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Time Local Well-posedness for the Benjamin-Ono Equation with Large Initial Data

By

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Abstract

This paper studies the time local well-posedness of the solution to the Benjamin-Ono equation. Our aim is to remove smallness condition on the initial data which was imposed in Kenig-Ponce-Vega's work [13].

§1. Introduction

We consider the initial value problem for the Benjamin-Ono equation:

$$\text{(1.1)} \qquad \begin{cases} \partial_t u + \mathcal{H}_x \partial_x^2 u + u\partial_x u = 0, & x, t \in \mathbf{R}, \\ u(x,0) = u_0(x), & x \in \mathbf{R}, \end{cases}$$

where $\mathcal{H}_x$ denotes the Hilbert transform, i.e., $\mathcal{H}_x = \mathcal{F}^{-1}(-i\xi/|\xi|)\mathcal{F}$. The equation (1.1) arises in the study of long internal gravity waves in deep stratified fluid. For the physical background, see Benjamin [3] and Ono [18].

We present the time local well-posedness of (1.1). Namely, we prove the existence, uniqueness of the solution and the continuous dependence on the initial data. There are several known results about this problem. One of their concern is to overcome the regularity loss arising from the nonlinearity. Because of this difficulty, the contraction mapping principle via the associated integral

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equation does not work as long as we consider the estimates only in the Sobolev space $H_x^{s,0}$, where $H_x^{s,\alpha}$ is defined by

$$H_x^{s,\alpha} = \{f \in \mathcal{S}'(\mathbf{R}); \|f\|_{H_x^{s,\alpha}} < \infty\}$$

with $\|f\|_{H_x^{s,\alpha}} = \|\langle x\rangle^\alpha \langle D_x\rangle^s f\|_{L_x^2}$, $\langle x\rangle^\alpha = (1+x^2)^{\alpha/2}$ and $\langle D_x\rangle^s = \mathcal{F}^{-1}\langle\xi\rangle^s\mathcal{F}$. Indeed, Molinet-Saut-Tzvetkov [17] negatively proved the solvability of the integral equation in $H_x^{s,0}$ for any $s \in \mathbf{R}$.

Saut [21] proved the global well-posedness for (1.1) in H_x^3. Abdelouhab-Bona-Felland-Saut [1] and Iorio [8] proved the time global existence and uniqueness of the solution in $H_x^{s,0}$ with $s > 3/2$. Their proofs are based on the energy method in which the estimate of $\|\partial_x u\|_{L_T^\infty(L_x^\infty)}$ gives the regularity constraint of the initial data. Ponce [19] obtained the global unique solution in $H_x^{3/2,0}$ by the combination of energy method and dispersive structure of linear part in (1.1). More recently, Koch-Tzvetkov [14] have studied the local well-posedness with $s > 5/4$ due to the cut off technique of $\mathcal{F}u(\xi)$. Furthermore, Kenig-Koenig [10] proved the local well-posedness with $s > 9/8$. We remark here that it is possible to minimize the regularity of u_0 by inducing another kind of function space. In fact, Kenig-Ponce-Vega [13] construct a time local solution via the integral equation by applying the smoothing property like

$$\|D_x \int_0^t V(t-t')F(t')dt'\|_{L_x^\infty(L_T^2)} \le C\|F\|_{L_x^1(L_T^2)},$$

where $\|u\|_{L_x^p(L_T^r)} = \|(\|u\|_{L^r[0,T]})\|_{L_x^p(\mathbf{R})}$, $D_x = \mathcal{F}^{-1}|\xi|\mathcal{F}$ and $V(t) = \exp(-t\mathcal{H}_x\partial_x^2)$. They obtained the time local well-posedness in $H_x^{s,0}$ $(s > 1)$ for the cubic nonlinearity (Their argument is also applicable to the quadratic case if u_0 satisfies $u_0 \in H_x^{s,0}$ $(s > 1)$ and the additional weight condition). In their result, however, the smallness of the initial data is required. This is because the inclusion $L_x^1(L_T^\infty) \cdot L_x^\infty(L_T^2) \subset L_x^1(L_T^2)$ yields $\|u\|_{L_x^1(L_T^\infty)}$ in the nonlinearity and we can not expect that $\|u\|_{L_x^1(L_T^\infty)} \to 0$ even when $T \to 0$.

Our concern in this paper is to remove this smallness condition of u_0. Before presenting the rough sketch of our idea, we introduce the function space Y_T in which the solution is constructed:

$$Y_T = \{u : [0,T] \times \mathbf{R} \to \mathbf{R}; \|\!|u|\!\|_{Y_T} < \infty\},$$

where $\|\!|u|\!\|_{Y_T} = \|u\|_{L_T^\infty(H_x^{s,0}\cap H_x^{s_1,\alpha_1})} + \|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u\|_{L_x^{1/\varepsilon}(L_T^2)} + \|\langle D_x\rangle^\mu\langle x\rangle^{\alpha_1}u\|_{L_x^2(L_T^\infty)}$ with $\rho, \mu > 0$ sufficiently small and $0 < \varepsilon < \rho$. We first consider the modified equation such that

$$\begin{cases} \partial_t u_\nu + \mathcal{H}_x\partial_x^2 u_\nu + u_\nu\partial_x\eta_\nu * u_\nu = 0, \\ u_\nu(0,x) = u_0(x), \end{cases} \tag{1.2}$$

where $\eta_\nu(x) = \nu^{-1}\eta(x/\nu)$ with $\eta \in C_0^\infty, \int \eta(x)dx = 1$ and $\nu \in (0,1]$. Then, the existence of u_ν in Y_T easily follows and it is continuated as long as $\|u_\nu(t)\|_{H_x^{s,0}\cap H_x^{s_1,\alpha_1}} < \infty$. Note that $\|u_\nu\|_{Y_T}$ is continuous with respect to T. To seek for the a priori estimate of $\|u_\nu\|_{Y_T}$, we deform (1.2). Let $\varphi \in C_0^\infty(\mathbf{R})$ and write $u_\nu\partial_x\eta_\nu * u_\nu = \varphi\partial_x\eta_\nu * u_\nu + (u_\nu - \varphi)\partial_x\eta_\nu * u_\nu$. Note here that, if φ is close to u_0, one can make $u_\nu - \varphi$ sufficiently small when $t \to 0$. To control the heavy term $\varphi\partial_x\eta_\nu * u_\nu$, we employ the gauge transform (see section 2) so that this quantity is, roughly speaking, absorbed in the linear operator. Then, our desired a priori estimate follows via the integral equation. As for the convergence of nonlinearity $u_\nu\partial_x\eta_\nu * u_\nu \to u\partial_x u$, we also consider the estimate of $u_\nu - u_{\nu'}$ in Y'_T which is slightly weaker than Y_T (see Proposition 6.1). Let us now state our main theorem.

Theorem 1.1. (i) *Let* $u_0 \in H_x^{s,0}\cap H_x^{s_1,\alpha_1} \equiv X^s$ *with* $s_1+\alpha_1 < s$, $1/2 < s_1$ *and* $1/2 < \alpha_1 < 1$. *Then, for some* $T = T(u_0) > 0$, *there exists a unique solution to* (1.1) *such that* $u \in C([0,T]; X^s) \cap Y_T$.
(ii) *Let* $u'(t)$ *be the solution to* (1.1) *with the initial data* u'_0 *satisfying* $\|u'_0 - u_0\|_{X^s} < \delta$. *If* $\delta > 0$ *is sufficiently small, then there exist some* $T' \in (0,T)$ *and* $C > 0$ *such that*

$$\|u' - u\|_{L^\infty_{T'}(X^s)} \le C\|u'_0 - u_0\|_{X^s},$$
$$\|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}(u' - u)\|_{L_x^{1/\varepsilon}(L^2_{T'})} \le C\|u'_0 - u_0\|_{X^s}.$$

In Theorem 1.1, the conditions on the initial data are determined by the estimate of maximal function, where, we call $\|f(\cdot,x)\|_{L^\infty_T}$ the maximal function of $f(t,x)$. Concretely speaking, the quantity $\|u\|_{L^1_x(L^\infty_T)}$ is bounded by $C(\|u_0\|_{H_x^{s,0}} + \|u_0\|_{H_x^{s_1,\alpha_1}})$ (see Lemma 4.2).

Remark 1. Only for the existence, one can further minimize the regularity of the initial data. Abdelouhab-Bona-Fell-Saut [1], Ginibre-Velo [7], Saut [21] and Tom [23] proved the global existence of weak solutions in L^2_x, $H_x^{1/2,0}$ and $H_x^{1,0}$, respectively. Recently, Tao [22] has studied the global well-posedness in $H_x^{1,0}$ but the L^2-stability of the data-to-solution map holds while the initial data belongs to $H_x^{1,0}$, i.e., $\|u'(t)-u(t)\|_{L^2} \le C\|u'_0-u_0\|_{L^2}$. More recently, Kato [9] has proved the well-posedness by supposing that $u_0 \in H_x^s$ with $s > 1/2$ and roughly speaking, u_0 satisfies the zero average condition $\int u_0(x)dx = 0$.

We also remark that Koch-Tzvetkov [17] and Biagioni-Linares [5] negatively proved the strong stability like

$$\|u'(t) - u(t)\|_{H_x^{s,0}} \le C\|u'_0 - u_0\|_{H_x^{s,0}} \quad \text{for } s > 0,$$

if there is no weight condition on u_0 and u_0'. Though our result requires slightly large regularity in comparison with Tao's work, it suggests that the additional weight condition yields the strong stability of the data-to-solution map in the sense that its target space coincides with that of initial data.

Remark 2. The upper bound of α_1 is required in the proof of weighted norm estimates (see section 4) and especially, in the estimate of $[\langle x\rangle^{\alpha_1}, \mathcal{H}_x\partial_x^2]$. It is possible to relax this weight condition. However, for the simplicity of our argument, we do not handle this kind of generalization in this paper. Let us also remark that the persistence of the solution fails if $\alpha_1 \geq 3$ (see Iorio [8]).

We close this section by introducing several notations and reviewing typical facts on the pseudo-differential operators. The Fourier transform $(2\pi)^{-1/2}\int e^{-ix\xi}f(x)dx$ is denoted by $\mathcal{F}f$ or $\hat{f}$. $\mathcal{B}(X;Y)$ stands for the class of bounded operators from X to Y. For simplicity, we often write $\mathcal{B}(X;X) = \mathcal{B}(X)$. The norm of the summation space $X+Y$ is given by $\|f\|_{X+Y} = \inf\{\|g\|_X + \|h\|_Y; g+h = f\}$. We call $a(x,\xi) \in C^\infty(\mathbf{R}\times\mathbf{R})$ belongs to the symbol class S^m if $\sup_{x,\xi}\langle\xi\rangle^{-m+j}|\partial_\xi^j\partial_x^k a(x,\xi)| < \infty$. For this symbol, the pseudo-differential operator $a(x,i^{-1}\partial_x)$ is defined by

$$a(x,i^{-1}\partial_x)f = (2\pi)^{-1/2}\int e^{ix\xi}a(x,\xi)\hat{f}(\xi)d\xi.$$

Let $\sigma(a(x,i^{-1}\partial_x))$ be the symbol of $a(x,i^{-1}\partial_x)$. It is well-known (cf. Kumano-go [16], Stein [20]) that, if $a(x,\xi) \in S^\ell$ and $b(x,\xi) \in S^m$, then we have

$$\sigma(a(x,i^{-1}\partial_x)b(x,i^{-1}\partial_x)) \in S^{\ell+m} \text{ and } \sigma([a(x,i^{-1}\partial_x), b(x,i^{-1}\partial_x)]) \in S^{\ell+m-1},$$

where $[A,B] = AB - BA$. These properties follow from the symbolic expansion formula like

$$\begin{aligned}(1.3)\qquad &\sigma(a(x,i^{-1}\partial_x)b(x,i^{-1}\partial_x))\\ &= \sum_{j=0}^{N-1}\frac{1}{2\pi j!i^j}\partial_\xi^j a(x,\xi)\partial_x^j b(x,\xi)\\ &\quad+\frac{1}{2\pi(N-1)!i^N}\text{Os-}\int\int e^{-i(x-y)(\xi-\zeta)}\partial_\zeta a(x,\zeta)\\ &\qquad\times\left(\int_0^1(1-\theta)^{N-1}\partial_x^N b(\theta y+(1-\theta)x,\xi d\theta\right)\,dyd\zeta,\end{aligned}$$

where Os-$\int\int$ stands for the oscillatory integral with respect to y and ζ. The expansion formula (1.3) is also applicable even in the case $a(x,\xi) = \langle\xi\rangle^\sigma$

and $b(x,\xi) = \langle x\rangle^\alpha$, which gives the equivalence of $\|\langle x\rangle^\alpha \langle D_x\rangle^\sigma f\|_{L^p_x}$ (resp. $\|\langle x\rangle^\alpha \langle D_x\rangle^\sigma f\|_{L^p_x(L^r_T)}$) and $\|\langle D_x\rangle^\sigma \langle x\rangle^\alpha f\|_{L^p_x}$ (resp. $\|\langle x\rangle^\alpha \langle D_x\rangle^\sigma f\|_{L^p_x(L^r_T)}$). For the symbol $a(x,\xi)\in S^m$, $|a|_N^{(m)}$ denotes the semi-norm defined by

$$|a|_N^{(m)} = \max_{j+k\le N} \sup_{x,\xi} \langle\xi\rangle^{-m+j} |\partial_\xi^j \partial_x^k a(x,\xi)|.$$

We note that, for $a(x,\xi)\in S^m$, $b(x,\xi)\in S^{m'}$ and arbitrary $N>0$, there exist some $N'>0$ and $C>0$ such that

$$|\sigma(a(x,i^{-1}\partial_x)b(x,i^{-1}\partial_x))|_N^{(m+m')} \le C|a|_{N'}^{(m)}|b|_{N'}^{(m')}.$$

Furthermore, for $a(x,\xi)\in S^m$, $a(x,i^{-1}\partial_x)\in\mathcal{B}(L^2_x)$ if $m\le 0$ and $a(x,i^{-1}\partial_x)\in\mathcal{B}(L^p_x)$ ($p\in[1,\infty]$) if $m<0$. In these estimates, we see that the operator norms $\|a(x,i^{-1}\partial_x)\|_{\mathcal{B}(L^p_x)}$ and $\|a(x,i^{-1}\partial_x)\|_{\mathcal{B}(L^p_x(L^r_T))}$ are estimated by $|a|_N^{(m)}$ for some $N>0$. We also denote $\int_0^t V(t-\tau)F(\tau)d\tau$ by $G(t)F$.

§2. Gauge Transform

In this section, we transform (1.2) appropriately for the a priori estimate. We write

$$\begin{aligned}(2.1)\qquad \partial_t u_\nu + \mathcal{H}_x\partial_x^2 u_\nu + \varphi\eta_\nu * \partial_x u_\nu &\\ + (u_\nu-\varphi)\eta_\nu * \partial_x u_\nu &= 0,\end{aligned}$$

with $\varphi\in C_0^\infty(\mathbf{R})$ to be chosen closely to u_0 in $H_x^{s,0}\cap H_x^{s_1,\alpha_1}$. We next define the gauge transformation of pseudo-differential operator with the symbol:

$$K_\nu(x,\xi) = \exp\left(\sqrt{\frac{\pi}{2}}\frac{i\xi}{|\xi|}(1-\psi(\xi))\hat\eta(\nu\xi)\int_{-\infty}^x \varphi(y)\,dy\right),$$

where $\psi\in C_0^\infty(\mathbf{R})$ satisfies

$$(2.2)\qquad \psi(\xi) = \begin{cases} 1, & \text{if } |\xi|<1,\\ 0, & \text{if } |\xi|>2.\end{cases}$$

Note that $K_\nu(x,\xi)\in S^0$ uniformly in $\nu\in(0,1]$. Applying $K_\nu\equiv K_\nu(x,i^{-1}\partial_x)$ to (2.1) and letting $v_\nu = K_\nu u_\nu$, we have

$$(2.3)\qquad \partial_t v_\nu + \mathcal{H}_x\partial_x^2 v_\nu + K_\nu(u_\nu-\varphi)\eta_\nu * \partial_x u_\nu + R_\nu(\varphi,u_\nu) = 0,$$

where $R_\nu(\varphi,u) = (K_\nu\varphi\eta_\nu * \partial_x + [K_\nu, \mathcal{H}_x\partial_x^2])u$. Note that the symbol of $K_\nu\varphi\eta_\nu * \partial_x + [K_\nu,(1-\psi(i^{-1}\partial_x))\mathcal{H}_x\partial_x^2]$ belongs to S^0 uniformly in $\nu\in(0,1]$ since the top symbol of the commutator is $-\partial_x K_\nu(x,\xi)\partial_\xi((1-\psi)\xi|\xi|)$. The desired a priori estimate of u_ν will be obtained in terms of v_ν by transforming (2.3) into the integral equation. Henceforth, we are led to the preliminaries about several estimates of $V(t)$ etc. (These are given in the next section.)

§3. Preliminary

We introduce several linear estimates. The first lemma gives the smoothing effects due to Kenig-Ponce-Vega [13] which overcome a loss of regularity in the nonlinearity.

Lemma 3.1. *Let $p \in [2,\infty]$. Then we have*

$$\|D_x^{1/2-1/p}V(t)\phi\|_{L_x^p(L_T^2)} \le CT^{1/p}\|\phi\|_{L_x^2}, \tag{3.1}$$

$$\|D_x^{1-1/p}G(t)F\|_{L_x^p(L_T^2)} \le CT^{1/p}\|F\|_{L_x^1(L_T^2)}, \tag{3.2}$$

$$\|D_x^{1/2}G(t)F\|_{L_T^\infty(L_x^2)} \le C\|F\|_{L_x^1(L_T^2)}. \tag{3.3}$$

Proof of Lemma 3.1. The case $p=\infty$ and (3.3) are given in [13]. By (3.1) and (3.3), it is easy to see that, for $\lambda \in \mathbf{R}$,

$$\|D_x^{i\lambda}V(t)\phi\|_{L_x^2(L_T^2)} \le T^{1/2}\|\phi\|_{L_x^2}, \tag{3.4}$$

$$\|D_x^{1/2+i\lambda}V(t)\phi\|_{L_x^\infty(L_T^2)} \le C\|\phi\|_{L_x^2}, \tag{3.5}$$

$$\|D^{1/2+i\lambda}G(t)F\|_{L_x^2(L_T^2)} \le CT^{1/2}\|F\|_{L_x^1(L_T^2)}. \tag{3.6}$$

where $C>0$ is independent of λ. Also, in [13], the following estimate appears:

$$\|D_x^{1+i\lambda}G(t)F\|_{L_x^\infty(L_T^2)} \le C\langle\lambda\rangle^N\|F\|_{L_x^1(L_T^2)}, \tag{3.7}$$

where N is a large positive integer. Then, applying Stein's interpolation for analytic families of operators to the pairs (3.4)–(3.5) and (3.6)–(3.7), we obtain the desired estimates for $p \ne \infty$. □

We next state the Strichartz estimates (for the proof, see [6, p. 377] and refer to [25]). These inequalities will be used for the weighted norm estimates.

Lemma 3.2. *Let p_j and r_j $(j=1,2)$ satisfy $0 \le 2/r_j = 1/2 - 1/p_j \le 1/2$. Then, we have*

$$\|V(t)\phi\|_{L_T^{r_1}(L_x^{p_1})} \le C\|\phi\|_{L_x^2}, \tag{3.8}$$

$$\|G(t)F\|_{L_T^{r_1}(L_x^{p_1})} \le C\|F\|_{L_T^{r_2'}(L_x^{p_2'})}, \tag{3.9}$$

where $1/p_2 + 1/p_2' = 1/r_2 + 1/r_2' = 1$.

We call $\|u(\cdot,x)\|_{L^\infty_T}$ the maximal function of u. Concerning the estimates of maximal functions, we have the following.

Lemma 3.3. *Let $\sigma > 1/2$ and $T \in [0,1]$. Then, we have*

$$\|V(t)\phi\|_{L^2_x(L^\infty_T)} \leq C\|\phi\|_{H^{\sigma,0}_x}, \tag{3.10}$$

$$\|G(t)F\|_{L^2_x(L^\infty_T)} \leq C\|F\|_{L^1_T(H^{\sigma,0}_x)}, \tag{3.11}$$

$$\|G(t)F\|_{L^2_x(L^\infty_T)} \leq C\|\langle D_x\rangle^\sigma F\|_{L^1_T(L^1_x)}. \tag{3.12}$$

Proof of Lemma 3.3. The estimate (3.10) is due to Vega [24]. From Minkowski's inequality, (3.11) follows. To prove (3.12), we first show that the integral kernel of $\langle D_x\rangle^{-\sigma}\exp(t\mathcal{H}_x\partial_x^2)$ (denoted by $K(t,x-y)$) is estimated as

$$|K(t,x-y)| \leq C\begin{cases} |x-y|^{-\sigma} & \text{if } |x-y|>1,\\ |x-y|^{-1+\sigma} & \text{if } |x-y|\leq 1,\end{cases} \tag{3.13}$$

where $C>0$ does not depend on $t\in(0,T]$. Let $z=x-y$ and write

$$\begin{aligned} K(t,z) &= (2\pi)^{-1}\int_{-\infty}^{\infty}\exp(-it\xi|\xi|+iz\xi)\langle\xi\rangle^{-\sigma}d\xi\\ &= (2\pi)^{-1}\int_0^\infty \exp(-it\xi^2+iz\xi)\langle\xi\rangle^{-\sigma}d\xi\\ &\quad + (2\pi)^{-1}\int_{-\infty}^0 \exp(it\xi^2+iz\xi)\langle\xi\rangle^{-\sigma}d\xi\\ &\equiv K_+(t,z)+K_-(t,z). \end{aligned}$$

We only consider the estimate of $K_+(t,z)$ since $K_-(t,z)$ is similarly estimated. Changing the integral variable, we can write

$$K_+(t,z) = \begin{cases} e^{iz^2/4t}z'\int_{-1}^{\infty} e^{-itz'^2\xi^2}\langle z'(\xi+1)\rangle^{-\sigma}d\xi & \text{if } z>0,\\ -e^{iz^2/4t}z'\int_{-\infty}^{-1} e^{-itz'^2\xi^2}\langle z'(\xi+1)\rangle^{-\sigma}d\xi & \text{if } z<0,\end{cases}$$

where $z'=z/2t$. Let us mainly consider the case $z>0$ step by step.

(*The case* $z>1$) The identity $\partial_\xi \xi e^{-itz'^2\xi^2} = (1-2itz'^2\xi^2)e^{-itz'^2\xi^2}$ and integration by parts give

$$\begin{aligned} |K_+(t,z)| &\leq z'|1-2itz'^2|^{-1} + z'\int_{-1}^{\infty}|\xi\partial_\xi((1-2itz'^2\xi^2)^{-1}\langle z'(\xi+1)\rangle^{-\sigma})|d\xi\\ &\leq Cz^{-\sigma}. \end{aligned}$$

(*The case* $0 < z \le 1$ *and* $tz'^2 > 1$) Let $\chi_{-1} \in C_0^\infty(\mathbf{R})$ with $\chi_{-1}(\xi) = 1$ near $\xi = -1$, and let $\tilde{\chi}_{-1} = 1 - \chi_{-1}$. Then, we see that

$$\begin{aligned}|K_+(t,z)| \le & \left| z' \int_{-1}^{\infty} e^{-itz'^2\xi^2} \chi_{-1}(\xi) \langle z'(\xi+1)\rangle^{-\sigma} d\xi \right| \\ & + \left| z' \int_{-1}^{\infty} e^{-itz'^2\xi^2} \tilde{\chi}_{-1}(\xi) \langle z'(\xi+1)\rangle^{-\sigma} d\xi \right| \\ \equiv & |K_{+,1}(t,z)| + |K_{+,2}(t,z)|.\end{aligned}$$

To estimate $|K_{+,1}(t,z)|$, we use the identity $\partial_\xi(e^{-itz'^2\xi^2} - e^{-itz'^2}) = -2itz'^2\xi e^{-itz'^2\xi^2}$ and integration by parts. This yields

(3.14)

$$\begin{aligned}|K_{+,1}(t,z)| \le & Ct^{-1}z'^{-1} \left(\int_{-1}^{\infty} (\partial_\xi \xi^{-1}\chi_{-1}) \langle z'(\xi+1)\rangle^{-\sigma} d\xi \right. \\ & \left. + \int_{-1}^{\infty} \xi^{-1}\chi_{-1} |e^{-itz'^2\xi^2} - e^{-itz'^2}| z' \langle z'(\xi+1)\rangle^{-\sigma-1} d\xi \right) \\ \le & Ct^{-1}z'^{-1-\sigma} + Ct^{-1}z'^{-1-\sigma} \int_{-1}^{0} |e^{-itz'^2\xi^2} - e^{-itz'^2}| (\xi+1)^{-\sigma-1} d\xi.\end{aligned}$$

The integral in (3.14) is bounded by $C(tz')^\sigma$ since, for $0 < R < 1$ and $\sigma' > \sigma$, we have

$$\begin{aligned}& \int_{-1}^{0} |e^{-itz'^2\xi^2} - e^{-itz'^2}| (\xi+1)^{-\sigma-1} d\xi \\ & \le C \int_{-1+R}^{0} (\xi+1)^{-\sigma-1} d\xi + C \int_{-1}^{-1+R} (tz'^2)^{\sigma'} (\xi+1)^{\sigma'-\sigma-1} d\xi \\ & \le C(R^{-\sigma} + (tz'^2)^{\sigma'} R^{\sigma'-\sigma})\end{aligned}$$

with $R = (\sigma/(\sigma'-\sigma))^{1/\sigma'}(tz'^2)^{-1}$. Thus, noting that $z > 2t^{1/2}$ in this case, we have $|K_{+,1}(t,z)| \le Cz^{-1+\sigma}$. The estimate $|K_{+,2}(t,z)| \le Cz^{-1+\sigma}$ follows from the identity $\partial_\xi \xi e^{-itz'^2\xi^2} = (1 - 2itz'^2\xi^2)e^{-itz'^2\xi^2}$ and integration by parts.

(*The case* $0 < z \le 1$ *and* $tz'^2 \le 1$) Changing the integral variable, we have another expression of $K_+(t,z)$ such that

$$\begin{aligned}K_+(t,z) = & e^{iz^2/4t} t^{-1/2} \int_{-t^{1/2}z'}^{\infty} e^{-i\xi^2} \langle t^{-1/2}(\xi + t^{1/2}z')\rangle^{-\sigma} d\xi \\ = & e^{iz^2/4t} t^{-1/2} \int_{-t^{1/2}z'}^{\infty} e^{-i\xi^2} \chi_0(\xi) \langle t^{-1/2}(\xi + t^{1/2}z')\rangle^{-\sigma} d\xi \\ & + e^{iz^2/4t} t^{-1/2} \int_{-t^{1/2}z'}^{\infty} e^{-i\xi^2} (1 - \chi_0(\xi)) \langle t^{-1/2}(\xi + t^{1/2}z')\rangle^{-\sigma} d\xi \\ \equiv & K_{+,3}(t,z) + K_{+,4}(t,z),\end{aligned}$$

where $\chi_0 \in C_0^\infty(\mathbf{R})$ with $\chi_0(\xi) = 1$ in the neighborhood of $[-1,0]$. It is easy to see that $|K_{+,3}(t,z)| \le Ct^{-(1-\sigma)/2} \le Cz^{-1+\sigma}$. Making use of the identity $\partial_\xi e^{-i\xi^2} = -2i\xi e^{-2i\xi^2}$ and integration by parts, we can show that $|K_{+,4}(t,z)| \le Ct^{-(1-\sigma)/2} \le Cz^{-1+\sigma}$. Thus, (3.13) follows. Hence, (3.13) and Young's inequality yield (3.12). □

In the nonlinear estimates, the fractional order differentiation will be applied to the quadratic term in (2.3). To handle this, we require the Leibnitz' type rule for the fractional order derivatives due to Kenig-Ponce-Vega [12, Appendix].

Lemma 3.4. *Let $\sigma \in (0,1)$ and $\sigma_0, \sigma_1 \in [0,\sigma]$ with $\sigma = \sigma_0 + \sigma_1$. Also, let $q \in [1,\infty)$ and $q_0, q_1, r_0, r_1 \in (1,\infty)$ with $1/q = 1/q_0 + 1/q_1$ and $1/2 = 1/r_0 + 1/r_1$. Then, we have*

(3.15)
$$\|D_x^\sigma(fg) - (D_x^\sigma f)g - f(D_x^\sigma g)\|_{L_x^q(L_T^2)} \le C\|D_x^{\sigma_1} f\|_{L_x^{q_0}(L_T^{r_0})}\|D_x^{\sigma_2} g\|_{L_x^{q_1}(L_T^{r_1})}.$$

When we apply the Leibnitz' rule for the fractional order derivative to the nonlinearity, we encounter the estimate of lower order derivatives like $D_x^{s-1/2}u$ and $\partial_x u$. The following lemma and its corollary help us control these quantities. In particular, we require the case $q_0 = 1, q_1 = \infty, r_0 = \infty$ and $r_1 = 2$ (the end point case of the interpolation).

Lemma 3.5. *Let $\sigma_0, \sigma_1 > 0$, $\alpha_0, \alpha_1 \in \mathbf{R}$ and $q_0, q_1, r_0, r_1 \in [1,\infty]$. Also, let $\sigma = (1-\theta)\sigma_0 + \theta\sigma_1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$ and $1/r = (1-\theta)/r_0 + \theta/r_1$ with $\theta \in [0,1]$. Then, for $f \in \mathcal{S}(\mathbf{R}; C^\infty[0,T])$, we have*

$$\begin{aligned}(3.16)\quad &\|D_x^\sigma \langle x\rangle^\alpha f\|_{L_x^q(L_T^r)} \\ &\le \left(\sup_{\lambda\in\mathbf{R}} e^{-\lambda^2}\|D_x^{\sigma_0+i\lambda(\sigma_1-\sigma_0)}\langle x\rangle^{\alpha_0+i\lambda(\alpha_1-\alpha_0)} f\|_{L_x^{q_0}(L_T^{r_0})}\right)^{1-\theta} \\ &\quad\times\left(\sup_{\lambda\in\mathbf{R}} e^{1-\lambda^2}\|D_x^{\sigma_1+i\lambda(\sigma_1-\sigma_0)}\langle x\rangle^{\alpha_1+i\lambda(\alpha_1-\alpha_0)} f\|_{L_x^{q_1}(L_T^{r_1})}\right)^{\theta}.\end{aligned}$$

Proof of Lemma 3.5. We first define the complex valued function $F(z)$ by

$$F(z) = e^{z^2}\int_{\mathbf{R}\times[0,T]} g_z(t,x)\ D_x^{\sigma(z)}\langle x\rangle^{\alpha(z)} f(t,x)\ dtdx,$$

where $\sigma(z) = (1-z)\sigma_0 + z\sigma_1$, $\alpha(z) = (1-z)\alpha_0 + z\alpha_1$ and

$$g_z(t,x) = \|g(\cdot,x)\|_{L_T^{r'}}^{q'/q'(z)-r'/r'(z)}|g(t,x)|^{r'/r'(z)}\ \mathrm{sgn} g(t,x)$$

with $g \in \mathcal{S}(\mathbf{R}; C^\infty[0,T])$, $1/q'(z) = (1-z)/q_0' + z/q_1'$ and $1/r'(z) = (1-z)/r_0' + z/r_1'$ (the prime $'$ denotes the Hölder conjugate). Then, $F(z)$ is holomorphic in the strip $S = \{z \in \mathbf{C}; 0 < \mathrm{Re} z < 1\}$ and continuous in $\overline{S}$. In addition, $\lim_{|\mathrm{Im} z| \to \infty} |F(z)| = 0$ in virtue of the multiplication e^{z^2}. According to the three line theorem, we see that

$$|F(z)| \le M_0^{1-\mathrm{Re} z} M_1^{\mathrm{Re} z}, \tag{3.17}$$

where $M_j = \sup_{\lambda \in \mathbf{R}} |F(j + i\lambda)|$ $(j = 0, 1)$. By applying Hölder's inequality,

(3.18)

$$M_j \le \|g\|_{L_x^{q'}(L_T^{r'})}^{q'/q_j'} \sup_{\lambda} \left(e^{j-\lambda^2} \|D_x^{\sigma_j + i\lambda(\sigma_1-\sigma_0)} \langle x \rangle^{\alpha_j + i\lambda(\alpha_1 - \alpha_0)} f\|_{L_x^{q_j}(L_T^{r_j})} \right).$$

Combining (3.17)–(3.18) and $(L_x^{q'}(L_T^{r'}))' \sim L_x^q(L_T^r)$ with $z = \theta$, we obtain Lemma 3.5. □

Corollary 3.6. *In addition to the assumptions in Lemma* 3.5, *let* $\mu > 0$. *Then, we have*

$$\begin{aligned} &\|\langle D_x \rangle^\sigma \langle x \rangle^\alpha f\|_{L_x^q(L_T^r)} \\ &\quad \le C \|\langle D_x \rangle^{\sigma_0+\mu} \langle x \rangle^{\alpha_0} f\|_{L_x^{q_0}(L_T^{r_0})}^{1-\theta} \|\langle D_x \rangle^{\sigma_1+\mu} \langle x \rangle^{\alpha_1} f\|_{L_x^{q_1}(L_T^{r_1})}^{\theta}. \end{aligned} \tag{3.19}$$

Proof of Corollary 3.6. By estimating the integral kernels of operators, we see, for instance, that

$$\|D_x^{\sigma_0 + i\lambda(\sigma_1 - \sigma_0)} \langle x \rangle^{i\lambda(\alpha_1-\alpha_0)} \langle D_x \rangle^{-(\sigma_0+\mu)}\|_{\mathcal{B}(L_x^{q_0}(L_T^{r_0}))} \le C \langle \lambda \rangle^N$$

with N sufficiently large. Then, Lemma 3.5 yields the desired result. □

In our argument, the pseudo-differential operator K_ν often appears. We note that $K_\nu \notin \mathcal{B}(L_x^1)$ uniformly in $\nu \in (0,1]$ since the symbol of K_ν contains the gap for $\xi = \pm\infty$ if $\nu = 0$. The following lemma states that $K_\nu \in \mathcal{B}(L_x^p)$ $(1 < p < \infty)$ and its operator norm is estimated in terms of $\|\varphi\|_{X^s}$.

Lemma 3.7. *Let* $p \in (1, \infty)$ *and* $\nu \in (0,1)$. *Then, we have*

$$\|K_\nu\|_{\mathcal{B}(L_x^p)} \le C, \tag{3.20}$$

where the positive constant C *is independent of* $\nu \in (0,1]$ *and does not diverge as* $\varphi \to u_0$ *in* X^s. *Furthermore, in the above inequalities, we may replace* L_x^p *by* $L_x^p(L_T^r)$ *with* $r \in (1, \infty)$.

Proof of Lemma 3.7. Note that $K_\nu(x,\xi) = L_{\nu,+}(x,\xi)\chi_+(\xi) + L_{\nu,-}\chi_-(\xi)$, where

$$L_{\nu,\pm}(x,\xi) = \exp\left(\pm i\sqrt{\frac{\pi}{2}}(1-\psi(\xi))\hat{\eta}(\nu\xi)\int_{-\infty}^{x}\varphi dy\right)$$

and $\chi_+(\xi)$ (resp. $\chi_-(\xi)$) is the characteristic function on $(0,\infty)$ (resp. $(-\infty,0)$). It is well-known that $\chi_\pm(i^{-1}\partial_x) \in \mathcal{B}(L^p_x)$, and thus it suffices to show that $L_{\nu,\pm}(x, i^{-1}\partial_x) \in \mathcal{B}(L^p_x)$. We write

$$\begin{aligned}(3.21)\qquad L_{\nu,\pm}(x,\xi) &= \psi(\xi/2)L_{\nu,\pm}(x,\xi) + (1-\psi(\xi/2))L_{\nu,\pm}(x,\xi)\\ &\equiv L_{\nu,\pm,1}(x,\xi) + L_{\nu,\pm,2}(x,\xi).\end{aligned}$$

By the integration by parts, the integral kernels of $L_{\nu,\pm,1}(x,i^{-1}\partial_x)$ (denoted by $L_{\nu,\pm,1}[x,y]$) are estimated as

$$\begin{aligned}|L_{\nu,\pm,1}[x,y]| &= (2\pi)^{-1}\left|\int e^{i(x-y)\xi}L_{\nu,\pm,1}(x,\xi)d\xi\right|\\ &\le C\exp(C\|\varphi\|_{X^s})\langle x-y\rangle^{-N},\end{aligned}$$

where $N > 0$ is sufficiently large. Also, note that

$$\begin{aligned}(3.22)\quad & L_{\nu,\pm,2}(x,\xi)\\ &= (1-\psi(\xi/2))\exp\left(\pm i(\pi/2)^{1/2}(1-\psi(\xi))\hat{\eta}(\nu\xi)\int_{-\infty}^{x}\varphi(y)dy\right)\\ &= (1-\psi(\xi/2))\\ &\quad +(1-\psi(\xi/2))\left(\exp\left(\pm i(\pi/2)^{1/2}\hat{\eta}(\nu\xi)\int_{-\infty}^{x}\varphi(y)dy\right)-1\right),\end{aligned}$$

where we remarked that, if $1-\psi(\xi/2)\neq 0$, then $1-\psi(\xi)=1$. Furthermore, the symbol $\exp(i^{-1}(\pi/2)^{1/2}\hat{\eta}(\nu\xi)\int_{-\infty}^{x}\varphi(y)dy) - 1$ yields the integral operator with the kernel bounded by $C\exp(C\|\varphi\|_{X^s})\nu^{-1}\langle(x-y)/\nu\rangle^{-N}$, and $1-\psi(\xi/2)$ yields L^p_x-bounded operator. Hence, we see that $L_{\nu,\pm,2}(x,i^{-1}\partial_x) \in \mathcal{B}(L^p_x)$ and (3.20) follows. □

§4. Weighted Norm Estimates

In this section, we derive several linear estimates in the weighted norms, which bring us the persistence of the solution.

Lemma 4.1. *Let $\sigma, \alpha \in [1/2, 1)$ and $\sigma' > \sigma + \alpha$. Then, for $T \in (0,1)$, we have*

$$\|V(t)\phi\|_{L_T^\infty(H_x^{\sigma,\alpha})} \le C(\|\phi\|_{H_x^{\sigma',0}} + \|\phi\|_{H_x^{\sigma,\alpha}}), \tag{4.1}$$

$$\|G(t)F\|_{L_T^\infty(H_x^{\sigma,\alpha})} \le CT^{1/2}\|D_x^{\sigma'-1/2}F\|_{L_x^1(L_T^2)} + C\|\langle D_x\rangle^\sigma \langle x\rangle^\alpha F\|_{L_T^1(L_x^2)+L_T^{4/3}(L_x^1)}. \tag{4.2}$$

Proof of Lemma 4.1. Let $w = V(t)\phi$ and multyply $\langle x\rangle^\alpha \langle D_x\rangle^\sigma$ on both hand sides of $(\partial_t + \mathcal{H}_x\partial_x^2)w = 0$ with $w(0,x) = \phi$, we have

$$(\partial_t + \mathcal{H}_x\partial_x^2)\langle x\rangle^\alpha\langle D_x\rangle^\sigma w = [\mathcal{H}_x\partial_x^2, \langle x\rangle^\alpha]\langle D_x\rangle^\sigma w.$$

Thus, Duhamel's principle gives

$$\langle x\rangle^\alpha\langle D_x\rangle^\sigma w = V(t)\langle x\rangle^\alpha\langle D_x\rangle^\sigma\phi + G(t)[\mathcal{H}_x\partial_x^2, \langle x\rangle^\alpha]\langle D_x\rangle^\sigma w. \tag{4.3}$$

Note that the symbolic calculation of the pseudo-differential operators gives

$$[\mathcal{H}_x\partial_x^2, \langle x\rangle^\alpha] = 2(\partial_x\langle x\rangle^\alpha)(1-\psi(i^{-1}\partial_x))D_x + (\mathcal{B}(L^2) \text{ operator}),$$

where $\psi(i^{-1}\partial_x) = \mathcal{F}^{-1}\psi(\xi)\mathcal{F}$ where $\psi(\xi)$ is defined by (2.2). Then, applying Lemma 3.2 and Hölder's inequality to (4.3), we have

$$\|V(t)\phi\|_{L_T^\infty(H_x^{\sigma,\alpha})} \le C\|\phi\|_{H_x^{\sigma,\alpha}} + CT^{1/2}\|(1-\psi(i^{-1}\partial_x))D_xV(t)\langle D_x\rangle^\sigma\phi\|_{L_x^q(L_T^2)},$$

where $1/q > \alpha - 1/2$. Also, Lemma 3.1 (3.1) gives

$$\begin{aligned}\|V(t)\phi\|_{L_T^\infty(H_x^{\sigma,\alpha})} &\le C\|\phi\|_{H_x^{\sigma,\alpha}} + CT^{1/2}\|D_x^{1/q+1/2}\langle D_x\rangle^\sigma\phi\|_{L_x^2}\\ &\le C(\|\phi\|_{H_x^{\sigma,\alpha}} + \|\phi\|_{H_x^{\sigma',0}}).\end{aligned}$$

We next prove (4.2). Similarly to the derivation of (4.3), we have

$$\begin{aligned}\langle x\rangle^\alpha\langle D_x\rangle^\sigma G(t)F &= G(t)\langle x\rangle^\alpha\langle D_x\rangle^\sigma F + G(t)[\mathcal{H}_x\partial_x^2, \langle x\rangle^\alpha]G(t)\langle D_x\rangle^\sigma F\\ &\equiv I_1 + I_2.\end{aligned} \tag{4.4}$$

By Lemma 3.2 (3.9), I_1 is estimated as

$$\|I_1\|_{L_T^\infty(L_x^2)} \le C\|\langle x\rangle^\alpha\langle D_x\rangle^\sigma F\|_{L_T^1(L_x^2)+L_T^{4/3}(L_x^1)}.$$

As for I_2, we see that

$$\begin{aligned}\|I_2\|_{L_T^\infty(L_x^2)} &\le C\|(\partial_x\langle x\rangle^\alpha)(1-\psi(i^{-1}\partial_x))D_xG(t)\langle D_x\rangle^\sigma F\|_{L_T^1(L_x^2)}\\ &\quad + C\|G(t)\langle D_x\rangle^\sigma F\|_{L_T^1(L_x^2)}\\ &\le CT^{1/2}\|D_xG(t)\langle D_x\rangle^\sigma F\|_{L_x^q(L_T^2)} + C\|\langle x\rangle^\alpha\langle D_x\rangle^\sigma F\|_{L_T^1(L_x^2)+L_T^{4/3}(L_x^1)}.\end{aligned} \tag{4.5}$$

Note that, to obtain the last inequality of (4.5), we used $1-\psi(i^{-1}\partial_x) \in \mathcal{B}(L_x^q(L_T^2))$ and Lemma 3.2 (3.9). Lemmas 3.1, 3.2 and $D_x(\langle D_x\rangle^\sigma - D_x^\sigma) \in \mathcal{B}(L^p)$ $(1 \le p \le \infty)$ yield

$$\begin{aligned}\|D_xG(t)\langle D_x\rangle^\sigma F\|_{L_x^q(L_T^2)} &\le C\|D_x^{1/q+\sigma}F\|_{L_x^1(L_T^2)} + C\|F\|_{L_T^1(L_x^2)+L_T^{4/3}(L_x^1)}\\ &\le C\|D_x^{\sigma'-1/2}F\|_{L_x^1(L_T^2)} + C\|\langle x\rangle^\alpha\langle D_x\rangle^\sigma F\|_{L_T^1(L_x^2)+L_T^{4/3}(L_x^1)}.\end{aligned}$$

Hence, Lemma 4.1 follows. □

The lemma given below is concerning the estimates of maximal function and determines the regularity and weight conditions on the initial data.

Lemma 4.2. *Let $\mu \in [0,1)$, $\alpha \in [1/2,1)$, $\sigma > \mu + 1/2$ and $\sigma' > \mu+\alpha+1/2$. Then, for $T \in (0,1)$, we have*

$$\|\langle D_x\rangle^\mu\langle x\rangle^\alpha V(t)\phi\|_{L_x^2(L_T^\infty)} \le C(\|\phi\|_{H_x^{\sigma',0}} + \|\phi\|_{H_x^{\sigma,\alpha}}), \tag{4.6}$$

$$\begin{aligned}\|\langle D_x\rangle^\mu\langle x\rangle^\alpha G(t)F\|_{L_x^2(L_T^\infty)} \le{}& CT^{1/2}\|D_x^{\sigma'-1/2}F\|_{L_x^1(L_T^2)}\\ &+C\|\langle D_x\rangle^\sigma\langle x\rangle^\alpha F\|_{L_T^1(L_x^2)+L_T^{4/3}(L_x^1)}.\end{aligned} \tag{4.7}$$

Proof of Lemma 4.2. We only prove (4.7) since (4.6) follows more easily. Applying Lemmas 3.2 and 3.3 to (4.4), we have

$$\begin{aligned}&\|\langle D_x\rangle^\mu\langle x\rangle^\alpha G(t)F\|_{L_x^2(L_T^\infty)}\\ &\quad\le \|\langle D_x\rangle^\mu G(t)\langle x\rangle^\alpha F\|_{L_x^2(L_T^\infty)} + \|\langle D_x\rangle^\mu G(t)[\mathcal{H}_x\partial_x^2, \langle x\rangle^\alpha]G(\cdot)F\|_{L_x^2(L_T^\infty)}\\ &\quad\le C\|\langle D_x\rangle^\sigma\langle x\rangle^\alpha F\|_{L_T^1(L_T^2)+L_T^1(L_x^1)} + CT\|\langle D_x\rangle^{\mu+1/2+\epsilon}P_1G(t)F\|_{L_T^\infty(L_x^2)}\\ &\qquad+CT^{1/2}\|\langle D_x\rangle^{\mu+1/2+\epsilon}P_2G(t)F\|_{L_x^2(L_T^2)},\end{aligned} \tag{4.8}$$

where $\mu+1/2+\epsilon<\sigma$, $P_1 = [\psi(i^{-1}\partial_x)\mathcal{H}_x\partial_x^2, \langle x\rangle^\alpha]$ and $P_2 = [(1-\psi(i^{-1}))\mathcal{H}_x\partial_x^2, \langle x\rangle^\alpha]$ with $\psi \in C_0^\infty(\mathbf{R})$. By Hölder's inequality, the first term on the right hand side of (4.8) is estimated as

$$\|\langle D_x\rangle^\sigma\langle x\rangle^\alpha F\|_{L_T^1(L_x^2)+L_T^1(L_x^1)} \le C\|\langle D_x\rangle^\sigma\langle x\rangle^\alpha F\|_{L_T^1(L_x^2)+L_T^{4/3}(L_x^1)}. \tag{4.9}$$

Since $P_1f = \displaystyle\int P_1(x,y)f(y)\,dy$, where

$$P_1(x,y) = -(2\pi)^{-1}\int e^{i(x-y)\xi}\partial_\xi(\psi(\xi)\xi|\xi|)d\xi\int_0^1 \partial_w\langle w\rangle^\alpha|_{w=\theta y+(1-\theta)x}\,d\theta$$

and $|\partial_x^k P_1(x,y)| \le C\langle x-y\rangle^{-2}$ $(k=0,1,2)$, the second term in (4.8) is estimated as

(4.10)
$$\begin{aligned}\|\langle D_x\rangle^{\mu+1/2+\epsilon}P_1G(t)F\|_{L^\infty_T(L^2_x)} &\le C\|G(t)F\|_{L^\infty_T(L^2_x)}\\ &\le C\|F\|_{L^1_T(L^2_x)+L^{4/3}_T(L^1_x)} \quad \text{(by Lemma 3.2)}\\ &\le C\|\langle D_x\rangle^\sigma\langle x\rangle^\alpha F\|_{L^1_T(L^2_x)+L^{4/3}_T(L^1_x)}.\end{aligned}$$

As for the third term in (4.8), we note that

$$\begin{aligned}P_2 = &\, 2\partial_x\langle x\rangle^\alpha(1-\psi(i^{-1}\partial_x))D_x\\ &+(\text{pseudo-differential operator with symbol in } S^0).\end{aligned}$$

Then, it follows that

$$\begin{aligned}&\|\langle D_x\rangle^{\mu+1/2+\epsilon}P_2G(t)F\|_{L^2_x(L^2_T)}\\ &\le C\|\langle D_x\rangle^{\mu+1/2+\epsilon}(\partial_x\langle x\rangle^\alpha)(1-\psi(i^{-1}\partial_x))D_xG(t)F\|_{L^\infty_T(L^2_x)}\\ &\quad +CT^{1/2}\|\langle D_x\rangle^{\mu+1/2+\epsilon}G(t)F\|_{L^\infty_T(L^2_x)}.\end{aligned}$$

Since the symbol of $[\langle D_x\rangle^{\mu+1/2+\epsilon}, \partial_x\langle x\rangle^\alpha]$ belongs to $S^{\mu-1/2+\epsilon}$, Lemmas 3.1 and 3.2 give

(4.11)
$$\begin{aligned}&\|\langle D_x\rangle^{\mu+1/2+\epsilon}P_2G(t)F\|_{L^2_x(L^2_T)}\\ &\le C\|D_x^{\mu+3/2+\epsilon}GF\|_{L^q_x(L^2_T)} + CT^{1/2}\|G\langle D_x\rangle^\sigma F\|_{L^\infty_T(L^2_x)}\\ &\le C\|D_x^{\sigma'-1/2}F\|_{L^1_x(L^2_T)} + C\|\langle D_x\rangle^\sigma\langle x\rangle^\alpha F\|_{L^1_T(L^2_x)+L^{4/3}_T(L^1_x)}\end{aligned}$$

where $1/q > \alpha - 1/2$. Combining (4.8)–(4.11), we obtain Lemma 4.2. □

§5. Nonlinear Estimates

In what follows, we only consider the case $s \in (1, 3/2)$ since the other cases are verified without major modification. For a brief description, we let

$$\begin{aligned}\|u\|_{initial} &= \|u\|_{L^\infty_T(X^s)},\\ \|u\|_{smooth} &= \|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u\|_{L^{1/\varepsilon}_x(L^2_T)},\\ \|u\|_{maxim} &= \|\langle D_x\rangle^\mu\langle x\rangle^{\alpha_1}u\|_{L^2_x(L^\infty_T)},\end{aligned}$$

and thus $\|u\|_{Y_T}$ (introduced in section 1) equals to $\|u\|_{initial} + \|u\|_{smooth} + \|u\|_{maxim}$.

Lemma 5.1. *There exist some $C, C_\varphi, \beta > 0$ and $\theta \in (0,1)$ such that*

$$\begin{aligned}(5.1)\quad & \|\langle D_x\rangle^{s-1/2} K_\nu(f\partial_x g)\|_{L^1_x(L^2_T)} \\ & \le C |\!|\!| f |\!|\!|_{maxim} |\!|\!| g |\!|\!|_{smooth} + C |\!|\!| f |\!|\!|^{\theta}_{maxim} |\!|\!| f |\!|\!|^{1-\theta}_{smooth} |\!|\!| g |\!|\!|^{1-\theta}_{maxim} |\!|\!| g |\!|\!|^{\theta}_{smooth} \\ & \quad + C_\varphi T^\beta |\!|\!| f |\!|\!|_{Y_T} |\!|\!| g |\!|\!|_{Y_T},\end{aligned}$$

$$(5.2)\quad \|\langle D_x\rangle^{s_1} \langle x\rangle^{\alpha_1} (f\partial_x g)\|_{L^1_T(L^2_x) + L^{4/3}_T(L^1_x)} \le C T^\beta |\!|\!| f |\!|\!|_{Y_T} |\!|\!| g |\!|\!|_{Y_T},$$

where C_φ may diverge as $\varphi \to u_0$ in X^s.

Proof of Lemma 5.1. For small $\kappa > 0$, we write

$$\begin{aligned}\langle D_x\rangle^{s-1/2} K_\nu(f\partial_x g) &= \langle D_x\rangle^{s-1/2} K_\nu \langle x\rangle^{-\kappa}(\langle x\rangle^{\kappa} f\partial_x g) \\ &= \langle x\rangle^{-\kappa} K_\nu \langle D_x\rangle^{s-1/2}(\langle x\rangle^{\kappa} f\partial_x g) \\ &\quad + \langle x\rangle^{-\kappa}[\langle D_x\rangle^{s-1/2}, K_\nu](\langle x\rangle^{\kappa} f\partial_x g) \\ &\quad + [\langle D_x\rangle^{s-1/2} K_\nu, \langle x\rangle^{-\kappa}](\langle x\rangle^{\kappa} f\partial_x g).\end{aligned}$$

Applying Hölder's inequality to the first term on the right hand side and noting that $[\langle D_x\rangle^{s-1/2}, K_\nu] \in \mathcal{B}(L^1_x(L^2_T))$ and $[\langle D_x\rangle^{s-1/2} K_\nu, \langle x\rangle^{-\kappa}] \in \mathcal{B}(L^1_x(L^2_T))$ since the symbol of these commutators belong to $S^{s-3/2}$ with $s - 3/2 < 0$, we have

$$\begin{aligned}\|\langle D_x\rangle^{s-1/2} K_\nu(f\partial_x g)\|_{L^1_x(L^2_T)} &\le C\|K_\nu \langle D_x\rangle^{s-1/2}(\langle x\rangle^{\kappa} f\partial_x g)\|_{L^p_x(L^2_T)} \\ &\quad + C_\varphi \|\langle x\rangle^{\kappa} f\partial_x g\|_{L^1_x(L^2_T)} \\ &\equiv I_1 + I_2,\end{aligned}$$

where $1 < p < 1/(1-\kappa)$. As for I_2, the Hölder and Sobolev inequalities yield

$$\begin{aligned}I_2 &\le C_\varphi \|\langle x\rangle^{\alpha_1} f\partial_x g\|_{L^2_x(L^2_T)} \\ &\le C_\varphi T^{1/2} \|f\|_{L^\infty_T(H^{s_1,\alpha_1}_x)} \|g\|_{L^\infty_T(H^{s,0}_x)}.\end{aligned}$$

By Lemma 3.7 and $\langle D_x\rangle^{s-1/2} - D_x^{s-1/2} \in \mathcal{B}(L^p_x(L^2_T))$, we see that

$$\begin{aligned}I_1 &\le C\|D_x^{s-1/2}(\langle x\rangle^{\kappa} f\partial_x g)\|_{L^p_x(L^2_T)} + C\|\langle x\rangle^{\kappa} f\partial_x g\|_{L^p_x(L^2_T)} \\ &\equiv I_{11} + I_{12}.\end{aligned}$$

I_{12} is estimated as

$$\begin{aligned}I_{12} &\le C\|\langle x\rangle^{\kappa+1/2} f\partial_x g\|_{L^2_x(L^2_T)} \\ &\le C T^{1/2} \|\langle x\rangle^{\alpha_1} f\|_{L^\infty_T(L^\infty_x)} \|g\|_{L^\infty_T(H^{1,0}_x)} \\ &\le C T^{1/2} \|f\|_{L^\infty_T(H^{s_1,\alpha_1}_x)} \|g\|_{L^\infty_T(H^{s,0}_x)},\end{aligned}$$

since $\kappa+1/2<\alpha_1$. By Lemma 3.4, we have

$$\begin{aligned}I_{11} &\le \|\langle x\rangle^{\kappa} f D_x^{s-1/2}\partial_x g\|_{L_x^p(L_T^2)}\\ &\quad +C\|D_x^{s-1/2}\langle x\rangle^{\kappa} f\|_{L_x^{p_1}(L_T^{r_1})}\|\partial_x g\|_{L_x^{p_2}(L_T^{r_2})}\\ &\equiv I_{111}+I_{112},\end{aligned}$$

where

$$\begin{aligned}1/p_1 &= \theta(1/p-\varepsilon)+(1-\theta)\varepsilon,\\ 1/r_1 &= \theta/\infty+(1-\theta)/2,\\ 1/p_2 &= (1-\theta)(1/p-\varepsilon)+\theta\varepsilon,\\ 1/r_2 &= (1-\theta)/\infty+\theta/2,\end{aligned}$$

with $\theta\mu/2+(1-\theta)(s+1/2-\mu/2)=s-1/2$, $\theta\in(0,1)$ and $\mu>0$ small enough. By Hölder's inequality and $\mathcal{H}_x\in\mathcal{B}(L_x^{1/\varepsilon}(L_T^2))$, we see that

$$\begin{aligned}I_{111} &\le \|\langle x\rangle^{\kappa+\rho} f\|_{L_x^{p/(1-\varepsilon p)}(L_T^\infty)}\|\langle x\rangle^{-\rho} D_x^{s-1/2}\partial_x g\|_{L_x^{1/\varepsilon}(L_T^2)}\\ &\le C|\!|\!|f|\!|\!|_{maxim}(|\!|\!|g|\!|\!|_{smooth}+T^{1/2}|\!|\!|g|\!|\!|_{initial}).\end{aligned}$$

To estimate I_{112} by the interpolation (Corollary 3.6), we choose $\kappa'>\kappa$ and $\rho'>0$ so that

$$\begin{aligned}\kappa' &= \theta\rho'+(1-\theta)(-\rho),\\ 0 &= (1-\theta)\rho'+\theta(-\rho).\end{aligned}$$

Then, we see that

(5.3)
$$\begin{aligned}&\|D_x^{s-1/2}\langle x\rangle^{\kappa} f\|_{L_x^{p_1}(L_T^{r_1})}\\ &\le \|\langle D_x\rangle^{s-1/2}\langle x\rangle^{\kappa'}\langle x\rangle^{-(\kappa'-\kappa)} f\|_{L_x^{p_1}(L_T^{r_1})}\\ &\le C\|\langle D_x\rangle^{\mu}\langle x\rangle^{\rho'-\kappa'+\kappa} f\|^{\theta}_{L_x^{p/(1-\varepsilon p)}(L_T^\infty)}\|\langle D_x\rangle^{s+1/2}\langle x\rangle^{-\rho-\kappa'+\kappa} f\|^{1-\theta}_{L_x^{1/\varepsilon}(L_T^2)}\\ &\le C\|\langle x\rangle^{1/2}\langle D_x\rangle^{\mu}\langle x\rangle^{\rho'-\kappa'+\kappa} f\|^{\theta}_{L_x^2(L_T^\infty)}\|\langle D_x\rangle^{s+1/2}\langle x\rangle^{-\rho-\kappa'+\kappa} f\|^{1-\theta}_{L_x^{1/\varepsilon}(L_T^2)}.\end{aligned}$$

We write

$$\begin{aligned}\langle x\rangle^{1/2}\langle D_x\rangle^{\mu}\langle x\rangle^{\rho'-\kappa'+\kappa} f &= \langle x\rangle^{\rho'-\kappa'+\kappa-\alpha_1+1/2}\langle D_x\rangle^{\mu}\langle x\rangle^{\alpha_1} f\\ &\quad +[\langle D_x\rangle^{\mu},\langle x\rangle^{\rho'-\kappa'+\kappa-\alpha_1+1/2}]\langle x\rangle^{\alpha_1} f\\ &\quad +[\langle x\rangle^{1/2},\langle D_x\rangle^{\mu}]\langle x\rangle^{\rho'-\kappa'+\kappa} f.\end{aligned}$$

Since ρ and κ are taken so small that $\rho' - \kappa' + \kappa - \alpha_1 + 1/2 < 0$ and all the commutators in the above are $L^2_x(L^\infty_T)$-bounded, we have

$$\|\langle x\rangle^{1/2}\langle D_x\rangle^{\mu}\langle x\rangle^{\rho'-\kappa'+\kappa} f\|_{L^2_x(L^\infty_T)} \le C|\!|\!| f |\!|\!|_{maxim}. \tag{5.4}$$

On the other hand, we write

$$\langle D_x\rangle^{s+1/2}\langle x\rangle^{-\rho-\kappa'+\kappa} f = \langle x\rangle^{-\rho-\kappa'+\kappa}\langle x\rangle^{s+1/2} f + [\langle D_x\rangle^{s+1/2}, \langle x\rangle^{-\rho-\kappa'+\kappa}] f.$$

By $\sigma([\langle D_x\rangle^{s+1/2}, \langle x\rangle^{-\rho-\kappa'+\kappa}]) \in S^{s-1/2}$ and the Sobolev embedding $H^{1/2,0}_x \subset L^{1/\varepsilon}_x$,

$$\begin{aligned} \|\langle D_x\rangle^{s+1/2}\langle x\rangle^{-\rho-\kappa'+\kappa} f\|_{L^{1/\varepsilon}_x(L^2_T)} \le& \|\langle x\rangle^{-\rho-\kappa'+\kappa}\langle x\rangle^{s+1/2} f\|_{L^{1/\varepsilon}_x(L^2_T)} \\ &+ T^{1/2}\|[\langle D_x\rangle^{s+1/2}, \langle x\rangle^{-\rho-\kappa'+\kappa}] f\|_{L^\infty_T(H^{1/2,0}_x)} \\ \le& C(|\!|\!| f |\!|\!|_{smooth} + T^{1/2}|\!|\!| f |\!|\!|_{initial}). \end{aligned} \tag{5.5}$$

By (5.3)–(5.5), we see that

$$\begin{aligned} &\|D^{s-1/2}_x\langle x\rangle^{\kappa} f\|_{L^{p_1}_x(L^{r_1}_T)} \\ &\le C|\!|\!| f |\!|\!|^{\theta}_{maxim}(|\!|\!| f |\!|\!|_{smooth} + T^{1/2}|\!|\!| f |\!|\!|_{initial})^{1-\theta} \\ &\le C|\!|\!| f |\!|\!|^{\theta}_{maxim}|\!|\!| f |\!|\!|^{1-\theta}_{smooth} + CT^{\beta}|\!|\!| f |\!|\!|_{Y_T}. \end{aligned} \tag{5.6}$$

As for $\|\partial_x g\|_{L^{p_2}_x(L^{r_2}_T)}$, we use $\mathcal{H}_x \in \mathcal{B}(L^{p_2}_x(L^{r_2}_T))$ and Corollary 3.6. Then,

$$\begin{aligned} \|\partial_x g\|_{L^{p_2}_x(L^{r_2}_T)} &\le C\|\langle D_x\rangle g\|_{L^{p_2}_x(L^{r_2}_T)} \\ &\le C|\!|\!| g |\!|\!|^{1-\theta}_{maxim}(|\!|\!| g |\!|\!|_{smooth} + T^{1/2}|\!|\!| g |\!|\!|_{initial})^{\theta} \\ &\le C|\!|\!| g |\!|\!|^{1-\theta}_{maxim}|\!|\!| g |\!|\!|^{\theta}_{smooth} + CT^{\beta}|\!|\!| g |\!|\!|_{Y_T}. \end{aligned} \tag{5.7}$$

Thus, combining (5.6) and (5.7), we obtan (5.1). We next prove (5.2). It suffices to estimate $\|D^{s_1}_x\langle x\rangle^{\alpha_1} f\partial_x g\|_{L^1_T(L^2_x)+L^{4/3}_T(L^1_x)}$. We write

$$\begin{aligned} D^{s_1}_x(\langle x\rangle^{\alpha_1} f\partial_x g) =& \langle x\rangle^{\alpha_1} f D^{s_1}_x\partial_x g + (D^{s_1}_x\langle x\rangle^{\alpha_1} f)\partial_x g \\ &+ (D^{s_1}_x(\langle x\rangle^{\alpha_1} f\partial_x g) - \langle x\rangle^{\alpha_1} f D^{s_1}_x\partial_x g - (D^{s_1}_x\langle x\rangle^{\alpha_1} f)\partial_x). \end{aligned}$$

Then, $L^2_T(L^2_x) \subset L^1_T(L^2_x)$ and Lemma 3.4 yield

$$\begin{aligned} &\|D^{s_1}_x\langle x\rangle^{\alpha_1} f\partial_x g\|_{L^1_T(L^2_x)+L^{4/3}_T(L^1_x)} \\ &\le CT^{1/2}\|\langle x\rangle^{\alpha_1} f\|_{L^{\tilde p_1}_x(L^{\tilde r_1}_T)}\|\langle D_x\rangle^{s_1}\partial_x g\|_{L^{\tilde p_2}_x(L^{\tilde r_2}_T)} + \|(D^{s_1}_x\langle x\rangle^{\alpha_1} f)\partial_x g\|_{L^{4/3}_T(L^1_x)} \\ &\le CT^{1/2}\|\langle x\rangle^{\alpha_1} f\|_{L^{\tilde p_1}_x(L^{\tilde r_1}_T)}\|\langle D_x\rangle^{s_1+1} g\|_{L^{\tilde p_2}_x(L^{\tilde r_2}_T)} + CT^{3/4}|\!|\!| f |\!|\!|_{initial}|\!|\!| g |\!|\!|_{initial}, \end{aligned} \tag{5.8}$$

where

$$1/\tilde{p}_1 + 1/\tilde{p}_2 = 1/2, \quad 1/\tilde{r}_1 + 1/\tilde{r}_2 = 1/2$$
$$1/\tilde{p}_2 = (1-\tilde{\theta})/2 + \tilde{\theta}\varepsilon, \quad 1/\tilde{r}_2 = (1-\tilde{\theta})/\infty + \tilde{\theta}/2,$$

with $s_1 + 1 = (1-\tilde{\theta})\tilde{\mu}/2 + \tilde{\theta}(s_1 + 1 + \tilde{\mu}/2)$ for small $\tilde{\mu} \in (0, \mu)$. By the interpolation (Corollary 3.6), we have

$$\|\langle D_x\rangle^{s_1+1} g\|_{L_x^{\tilde{p}_2}(L_T^{\tilde{r}_2})} \le C\|\langle D_x\rangle^{\tilde{\mu}}\langle x\rangle^{\tilde{\rho}} g\|_{L_x^2(L_T^\infty)}^{1-\tilde{\theta}} \|\langle D_x\rangle^{s_1+1+\tilde{\mu}}\langle x\rangle^{-\rho} g\|_{L_x^{1/\varepsilon}(L_T^2)}^{\tilde{\theta}},$$

where $\tilde{\rho}$ satisfies $0 = (1-\tilde{\theta})\tilde{\rho} + \tilde{\theta}(-\rho)$. Since the smallness of ρ and $\tilde{\mu}$ allows $\tilde{\rho} < \alpha_1$ and $s_1 + 1 + \tilde{\mu} < s + 1/2$, we see that

$$(5.9) \quad \|\langle D_x\rangle^{s_1+1} g\|_{L_x^{\tilde{p}_2}(L_T^{\tilde{r}_2})} \le C|\!|\!| g|\!|\!|_{maxim}^{1-\tilde{\theta}} (|\!|\!| g|\!|\!|_{smooth} + T^{1/2}|\!|\!| g|\!|\!|_{initial})^{\tilde{\theta}} \le C|\!|\!| g|\!|\!|_{Y_T}.$$

As for $\|\langle x\rangle^{\alpha_1} f\|_{L_x^{\tilde{p}_1}(L_T^{\tilde{r}_1})}$, we remark that $1/2 - 1/\tilde{p}_1 = 1/\tilde{p}_2 = \tilde{\mu}/4(s_1+1) + \varepsilon(s_1+1-\tilde{\mu}/2)/(s_1+1) < \mu$ for small $\tilde{\mu}$ and ε. Therefore, the Hardy-Littlewood-Sobolev inequality gives

$$(5.10) \quad \|\langle x\rangle^{\alpha_1} f\|_{L_x^{\tilde{p}_1}(L_T^{\tilde{r}_1})} \le C\|\langle D_x\rangle^{1/\tilde{p}_2}\langle x\rangle^{\alpha_1} f\|_{L_x^2(L_T^{\tilde{r}_1})} \le CT^{1/\tilde{r}_1}|\!|\!| f|\!|\!|_{maxim} \le C|\!|\!| f|\!|\!|_{Y_T}$$

Combining (5.8)–(5.10), we obtain (5.2). □

The lemma given below is concerning the estimates of the remainder term.

Lemma 5.2. *Let s' slightly less than s. Then, there exist $C_\varphi > 0$ and small positive constant β such that*

$$(5.11) \quad \|R_\nu(\varphi, u_\nu)\|_{L_T^1(H_x^{s,0})} \le C_\varphi T|\!|\!| u_\nu|\!|\!|_{Y_T},$$

$$(5.12) \quad \|R_\nu(\varphi, u_\nu) - R_{\nu'}(\varphi, u_{\nu'})\|_{L_T^1(H_x^{s',0})} \le C_\varphi T\|u_\nu - u_{\nu'}\|_{L_T^\infty(H_x^{s',0})} + C_\varphi T(\nu^\beta + {\nu'}^\beta)(|\!|\!| u_\nu|\!|\!|_{Y_T} + |\!|\!| u_{\nu'}|\!|\!|_{Y_T}).$$

Proof of Lemma 5.2. Since $K_\nu\varphi\eta_\nu * \partial_x + [K_\nu, (1-\psi(i^{-1}\partial_x))\mathcal{H}_x\partial_x^2] \in S^0$, this is $H_x^{s,0}$-bounded and its operator norm is estimated in terms of large order derivatives of φ. Also, $[\psi(i^{-1}\partial_x)\mathcal{H}_x\partial_x^2, K_\nu] \in \mathcal{B}(H_x^{s,0})$ since $\psi(i^{-1}\partial_x)\mathcal{H}_x\partial_x^2 \in \mathcal{B}(H_x^{s,0})$. Hence, we obtain (5.11). The estimate (5.12) likewise follows. We note that ν^β and ${\nu'}^\beta$ appear in the estimates of $K_\nu - K_{\nu'}$ and $\eta_\nu - \eta_{\nu'}$ (The slight loss of regularity occurs in these estimates). □

§6. Proof of Theorem 1.1

The local existence of the modified solution u_ν to (1.2) follows from Lemmas 3.1, 4.1, 4.2 and the strong smoothing property of $\eta_\nu *$. Furthermore, this local solution is continuated as long as $\|u_\nu(t)\|_{X^s} < \infty$. We note that $\|\!|u_\nu|\!\|_{Y_T}$ is continuous with respective to T. Let $\|u_0\|_{X^s} \le \delta_0$ and $T_\nu = \sup\{T';\ \|\!|u_\nu|\!\|_{Y_\tau} < 2\tilde{C}_\varphi \delta_0$ for $0 < \tau < T'\}$ (The large positive constant $\tilde{C}_\varphi$ may diverge as $\varphi \to u_0$ in X^s, and it will be specified later). The modified solutions u_ν and $v_\nu = K_\nu u_\nu$ respectively satisfy

$$u_\nu = V(t)u_0 - G(t)(u_\nu \eta_\nu * \partial_x u_\nu), \tag{6.1}$$

$$v_\nu = V(t)K_\nu u_0 - G(t)K_\nu((u_\nu - \varphi)\eta_\nu * \partial_x u_\nu) - G(t)R_\nu(\varphi, u_\nu). \tag{6.2}$$

The uniform lower bound of T_ν and convergence of u_ν as $\nu \downarrow 0$ are obtained by the following proposition.

Proposition 6.1. *The following assertions hold.*

1. *There exists some $T_0 > 0$ such that $\inf_{\nu>0} T_\nu \ge T_0$.*
2. *For some $T \in (0, T_0)$, we have*

$$\|\!|u_\nu|\!\|_{Y_T} \le 2C_0\delta_0, \tag{6.3}$$

$$\|\!|u_\nu - u_{\nu'}|\!\|_{Y'_T} \le C_\varphi(\nu^\beta + \nu'^\beta), \tag{6.4}$$

where $\|\!|\cdot|\!\|_{Y'_T}$ is given by

$$\|\!|f|\!\|_{Y'_T} = \|f\|_{L^\infty_T(H^{s',0}_x \cap H^{s'_1,\alpha_1}_x)} + \|\langle x\rangle^{-\rho}\langle D_x\rangle^{s'+1/2} f\|_{L^{1/\varepsilon}_x(L^2_T)}$$
$$+ \|\langle D_x\rangle^{\mu'}\langle x\rangle^{\alpha_1} f\|_{L^2_x(L^\infty_T)}$$

with s' (resp. s'_1, μ') slightly less than s (resp. s_1, μ).

To prove Proposition 6.1, we need two lemmas. The first lemma suggests that the estimates of v_ν gives those of u_ν.

Lemma 6.2. *Let $s' < s$ and $T \in (0, T_\nu)$. Then, there exist positive constants C, C_φ and β such that*

$$\|u_\nu\|_{L^\infty_T(H^{s,0}_x)} \le C\|v_\nu\|_{L^\infty_T(H^{s,0}_x)} + C_\varphi\|u_\nu\|_{L^\infty_T(L^2_x)}, \tag{6.5}$$

$$\|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u_\nu\|_{L^{1/\varepsilon}_x(L^2_T)} \le C\|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}v_\nu\|_{L^{1/\varepsilon}_x(L^2_T)} + C_\varphi T^\beta\|u_\nu\|_{L^\infty_T(H^{s,0}_x)}, \tag{6.6}$$

$$
\begin{aligned}
(6.7)\ \ &\|u_\nu - u_{\nu'}\|_{L_T^\infty(H_x^{s',0})} \\
&\quad \le C\|v_\nu - v_{\nu'}\|_{L_T^\infty(H_x^{s',0})} + C_\varphi \|u_\nu - u_{\nu'}\|_{L_T^\infty(L_x^2)} \\
&\qquad + C_\varphi(\nu^\beta + \nu'^\beta)(|\!|\!|u_\nu|\!|\!|_{Y_T} + |\!|\!|u_{\nu'}|\!|\!|_{Y_T}), \\
(6.8)\ \ &\|\langle x\rangle^{-\rho}\langle D_x\rangle^{s'+1/2}(u_\nu - u_{\nu'})\|_{L_x^{1/\varepsilon}(L_T^2)} \\
&\quad \le C\|\langle x\rangle^{-\rho}\langle D_x\rangle^{s'+1/2}(v_\nu - v_{\nu'})\|_{L_x^{1/\varepsilon}(L_T^2)} \\
&\qquad + C_\varphi T^\beta \|u_\nu - u_{\nu'}\|_{L_T^\infty(H_x^{s',0})} + C_\varphi(\nu^\beta + \nu'^\beta)(|\!|\!|u_\nu|\!|\!|_{Y_T} + |\!|\!|u_{\nu'}|\!|\!|_{Y_T}),
\end{aligned}
$$

where C_φ is allowed to diverge as $\varphi \to u_0$ in X^s.

Proof of Lemma 6.2. Since $\partial_x K_\nu = K_\nu \partial_x - (\pi/2)^{1/2}\varphi K_\nu \mathcal{H}_x(1 - \psi(i^{-1}\partial_x))\eta_\nu *$, we see that

$$
\langle D_x\rangle^{s-1}\partial_x v_\nu = K_\nu \langle D_x\rangle_x^{s-1}\partial_x u_\nu + r(u_\nu), \tag{6.9}
$$

where $r(u_\nu)$ is a remainder with $\|r(u_\nu)\|_{L_T^\infty(L_x^2)}$ bounded by $C_\varphi\|u_\nu\|_{L_T^\infty(L_x^2)} + \kappa\|u_\nu\|_{L_T^\infty(H_x^{s,0})}$ with $\kappa > 0$ small. Let $\widetilde{K}_\nu(\varphi)(= \widetilde{K}_\nu)$ be the pseudo-differential operator of the symbol:

$$
\widetilde{K}_\nu(x,\xi) = \exp\left(-\sqrt{\frac{\pi}{2}}\frac{i\xi}{|\xi|}(1-\psi(\xi))\hat{\eta}(\nu\xi)\int_{-\infty}^x \varphi(y)\,dy\right).
$$

We here note that $\widetilde{K}_\nu$ plays a role like an inverse of K_ν and, precisely speaking, $I - \widetilde{K}_\nu K_\nu \in S^{-1}$ uniformly in $\nu \in (0,1]$. Applying $\widetilde{K}_\nu$ to (6.9), we have

$$
\langle D_x\rangle^{s-1}\partial_x u_\nu = \widetilde{K}_\nu\langle D_x\rangle_x^{s-1}\partial_x v_\nu + (I - \widetilde{K}_\nu K_\nu)\langle D_x\rangle_x^{s-1}\partial_x u_\nu - \widetilde{K}_\nu r(u_\nu).
$$

Thus, it follows from Lemma 3.7 that

$$
\begin{aligned}
\|\langle D_x\rangle^{s-1}\partial_x u_\nu\|_{L_T^\infty(L_x^2)} \le{}& C\|v_\nu\|_{L_T^\infty(H_x^{s,0})} + C_\varphi\|u_\nu\|_{L_T^\infty(H_x^{s-1,0})} \\
&+ (C_\varphi\|u_\nu\|_{L_T^\infty(L_x^2)} + \kappa\|u_\nu\|_{L_T^\infty(H_x^{s,0})}).
\end{aligned}
$$

Using $\|u_\nu\|_{L_T^\infty(H_x^{s-1,0})} \le C\|u_\nu\|_{L_T^\infty(L_x^2)} + \kappa\|u_\nu\|_{L_T^\infty(H_x^{s,0})}$ and taking κ sufficiently small, we obtain (6.5). We next prove (6.6). Write

$$
\begin{aligned}
\langle D_x\rangle^{s+1/2}u_\nu ={}& \widetilde{K}_\nu\langle D_x\rangle^{s+1/2}v_\nu + (I - \widetilde{K}_\nu K_\nu)\langle D_x\rangle^{s+1/2}u_\nu \\
&- \widetilde{K}_\nu[\langle D_x\rangle^{s+1/2}, K_\nu]u_\nu.
\end{aligned}
$$

Then, Lemma 3.7, $L^2_T(L^{1/\varepsilon}_x) \subset L^{1/\varepsilon}_x(L^2_T)$ and $H^{1/2,0}_x \subset L^{1/\varepsilon}_x$ yield

$$\begin{aligned}\|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u_\nu\|_{L^{1/\varepsilon}_x(L^2_T)} &\le \|\widetilde{K}_\nu\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}v_\nu\|_{L^{1/\varepsilon}_x(L^2_T)} \\ &\quad +\|[\langle x\rangle^{-\rho},\widetilde{K}_\nu]\langle D_x\rangle^{s+1/2}v_\nu\|_{L^2_T(L^{1/\varepsilon}_x)} \\ &\quad +\|\langle x\rangle^{-\rho}(I-\widetilde{K}_\nu K_\nu)\langle D_x\rangle^{s+1/2}u_\nu\|_{L^2_T(L^{1/\varepsilon}_x)} \\ &\quad +\|\langle x\rangle^{-\rho}\widetilde{K}_\nu[\langle D_x\rangle^{s+1/2},K_\nu]u_\nu\|_{L^2_T(L^{1/\varepsilon}_x)} \\ &\le C\|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}v_\nu\|_{L^{1/\varepsilon}_x(L^2_T)} \\ &\quad +CT^{1/2}\|[\langle x\rangle^{-\rho},\widetilde{K}_\nu]\langle D_x\rangle^{s+1/2}v_\nu\|_{L^\infty_T(H^{1/2,0}_x)} \\ &\quad +CT^{1/2}\|(I-\widetilde{K}_\nu K_\nu)\langle D_x\rangle^{s+1/2}u_\nu\|_{L^\infty_T(H^{1/2,0}_x)} \\ &\quad +CT^{1/2}\|\widetilde{K}_\nu[\langle D_x\rangle^{s+1/2},K_\nu]u_\nu\|_{L^\infty_T(H^{1/2,0}_x)}.\end{aligned}$$

Since $\sigma([\langle x\rangle^{-\rho},\widetilde{K}_\nu]) \in S^{-1}$ and $\sigma([\langle D_x\rangle^{s+1/2},K_\nu]) \in S^{s-1/2}$, we see that

$$\begin{aligned}\|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u_\nu\|_{L^{1/\varepsilon}_x(L^2_T)} &\le C\|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}v_\nu\|_{L^{1/\varepsilon}_x(L^2_T)} \\ &\quad +C_\varphi T^{1/2}(\|v_\nu\|_{L^\infty_T(H^{s,0}_x)}+\|u_\nu\|_{L^\infty_T(H^{s,0}_x)}).\end{aligned}$$

Since $\|v_\nu\|_{L^\infty_T(H^{s,0}_x)} \le C_\varphi\|u_\nu\|_{L^\infty_T(H^{s,0}_x)}$, we obtain (6.6). The estimates (6.7) and (6.8) follow in the similar way. We note that, to derive $\nu^\beta+\nu'^\beta$ in the estimate of $K_\nu - K_{\nu'}$, the slight loss of regularity occurs. □

The second lemma allows to make the nonlinearity in (6.2) small enough by taking φ close to u_0 and $T>0$ small.

Lemma 6.3. *There exist positive constants C, C_φ and β such that*

(6.10) $$|||u_\nu-\varphi|||_{maxim} \le C\|u_0-\varphi\|_{X^s}+C_\varphi T^\beta(1+|||u_\nu|||_{Y_T})^2,$$

where C_φ may diverge as $\varphi \to u_0$ in X^s. Furthermore, taking φ close to u_0 in X^s, we have

(6.11) $$|||u_\nu|||_{smooth} \le C\|u_0-\varphi\|_{X^s}+C_\varphi T^\beta(1+|||u_\nu|||_{Y_T})^3.$$

Proof of Lemma 6.3. We first note that $|||\eta_\nu * u_\nu|||_{Y_T} \le C|||u_\nu|||_{Y_T}$. In fact, by regarding $\eta_\nu * = \sqrt{2\pi}\hat{\eta}(\nu i^{-1}\partial_x)$, the symbol of $[\langle x\rangle^\alpha,\hat{\eta}(\nu i^{-1}\partial_x)]$ belongs to S^{-1} uniformly in $\nu\in(0,1]$ if $\alpha\le 1$, which yields $[\langle x\rangle^\alpha,\hat{\eta}(\nu i^{-1}\partial_x)] \in \mathcal{B}(L^r_T(L^p_x))\cap\mathcal{B}(L^p_x(L^r_T))$ with the operator norms independent of ν. Then, it is easy to see that

$$\|\eta_\nu * u_\nu\|_{initial} \le C\|u_\nu\|_{initial},$$
$$\|\eta_\nu * u_\nu\|_{maxim} \le C\|u_\nu\|_{maxim}.$$

The estimate of $\|\eta_\nu * u_\nu\|_{smooth}$ follows from

$$\begin{aligned}\|\eta_\nu * u_\nu\|_{smooth} &\le \|\langle x\rangle^{-\rho}\hat{\eta}(\nu i^{-1}\partial_x)\langle x\rangle^{\rho}\cdot\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u_\nu\|_{L_x^{1/\varepsilon}(L_T^2)}\\ &\le \|\hat{\eta}(\nu i^{-1}\partial_x)\cdot\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u_\nu\|_{L_x^{1/\varepsilon}(L_T^2)}\\ &\quad+\|\langle x\rangle^{-\rho}[\hat{\eta}(\nu i^{-1}\partial_x),\langle x\rangle^{\rho}]\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u_\nu\|_{L_x^{1/\varepsilon}(L_T^2)}\\ &\le C\|u_\nu\|_{smooth}.\end{aligned}$$

Therefore, we have $\|\eta_\nu * u_\nu\|_{Y_T} \le C\|u_\nu\|_{Y_T}$.

Applying Lemmas 4.2 and 5.1 to (6.1), we have

(6.12)
$$\begin{aligned}\|u_\nu-\varphi\|_{maxim} &\le \|V(t)u_0-\varphi\|_{maxim} + CT^{1/2}\|D_x^{s-1/2}(u_\nu\eta_\nu * \partial_x u_\nu)\|_{L_x^1(L_T^2)}\\ &\quad+C\|\langle D_x\rangle^{s_1}\langle x\rangle^{\alpha_1}(u_\nu\eta_\nu * \partial_x u_\nu)\|_{L_T^1(L_x^2)+L_T^{4/3}(L_x^1)}\\ &\le \|V(t)u_0-\varphi\|_{maxim} + CT^{\beta}\|u_\nu\|_{Y_T}^2.\end{aligned}$$

In virtue of Lemma 4.2, the first term in (6.12) is estimated as

$$\begin{aligned}\|V(t)u_0-\varphi\|_{maxim} &\le \|V(t)(u_0-\varphi)\|_{maxim} + \|V(t)\varphi-\varphi\|_{maxim}\\ &\le C\|u_0-\varphi\|_{X^s} + CT\|\varphi\|_{H_x^{m,n}}\end{aligned} \tag{6.13}$$

with $m, n > 0$ large. Thus, combining (6.12) and (6.13), we obtain (6.10). We next prove (6.11). Applying Hölder's inequality and Lemma 3.1 to (6.2), we have

$$\begin{aligned}\|v_\nu\|_{smooth} &\le C\|\langle D_x\rangle^{s+1/2}V(t)K_\nu u_0\|_{L_x^\infty(L_T^2)}\\ &\quad+C\|\langle D_x\rangle^{s-1/2}K_\nu((u_\nu-\varphi)\eta_\nu * \partial_x u_\nu)\|_{L_x^1(L_T^2)}\\ &\quad+C\|R_\nu(\varphi,u_\nu)\|_{L_T^1(H_x^{s,0})}.\end{aligned} \tag{6.14}$$

Note that, to obtain the estimate of $R_\nu(\varphi, u_\nu)$ in the above inequality, we used

$$\begin{aligned}&\|\langle D_x\rangle^{s+1/2}GR_\nu(\varphi,u_\nu)\|_{L_x^\infty(L_T^2)}\\ &\le \int_0^T \|\langle D_x\rangle^{s+1/2}V(t)V(-\tau)R_\nu(\varphi,u_\nu)\|_{L_x^\infty(L_T^2)}\,d\tau\\ &\le C\int_0^T \|\langle D_x\rangle^{s}V(-\tau)R_\nu(\varphi,u_\nu)\|_{L_x^2}\,d\tau.\end{aligned}$$

Lemmas 3.1, 3.2 and $L_T^2(L_x^\infty) \subset L_x^\infty(L_T^2)$ give

$$
\begin{aligned}
(6.15) \qquad & (\text{The first term in (6.14)}) \\
& \leq \|\langle D_x\rangle^{1/2} V(t) K_\nu \langle D_x\rangle^s (u_0-\varphi)\|_{L_x^\infty(L_T^2)} \\
& \quad + T^{1/4}\|\langle D_x\rangle^{1/2} V(t) K_\nu \langle D_x\rangle^s \varphi\|_{L_T^4(L_x^\infty)} \\
& \quad + T^{1/4}\|V(t)\langle D_x\rangle^{1/2}[\langle D_x\rangle^s, K_\nu] u_0\|_{L_T^4(L_x^\infty)} \\
& \leq C\|u_0-\varphi\|_{H_x^{s,0}} + C_\varphi T^{1/4} + C_\varphi T^{1/4}\|u_0\|_{H_x^{s-1/2,0}} \\
& \leq C\|u_0-\varphi\|_{H_x^{s,0}} + C_\varphi T^{1/4}.
\end{aligned}
$$

Applying Lemmas 5.1, 6.3 (6.10) and

$$
\begin{aligned}
\|\!|\eta_\nu * u_\nu\|\!|_{maxim} &\leq C\|\!|u_\nu\|\!|_{maxim}, \\
\|\!|u_\nu\|\!|_{maxim} &\leq \|\!|u_\nu-\varphi\|\!|_{maxim} + C\|\varphi\|_{H_x^{s_1,\alpha_1}}, \\
\|\!|\eta_\nu * u_\nu\|\!|_{smooth} &\leq \|\!|u_\nu\|\!|_{smooth} + CT^{1/2}\|\!|u_\nu\|\!|_{initial},
\end{aligned}
$$

we have

$$
\begin{aligned}
(6.16) \qquad & (\text{The second term in (6.14)}) \\
& \leq C\|\!|u_\nu-\varphi\|\!|_{maxim}\|\!|u_\nu\|\!|_{smooth} \\
& \quad + C\|\!|u_\nu-\varphi\|\!|_{maxim}^{\theta}\|\!|u_\nu-\varphi\|\!|_{smooth}^{1-\theta}\|\!|u_\nu\|\!|_{maxim}^{1-\theta}\|\!|u_\nu\|\!|_{smooth}^{\theta} \\
& \quad + C_\varphi T^\beta(1+\|\!|u_\nu\|\!|_{Y_T})^2 \\
& \leq C(\|u_0-\varphi\|_{X^s} + \|u_0-\varphi\|_{X^s}^{\theta})\|\!|u_\nu\|\!|_{smooth} \\
& \quad + C_\varphi T^\beta(1+\|\!|u_\nu\|\!|_{Y_T})^3.
\end{aligned}
$$

Also, Lemma 5.2 gives

$$
(6.17) \qquad (\text{The third term in (6.14)}) \leq C_\varphi T\|\!|u_\nu\|\!|_{Y_T}.
$$

Combining (6.14)–(6.17) and applying Lemma 6.2, we obtain (6.11). □

Proof of Proposition 6.1. Applying Lemmas 4.1, 4.2 and the nonlinear estimates as in Lemma 5.1 to (6.1), we have

$$
\begin{aligned}
(6.18) \qquad & \|u_\nu\|_{L_T^\infty(H_x^{s_1,\alpha_1})} + \|\!|u_\nu\|\!|_{maxim} \\
& \leq C\|u_0\|_{X^s} + CT^{1/2}\|D_x^{s-1/2}(u_\nu \eta_\nu * \partial_x u_\nu)\|_{L_x^1(L_T^2)} \\
& \quad + C\|\langle D_x\rangle^{s_1}(\langle x\rangle^{\alpha_1} u_\nu \eta_\nu * \partial_x u_\nu)\|_{L_T^1(L_x^2)+L_T^{4/3}(L_x^1)} \\
& \leq C\|u_0\|_{X^s} + CT^\beta\|\!|u_\nu\|\!|_{Y_T}^2.
\end{aligned}
$$

We next apply Lemmas 3.1 and 5.2 to (6.2). Then, it follows from Lemmas 5.1 and 5.2 that

$$\begin{aligned}
&\|v_\nu\|_{L_T^\infty(H_x^{s,0})} + \|\!|v_\nu\|\!|_{smooth} \\
&\quad \le \|v_\nu\|_{L_T^\infty(H_x^{s,0})} + C\|\langle D_x\rangle^{s+1/2} v_\nu\|_{L_x^\infty(L_T^2)} \\
&\quad \le C\|u_0\|_{X^s} + C\|K_\nu((u_\nu-\varphi)\eta_\nu * \partial_x u_\nu)\|_{L_x^1(L_T^2)} \\
&\qquad + C\|R_\nu(\varphi,u_\nu)\|_{L_T^1(H_x^{s,0})} \\
&\quad \le C\|u_0\|_{X^s} + C\|\!|u_\nu-\varphi\|\!|_{maxim}\|\!|u_\nu\|\!|_{smooth} \\
&\qquad + C\|\!|u_\nu-\varphi\|\!|_{maxim}^\theta \|\varphi\|_{H_x^{s_1,\alpha_1}}^{1-\theta}\|\!|u_\nu\|\!|_{smooth} + C_\varphi T^\beta(1+\|\!|u_\nu\|\!|_{Y_T})^2.
\end{aligned}$$

Thus, by Lemmas 6.2 and 6.3 (6.10),

$$\begin{aligned}
(6.19)\qquad &\|u_\nu\|_{L_T^\infty H_x^{s,0}} + \|\!|u_\nu\|\!|_{smooth} \\
&\quad \le C_\varphi\|u_0\|_{X^s} + C(\|u_\nu-\varphi\|_{X^s} + \|u_\nu-\varphi\|_{X^s}^\theta)\|\!|u_\nu\|\!|_{Y_T} \\
&\qquad + C_\varphi T^\beta(1+\|\!|u_\nu\|\!|_{Y_T})^3.
\end{aligned}$$

Combining (6.18) and (6.19), we have

$$\begin{aligned}
\|\!|u_\nu\|\!|_{Y_T} \le{}& \tilde{C}_\varphi\delta_0 + C(\|u_0-\varphi\|_{X^s} + \|u_0-\varphi\|_{X^s}^\theta)\cdot 2\tilde{C}_\varphi\delta_0 \\
&+ C_\varphi T^\beta(1+2\tilde{C}_\varphi\delta_0)^3.
\end{aligned}$$

Let $T \uparrow T_\nu$ and $\varphi \in C_0^\infty(\mathbf{R})$ sufficiently close to u_0 in X^s. Then, we have

$$2\tilde{C}_\varphi\delta_0 \le \tilde{C}_\varphi\delta_0 + (1/2)\tilde{C}_\varphi\delta_0 + C_\varphi T_\nu^\beta(1+2\tilde{C}_\varphi\delta_0)^3.$$

Hence, for any sequence $\{\nu_n\}$ such that $\nu_n \to 0$ as $n \to \infty$, $\liminf_{n\to\infty} T_{\nu_n} = 0$ causes the contradiction. This is the proof of the first statement and (6.3) in Proposition 6.1. We next prove (6.4). Let $u_{\nu,\nu'} = u_\nu - u_{\nu'}$ and $v_{\nu,\nu'} = v_\nu - v_{\nu'}$. We see that

$$\begin{aligned}
(6.20)\quad u_{\nu,\nu'} ={}& -G(t)(u_{\nu,\nu'}\eta_\nu * \partial_x u_\nu) - G(t)(u_{\nu'}\eta_{\nu'} * \partial_x u_{\nu,\nu'}) \\
&- G(t)u_{\nu'}(\eta_\nu - \eta_{\nu'}) * \partial_x u_\nu, \\
(6.21)\quad v_{\nu,\nu'} ={}& V(t)(K_\nu - K_{\nu'})u_0 \\
&- G(t)K_{\nu'}(u_{\nu,\nu'}\eta_\nu * \partial_x u_\nu) - G(t)K_{\nu'}(u_{\nu'}-\varphi)\eta_{\nu'} * \partial_x u_{\nu,\nu'} \\
&- G(t)(K_\nu - K_{\nu'})((u_\nu-\varphi)\eta_\nu * \partial_x u_\nu) \\
&- G(t)K_{\nu'}((u_{\nu'}-\varphi)(\eta_\nu-\eta_{\nu'}) * \partial_x u_\nu) \\
&- G(t)(R_\nu(\varphi,u_\nu) - R_{\nu'}(\varphi,u_{\nu'})).
\end{aligned}$$

Applying Lemmas 6.1, 4.2 and the nonlinear estimates as in Lemma 5.1 to (6.20), we have

$$\begin{aligned}(6.22)\qquad &\|u_{\nu,\nu'}\|_{L_T^\infty(H_x^{s_1',\alpha_1})} + \|\langle D_x\rangle^{\mu'}\langle x\rangle^{\alpha_1}u_{\nu,\nu'}\|_{L_x^2(L_T^\infty)}\\ &\le CT^\beta(\|u_\nu\|_{Y_T'} + \|u_{\nu'}\|_{Y_T'})\|u_{\nu,\nu'}\|_{Y_T'}\\ &\quad + CT^\beta\|u_{\nu'}\|_{Y_T'}\|(\eta_\nu - \eta_{\nu'}) * u_\nu\|_{Y_T'}.\end{aligned}$$

In order to estimate $\|(\eta_\nu - \eta_{\nu'}) * u_\nu\|_{Y_T'}$ in (6.22), it suffices to show that

$$\langle x\rangle^\alpha\langle D_x\rangle^\sigma(\eta_\nu - \eta_{\nu'}) * \langle D_x\rangle^{-\tilde\sigma}\langle x\rangle^{-\alpha} \in \mathcal{B}(L_T^r(L_x^p)) \cap \mathcal{B}(L_x^p(L_T^r)),$$

with the operator norms bounded by $C(\nu^\beta + \nu'^\beta)$, where $-1 < \alpha < 1$, $0 < \sigma < \tilde\sigma < 2$ and $1 \le p, r \le \infty$. Note that $\eta_\nu * = \sqrt{2\pi}\hat\eta(\nu i^{-1}\partial_x)$ and write

$$\begin{aligned}&\langle x\rangle^\alpha\langle D_x\rangle^\sigma(\eta_\nu - \eta_{\nu'}) * \langle D_x\rangle^{-\tilde\sigma}\langle x\rangle^{-\alpha}\\ &\quad = \sqrt{2\pi}\langle x\rangle^\alpha(\hat\eta(\nu i^{-1}\partial_x) - \hat\eta(\nu' i^{-1}\partial_x))\langle D_x\rangle^{-(\tilde\sigma-\sigma)}\langle x\rangle^{-\alpha}\\ &\quad = \sqrt{2\pi}(\hat\eta(\nu i^{-1}\partial_x) - \hat\eta(\nu' i^{-1}\partial_x))\langle D_x\rangle^{-(\tilde\sigma-\sigma)}\\ &\qquad + \sqrt{2\pi}[\langle x\rangle^\alpha, (\hat\eta(\nu i^{-1}\partial_x) - \hat\eta(\nu' i^{-1}\partial_x))\langle D_x\rangle^{-(\tilde\sigma-\sigma)}]\langle x\rangle^{-\alpha}\\ &\quad \equiv P_{1,\nu,\nu'}(x, i^{-1}\partial_x) + P_{2,\mu,\mu'}(x, i^{-1}\partial_x)\langle x\rangle^{-\alpha}.\end{aligned}$$

Let us only consider the case $\alpha > 0$ since, in this case, the multiplication of $\langle x\rangle^{-\alpha}$ is bounded on $L_T^r(L_x^p)$ $(L_x^p(L_T^r))$ and the other case $\alpha \le 0$ is also verified by taking the commutator of $(\hat\eta(\nu i^{-1}\partial_x) - \hat\eta(\nu' i^{-1}\partial_x))\langle D_x\rangle^{-(\tilde\sigma-\sigma)}$ and $\langle x\rangle^{-\alpha}$. It is easy to see that $P_{1,\nu,\nu'}(x,\xi) \in S^{-(\tilde\sigma-\sigma-\beta)}$ with $0 < \beta < \min\{1, \tilde\sigma - \sigma\}$ and

$$\begin{aligned}|P_{1,\nu,\nu'}|_N^{(-(\tilde\sigma-\sigma-\beta))} &\le C_N|\hat\eta(\nu\xi) - \hat\eta(\nu'\xi)|_{N'}^{(\beta)}\\ &\le C_N(|\hat\eta(\nu\xi) - 1|_{N'}^{(\beta)} + |\hat\eta(\nu'\xi) - 1|^{(\beta)_{N'}})\\ &\le C_N(\nu^\beta + \nu'^\beta),\end{aligned}$$

where, to obtain the last inequality in the above estimate, we used $|\hat\eta(\nu\xi) - 1| \le C\nu^\beta\langle\xi\rangle^\beta$ and $|\nu^j(\partial_\xi^j\hat\eta)(\nu\xi)| \le C_j\nu^\beta\langle\xi\rangle^{-j+\beta}$. As for $P_{2,\nu,\nu'}(x, i^{-1}\partial_x)$, we first note that

$$\begin{aligned}P_{2,\nu,\nu'}(x,\xi) = &\frac{i}{\sqrt{2\pi}}\iint e^{-i(x-y)(\xi-\zeta)}\partial_\zeta(\hat\eta(\nu\zeta)\\ &\qquad -\hat\eta(\nu'\zeta))\langle\zeta\rangle^{-(\tilde\sigma-\sigma)}\int_0^1 \partial_w\langle w\rangle^\alpha|_{w=\theta y+(1-\theta)x}\, d\theta dy d\zeta\\ &\in S^{-(1+\tilde\sigma-\sigma)}.\end{aligned}$$

In particular, by regarding $P_2(x,\xi) \in S^{-(1+\tilde\sigma-\sigma-\beta)}$,

$$\begin{aligned}|P_{2,\nu,\nu'}|_N^{(-(1+\tilde\sigma-\sigma-\beta))} &\le C_N|\partial_\xi(\hat\eta(\nu\xi)-\hat\eta(\nu'\xi))\langle\xi\rangle^{-(\tilde\sigma-\sigma)}|_{N'}^{(-(1+\tilde\sigma-\sigma-\beta))}\\ &\le C_N|\hat\eta(\nu\xi)-\hat\eta(\nu'\xi)|_{N''}^{(\beta)}\\ &\le C_N(|\hat\eta(\nu\xi)-1|_{N''}^{(\beta)}+|\hat\eta(\nu'\xi)-1|_{N''}^{(\beta)})\\ &\le C_N(\nu^\beta+\nu'^\beta).\end{aligned}$$

Therefore, observing the integral kernels of $P_{j,\nu,\nu'}(x,i^{-1}\partial_x)$ $(j=1,2)$ and noting that these kernels are estimated in terms of some semi-norms of $P_{j,\nu,\nu'}(x,\xi)$, we see that

$$\begin{aligned}(6.23)\qquad &\|\langle x\rangle^\alpha\langle D_x\rangle^\sigma(\eta_\nu-\eta_{\nu'})*\langle D_x\rangle^{-\tilde\sigma}\langle x\rangle^{-\alpha}\|_{\mathcal{B}(L_T^r(L_x^p))\cap\mathcal{B}(L_x^p(L_T^r))}\\ &\le C(\nu^\beta+\nu'^\beta).\end{aligned}$$

This implies that $|\!|\!|(\eta_\nu-\eta_{\nu'})*u_\nu|\!|\!|_{Y_T'} \le C(\nu^\beta+\nu'^\beta)|\!|\!|u_\nu|\!|\!|_{Y_T}$. Hence, by (6.22), we have

$$\begin{aligned}(6.24)\qquad &\|u_{\nu,\nu'}\|_{L^\infty(H_x^{s_1',\alpha_1})}+\|\langle D_x\rangle^{\mu'}\langle x\rangle^{\alpha_1}u_{\nu,\nu'}\|_{L_x^2(L_T^\infty)}\\ &\le CT^\beta(|\!|\!|u_\nu|\!|\!|_{Y_T}+|\!|\!|u_{\nu'}|\!|\!|_{Y_T})|\!|\!|u_{\nu,\nu'}|\!|\!|_{Y_T}\\ &\quad+C(\nu^\beta+\nu^\beta)|\!|\!|u_\nu|\!|\!|_{Y_T}|\!|\!|u_{\nu'}|\!|\!|_{Y_T}.\end{aligned}$$

Applying Lemma 3.1 and the nonlinear estimates as in Lemmas 5.1–5.2 with s replaced by s' to (6.21) and making use of the estimates similar to (6.23) with $(\eta_\nu-\eta_{\nu'})*$ replaced by $K_\nu-K_{\nu'}$, we see that

$$\begin{aligned}(6.25)\qquad &\|v_{\nu,\nu'}\|_{L_T^\infty(H_x^{s',0})}+\|\langle x\rangle^{-\rho}\langle D_x\rangle^{s'+1/2}v_{\nu,\nu'}\|_{L_x^{1/\varepsilon}(L_T^2)}\\ &\le CL(\varphi,u_\nu,u_{\nu'})|\!|\!|u_{\nu,\nu'}|\!|\!|_{Y_T'}\\ &\quad+C_\varphi T^\beta(1+|\!|\!|u_\nu|\!|\!|_{Y_T}+|\!|\!|u_{\nu'}|\!|\!|_{Y_T})|\!|\!|u_{\nu,\nu'}|\!|\!|_{Y_T'}\\ &\quad+C_\varphi(\nu^\beta+\nu'^\beta)(1+|\!|\!|u_\nu|\!|\!|_{Y_T}+|\!|\!|u_{\nu'}|\!|\!|_{Y_T})^2,\end{aligned}$$

where

$$\begin{aligned}L(\varphi,u_\nu,u_{\nu'}) &= |\!|\!|u_\nu|\!|\!|_{smooth}+|\!|\!|u_{\nu'}-\varphi|\!|\!|_{maxim}\\ &\quad+|\!|\!|u_\nu|\!|\!|_{maxim}^{1-\theta'}|\!|\!|u_\nu|\!|\!|_{smooth}^{\theta'}+|\!|\!|u_{\nu'}-\varphi|\!|\!|_{maxim}^{\theta'}|\!|\!|u_{\nu'}|\!|\!|_{smooth}^{1-\theta'}\end{aligned}$$

with $\theta' \in (0,1)$ determined by $s'-1/2 = \theta'(\mu'/2)+(1-\theta')(s'+1/2-\mu'/2)$ for small $\mu'>0$. Then, it follows from Lemmas 6.2–6.3 that, by taking φ close to u_0 and $T>0$ small in (6.24)–(6.25),

$$|\!|\!|u_{\nu,\nu'}|\!|\!|_{Y_T'} \le \frac{1}{2}|\!|\!|u_{\nu,\nu'}|\!|\!|_{Y_T'}+C_\varphi(\nu^\beta+\nu'^\beta).$$

This completes the proof of (6.4). □

We are now in the position to prove our main result.

Proof of Theorem 1.1. By Proposition 6.1 (6.3), there exist a function $u \in Y_T$ and a sequence $\{u_{\nu_n}\}$ such that

$$\begin{aligned}
&\lim_{n\to\infty} u_{\nu_n} = u \quad \text{weakly-}* \text{ in } L^\infty_T(H^{s,0}_x) \text{ and } L^\infty_T(H^{s_1,\alpha_1}_x),\\
&\lim_{n\to\infty} \langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u_{\nu_n} = \langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u \quad \text{weakly in } L^{1/\varepsilon}_x(L^2_T),\\
&\lim_{n\to\infty} \langle D_x\rangle^{\mu}\langle x\rangle^{\alpha_1}u_{\nu_n} = \langle D_x\rangle^{\mu}\langle x\rangle^{\alpha_1}u \quad \text{weakly-}* \text{ in } L^2_x(L^\infty_T).
\end{aligned}$$

Note that, in the above convergence, we identify $L^\infty_T(H^{s,0}_x)$ (resp. $L^\infty_T(H^{s_1,\alpha_1}_x)$, $L^2_x(L^\infty_T)$) with the dual of $L^1_T(H^{-s,0}_x)$ (resp. $L^1_T(H^{-s_1,-\alpha_1}_x), L^2_x(L^1_T)$). Furthermore, (6.4) implies that $\lim_{n\to\infty} u_{\nu_n} = u$ strongly in Y'_T. Thus, the nonlinearity $u_{\nu_n}\eta_{\nu_n} * \partial_x u_{\nu_n}$ tends to $u\partial_x u$, for instance, in $L^\infty_T(L^2_x)$ and u satisfies

$$\partial_t u + \mathcal{H}_x\partial_x^2 u + u\partial_x u = 0 \quad \text{in } L^\infty_T(H^{s-2,0}_x). \tag{6.26}$$

Following the transformation as in section 2, we can write

$$u = V(t)u_0 - G(t)u\partial_x u, \tag{6.27}$$

$$v = V(t)K_0u_0 - G(t)K_0((u-\widetilde{\varphi})\partial_x u) - GR_0(\widetilde{\varphi}, u), \tag{6.28}$$

where $\widetilde{\varphi}$ is arbitrary C^∞_0-function, $v = K_0u$, K_0 is the pseudo-differential operator of the symbol like

$$K_0(x,\xi) = \exp\left(\sqrt{\frac{\pi}{2}}\frac{i\xi}{|\xi|}(1-\psi(\xi))\int_{-\infty}^x \widetilde{\varphi}(y)dy\right),$$

and $R_0(\widetilde{\varphi}, u) = K_0\widetilde{\varphi}\partial_x u + [K_0, \mathcal{H}_x\partial_x^2]u$.

Continuity in time. We next prove $u \in C([0,T]; H^{s,0}_x \cap H^{s_1,\alpha_1}_x)$. To this end, we use (6.27). Since $V(t)u_0 \in C([0,T]; H^{s,0}_x)$, it sufficies to show that $G(t)u\partial_x u \in C([0,T]; H^{s,0}_x)$. Write

$$\begin{aligned}
&\int_0^{t+h} V(t+h-\tau)u\partial_x u d\tau - \int_0^t V(t-\tau)u\partial_x u d\tau\\
&\quad = \int_t^{t+h} V(t+h-\tau)u\partial_x u d\tau + (V(h)-I)\int_0^t V(t-\tau)u\partial_x u d\tau\\
&\quad \equiv I_1(t,h) + I_2(t,h).
\end{aligned}$$

By Lemma 3.1, we have

$$\|D^s_x I_1(t,h)\|_{L^2_x} \le C\|D^{s-1/2}_x(u\partial_x u)\|_{L^1_x(L^2[t,t+h])}.$$

Therefore, Lebesgue's convergence theorem yields $\lim_{h\to 0} I_1(t,h) = 0$. Since Lemma 3.1 also yields $\int_0^t V(t-\tau)u\partial_x u d\tau \in H_x^{s,0}$, the strong continuity of $V(t)$ in L_x^2 gives $\lim_{h\to 0} I_2(t,h) = 0$. Thus, $u \in C([0,T]; H_x^{s,0})$. The continuity of u in $H_x^{s_1,\alpha_1}$ follows by referring to Lemma 4.1.

Uniqueness in Y_T. Let $u_1, u_2 \in Y_T$ be the solutions to (6.27) with the same initial data u_0. Note that u_j $(j = 1, 2)$ satisfies

$$
\begin{aligned}
u_j &= V(t)u_0 - G(t)u_j\partial_x u_j, \\
v_j &= V(t)K_0u_0 - G(t)K_0((u_j - \widetilde{\varphi})\partial_x u_j) - G(t)R_0(\widetilde{\varphi}, u_j).
\end{aligned}
$$

Let $Y_{[0,h]} = \{u(t,x); \|u\|_{Y_{[0,h]}} < \infty\}$, where

$$
\begin{aligned}
\|u\|_{Y_{[0,h]}} = \|u\|_{L^\infty([0,h];X^s)} &+ \|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u\|_{L_x^{1/\varepsilon}(L^2[0,h])} \\
&+ \|\langle D_x\rangle^{\mu}\langle x\rangle^{\alpha_1}u\|_{L_x^2(L^\infty[0,h])}.
\end{aligned}
$$

Applying the analogy to derive Proposition 6.1 (6.3) and taking $\widetilde{\varphi} \in C_0^\infty(\mathbf{R})$ close to u_0 and $h > 0$ sufficiently small, we have $\|u_1 - u_2\|_{Y_{[0,h]}} \le 0$ i.e., $u_1(t) = u_2(t)$ on $0 \le t \le h$. By similar argument, we have $u_1(t) = u_2(t)$ on $0 \le t \le h + h'$ for some $h' > 0$. Thus, the solution is unique in Y_T.

Stability. Let u' be the solution to (6.26) with $u_0' \in X^s$ as initial data. Then, applying the similar argument for Proposition 6.1 to (6.27) and (6.28) with the common $\widetilde{\varphi} \in C_0^\infty(\mathbf{R})$ close to u_0 in X^s, we see that

$$
\|u' - u\|_{Y_{T'}} \le C\|u_0' - u_0\|_{X^s} + M(T', \widetilde{\varphi}, u', u)\|u' - u\|_{Y_{T'}},
$$

where

$$
\begin{aligned}
M(T', \widetilde{\varphi}, u', u) = \|u'\|_{smooth} &+ \|u\|_{smooth} + \|u'\|_{smooth}^{\theta}\|u'\|_{maxim}^{1-\theta} \\
&+ \|u - \widetilde{\varphi}\|_{maxim} + C_{\widetilde{\varphi}}{T'}^{\beta}(1 + \|u'\|_{Y_{T'}} + \|u\|_{Y_{T'}})
\end{aligned}
$$

for some $\theta \in (0,1)$. By Lemma 6.3, $M(T', \widetilde{\varphi}, u', u)$ is estimated as

$$
\begin{aligned}
M(T', \widetilde{\varphi}, u', u) \le C(&\|u_0' - \widetilde{\varphi}\|_{X^s} + \|u_0 - \widetilde{\varphi}\|_{X^s}) \\
&+ C_{\widetilde{\varphi}}{T'}^{\beta'}(1 + \|u'\|_{Y_{T'}} + \|u\|_{Y_{T'}})^N,
\end{aligned}
$$

where $\beta', N > 0$ and $C_{\widetilde{\varphi}} > 0$ may diverge as $\widetilde{\varphi} \to u_0$ in X^s. If u_0' is sufficiently close to u_0, then, by taking $\widetilde{\varphi} \in C_0^\infty$ close to u_0 and $T' > 0$ small enough, we can make $M(T', \widetilde{\varphi}, u', u)$ small. This implies Theorem 1.1(ii). □

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