $H^1(\mathcal{I}_C(3)) \to H^1(\mathcal{N}_C^\vee(3)(2Z))$. Then we have the following:*

(1) $\mathbf{t}$ *is contained in $H^1(\mathcal{O}_C(2Z)) \subset H^1(\mathcal{N}_C^\vee(3)(2Z))$. Moreover, the cup product by $\cup_4$ corresponding to φ equals the cup product $\mathbf{t} \cup u$ by*

$$\cup_5 : H^1(\mathcal{O}_C(2Z)) \times H^0(\mathcal{O}_C(3)) \overset{\cup}{\longrightarrow} H^1(\mathcal{O}_C(3)(2Z)).$$

(2) *Let $p : C \to \mathbb{P}^1$ be the projection from Z, and let x', y' be two linearly independent global sections of $p^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_C(\mathbf{h} - Z)$ corresponding to x, y. Then $\mathbf{t}$ is represented by a 1-cocycle*

$$\frac{\xi}{xy} \in C^1(\mathfrak{U}_1, \mathcal{O}_C(2Z)) = \Gamma(D(x') \cap D(y'), \mathcal{O}_C(2Z))$$

with respect to the open affine covering $\mathfrak{U}_1 = \{D(x'), D(y')\}$ of C.

Proof. We compute the coboundary $\bar{\delta}(u)$ in $H^1(\mathcal{I}_C(3))$. We recall the cubic equation $f = Ax + By$ defining S (cf. §2.3). By the smoothness of S,

$$\mathfrak{U}_2 := \{D(x), D(y), D(A), D(B)\}$$

is an open affine covering of $\mathbb{P}^3$. We compute $\bar{\delta}(u)$ by the Čech cohomology with respect to $\mathfrak{U}_2$. By (3.2) u is represented by η_1/x over $D(x)$ and η_2/y over $D(y)$, where η_1, η_2 are quartic polynomials such that $x\eta_2 - y\eta_1 = \xi f$. Therefore, $\bar{\delta}(u)$ in $H^1(\mathcal{I}_C(3))$ is represented by

$$\bar{\delta}(u) = \frac{\eta_2}{y} - \frac{\eta_1}{x} = \frac{x\eta_2 - y\eta_1}{xy} = \frac{\xi}{xy}f$$

over $D(x) \cap D(y)$. Thus $\bar{\delta}(u)$ is contained in the subsheaf $\mathcal{I}_S(3) \subset \mathcal{I}_C(3)$ over $D(x) \cap D(y)$. Restricting it to C, we see that $\mathbf{t}$ is contained in the subsheaf $\mathcal{O}_C \subset \mathcal{N}_C^\vee(3)$ over $D(x) \cap D(y)$ and represented by ξ/xy there.

On the other hand, the subcovering $\{D(x), D(y)\}$ of $\mathfrak{U}_2$ covers whole C except for Z. Indeed, the two linear forms $\{x, y\}$ is a basis of the pencil $P :=$

$H^0(\mathcal{O}_C(1)(-Z))$ and the fixed part of P is exactly Z. Therefore $D(x) = D(x') \setminus Z$, $D(y) = D(y') \setminus Z$, and ξ/xy gives a section of $\mathcal{O}_C(2Z)$ over $D(x') \cap D(y')$. Now we make a change on the coverings of C. We consider another open affine covering

$$\mathfrak{U}_3 := \{D(x'), D(y'), D(A), D(B)\}$$

of C. Then both $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are refinements of $\mathfrak{U}_3$. There are isomorphisms between all Čech cohomology groups

$$\check{H}^1(\mathfrak{U}_i, \mathcal{O}_C(2Z)) \quad (1 \leq i \leq 3)$$

induced by natural maps

$$C^\bullet(\mathfrak{U}_1, \mathcal{O}_C(2Z)) \longleftarrow C^\bullet(\mathfrak{U}_3, \mathcal{O}_C(2Z)) \longrightarrow C^\bullet(\mathfrak{U}_2, \mathcal{O}_C(2Z))$$

of Čech complexes with respect to $\mathfrak{U}_i$ $(1 \leq i \leq 3)$. Moreover, by the above computation, we see that the 1-cocycle representing $\mathbf{t}$ can be taken from the one in $C^1(\mathfrak{U}_3, \mathcal{O}_C(2Z))$, and mapped to ξ/xy in $C^1(\mathfrak{U}_1, \mathcal{O}_C(2Z))$. Hence we have proved (2) and $\mathbf{t} \in H^1(\mathcal{O}_C(2Z))$. Finally, we prove (1). By the definition of u, the restriction of φ to $\mathcal{O}_C \subset \mathcal{N}_C^\vee(3)$ as a homomorphism $\mathcal{N}_C^\vee(3) \to \mathcal{O}_C(3)$ is the multiplication map by u. The desired cup product is $\mathbf{t} \cup u$ by $\cup_5$. $\square$

§3.2.

In this subsection, we show that the cup product $\mathbf{t} \cup u$ obtained in Lemma 3.5 is non-zero. For this purpose, first we show $\mathbf{t} \neq 0$.

Claim 3.6. $\mathbf{t} \neq 0$ in $H^1(\mathcal{O}_C(2Z))$.

Proof.   Since $\{x, y\}$ is a basis of $H^0(\mathcal{O}_S(1)(-E))$, by the base point free pencil trick, there exists an exact sequence

$$\mathbb{K} : 0 \longrightarrow \mathcal{O}_S(2E) \xrightarrow{[y \ -x]} \mathcal{O}_S(1)(E)^{\oplus 2} \xrightarrow{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]} \mathcal{O}_S(2) \longrightarrow 0$$

of Koszul type. The restriction of $\mathbb{K}$ to C is the exact sequence

$$\mathbb{K}_C : 0 \longrightarrow \mathcal{O}_C(2Z) \xrightarrow{[y \ -x]} \mathcal{O}_C(1)(Z)^{\oplus 2} \xrightarrow{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]} \mathcal{O}_C(2) \longrightarrow 0.$$

The restriction map $\mathbb{K} \to \mathbb{K}_C$ induces

$$
\begin{array}{ccccc}
H^0(\mathcal{O}_S(1)(E)^{\oplus 2}) & \xrightarrow{\sigma} & H^0(\mathcal{O}_S(2)) & \xrightarrow{\delta''} & H^1(\mathcal{O}_S(2E)) \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle res} \\
H^0(\mathcal{O}_C(1)(Z)^{\oplus 2}) & \xrightarrow{\gamma} & H^0(\mathcal{O}_C(2)) & \xrightarrow{\delta'} & H^1(\mathcal{O}_C(2Z)).
\end{array}
$$

By the definition of the Čech coboundary map and the description of $\mathbf{t}$ obtained in Lemma 3.5 (2), we have $\delta'(\xi|_C) = \mathbf{t}$. Put $\hat{\mathbf{t}} := \delta''(\xi|_S)$. Then $\hat{\mathbf{t}}$ is an element of $H^1(\mathcal{O}_S(2E))$ such that $\hat{\mathbf{t}} = \xi/xy$ over $D(x') \cap D(y')$ and $\hat{\mathbf{t}}|_C = \mathbf{t}$.

On the other hand, we obtain the exact sequence

$$\mathbb{K}_E : 0 \longrightarrow \mathcal{O}_E(-2) \xrightarrow{[\,A\ B\,]} \mathcal{O}_E^{\oplus 2} \xrightarrow{\left[\begin{smallmatrix} -B \\ A \end{smallmatrix}\right]} \mathcal{O}_E(2) \longrightarrow 0$$

as the restriction of $\mathbb{K}$ to E. Here A, B denote the quadratic polynomials in the equation $f = Ax + By$ of S. The restriction map $\mathbb{K} \to \mathbb{K}_E$ induces

$$
\begin{array}{ccccc}
H^0(\mathcal{O}_S(1)(E)^{\oplus 2}) & \xrightarrow{\ \sigma\ } & H^0(\mathcal{O}_S(2)) & \xrightarrow{\ \delta''\ } & H^1(\mathcal{O}_S(2E)) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(\mathcal{O}_E^{\oplus 2}) & \xrightarrow{\ \varepsilon\ } & H^0(\mathcal{O}_E(2)) & \longrightarrow & H^1(\mathcal{O}_E(-2)).
\end{array}
$$

It follows from $H^1(\mathcal{O}_S(2E - C)) = 0$ that the restriction map $H^1(\mathcal{O}_S(2E)) \xrightarrow{res} H^1(\mathcal{O}_C(2Z))$ is injective. Hence it suffices to prove $\hat{\mathbf{t}} \neq 0$ for the claim. Suppose that $\hat{\mathbf{t}} = 0$ for contradiction. Then $\xi|_S \in \operatorname{im} \sigma$ and hence $\xi|_E \in \operatorname{im} \varepsilon$. This implies that $\xi|_E$ is a linear combination of $A|_E$ and $B|_E$. When we consider the divisors of zeros corresponding to $\xi|_E$ and $\langle A|_E, B|_E \rangle$, this means $Z = \operatorname{div}(\xi|_E)$ (by (3.4)) belongs to the restriction $\Lambda|_E$ of the conic pencil $\Lambda = |\mathbf{h} - E|$ to E (cf. Remark 2.7). This contradicts Lemma 3.3. Thus $\hat{\mathbf{t}} \neq 0$. $\qquad\square$

We next prepare an effective divisor Δ on S which fills a gap between Mumford's case ($C \sim 4\mathbf{h} + 2E$) and our general case. Consider the linear system $|C - 4\mathbf{h}|$ ($\neq \emptyset$ by assumption) on S. Since $(C - 4\mathbf{h} - mE) \cdot E = -2 + m < 0$ if and only if $m < 2$, $\operatorname{Bs}|C - 4\mathbf{h}|$ contains E with multiplicity two. We take a member $\Delta^\dagger$ of $|C - 4\mathbf{h} - 2E|$ which is disjoint from E and fix it. Then there exists a cup product map

$$\cup_6 : H^1(\mathcal{O}_C(2Z)) \times H^0(\mathcal{O}_C(3)(\Delta)) \longrightarrow H^1(\mathcal{O}_C(3)(2Z + \Delta)),$$

which is compatible with $\cup_5$ via natural maps. The last sheaf $\mathcal{O}_C(3)(2Z + \Delta)$ is isomorphic to the canonical line bundle $\mathcal{O}_C(K_C)$ by

$$\mathcal{O}_C(3)(2Z + \Delta) \cong \mathcal{O}_S(3\mathbf{h} + 2E + \Delta)\big|_C \cong \mathcal{O}_S(C - \mathbf{h})\big|_C \cong \mathcal{O}_C(K_C).$$

Thus $\cup_6$ is the Serre duality cup pairing for $\mathcal{O}_C(2Z)$. Now we consider an exact sequence

$$(3.6) \qquad 0 \longrightarrow \underbrace{\mathcal{O}_S(-\mathbf{h})}_{\cong \mathcal{O}_S(K_S)} \longrightarrow \mathcal{O}_S(C - \mathbf{h}) \longrightarrow \underbrace{\mathcal{O}_S(C - \mathbf{h})\big|_C}_{\cong \mathcal{O}_C(K_C)} \longrightarrow 0.$$

---

†When $\Delta = 0$, then $C \sim 4\mathbf{h} + 2E$ on S. This is exactly the case of Mumford's example ([10]). Taking $\Delta = 0$ in our proof, we have a proof for his case. Thus Proposition 3.1 is a natural generalization of his example.

Then the cup product map with its extension class $\mathbf{e}$, which is the coboundary map of (3.6), induces the next commutative diagram:

$$
\begin{array}{ccc}
H^1(\mathcal{O}_C(2Z)) \times \ \ H^0(\mathcal{O}_C(3)(\Delta))) & \xrightarrow{\ \cup_6\ } & H^1(\mathcal{O}_C(3)(2Z+\Delta)) \quad \cong H^1(K_C) \\
{\scriptstyle res}\uparrow \qquad\qquad\qquad \downarrow{\scriptstyle \cup\,\mathbf{e}} & & \qquad\qquad \downarrow{\scriptstyle \cup\,\mathbf{e}} \\
H^1(\mathcal{O}_S(2E)) \times H^1(\mathcal{O}_S(3)(\Delta - C))) & \xrightarrow{\ \cup_7\ } & H^2(\mathcal{O}_S(3)(2E+\Delta-C)) \cong H^2(K_S),
\end{array}
$$

where res is the restriction map in the proof of Claim 3.6. The last cup product $\cup_7$ is the Serre duality cup pairing for $\mathcal{O}_S(2E)$.

We have already got the non-zero element $\hat{\mathbf{t}}$ of $H^1(\mathcal{O}_S(2E))$ such that $\hat{\mathbf{t}}\big|_C = \mathbf{t}$ in the proof of Claim 3.6. By the commutativity of the diagram, we have $(\mathbf{t} \cup u) \cup \mathbf{e} = \hat{\mathbf{t}} \cup (u \cup \mathbf{e})$. Since $H^1(\mathcal{O}_S(2E))$ is of dimension one, by the Serre duality, we have only to show that $u \cup \mathbf{e} \neq 0$ in $H^1(\mathcal{O}_S(3)(\Delta - C)))$ instead of $\mathbf{t} \cup u \neq 0$ in $H^1(K_C)$.

Claim 3.7. $u \cup \mathbf{e} \neq 0$ *in* $H^1(\mathcal{O}_S(3)(\Delta - C))$.

Proof. Suppose that $u \cup \mathbf{e} = 0$ for contradiction. Since the cup product map with $\mathbf{e}$ is the coboundary map of the exact sequence

$$
0 \longrightarrow \mathcal{O}_S(3)(\Delta - C) \longrightarrow \mathcal{O}_S(3)(\Delta) \longrightarrow \mathcal{O}_C(3)(\Delta) \longrightarrow 0,
$$

there exists $\hat{u}'$ in $H^0(\mathcal{O}_S(3)(\Delta))$ such that $\hat{u}'\big|_C = u$. Since Δ and E are disjoint, the image of $\hat{u}'$ by the restriction map

$$
H^0(\mathcal{O}_S(3)(\Delta + E)) \xrightarrow{\ r_E\ } H^0(\mathcal{O}_E(2))
$$

is zero. Now we recall that u has a lift $\hat{u}$ in $H^0(\mathcal{O}_S(3)(E))$ such that $r_E(\hat{u}) \neq 0$ by (3.3). Since $H^0(\mathcal{O}_S(3)(\Delta + E - C)) \cong H^0(S, -\mathbf{h} - E) = 0$, we deduce $\hat{u}' = \hat{u}$ from $\hat{u}'\big|_C = \hat{u}\big|_C = u$. This is a contradiction.   □

Therefore we complete the proof of Proposition 3.1.   □

§3.3.

In this subsection, we give a technical remark to Proposition 3.1. In the proof of this proposition, the assumption that C is a general member of $|\mathbf{D}|$ was used only to prove Lemma 3.3. We characterize the members C that do not satisfy $Z = C \cap E \notin \Lambda\big|_E$, where Λ is the conic pencil $|\mathbf{h} - E|$ on S.

Proposition 3.8.   *Let* C, E, Z, *and* Λ *be as above. Then the following two conditions are equivalent:* (1) $Z \notin \Lambda\big|_E$; (2) $H^0(\mathcal{O}_C(1)(-2Z)) = 0$.

Proof. Let us consider the commutative diagram of restriction maps:

$$
\begin{array}{ccc}
H^0(\mathcal{O}_S(1)(-E)) & \xrightarrow{r_1} & H^0(\mathcal{O}_E(2)) \\
\downarrow{\scriptstyle v_1} & & \downarrow{\scriptstyle v_2} \\
H^0(\mathcal{O}_C(1)(-Z)) & \xrightarrow{r_2} & H^0(\mathcal{O}_Z).
\end{array}
$$

The first condition is equivalent to the injectivity of the the composite $v_2 \circ r_1$. On the other hand, the second condition is equivalent to the injectivity of r_2. Therefore, it suffices to show that v_1 is an isomorphism. In fact, we can easily check that $C - \mathbf{h} + E$ is nef and big. This implies that $H^i(\mathcal{O}_S(1)(-E-C)) = 0$ for $i = 0, 1$. Thus we have the equivalence. $\square$

Suppose that $H^0(\mathcal{O}_C(1)(-2Z)) \neq 0$. Then there exists a plane H which is tangential to C at Z. Let $Z = p + q$ where $p, q \in C$. Then the tangents to C at p and q are coplanar. (See Figure 1.)
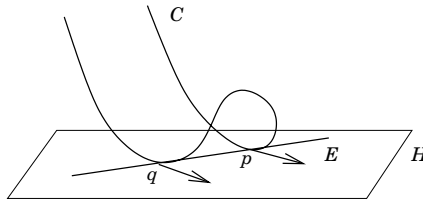

Figure 1. Two tangents on a plane

When $Z \in \Lambda\big|_E$, what can we say about the obstruction? Let C be such a special member of $|\mathbf{D}|$. Then the reverse diagram chase in the proof of Claim 3.6 shows $\mathbf{t} = 0$. Thus the cup product $\mathbf{t} \cup u$ by $\cup_5$ is zero. Since $H^1(\mathcal{N}_C), H^1(\mathcal{O}_C(3))$ and $H^1(\mathcal{O}_C(3)(2Z))$ are all isomorphic via natural maps, we deduce all the previous cup products by $\cup_i$ $(i \leq 4)$ are zero. Hence $\varphi \in H^0(\mathcal{N}_C)$ corresponding to $\mathbf{t} = 0$ is not obstructed at the second order. However, we will later see that C corresponds to a non-reduced point of the Hilbert scheme (cf. Proposition 4.5). This implies that φ is obstructed at the n-th order for some $n \geq 3$.

§4. An Application to Non-reduced Components of the Hilbert Scheme

In this section, we apply Proposition 3.1 to a problem on non-reduced components of the Hilbert scheme, and prove the main theorem. The theorem shows that a special case of Conjecture 4.7 of Kleppe and Ellia is true.

Let W be an irreducible closed subset of the Hilbert scheme $H_{d,g}^S$ with $d \geq 3$. Suppose that W is maximal among all the irreducible closed subsets of $H_{d,g}^S$ whose general member is contained in a smooth cubic surface. Let C be a general member of W and let S be a general cubic surface containing C. Then we obtain a 7-tuple $(a; b_1, \ldots, b_6)$ of integers satisfying

$$(4.1) \qquad \begin{cases} a > b_1 \geq b_2 \geq \cdots \geq b_6 \geq 0, \quad a \geq b_1 + b_2 + b_3, \\ d = 3a - \displaystyle\sum_{i=1}^{6} b_i, \quad \text{and} \quad g = \binom{a-1}{2} - \displaystyle\sum_{i=1}^{6} \binom{b_i}{2} \end{cases}$$

as the $\mathbb{E}$-multidegree of C. (See §2.2 for more detail.)

Conversely, suppose that a 7-tuple $(a; b_1, \ldots, b_6)$ satisfying (4.1) is given. If $\mathcal{L}$ is an invertible sheaf of this multidegree on a smooth cubic surface S, then every general member of $|\mathcal{L}|$ is a smooth connected curve by the conditions $a > b_1$ and $b_6 \geq 0$. Thus we have a non-empty irreducible closed subset W of $H_{d,g}^S$ by

(4.2)

$$W := \left\{ C \in H_{d,g}^S \,\middle|\, C \subset S \colon \text{a smooth cubic, } \mathcal{O}_S(C) \cong \mathcal{O}_S(a; b_1, \ldots, b_6) \in \operatorname{Pic} S \right\}^{-},$$

where $^{-}$ denotes the closure in $(H_{d,g}^S)_{\mathrm{red}}$.

Definition 4.1. For a 7-tuple $(a; b_1, \ldots, b_6)$ of integers satisfying (4.1), we denote the above subset W of $H_{d,g}^S$ by $W_{(a; b_1, \ldots, b_6)}$.

When $d > 9$, any general member C of W is contained in the unique cubic surface S, and furthermore, the above construction gives one-to-one correspondence $(a; b_1, \ldots, b_6) \leftrightarrow W_{(a; b_1, \ldots, b_6)}$ between the 7-tuples satisfying (4.1) and the maximal irreducible closed subsets W of $H_{d,g}^S$ whose general member is contained in a smooth cubic surface (cf. [6, Remark 2]). Thus to determine all irreducible components of $H_{d,g}^S$ whose general member is contained in a smooth cubic, it suffices to solve the next problem:

Problem 4.2. *Determine all* $W_{(a; b_1, \ldots, b_6)}$ *that are irreducible components of* $(H_{d,g}^S)_{\mathrm{red}}$.

The above problem makes sense only when $g \geq 3d - 18$. This is because, as is found in [6], $\dim W_{(a; b_1, \ldots, b_6)} = d + g + 18$ when $d > 9$, while every irreducible component of $H_{d,g}^S$ is of dimension at least $4d \ (= \chi(\mathcal{N}_C))$ from a general theory. In what follows, we consider the above problem in the range

$$\Omega := \{ (d, g) \in \mathbb{Z}^2 \,|\, d > 9, \ g \geq 3d - 18 \}.$$

Let $(d, g) \in \Omega$, let $W = W_{(a; b_1, \ldots, b_6)}$, and let C be a general member of W. Then, we have natural inequalities

$$(4.3) \qquad \dim W \leq \dim_{[C]} H_{d,g}^S \leq h^0(\mathcal{N}_C).$$

If $\dim W = \dim_{[C]} H_{d,g}^S$, then W is an irreducible component of $(H_{d,g}^S)_{\mathrm{red}}$. $H_{d,g}^S$ is smooth at $[C]$ if and only if $\dim_{[C]} H_{d,g}^S = h^0(\mathcal{N}_C)$. The exact sequence (3.5) induces $H^1(\mathcal{N}_C) \cong H^1(\mathcal{O}_C(3))$ because we have $H^1(\omega_C(1)) = 0$. Therefore we get

$$(4.4) \qquad h^0(\mathcal{N}_C) - \dim W = (4d + h^1(\mathcal{O}_C(3))) - (d + g + 18) = h^1(\mathcal{I}_C(3)).$$

Here the last equality follows from the exact sequence (2.4). By the same equality, in our case where $\dim W \geq 4d$, we always have

$$(4.5) \qquad h^1(\mathcal{O}_C(3)) \geq h^1(\mathcal{I}_C(3)).$$

Let S be the cubic surface containing C and let $\mathbf{h}$ be the class of hyperplane sections of S. Then, as we saw in §2, the dimension $h^1(\mathcal{I}_C(3))$ can be computed from the fixed part F of the linear system $\Lambda_3 := |C - 3\mathbf{h}|$ on S. By the formula (2.7), F is empty (i.e. Λ_3 is free), or a union of three kinds of (multiple) lines: single, double, or triple.

Lemma 4.3. *Let $(d, g) \in \Omega$, let $W = W_{(a; b_1, \ldots, b_6)} \subset H_{d,g}^S$, and let C be as above.*

(1) *If $d < 12$, then $H^1(\mathcal{I}_C(3)) = 0$.*

(2) *If $d \geq 12$, then we have*

$$h^1(\mathcal{I}_C(3)) = \sharp \{i | b_i = 2\} + 3(\sharp \{i | b_i = 1\}) + 6(\sharp \{i | b_i = 0\}),$$

where $\sharp$ denotes the cardinality of a set. In particular, $H^1(\mathcal{I}_C(3)) = 0$ if and only if $b_6 \geq 3$.

Proof. Let S, $\mathbf{h}$, and Λ_3 be as above. By the Serre duality, we have

$$(4.6) \qquad H^1(\mathcal{O}_C(3))^\vee \cong H^2(\mathcal{O}_S(3\mathbf{h} - C))^\vee \cong H^0(\mathcal{O}_S(C - 4\mathbf{h})).$$

Suppose $d < 12$. Then the last cohomology group vanishes because $(C - 4\mathbf{h}) \cdot \mathbf{h} = d - 12$. This implies $H^1(\mathcal{I}_C(3)) = 0$ by (4.5). Thus we proved (1). Suppose $d \geq 12$. Then by the Riemann-Roch theorem on S, we have $\chi(C - 3\mathbf{h}) = g - 2d + 9 \geq d - 9 > 0$, while $H^2(C - 3\mathbf{h}) \cong H^0(2\mathbf{h} - C)^\vee = 0$. Therefore

$C - 3\mathbf{h}$ is effective. Similarly, we have $(C - 3\mathbf{h})^2 = 2g - 5d + 25 \geq d - 11 > 0$. By applying Corollary 2.5 to Λ_3, we get the conclusion. $\square$

When $b_6 \geq 3$ (i.e. Λ_3 is free), the lemma shows $H^1(\mathcal{I}_C(3)) = 0$. This implies $h^0(\mathcal{N}_C) = \dim W$ by (4.4). Thus the following is obvious.

Proposition 4.4 (Kleppe [6])**.** *Let* $(d, g) \in \Omega$ *and let* $W = W_{(a;b_1,\ldots,b_6)} \subset H^S_{d,g}$. *If* $b_6 \geq 3$, *then* $H^S_{d,g}$ *is generically non-singular along* W. *Moreover,* W *is an irreducible component of* $H^S_{d,g}$.

When $d \geq 12$ and $b_6 \leq 2$ (i.e. Λ_3 is non-free), we have $h^1(\mathcal{I}_C(3)) \neq 0$ by Lemma 4.3. So there may be some irreducible component V which strictly contains W. In this case, Problem 4.2 becomes non-trivial. However, as long as we study the case where $h^1(\mathcal{I}_C(3)) = 1$, the dichotomy between (A) and (B) described in the introduction (cf. §1) makes the situation simple. Now we give a proof of Theorem 1.2.

Proof of Main Theorem   Let W be as in the statement. Then $W = W_{(a;b_1,\ldots,b_6)}$ for some 7-tuple $(a; b_1, \ldots, b_6)$ satisfying (4.1). Lemma 4.3 shows that we have $h^1(\mathcal{I}_C(3)) = 1$ if and only if $d \geq 12$, $b_6 = 2$ and $b_5 \geq 3$. Thus the proof of the theorem reduces to the next proposition which is an application of Proposition 3.1.

Proposition 4.5.   *Let* $d \geq 12$, *let* $g \geq 3d - 18$, *and let* $W = W_{(a;b_1,\ldots,b_6)} \subset H^S_{d,g}$. *If* $b_6 = 2$ *and* $b_5 \geq 3$, *then* $H^S_{d,g}$ *is generically singular along* W. *Moreover,* W *is an irreducible component of* $(H^S_{d,g})_{\mathrm{red}}$. *Hence* $H^S_{d,g}$ *is non-reduced along* W.

Proof.   We check that any general member C of W satisfies the two conditions (i) and (ii) of Proposition 3.1. The condition (i) is clearly satisfied with $E = E_6$ because of Lemma 2.2 (iii). Since $h^1(\mathcal{I}_C(3)) = 1$, we have $H^1(\mathcal{O}_C(3)) \neq 0$ by (4.5). Therefore, the condition (ii) follows from (4.6).

Since C has an obstructed deformation by Proposition 3.1, $H^S_{d,g}$ is singular at $[C]$ and we have $\dim_{[C]} H^S_{d,g} < h^0(\mathcal{N}_C)$. Consequently, we have $\dim W = \dim_{[C]} H^S_{d,g}$ in (4.3) from $h^1(\mathcal{I}_C(3)) = 1$. Hence W is an irreducible component of $(H^S_{d,g})_{\mathrm{red}}$. Moreover, since $H^S_{d,g}$ is singular at any general point of W, $H^S_{d,g}$ is non-reduced along W. $\square$

Therefore the proof of Theorem 1.2 is completed. $\square$

We give some example of non-reduced components of the Hilbert scheme.

Example 4.6.   Let $\lambda \geq 0$ be an integer. Then the subsets

$$W_{(\lambda+12;\lambda+3,3,3,3,3,2)} \subset H^S_{d,4d-37} \quad (d = 2\lambda + 19) \quad \text{and}$$

$$W_{(\lambda+12;\lambda+4,3,3,3,3,2)} \subset H^S_{d,\frac{7}{2}d-27} \quad (d = 2\lambda + 18)$$

are irreducible components of $(H^S_{\mathbb{P}^3})_{\mathrm{red}}$. Moreover, $H^S_{\mathbb{P}^3}$ is non-reduced along each of them.

Theorem 1.2 shows that the next conjecture is true whenever $h^1(\mathcal{I}_C(3)) = 1$ without the assumption that $H^1(\mathcal{I}_C(1)) = 0$. In fact, $H^1(\mathcal{I}_C(1)) = 0$ follows from $h^1(\mathcal{I}_C(3)) = 1$.

Conjecture 4.7 (Kleppe [6], Ellia [3])**.**   Let $(d,g) \in \Omega$ and let W be an irreducible closed subset of $H^S_{d,g}$ whose general member C is contained in a smooth cubic surface. Suppose that W is maximal among all such subsets. If $H^1(\mathcal{I}_C(3)) \neq 0$ and $H^1(\mathcal{I}_C(1)) = 0$, then W is an irreducible component of $(H^S_{d,g})_{\mathrm{red}}$ of dimension $d + g + 18$. Moreover, $H^S_{d,g}$ is non-reduced along W.

Remark 4.8.   This was originally conjectured by Kleppe in [6] without the assumption of linearly normality $(H^1(\mathcal{I}_C(1)) = 0)$. He proved that the conjecture is true in the following two ranges: $g > 7 + (d-2)^2/8$ for $d \geq 18$, $g > -1 + (d^2 - 4)/8$ for $14 \leq d \leq 17$. When $d < 14$, we have $H^1(\mathcal{I}_C(3)) = 0$ by e.g. Lemma 4.3 (1) or [6, Corollary 17]. Hence he considered the conjecture in the range $d \geq 14$. Later, Ellia [3] proved the conjecture for the wider range that $g > G(d,5)$ for $d \geq 21$. Here $G(d,5)$ denotes the maximal genus of curves of degree d, not contained in a quartic surface. $G(d,5)$ nearly equals $d^2/10$ for $d \gg 0$. Moreover, he gave a counterexample for linearly non-normal curves, and suggested restricting the conjecture to linearly normal curves.

After the original version of this paper was submitted, the author learned that Kleppe [7] had made further progress in proving the conjecture: his result consists of a proof of the conjecture for part of the case $h^1(\mathcal{I}_C(3)) = 1$ and that for part of the case $h^1(\mathcal{I}_C(3)) = 3$, but does not cover our result (cf. Example 4.6). The method of his proofs is different from ours (cf. Remark 4.9).

Remark 4.9.   To prove Conjecture 4.7 for a given $W = W_{(a;b_1,\ldots,b_6)} \subset H^S_{d,g}$, it suffices to prove that W is a component of $(H^S_{d,g})_{\mathrm{red}}$ because $H^S_{d,g}$ is automatically non-reduced along W by the assumption $H^1(\mathcal{I}_C(3)) \neq 0$. In [6],[3] and [7], the authors proved that W is a component of $(H^S_{d,g})_{\mathrm{red}}$ by contradiction. First they assumed that a general member C of W is a specialization

of curves contained not in a cubic but in a surface of degree greater than three. Then they got a contradiction by using a dimension count of a certain family of curves on a quartic ([6], [3]), or using the fact that the dimension of cohomology groups can only increase under specialization by semicontinuity ([7]).

Finally we remark that W in Conjecture 4.7 is not an irreducible component of $(H_{d,g}^S)_{\mathrm{red}}$ provided that $h^1(\mathcal{I}_C(1)) \neq 0$. This fact is obtained from the following, whose proof is essentially given by [3, Remark VI.6] and [2, Remark 2.10].

Proposition 4.10 (Ellia [3], Dolcetti-Pareschi [2]). *Let $(d,g) \in \Omega$ and let $W = W_{(a;b_1,...,b_6)} \subset H_{d,g}^S$. Suppose that $b_6 = 0$. Then W is not an irreducible component of $(H_{d,g}^S)_{\mathrm{red}}$.*

Appendix (Irreducible components of $H_{d,g}^S$ whose general member is contained in a smooth quadric)

We can naturally consider the same problem as Problem 4.2 for curves contained in a smooth quadric surface $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ with bidegree $(a,b) \in \mathrm{Pic}\, Q \cong \mathbb{Z}^2$. This problem is easier than Problem 4.2. One of the reason for this is that we have $H^1(Q, D) = 0$ for any effective divisor D on Q.

Let $d > 4$ and $g \geq 0$ be two integers. For a pair (a,b) of non-negative integers satisfying $a + b = d$, $(a-1)(b-1) = g$ and $a \geq b > 0$, we define an irreducible closed subset $W_{(a,b)}$ of $H_{d,g}^S$ as follows:

$$W_{(a,b)} := \overline{\left\{ C \in H_{d,g}^S \mid C \subset Q: \text{a smooth quadric}, \, \mathcal{O}_Q(C) \cong \mathcal{O}_Q(a,b) \in \mathrm{Pic}\, Q \right\}}.$$

Then $W_{(a,b)}$ is an irreducible closed subset of $H_{d,g}^S$ whose general member is contained in a smooth quadric surface and maximal among all such subsets. We can easily see that $\dim W_{(a,b)} = 2d + g + 8$. The next proposition shows that $W_{(a,b)}$ is an irreducible component of $H_{d,g}^S$ if and only if $g \geq 2d - 8$.

Proposition 4.11. *Let $d > 4$ and $g \geq 0$ be two integers, and let $W_{(a,b)} \subset H_{d,g}^S$. Then $H_{d,g}^S$ is generically non-singular along $W_{(a,b)}$. Moreover, if $g \geq 2d - 8$, then $W_{(a,b)}$ is an irreducible component of $H_{d,g}^S$. Otherwise, $W_{(a,b)}$ is a subvariety of $H_{d,g}^S$ of codimension $2d - 8 - g$.*

Proof. Let C be a general member of $W_{(a,b)}$ which is contained in a smooth quadric surface Q, and let $\mathbf{h}$ be the class of hyperplane sections of

Q. Then the exact sequence $0 \to \mathcal{I}_Q(2) \to \mathcal{I}_C(2) \to \mathcal{I}_{C/Q}(2) \to 0$ induces $H^i(\mathcal{I}_C(2)) \cong H^i(\mathcal{I}_{C/Q}(2))$ for $i = 1, 2$. Therefore, we obtain

$$(4.7) \qquad H^i(\mathcal{I}_C(2)) \cong H^{2-i}(\mathcal{O}_Q(a-4, b-4))^\vee \qquad (i = 1, 2)$$

by $H^i(\mathcal{I}_{C/Q}(2)) \cong H^i(\mathcal{O}_Q(2\mathbf{h} - C)) \cong H^{2-i}(\mathcal{O}_Q(C - 4\mathbf{h}))^\vee$.

First we assume that $g \geq 2d - 8$. Since $g - 2d + 8 = (a-3)(b-3)$, we have $a \geq b > 3$ when $g > 2d - 8$, and we have $a = 3$ or $b = 3$ when $g = 2d - 8$. Thus it follows from (4.7) that $H^1(\mathcal{I}_C(2)) = 0$. By [6, Theorem 1 (a)], $W_{(a,b)}$ is a reduced component of $H^S_{d,g}$. Next we assume that $g < 2d - 8$. This implies $b < 3$ and hence we have $H^1(\mathcal{O}_C(2)) \cong H^2(\mathcal{I}_C(2)) = 0$ by (4.7). By [6, Theorem 1 (b)], $H^S_{d,g}$ is generically non-singular along $W_{(a,b)}$, and the codimension of $W_{(a,b)}$ in $H^S_{d,g}$ is equal to $h^1(\mathcal{I}_C(2)) = 2d - 8 - g$.

Thus we conclude that $H^S_{d,g}$ is generically non-singular along $W_{(a,b)}$. $\square$

References

[1] Curtin, D., Obstructions to deforming a space curve, *Trans. Amer. Math. Soc.*, **267** (1981), 83-94.

[2] Dolcetti, A. and Pareschi, G., On linearly normal space curves, *Math. Z.*, **198** (1988), no. 1, 73-82.

[3] Ellia, P., D'autres composantes non réduites de Hilb $\mathbb{P}^3$, *Math. Ann.*, **277** (1987), 433-446.

[4] Fløystad, G., Determining obstructions for space curves, with applications to non-reduced components of the Hilbert scheme, *J. Reine Angew. Math.*, **439** (1993), 11-44.

[5] Geramita, A. V., *Lectures on the nonsingular cubic surface in $\mathbb{P}^3$ - II (The degree and genus of curves on this surface)*, Queen's papers in pure and applied mathematics, The curves seminar at queen's volume, 1990, VII. A1-A81.

[6] Kleppe, J. O., *Non-reduced components of the Hilbert scheme of smooth space curves*, Proc. Rocca di Papa 1985, Lecture Notes in Math., **1266**, Springer-Verlag, Berlin, 1987, pp.181-207.

[7] ———, The Hilbert scheme of space curves of small Rao module with an appendix on non-reduced components, preprint, 1996.

[8] Kollár, J., *Rational curves on algebraic varieties*, Springer-Verlag, Berlin, 1996.

[9] Martin-Deschamps, M. and Perrin, D., Le schéma de Hilbert des courbes gauches locale-ment Cohen-Macaulay n'est (presque) jamais réduit, *Ann. Sci. École Norm. Sup. (4)*, **29** (1996), 757-785.

[10] Mumford, D., Further pathologies in algebraic geometry, *Amer. J. Math.*, **84** (1962), 642-648.

[11] Nasu, H., Classification of space curves of degree 16 and genus 30, preprint, 2004.