

Pole Structure of Topological String Free Energy

By

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Abstract

We show that the free energy of the topological string admits a certain pole structure by using the operator formalism. Combined with the results of Peng that proved the integrality, this gives a combinatoric proof of the Gopakumar–Vafa conjecture.

§1. Introduction and Main Theorem

Recent developments in the string duality make it possible to express the partition function and the free energy of the topological string on a toric Calabi–Yau threefold in terms of the symmetric functions (see [AKMV]). In mathematical terms, the free energy is none other than the generating function of the Gromov–Witten invariants [LLLZ]. In this paper we treat the case of the canonical bundle of a toric surface and its straightforward generalization.

The partition function is given as follows. Let $r \geq 2$ be an integer and $\gamma = (\gamma_1, \dots, \gamma_r)$ be an r -tuple of integers which will be fixed from here on. Let $\vec{Q} = (Q_1, \dots, Q_r)$ be an r -tuple of (formal) variables and q a variable.

Definition 1.1.

$$\mathcal{Z}^\gamma(q; \vec{Q}) = 1 + \sum_{\vec{d} \in \mathbb{Z}_{\geq 0}^r, \vec{d} \neq \vec{0}} \mathcal{Z}_{\vec{d}}^\gamma(q) \vec{Q}^{\vec{d}},$$
$$\mathcal{Z}_{\vec{d}}^\gamma(q) = (-1)^{\gamma \cdot \vec{d}} \sum_{\substack{(\lambda^1, \dots, \lambda^r) \\ \lambda^i \in \mathcal{P}_{d_i}}} \prod_{i=1}^r q^{\frac{\gamma_i \kappa(\lambda^i)}{2}} W_{\lambda^i, \lambda^{i+1}}(q),$$

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where $\vec{Q}^{\vec{d}} = Q_1^{d_1} \cdots Q_r^{d_r}$ for $\vec{d} = (d_1, \dots, d_r)$, \mathcal{P}_d is the set of partitions of d , $\lambda^{r+1} = \lambda^1$ and

$$\kappa(\lambda) = \sum_{i \geq 1} \lambda_i(\lambda_i - 2i + 1) \quad (\lambda: \text{a partition}).$$

For a pair of partitions (μ, ν) , $W_{\mu, \nu}(q)$ is defined as follows.

$$W_{\mu, \nu}(q) = (-1)^{|\mu|+|\nu|} q^{\frac{\kappa(\mu)+\kappa(\nu)}{2}} \sum_{\eta \in \mathcal{P}} s_{\mu/\eta}(q^{-\rho}) s_{\nu/\eta}(q^{-\rho})$$

where \mathcal{P} is the set of partitions and $s_{\mu/\eta}(q^{-\rho})$ is the skew-Schur function associated to partitions μ, η with variables specialized to $q^{-\rho} = (q^{i-\frac{1}{2}})_{i \geq 1}$.

We also define the free energy to be the logarithm of the partition function:

$$\mathcal{F}^\gamma(q; \vec{Q}) = \log \mathcal{Z}^\gamma(q; \vec{Q}).$$

The logarithm should be considered as a formal power series in the variables Q_1, \dots, Q_r . The coefficient of $\vec{Q}^{\vec{d}}$ is denoted by $\mathcal{F}_d^\gamma(q)$.

The free energy is related to the Gromov–Witten invariants in the following way. The target manifold is the total space X of the canonical bundle on a smooth, complete toric surface S . Recall that a toric surface S is given by a two-dimensional complete fan; its one-dimensional cones (1-cones) correspond to toric invariant rational curves. Let r_S be the number of 1-cones, $\vec{C} = (C_1, \dots, C_{r_S})$ the set of the rational curves and γ_S the set of the self-intersection numbers:

$$\gamma_S = (C_1^2, \dots, C_{r_S}^2).$$

For example, if S is \mathbf{P}^2 , \mathbf{F}_0 , \mathbf{F}_1 , \mathcal{B}_2 or \mathcal{B}_3 , γ_S is $(1, 1, 1)$, $(0, 0, 0, 0)$, $(1, 0, -1, 0)$, $(0, 0, -1, -1, -1)$ or $(-1, -1, -1, -1, -1, -1)$. Then the generating function of the Gromov–Witten invariants $N_\beta^g(X)$ of $X = K_S$ with fixed degree β ,

$$\mathcal{F}_\beta(X) = \sum_{g \geq 0} N_\beta^g(X) g_s^{2g-2},$$

is exactly equal to the sum [Z1][LLZ2]:

$$\mathcal{F}_\beta(X) = \sum_{\vec{d}; [\vec{d}, \vec{C}] = \beta} \mathcal{F}_d^{\gamma_S}(q) \Big|_{q=e^{\sqrt{-1}g_s}}.$$

Actually, in the localization calculation, each $\mathcal{F}_d^{\gamma_S}(q) \Big|_{q=e^{\sqrt{-1}g_s}}$ is the contribution from the fixed point loci in the moduli of stable maps of which the image curves are \vec{d}, \vec{C} (see [Z1]).

Note that some pairs (r, γ) do not correspond to toric surfaces. One of the simplest cases is $r = 2$, since for any two dimensional fan to be complete, it must has at least three 1-cones. In this article, we also deal with such non-geometric cases.

One problem concerning the Gromov–Witten invariants is the Gopakumar–Vafa conjecture [GV]. (see also [BP][HST]). Let us define the numbers $\{n_\beta^g(X)\}_{g,\beta}$ by rewriting $\{\mathcal{F}_\beta(X)\}_{\beta \in H_2(X;\mathbb{Z})}$ in the form below.

$$(1) \quad \mathcal{F}_\beta(X) = \sum_{g \geq 0} \sum_{k; k|\beta} \frac{n_{\beta/k}^g(X)}{k} \left(2 \sin \frac{kg_s}{2}\right)^{2g-2}.$$

Then the conjecture states the followings.

1. $n_\beta^g(X) \in \mathbb{Z}$ and $n_\beta^g(X) = 0$ for every fixed β and $g \gg 1$.
2. Moreover, $n_\beta^g(X)$ is equal to the number of certain BPS states in M-theory (see [HST] for a mathematical formulation).

By the Möbius inversion formula [BP], (1) is equivalent to

$$(2) \quad \sum_{k; k|\beta} \frac{\mu(k)}{k} \mathcal{F}_{\beta/k}(X)|_{g_s \rightarrow kg_s} = \sum_{g \geq 1} n_\beta^g(X) \left(2 \sin \frac{g_s}{2}\right)^{2g-2}$$

where $\mu(k)$ is the Möbius function. Therefore the first part is equivalent to the LHS being a polynomial in $t = -(2 \sin \frac{g_s}{2})^2$ with integer coefficients divided by t .

In this article, we deal with the first part of the Gopakumar–Vafa conjecture. As it turns out, it holds not only for a class $\beta \in H_2(X;\mathbb{Z})$ but also for each torus equivariant class $\vec{d} \cdot \vec{C}$. Moreover, it also holds in non-geometric cases.

Definition 1.2. We define

$$G_{\vec{d}}^\gamma(q) = \sum_{k'; k'|k} \frac{k'}{k} \mu\left(\frac{k}{k'}\right) \mathcal{F}_{k'\vec{d}/k}^\gamma(q^{k/k'}) \quad (k = \text{gcd}(\vec{d})).$$

The main result of the paper is the following theorem. We set $t = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2$.

Theorem 1.3. $t \cdot G_{\vec{d}}^\gamma(q) \in \mathbb{Q}[t]$.

Peng proved that $G_{\vec{d}}^\gamma(q)$ is a rational function in t such that its numerator and denominator are polynomials with integer coefficients and the denominator is monic [P]. So we have

Corollary 1.4. $t \cdot G_d^\gamma(q) \in \mathbb{Z}[t]$.

In the geometric case corresponding to $X = K_S$, the corollary implies the integrality of $n_\beta^g(X)$ and its vanishing at higher genera. This is because the LHS of (2) is equal to

$$\sum_{\vec{d}; [\vec{d}, \vec{C}] = \beta} G_d^\gamma(q) \Big|_{q=e^{\sqrt{-1}g_s}}.$$

The organization of the paper is as follows. In Section 2, we introduce the infinite-wedge space (Fock space) and the fermion operator algebra and write the partition function in terms of matrix elements of a certain operator. In Section 3, we express the matrix elements as the sum of certain quantities - amplitudes - over a set of graphs. Then we rewrite the partition function in terms of graph amplitudes. In Section 4, we take the logarithm of the partition function and obtain the free energy. The key idea is to use the exponential formula, which is the relation between log/exp and connected/disconnected graph sums. We give an outline of the proof of the main theorem in Section 5. Then, in Section 6, we study the pole structure of the amplitudes and finish the proof. The rigorous formulation of the exponential formula and the proof of the free energy appear in Appendices A and B. Appendix C contains the proof of a lemma.

§2. Partition Function in Operator Formalism

The goal of this section is to express the partition function in terms of the matrix elements of certain operator in the fermion operator algebra. We first introduce the infinite wedge space [OP1] (also called the Fock space in [KNTY]) and an action of the fermion operator algebra in Subsection 2.1. Then we see that the skew-Schur function and other quantities of partitions ($|\lambda|$ and $\kappa(\lambda)$) are the matrix elements of operators. In Subsection 2.2, we rewrite $W_{\mu,\nu}(q)$ and the partition function.

§2.1. Operator formalism

In this subsection we first briefly explain notations (mainly) on partitions. Secondly we introduce the infinite wedge space, the fermion operator algebra and define some operators. Then we restrict ourselves to a subspace of the infinite wedge space. We see that the canonical basis is naturally associated to the set of partitions and that the skew-Schur function, $|\lambda|$ and $\kappa(\lambda)$ are

the matrix elements of certain operators with respect to the basis. Finally we introduce a new basis which will play an important role in the later calculations.

2.1.1. Partitions

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers containing only finitely many nonzero terms. The nonzero λ_i 's are called the parts. The number of parts is the *length* of λ , denoted by $l(\lambda)$. The sum of the parts is the *weight* of λ , denoted by $|\lambda|$: $|\lambda| = \sum_i \lambda_i$. If $|\lambda| = d$, λ is a partition of d . The set of all partitions of d is denoted by \mathcal{P}_d and the set of all partitions by \mathcal{P} . Let $m_k(\lambda) = \#\{\lambda_i : \lambda_i = k\}$ be the *multiplicity* of k where $\#$ denotes the number of elements of a finite set. Let $\text{aut}(\lambda)$ be the symmetric group acting as the permutations of the equal parts of λ : $\text{aut}(\lambda) \cong \prod_{k \geq 1} \mathfrak{S}_{m_k(\lambda)}$. Then $\#\text{aut}(\lambda) = \prod_{k \geq 1} m_k(\lambda)!$. We define

$$z_\lambda = \prod_{i=1}^{l(\lambda)} \lambda_i \cdot \#\text{aut}(\lambda),$$

which is the number of the centralizers of the conjugacy class associated to λ .

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is identified as the Young diagram with λ_i boxes in the i -th row ($1 \leq i \leq l(\lambda)$). The Young diagram with λ_i boxes in the i -th column is its *transposed* Young diagram. The corresponding partition is called the *conjugate partition* and denoted by λ^t . Note that $\lambda_i^t = \sum_{k \geq i} m_k(\lambda)$.

We define

$$\kappa(\lambda) = \sum_{i=1}^{l(\lambda)} \lambda_i(\lambda_i - 2i + 1).$$

This is equal to twice the sum of contents $\sum_{x \in \lambda} c(x)$ where $c(x) = j - i$ for the box x at the (i, j) -th place in the Young diagram λ . Thus, $\kappa(\lambda)$ is always even and satisfies $\kappa(\lambda^t) = -\kappa(\lambda)$.

$\mu \cup \nu$ denotes the partition whose parts are $\mu_1, \dots, \mu_{l(\mu)}, \nu_1, \dots, \nu_{l(\nu)}$ and $k\mu$ the partition $(k\mu_1, k\mu_2, \dots)$ for $k \in \mathbb{N}$.

We define

$$[k] = q^{\frac{k}{2}} - q^{-\frac{k}{2}} \quad (k \in \mathbb{Q}),$$

which is called the *q-number*. For a partition λ , we use the shorthand notation

$$[\lambda] = \prod_{i=1}^{l(\lambda)} [\lambda_i].$$

2.1.2. Infinite wedge space (Fock Space)

A subset S of $\mathbb{Z} + \frac{1}{2}$ is called a *Maya diagram* if both $S_+ := S \cap \{k \in \mathbb{Z} + \frac{1}{2} | k > 0\}$ and $S_- := S^c \cap \{k \in \mathbb{Z} + \frac{1}{2} | k < 0\}$ are finite sets. The *charge* $\chi(S)$ is defined by

$$\chi(S) = \#S_+ - \#S_-.$$

The set of all Maya diagrams of charge p is denoted by \mathcal{M}_p . We write a Maya diagram S in the decreasing sequence $S = (s_1, s_2, s_3, \dots)$. Note that if $\chi(S) = p$, $s_i \geq p - i + \frac{1}{2}$ for all $i \geq 1$ and $s_i = p - i + \frac{1}{2}$ for $i \gg 1$. Therefore

$$\lambda^{(p)}(S) = \left(s_1 - p + \frac{1}{2}, s_2 - p + \frac{3}{2}, \dots \right)$$

is a partition. $\lambda^{(p)} : \mathcal{M}_p \xrightarrow{\sim} \mathcal{P}$ is a canonical bijection for each p where the inverse $(\lambda^{(p)})^{-1}(\mu) = S$ ($\mu \in \mathcal{P}$) is given by

$$S_+ = \left\{ \mu_i - i + \frac{1}{2} + p \mid 1 \leq i \leq k \right\}, \quad S_- = \left\{ \mu_i^t - i + \frac{1}{2} + p \mid 1 \leq i \leq k \right\}.$$

Here $k = \#\{\mu_i | \mu_i = i\}$ is the number of diagonal boxes in the Young diagram of μ . We define

$$d(S) = |\lambda^{(p)}(S)|.$$

Let V be an infinite dimensional linear space over \mathbb{C} equipped with a basis $\{e_k\}_{k \in \mathbb{Z} + \frac{1}{2}}$ satisfying the following condition: every element $v \in V$ is expressed as $v = \sum_{k > m} v_k e_k$ with some $m \in \mathbb{Z}$. Let $\bar{V} = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ be the topological dual space and $\{\bar{e}_k\}_{k \in \mathbb{Z} + \frac{1}{2}}$ the dual basis: $\bar{e}_l(e_k) = \delta_{k,l}$.

For each Maya diagram $S = (s_1, s_2, s_3, \dots)$, $|v_S\rangle$ denotes the symbol

$$|v_S\rangle = e_{s_1} \wedge e_{s_2} \wedge e_{s_3} \wedge \dots$$

The infinite wedge space of charge p , $\Lambda_p^{\infty} V$ is the vector space over \mathbb{C} spanned by $\{|v_S\rangle\}_{S \in \mathcal{M}_p}$, and the infinite wedge space $\Lambda^{\infty} V$ is the direct sum of the charge p spaces ($p \in \mathbb{Z}$):

$$\Lambda_p^{\infty} V = \prod_{S \in \mathcal{M}_p} \mathbb{C}|v_S\rangle, \quad \Lambda^{\infty} V = \bigoplus_{p \in \mathbb{Z}} \Lambda_p^{\infty} V.$$

We define the charge operator J_0 and the mass operator M on $\Lambda^{\infty} V$ by

$$J_0 v_S = \chi(S) v_S, \quad M v_S = d(S) v_S.$$

Since J_0 and M commute, $\Lambda_p^{\frac{\infty}{2}} V$ decomposes into the eigenspace $\Lambda_p^{\frac{\infty}{2}} V(d)$ of M with eigenvalue d :

$$\Lambda_p^{\frac{\infty}{2}} V(d) = \bigoplus_{\chi(S)=p, d(S)=d} \mathbb{C}|v_S\rangle.$$

The dimension of the eigenspace is equal to the number $p(d)$ of partitions of d .

For a Maya diagram $S = (s_1, s_2, \dots)$, we define the symbol $\langle v_S |$ by

$$\langle v_S | = \dots \wedge \bar{e}_{s_2} \wedge \bar{e}_{s_1}.$$

The dual infinite wedge space is defined by

$$\Lambda^{\frac{\infty}{2}} \bar{V} = \bigoplus_{p \in \mathbb{Z}} \Lambda_p^{\frac{\infty}{2}} \bar{V}, \quad \Lambda_p^{\frac{\infty}{2}} \bar{V} = \bigoplus_{S \in \mathcal{M}_p} \mathbb{C}\langle v_S |.$$

The dual pairing is denoted by $\langle \ | \ \rangle$:

$$\langle v_{S'} | v_S \rangle = \delta_{S, S'}.$$

We set

$$|p\rangle = e_{p-\frac{1}{2}} \wedge e_{p-\frac{3}{2}} \wedge \dots, \quad \langle p| = \dots \wedge \bar{e}_{p-\frac{3}{2}} \wedge \bar{e}_{p-\frac{1}{2}}.$$

$|p\rangle$ is called the *vacuum state* of the charge p . Note that every $|p\rangle$ ($p \in \mathbb{Z}$) corresponds to the empty partition. It is the basis of the subspace with charge p and degree zero: $\Lambda_p^{\frac{\infty}{2}} V(0) = \mathbb{C}|p\rangle$.

2.1.3. Fermion operator algebra

Now we introduce the fermion operators algebra. It is the associative algebra with 1 generated by ψ_k, ψ_k^* ($k \in \mathbb{Z} + \frac{1}{2}$), with the relations:

$$\{\psi_k, \psi_l^*\} = \delta_{k, l}, \quad \{\psi_k, \psi_l\} = \{\psi_k^*, \psi_l^*\} = 0$$

for all $k, l \in \mathbb{Z} + \frac{1}{2}$. Here $\{A, B\} = AB + BA$ is the anti-commutator. We define the actions of ψ_k, ψ_l^* on $\Lambda^{\frac{\infty}{2}} V$ and $\Lambda^{\frac{\infty}{2}} \bar{V}$ as follows:

$$\begin{aligned} \psi_k &= e_k \wedge, & \psi_k^* &= \frac{\partial}{\partial e_k} & (\text{left action on } \Lambda^{\frac{\infty}{2}} V), \\ \psi_k &= \frac{\partial}{\partial \bar{e}_k}, & \psi_k^* &= \wedge \bar{e}_k & (\text{right action on } \Lambda^{\frac{\infty}{2}} \bar{V}). \end{aligned}$$

These are compatible with the dual pairing. So any operator A of the fermion algebra satisfies $\langle v_{S'} | (A|v_S\rangle) = (\langle v_{S'} | A) |v_S\rangle$. We call it the matrix element of A with respect to $\langle v_{S'} |$ and $|v_S\rangle$ and write it as

$$\langle v_{S'} | A |v_S\rangle.$$

The operators ψ_k and ψ_k^* satisfy:

$$\begin{aligned}\psi_k|p\rangle &= 0(k < p), & \psi_k^*|p\rangle &= 0(k > p), \\ \langle p|\psi_k &= 0(k > p), & \langle p|\psi_k^* &= 0(k < p).\end{aligned}$$

Let us define some operators. We first define, for $i, j \in \mathbb{Z} + \frac{1}{2}$,

$$E_{i,j} =: \psi_i \psi_j^* :, \quad : \psi_i \psi_j^* := \begin{cases} \psi_i \psi_j^* & (j > 0) \\ -\psi_j^* \psi_i & (j < 0). \end{cases}$$

The commutation relation is

$$[E_{i,j}, E_{k,l}] = \delta_{j,k} E_{i,l} - \delta_{i,l} E_{k,j} + \delta_{i,l} \delta_{j,k} (\theta(l < 0) - \theta(j < 0))$$

where $\theta(l < 0) = 0$ if $l > 0$ and 1 if $l < 0$. $\theta(j < 0)$ is defined similarly. Next we define

$$C = \sum_{k \in \mathbb{Z} + \frac{1}{2}} E_{k,k}, \quad H = \sum_{k \in \mathbb{Z} + \frac{1}{2}} k E_{k,k}, \quad \mathcal{F}_2 = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{k^2}{2} E_{k,k}.$$

These act on the state $|v_S\rangle$ of charge $\chi(S) = p$ as follows:

$$\begin{aligned}C|v_S\rangle &= p|v_S\rangle, & H|v_S\rangle &= \left(d(S) + \frac{p}{2}\right)|v_S\rangle, \\ \mathcal{F}_2|v_S\rangle &= \left(\frac{\kappa(\lambda^{(p)}(S))}{2} + p d(S) + \frac{p(4p^2 - 1)}{24}\right)|v_S\rangle.\end{aligned}$$

Since C is equal to the charge operator J_0 , it is also called the charge operator. We call H the *energy* operator.

We define

$$\alpha_m = \sum_{k \in \mathbb{Z} + \frac{1}{2}} E_{k-m,k} \quad (m \in \mathbb{Z} \setminus \{0\}).$$

Since these operators satisfy the commutation relations $[\alpha_m, \alpha_n] = m\delta_{m+n,0}$, they are called *bosons*. Note that $[C, \alpha_m] = 0$, $[H, \alpha_m] = -m\alpha_m$.

The operator

$$\Gamma_{\pm}(p) = \exp \left[\sum_{n \geq 1} \frac{p_n \alpha_{\pm n}}{n} \right].$$

is called the *vertex operators* where $p = (p_1, p_2, \dots)$ is a (possibly infinite) sequence. In the later calculation, the sequence p is taken to be the power sum functions of certain variables.

Next we define the operator (see [OP1]) which will play an important role later.

$$\mathcal{E}_c(n) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} q^{n(k - \frac{c}{2})} E_{k-c, k} + \frac{\delta_{c,0}}{[n]} \quad ((c, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}).$$

This is, in a sense, a deformation of the boson α_m by \mathcal{F}_2 since

$$(3) \quad \mathcal{E}_m(0) = \alpha_m \quad (m \neq 0) \quad \text{and} \quad q^{\mathcal{F}_2} \alpha_{-n} q^{-\mathcal{F}_2} = \mathcal{E}_{-n}(n) \quad (n \in \mathbb{N}).$$

The commutation relation is as follows

$$[\mathcal{E}_a(m), \mathcal{E}_b(n)] = \begin{cases} \left[\begin{array}{c|c} a & m \\ \hline b & n \end{array} \right] \mathcal{E}_{a+b}(m+n) & \text{if } (a+b, m+n) \neq (0, 0) \\ a & \text{if } (a+b, m+n) = (0, 0) \end{cases}$$

where $| \cdot |$ in the RHS means the determinant and $[\cdot]$ the q -number. Note that the operators $H, \mathcal{F}_2, \alpha_m, \Gamma_{\pm}(p)$ and $\mathcal{E}_c(n)$ preserve the charge of a state because they commute with the charge operator C .

2.1.4. Charge zero subspace

From here on we work only on the charge zero space $\Lambda_0^{\frac{\infty}{2}} V$.

The $\langle 0|A|0 \rangle$ is called the *vacuum expectation value* (VEV) of the operator A and denoted by $\langle A \rangle$.

We use the set of partitions \mathcal{P} to label the states instead of the set \mathcal{M}_0 of Maya diagrams: for a partition λ ,

$$|v_{\lambda}\rangle = e_{\lambda_1 - \frac{1}{2}} \wedge e_{\lambda_2 - \frac{3}{2}} \wedge \dots, \quad \langle v_{\lambda} | = \dots \wedge \bar{e}_{\lambda_2 - \frac{3}{2}} \wedge \bar{e}_{\lambda_1 - \frac{1}{2}}.$$

$|v_{\emptyset}\rangle = |0\rangle$, is the vacuum state and $\langle v_{\emptyset} | = \langle 0 |$.

With this notation, the action of the energy operator H and \mathcal{F}_2 are written as follows:

$$(4) \quad H|v_{\lambda}\rangle = |\lambda||v_{\lambda}\rangle, \quad \mathcal{F}_2|v_{\lambda}\rangle = \frac{\kappa(\lambda)}{2}|v_{\lambda}\rangle.$$

One of the important points is that the Schur function and the skew Schur function are the matrix elements of the vertex operators: let $x = (x_1, x_2, \dots)$ be variables, $p_i(x) = \sum_j (x_j)^i$ be the i -th power sum function and $p(x) = (p_1(x), p_2(x), \dots)$; then

$$(5) \quad \begin{aligned} \langle v_{\lambda} | \Gamma_{-}(p(x)) | 0 \rangle &= \langle 0 | \Gamma_{+}(p(x)) | v_{\lambda} \rangle = s_{\lambda}(x), \\ \langle v_{\lambda} | \Gamma_{-}(p(x)) | v_{\eta} \rangle &= \langle v_{\eta} | \Gamma_{+}(p(x)) | v_{\lambda} \rangle = s_{\lambda/\eta}(x). \end{aligned}$$

2.1.5. Bosonic basis

For a partition $\mu \in \mathcal{P}$, we write

$$\alpha_{\pm\mu} = \prod_{i=1}^{l(\mu)} \alpha_{\pm\mu_i}.$$

This is well-defined since it is a product of commuting operators. Now we introduce the different kind of states:

$$|\mu\rangle = \alpha_{-\mu}|0\rangle, \quad \langle\mu| = \langle 0|\alpha_{\mu} \quad (\mu \in \mathcal{P}).$$

The following relations are easily calculated from the commutation relations:

$$C|\mu\rangle = 0, \quad H|\mu\rangle = |\mu||\mu\rangle, \quad \langle\mu|\nu\rangle = z_{\mu}\delta_{\mu,\nu}.$$

The set $\{|\mu\rangle|\mu \in \mathcal{P}_d\}$ is a basis of the degree d subspace $\Lambda_0^{\infty}V(d)$ since it consists of $p(d) = \dim \Lambda_0^{\infty}V(d)$ linearly independent states of energy d .

$$\Lambda_0^{\infty}V(d) = \bigoplus_{\mu \in \mathcal{P}_d} \mathbb{C}|\mu\rangle.$$

We call the new basis $\{|\mu\rangle|\mu \in \mathcal{P}_d\}$ the bosonic basis and the old basis $\{|v_{\lambda}\rangle|\lambda \in \mathcal{P}_d\}$ the fermionic basis.

The relation between the bosonic and fermionic basis is written in terms of the character of the symmetric group. Every irreducible, finite dimensional representation of the symmetric group \mathfrak{S}_d corresponds, one-to-one, to a partition $\lambda \in \mathcal{P}_d$. On the other hand, every conjugacy class also corresponds, one-to-one, to a partition $\mu \in \mathcal{P}_d$. The character of the representation λ on the conjugacy class μ is denoted by $\chi_{\lambda}(\mu)$.

Lemma 2.1.

$$|v_{\lambda}\rangle = \sum_{\mu \in \mathcal{P}_{|\lambda|}} \frac{\chi_{\lambda}(\mu)}{z_{\mu}} |\mu\rangle \quad |\mu\rangle = \sum_{\lambda \in \mathcal{P}_{|\mu|}} \chi_{\lambda}(\mu) |v_{\lambda}\rangle.$$

Proof. The vertex operator is formerly expanded as follows:

$$(6) \quad \Gamma_{\pm}(p(x)) = \sum_{\mu \in \mathcal{P}} \frac{p_{\mu}}{z_{\mu}} \alpha_{\pm\mu}.$$

where $p_{\mu}(x) = \prod_{i=1}^{l(\mu)} p_{\mu_i}(x)$. Substituting into the first equation of (5), we obtain

$$s_{\lambda}(x) = \sum_{\mu \in \mathcal{P}_{|\lambda|}} \langle v_{\lambda}|\mu\rangle \frac{p_{\mu}(x)}{z_{\mu}}$$

Comparing with the Frobenius character formula

$$(7) \quad s_\lambda(x) = \sum_{\mu \in \mathcal{P}_{|\lambda|}} \chi_\lambda(\mu) \frac{p_\mu(x)}{z_\mu},$$

we obtain $\langle v_\lambda | \mu \rangle = \chi_\lambda(\mu)$. Then the lemma follows immediately. □

§2.2. Partition function in operator formalism

In this subsection, we first express $W_{\mu,\nu}(q)$ in terms of the fermion operator algebra. Then we rewrite the partition function in terms of the matrix elements of the operator $q^{\mathcal{F}_2}$ with respect to the bosonic basis.

Lemma 2.2.

$$W_{\mu,\nu}(q) = (-1)^{|\mu|+|\nu|} \sum_{\substack{\eta': |\eta'| \leq |\mu|, |\nu| \\ \mu': |\mu'| = |\mu| - |\eta'|, \\ \nu': |\nu'| = |\nu| - |\eta'|}} \sum_{\mu', \nu'} \frac{(-1)^{l(\mu') + l(\nu')}}{z_{\mu'} z_{\nu'} z_{\eta'} [\mu'] [\nu']} \\ \times \langle v_\mu | q^{\mathcal{F}_2} | \mu' \cup \eta' \rangle \langle v_\nu | q^{\mathcal{F}_2} | \nu' \cup \eta' \rangle.$$

Proof. We rewrite the skew-Schur function in the variables $x = (x_1, x_2, \dots)$. By (5) and (6),

$$s_{\mu/\eta}(x) = \sum_{\substack{\mu': |\mu'| = |\mu| - |\eta|, \\ \eta': |\eta'| = |\eta|}} \frac{p_{\mu'}(x)}{z_{\mu'} z_{\eta'}} \langle v_\mu | \mu' \cup \eta' \rangle \langle \eta' | v_\eta \rangle.$$

When $\eta = \emptyset$, this is nothing but the Frobenius character formula (7).

The power sum function $p_i(x)$ associated to the specialized variables $q^{-\rho}$ is

$$p_i(q^{-\rho}) = -\frac{1}{[i]}.$$

Therefore

$$\begin{aligned} & \sum_{\eta} s_{\mu/\eta}(q^{-\rho}) s_{\nu/\eta}(q^{-\rho}) \\ &= \sum_{d=0}^{\min\{|\mu|, |\nu|\}} \sum_{\eta \in \mathcal{P}_d} s_{\mu/\eta}(q^{-\rho}) s_{\nu/\eta}(q^{-\rho}) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{d=0}^{\min\{|\mu|, |\nu|\}} \sum_{\substack{\mu'; |\mu'|=|\mu|-d, \\ \nu'; |\nu'|=|\nu|-d, \\ \lambda, \lambda' \in \mathcal{P}_d}} \frac{(-1)^{l(\mu')+l(\nu')}}{z_{\mu'} z_{\nu'} z_{\lambda} z_{\lambda'} [\mu'] [\nu']} \\
 &\quad \times \underbrace{\langle v_{\mu} | \mu' \cup \lambda \rangle \langle v_{\nu} | \nu' \cup \lambda' \rangle \sum_{\eta \in \mathcal{P}_d} \langle \lambda | v_{\eta} \rangle \langle v_{\eta} | \lambda' \rangle}_{= \langle \lambda | \lambda' \rangle = z_{\lambda} \delta_{\lambda, \lambda'}} \\
 &= \sum_{\mu', \nu', \lambda} \frac{(-1)^{l(\mu')+l(\nu')}}{z_{\mu'} z_{\nu'} z_{\lambda} [\mu'] [\nu']} \langle v_{\mu} | \mu' \cup \lambda \rangle \langle v_{\nu} | \nu' \cup \lambda \rangle.
 \end{aligned}$$

By (4), the factor $q^{\frac{\kappa(\mu)+\kappa(\nu)}{2}}$ is written as follows:

$$q^{\frac{\kappa(\mu)+\kappa(\nu)}{2}} = \langle v_{\mu} | q^{\mathcal{F}_2} | v_{\mu} \rangle \langle v_{\nu} | q^{\mathcal{F}_2} | v_{\nu} \rangle.$$

Combining the above expressions, we obtain the lemma. □

Before rewriting the partition function, we explain some notations. We use the symbols $\vec{\mu}, \vec{\nu}, \vec{\lambda}$ to denote r -tuples of partitions. The i -th partition of $\vec{\mu}$ is denoted by μ^i and $\mu^{r+1} := \mu^1$. ν^i, λ^i ($1 \leq i \leq r+1$) are defined similarly. We define:

$$l(\vec{\mu}) = \sum_{i=1}^r l(\mu^i), \quad \text{aut}(\vec{\mu}) = \prod_{i=1}^r \text{aut}(\mu^i), \quad z_{\vec{\mu}} = \prod_{i=1}^r z_{\mu^i}, \quad [\vec{\mu}] = \prod_{i=1}^r [\mu^i].$$

$l(\vec{\nu}), l(\vec{\lambda}), \text{aut}(\vec{\nu}), \text{aut}(\vec{\lambda}), z_{\vec{\nu}}, z_{\vec{\lambda}}$ and $[\vec{\nu}], [\vec{\lambda}]$ are defined in the same manner.

A triple $(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ is an r -set if it satisfies

$$|\mu^i| + |\lambda^i| = |\nu^i| + |\lambda^{i+1}| \quad (1 \leq \forall i \leq r).$$

$(|\mu^i| + |\lambda^i|)_{1 \leq i \leq r}$ is called the degree of the r -set.

As the consequence of the above lemma, the partition function is written in terms of the matrix elements.

Proposition 2.3.

$$\mathcal{Z}_d^{\gamma}(q) = (-1)^{\gamma \cdot \vec{d}} \sum_{\substack{(\vec{\mu}, \vec{\nu}, \vec{\lambda}); \\ r\text{-set of} \\ \text{degree } \vec{d}}} \frac{(-1)^{l(\vec{\mu})+l(\vec{\nu})}}{z_{\vec{\mu}} z_{\vec{\nu}} z_{\vec{\lambda}} [\vec{\mu}] [\vec{\nu}]} \prod_{i=1}^r \langle \lambda^i \cup \mu^i | q^{(\gamma_i+2)\mathcal{F}_2} | \nu^i \cup \lambda^{i+1} \rangle.$$

Proof. Using Lemmas 2.2 and (4), we write $\mathcal{Z}_d^\gamma(q)$ as follows.

$$\begin{aligned}
 (-1)^{\gamma \cdot \vec{d}} \mathcal{Z}_d^\gamma(q) &= \sum_{\substack{(\vec{\mu}, \vec{\nu}, \vec{\eta}); \\ \gamma\text{-set of} \\ \text{degree } \vec{d}}} \frac{(-1)^{l(\vec{\mu})+l(\vec{\nu})}}{z_{\vec{\mu}} z_{\vec{\nu}} z_{\vec{\eta}} [\vec{\mu}] [\vec{\nu}]} \\
 &\times \prod_{i=1}^r \left(\sum_{\lambda^i; |\lambda^i|=d_i} \langle \mu^i \cup \eta^i | q^{\mathcal{F}_2} | v_{\lambda^i} \rangle \langle v_{\lambda^i} | q^{\gamma_i \mathcal{F}_2} | v_{\lambda^i} \rangle \langle v_{\lambda^i} | q^{\mathcal{F}_2} | \nu^i \cup \eta^{i+1} \rangle \right).
 \end{aligned}$$

The first factor in the bracket comes from $W_{\lambda^{i-1}, \lambda^i}(q)$, the second is the factor $q^{\gamma_i \mathcal{F}_2}$ and the third comes from $W_{\lambda^i, \lambda^{i+1}}(q)$. The bracket term is equal to

$$\langle \mu^i \cup \eta^i | q^{(\gamma_i+2)\mathcal{F}_2} | \nu^i \cup \eta^{i+1} \rangle.$$

If we replace the letter η with λ , we obtain the proposition. □

§3. Partition Function as Graph Amplitudes

In this section, we express the partition function as the sum of some amplitudes over possibly disconnected graphs. (The term an *amplitude* will be used to denote a map from a set of graphs to a ring.)

The graph description is achieved in two steps. Firstly, in Subsections 3.2 and 3.3, we study the matrix element appeared in the partition function:

$$(8) \quad \langle \mu | q^{a\mathcal{F}_2} | \nu \rangle \quad (a \in \mathbb{Z});$$

we introduce the graph set associated to μ, ν and a and describe the matrix element as the sum of certain amplitude over the set. Secondly, in Subsection 3.4 we turn to the whole partition function; we combine these graph sets and the amplitude to make another type of a graph set and amplitude; then we rewrite the partition function in terms of them.

§3.1. Notations

Let us briefly summarize the notations on graphs (see [GY]).

- A *graph* G is a pair of the vertex set $V(G)$ and the edge set $E(G)$ such that every edge has two vertices associated to it. We only deal with graphs whose vertex sets and edge sets are finite.
- A *directed* graph is a graph whose edges has directions.

- A *label* of a graph G is a map from the vertex set $V(G)$ or from the edge set $E(G)$ to a set.
- An *isomorphism* between two graphs G and F is a pair of bijections $V(G) \rightarrow V(F)$ and $E(G) \rightarrow E(F)$ preserving the incidence relations. An *isomorphism* of labeled graphs is a graph isomorphism that preserves the labels.
- An *automorphism* is an isomorphism of the (labeled) graph to itself.
- The *graph union* of two graphs G and F is the graph whose vertex set and edge set are the disjoint unions, respectively, of the vertex sets $V(G), V(F)$ and of the edge sets $E(G), E(F)$. It is denoted by $G \cup F$.
- If G is connected, the *cycle rank* (or *Betti number*) is $\#E(G) - \#V(G) + 1$. If G is not connected, its cycle rank is the sum of those of connected components.
- A connected graph with the cycle rank zero is a *tree* and the graph union of trees is a *forest*.

We also use the following notations.

- A set of not necessarily connected graphs is denoted by a symbol with the superscript \bullet and the subset of connected graphs by the same symbol with \circ (e.g. \mathbb{G}^\bullet and \mathbb{G}°).
- For any finite set of integers $s = (s_1, s_2, \dots, s_l)$,

$$|s| = \sum_i s_i.$$

Note that $|s|$ can be negative if s has negative elements.

When s has at least one nonzero element, we define

$$\text{gcd}(s) = \text{greatest common divisor of } \{|s_i|, s_i \neq 0\}$$

where $|s_i|$ is the absolute value of s_i .

§3.2. Graph description of VEV

By the relation (3), the matrix element (8) is rewritten as the vacuum expectation value:

$$(9) \quad \langle \mu | q^{a\mathcal{F}_2} | \nu \rangle = \langle \mathcal{E}_{\mu_1(\mu)}(0) \cdots \mathcal{E}_{\mu_1}(0) \mathcal{E}_{-\nu_1}(a\nu_1) \cdots \mathcal{E}_{-\nu_l(\nu)}(a\nu_l(\nu)) \rangle.$$

Therefore, in this subsection, we consider the VEV

$$(10) \quad \langle \mathcal{E}_{c_1}(n_1) \mathcal{E}_{c_2}(n_2) \cdots \mathcal{E}_{c_l}(n_l) \rangle$$

where

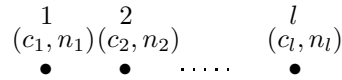
$$\vec{c} = (c_1, \dots, c_l), \quad \vec{n} = (n_1, \dots, n_l)$$

is a pair of ordered sets of integers of the same length l . We assume that $(c_i, n_i) \neq (0, 0)$ ($1 \leq \forall i \leq l$) because $\mathcal{E}_0(0)$ is not well-defined. We also assume that $|\vec{c}| = 0$ because (10) vanishes otherwise.

3.2.1. Set of graphs

When we compute the VEV (10), we make use of the commutation relation several times. We associate to this process a graph generating algorithm in a natural way.

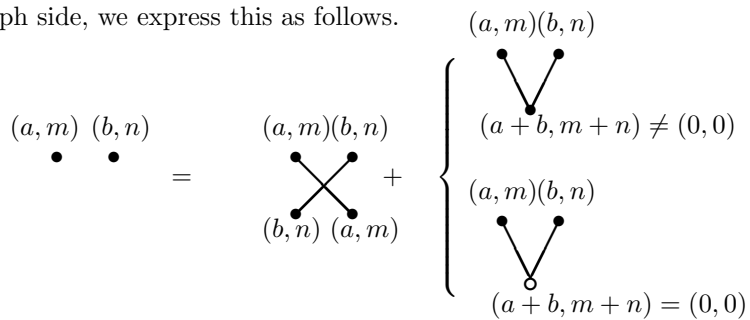
0. In the computation, we start with (10); there are l $\mathcal{E}_{c_i}(n_i)$'s in the angle bracket and we associate to each a black vertex; we draw l black vertices horizontally on \mathbb{R}^2 and assign i and (c_i, n_i) to the i -th vertex (counted from the left). The picture is as follows.



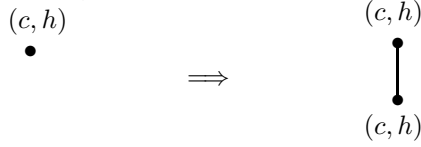
1. The first step in the computation is to take the rightmost adjacent pair $\mathcal{E}_a(m), \mathcal{E}_b(n)$ with $a \geq 0$ and $b < 0$ and apply the commutation relation

$$\mathcal{E}_a(m) \mathcal{E}_b(n) = \mathcal{E}_b(n) \mathcal{E}_a(m) + \begin{cases} \begin{bmatrix} a & m \\ b & n \end{bmatrix} \mathcal{E}_{a+b}(m+n) & (a+b, m+n) \neq (0, 0) \\ a & (a+b, m+n) = (0, 0) \end{cases}$$

On the graph side, we express this as follows.



For other black vertices, we do as follows.

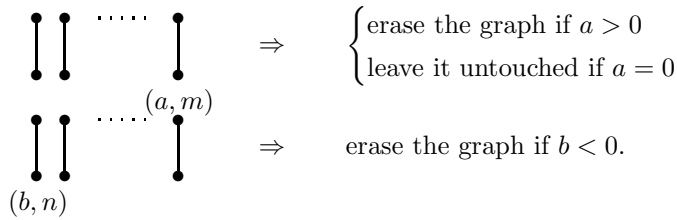


2. Secondly, we use the following relations if applicable:

$$\langle \cdots \mathcal{E}_a(m) \rangle = \begin{cases} 0 & (a > 0) \\ \langle \cdots \rangle_{\frac{1}{|m|}} & (a = 0) \end{cases}$$

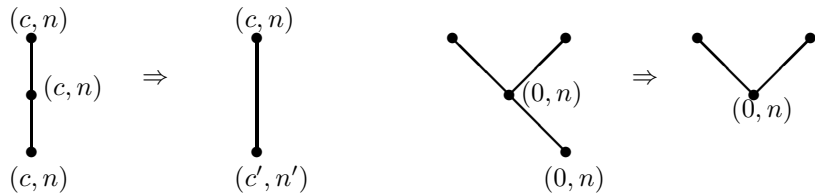
$$\langle \mathcal{E}_b(n) \cdots \rangle = 0 \quad (b < 0), \quad \text{and } \langle 1 \rangle = 1.$$

Accordingly, in the drawing, we do as follows.



3. We repeat these steps until all terms become constants. Although the number of terms may increase at first, it stabilizes in the end and the computation stops with finite processes.

4. Finally, we simplify the drawings:



For concreteness, we show the case of

$$\vec{c} = (c, c, -c, -c), \quad \vec{n} = (0, 0, 0, d) \quad (c > 0, d \neq 0).$$

We will omit the labels and irrelevant middle vertices. We start with

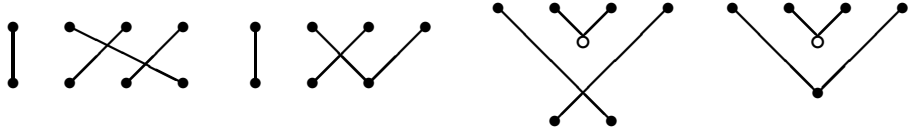
$$\langle \mathcal{E}_c(0) \underbrace{\mathcal{E}_c(0) \mathcal{E}_{-c}(0)} \mathcal{E}_{-c}(d) \rangle \quad \bullet \quad \bullet \quad \bullet \quad \bullet$$

We pick the pair with the underbrace and apply the step 1:

$$\langle \mathcal{E}_c(0)\mathcal{E}_{-c}(0) \underbrace{\mathcal{E}_c(0)\mathcal{E}_{-c}(d)} \rangle + c \underbrace{\langle \mathcal{E}_c(0)\mathcal{E}_{-c}(d) \rangle}$$

We skip the step 2 since there is no place to apply the relation. We proceed to the step 1 again and take the commutation relation of the underbraced pairs:

$$\langle \mathcal{E}_c(0)\mathcal{E}_{-c}(0)\mathcal{E}_{-c}(d)\mathcal{E}_c(0) \rangle + [cd]\langle \mathcal{E}_c(0)\mathcal{E}_{-c}(0)\mathcal{E}_0(d) \rangle + c\langle \mathcal{E}_{-c}(d)\mathcal{E}_c(0) \rangle + c[cd]\langle \mathcal{E}_0(d) \rangle$$



This time we apply the step 2. The first and the third term become zero. In the second and the fourth terms, $\mathcal{E}_0(d)$'s are replaced by $1/[d]$. The result is

$$\frac{[cd]}{[d]} \langle \mathcal{E}_c(0)\mathcal{E}_{-c}(0) \rangle + \frac{c[cd]}{[d]}$$

Applying the step 1 again, we obtain

$$\frac{[cd]}{[d]} \langle \mathcal{E}_{-c}(0)\mathcal{E}_c(0) \rangle + \frac{c[cd]}{[d]} + \frac{c[cd]}{[d]}$$

The first term is zero. Hence all the terms become constants and the process is finished. What we obtained are the two nonzero terms and the corresponding two graphs:



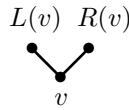
Definition 3.1. $\text{Graph}^\bullet(\vec{c}, \vec{n})$ is the set of graphs generated by the above recursion algorithm. $\text{Graph}^\circ(\vec{c}, \vec{n})$ is the subset consisting of all connected graphs.

Since $F \in \text{Graph}^\bullet(\vec{c}, \vec{n})$ is the graph union of trees, we call F a *VEV forest*. A univalent vertex at the highest level is called a *leaf*. The unique bivalent vertex at the lowest level of each connected component is called the *root*.

A VEV forest has two types of labels. The labels attached to leaves are referred to as the *leaf indices*. The two component labels on every vertex v is called the vertex-label of v and denoted by (c_v, n_v) .

We define \overline{F} to be the graph obtained from a VEV forest F by forgetting the leaf indices. Two VEV forests F and F' are *equivalent* if $\overline{F} \cong \overline{F'}$. The set of equivalence classes in $\text{Graph}^\bullet(\vec{c}, \vec{n})$ and $\text{Graph}^\circ(\vec{c}, \vec{n})$ are denoted by $\overline{\text{Graph}}^\bullet(\vec{c}, \vec{n})$ and $\overline{\text{Graph}}^\circ(\vec{c}, \vec{n})$. A connected component T of \overline{F} is called a *VEV tree*. Note that T is regarded as an element of $\overline{\text{Graph}}^\circ(\vec{c}', \vec{n}')$ with some (\vec{c}', \vec{n}') .

Let $V_2(F)$ be the set of vertices which have two adjacent vertices at the upper level. For a connected component T of F , $V_2(T)$ is defined similarly. $L(v)$ and $R(v)$ denote the upper left and right vertices adjacent to $v \in V_2(F)$.



A left leaf (right leaf) is a leaf which is $L(v)$ ($R(v)$) of some vertex v .

We summarize the properties of the vertex labels of a VEV tree T .

1. $c_{\text{root}} = \sum_{v:\text{leaves}} c_v = 0$.
2. If a vertex v is white, $(c_v, n_v) = (0, 0)$ and it is the root vertex.
3. If v is black, $(c_v, n_v) \neq (0, 0)$.
4. $c_{L(v)} \geq 0$ and $c_{R(v)} < 0$ for $v \in V_2(T)$.
5. $c_v = c_{L(v)} + c_{R(v)}$ and $n_v = n_{L(v)} + n_{R(v)}$ for $v \in V_2(T)$.

3.2.2. Amplitude

In the computation process, the birth of each black vertex $v \in V_2(F)$ means that the corresponding term is multiplied by the factor

$$(11) \quad \zeta_v = \begin{vmatrix} c_{L(v)} & n_{L(v)} \\ c_{R(v)} & n_{R(v)} \end{vmatrix} \quad (v \in V_2(F)).$$

Moreover if the vertex is the root, then the corresponding term is multiplied by $1/[n_v]$. On the other hand, the birth of a white vertex v means that the corresponding term is multiplied by the factor $c_{L(v)}$. Therefore we define the amplitude $\mathcal{A}(F)$ for $F \in \text{Graph}^\bullet(\vec{c}, \vec{n})$ by

$$(12) \quad \begin{aligned} \mathcal{A}(F) &= \prod_{T: \text{VEV tree in } F} \mathcal{A}(T), \\ \mathcal{A}(T) &= \begin{cases} \prod_{v \in V_2(T)} [\zeta_v] / [n_{\text{root}}] & (n_{\text{root}} \neq 0) \\ c_{L(\text{root})} \prod_{\substack{v \in V_2(T) \\ v \neq \text{root}}} [\zeta_v] & (n_{\text{root}} = 0) \end{cases} \end{aligned}$$

From the definition of $\text{Graph}^\bullet(\vec{c}, \vec{n})$, it is clear that the sum of all the amplitude $\mathcal{A}(F)$ is equal to the VEV (10).

Proposition 3.2.

$$\langle \mathcal{E}_{c_1}(n_1) \mathcal{E}_{c_2}(n_2) \cdots \mathcal{E}_{c_l}(n_l) \rangle = \sum_{F \in \text{Graph}^\bullet(\vec{c}, \vec{n})} \mathcal{A}(F).$$

§3.3. Matrix elements

In this subsection, we apply the result of preceding subsection to the matrix element (8).

3.3.1. Graphs

Let μ, ν be two partitions $|\mu| = |\nu| = d > 1, a \in \mathbb{Z}$.

Definition 3.3. $\text{Graph}_a^\bullet(\mu, \nu)$ is the set $\text{Graph}^\bullet(\vec{c}, \vec{n})$ with

$$\vec{c} = (\mu_{l(\mu)}, \dots, \mu_1, -\nu_1, \dots, -\nu_{l(\nu)}), \quad \vec{n} = (\underbrace{0, \dots, 0}_{l(\mu) \text{ times}}, a\nu_1, \dots, a\nu_{l(\nu)}).$$

The subset of connected graphs is denoted by $\text{Graph}_a^\circ(\mu, \nu)$. The set of equivalence classes of $\text{Graph}_a^\bullet(\mu, \nu)$ and $\text{Graph}_a^\circ(\mu, \nu)$ obtained by forgetting the leaf indices are denoted by $\overline{\text{Graph}}_a^\bullet(\mu, \nu)$ and $\overline{\text{Graph}}_a^\circ(\mu, \nu)$.

The next proposition follows immediately from (9) and Proposition 3.2.

Proposition 3.4.

$$\langle \mu | q^{a\mathcal{F}_2} | \nu \rangle = \sum_{F \in \text{Graph}_a^\bullet(\mu, \nu)} \mathcal{A}(F).$$

Example 3.5. Some examples of $\text{Graph}_a^\bullet(\mu, \nu)$ and the amplitudes of its elements are shown below.

$\mu = (\mu_1, \mu_2, \mu_3), \nu = (d)$ where $d = \mu_1 + \mu_2 + \mu_3$:

$a \neq 0$:

$$\frac{[ad\mu_1][ad\mu_2][ad\mu_3]}{[ad]}$$

$a = 0$:

$$0$$

$\mu = \nu = (c, c), a \neq 0$:

$$\frac{[ac^2]^2[2ac^2]}{[ac]} \quad \left(\frac{[ac^2]}{[ac]}\right)^2 \quad \left(\frac{[ac^2]}{[ac]}\right)^2$$

$\mu = \nu = (c, c)$ and $a = 0$.

$$c^2 \quad c^2$$

§3.4. Combined amplitude

In the previous section, we have written the matrix elements as the sum of amplitudes. In Section 2, we have expressed the partition function $\mathcal{Z}_d^\gamma(q)$ in terms of matrix elements. Combining these results, we obtain

$$\mathcal{Z}_d^\gamma(q) = (-1)^{\gamma \cdot \vec{d}} \sum_{\substack{(\vec{\mu}, \vec{\nu}, \vec{\lambda}); \\ r\text{-set of} \\ \text{degree } \vec{d}}} \frac{(-1)^{l(\vec{\mu})+l(\vec{\nu})}}{z_{\vec{\lambda}} z_{\vec{\mu}} z_{\vec{\nu}} [\vec{\mu}] [\vec{\nu}]} \sum_{\substack{(F_i)_{1 \leq i \leq r}; \\ F_i \in \text{Graph}_{(\gamma_i+2)}^\bullet(\mu^i \cup \lambda^i, \nu^i \cup \lambda^{i+1})}} \prod_{i=1}^r \mathcal{A}(F_i).$$

Next, we will associate the each term in the RHS a new type of graph and its amplitudes.

3.4.1. Combined forests

Let $(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ be an r -set. We construct a new graph from an r -tuple

$$(F_1, \dots, F_r), \quad F_i \in \text{Graph}_{(\gamma_i+2)}^\bullet(\mu^i \cup \lambda^i, \nu^i \cup \lambda^{i+1})$$

as follows.

1. Assign the label i to each F_i and make the graph union.
2. Recall that every left leaf of F_i is associated to a part of λ^i or μ^i and every right leaf to a part of ν^i or λ^{i+1} (see the remark below). Since there are two leaves associated to λ_j^i ($1 \leq j \leq l(\lambda^i)$), the one in F_{i-1} and the other in F_i , join them and assign the label λ_j^i to the new edge.

The resulting graph W is a set of VEV forests marked by $1, \dots, r$ and joined through leaves. We call W a *combined forest*. The new edges are called the *bridges*. The label of a bridge b is denoted by $h(b)$. Note that the number of bridges in W is equal to $l(\vec{\lambda})$.

Remark 3.6. To be precise, every left leaf corresponds to a part of $\mu^i \cup \lambda^i$. So a problem would arise if there are equal parts in μ^i and λ^i since they are indistinguishable in $\mu^i \cup \lambda^i$. To solve it, we set the following rule. Assume μ^i and λ^i have an equal part, say k . Then there are several left leaves with the vertex label $(k, 0)$. We regard the outer such leaves come from λ^i and the inner leaves come from μ^i . For right leaves, we set the same rule; outer leaves come from λ^{i+1} and the inner leaves come from ν^i .

Definition 3.7. The set $\text{Comb}_\gamma^\bullet(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ is the set of combined forests constructed by the above procedure. The set $\text{Comb}_\gamma^\circ(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ is the subset consisting of all connected combined forests.

We show some examples in the Figure 1.

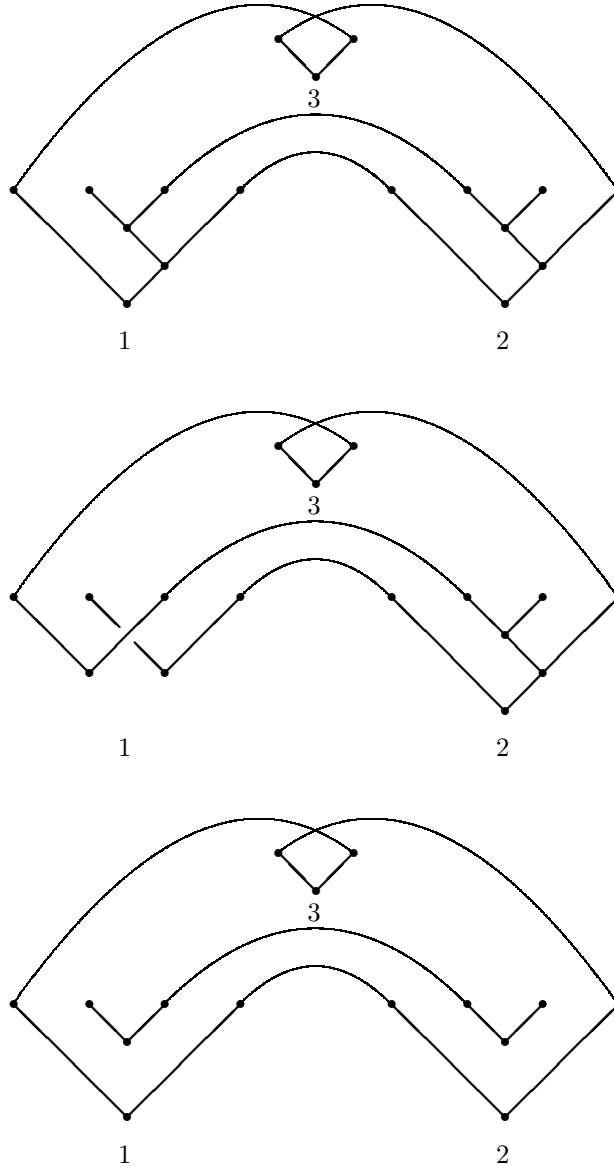


Figure 1. Examples of combined forests. The graphs shown are three of nine elements in the set $\text{Comb}_{\gamma}^{\bullet}(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ where $r = 3$, $\gamma_i + 2 \neq 0$ and $\lambda^1 = (1) = \lambda^3, \lambda^2 = (1, 1), \mu^1 = (1) = \nu^2, \mu^2 = \mu^3 = \nu^1 = \nu^3 = \emptyset$. The leaf indices and the vertex labels are omitted.

§3.5. Combined amplitude and partition function

We first define a slightly different amplitude \mathcal{B} . For a VEV tree $T \in \overline{\text{Graph}}_a^\circ(\mu, \nu)$,

$$\mathcal{B}(T) = \frac{\mathcal{A}(T)}{[\mu][\nu]}.$$

We define the amplitude \mathcal{H} of a combined forest $W \in \text{Comb}_\gamma^\bullet(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ by

$$\mathcal{H}(W) = (-1)^{L_1(W)+L_2(W)} \prod_{T: \text{VEV tree in } W} \mathcal{B}(T) \prod_{b: \text{bridge}} [h(b)]^2$$

where $L_1(W) = l(\vec{\mu}) + l(\vec{\nu})$ and $L_2(W) = \gamma \cdot \vec{d}$. Note that $L_1(W)$ and $L_2(W)$ are equal to:

$$L_1(W) = \#(\text{leaves}) - 2\#(\text{bridges}), \quad L_2(W) = \sum_{T: \text{VEV tree}} n_{\text{root}}.$$

\mathcal{H} is called the combined amplitude.

Then the partition function is expressed with $\text{Comb}_\gamma^\bullet(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ and \mathcal{H} :

Proposition 3.8.

$$\mathcal{Z}_d^\gamma(q) = \sum_{\substack{(\vec{\mu}, \vec{\nu}, \vec{\lambda}); \\ r\text{-set of} \\ \text{degree } \vec{d}}} \frac{1}{z_{\vec{\lambda}} z_{\vec{\mu}} z_{\vec{\nu}}} \sum_{W \in \text{Comb}_\gamma^\bullet(\vec{\mu}, \vec{\nu}, \vec{\lambda})} \mathcal{H}(W).$$

§4. Free Energy

In this section, we obtain the free energy as the sum over connected combined forests.

It is well-known, for example in the calculation of the Feynman diagram, that if one takes the logarithm of the sum of amplitudes over not necessarily connected graphs, then one obtains the sum over connected graphs. More precisely, the statement is written as follows.

$$\log \left[1 + \sum_{G \in \mathbb{G}^\bullet} \frac{1}{|\text{Aut}(G)|} \Psi(G) \right] = \sum_{G \in \mathbb{G}^\circ} \frac{1}{|\text{Aut}(G)|} \Psi(G)$$

where \mathbb{G}^\bullet is a set of graphs, Ψ is an amplitude on it and $\mathbb{G}^\circ \subset \mathbb{G}^\bullet$ is the subset of connected graphs. This formula is nothing but a variation of the exponential formula ([S], chapter 5). Therefore we call it the *exponential formula*.

Of course, for the exponential formula to hold, \mathbb{G}^\bullet and Ψ must satisfy certain conditions. The most important are that \mathbb{G}° generates \mathbb{G}^\bullet by the graph union operation and that Ψ is multiplicative with respect to the graph union. The rigorous formulation of the formula will appear in Appendix A.

In view of the exponential formula, we expect that the free energy is the sum over connected graphs. In fact, this is true.

Proposition 4.1.

$$\mathcal{F}_d^\gamma(q) = \sum_{\substack{(\vec{\mu}, \vec{\nu}, \vec{\lambda}); \\ r\text{-set of} \\ \text{degree } d}} \frac{1}{z_{\vec{\lambda}} z_{\vec{\mu}} z_{\vec{\nu}}} \sum_{W \in \text{Comb}_\gamma^\circ(\vec{\mu}, \vec{\nu}, \vec{\lambda})} \mathcal{H}(W).$$

We delegate the proof to Appendix B.

§5. Proof of Main Theorem: Outline

We sketch the proof of Theorem 1.3 in this section.

The combined forest which is the same as a combined forest W except that all the vertex-labels are multiplied by $k \in \mathbb{N}$, is denote by $W_{(k)}$.

We first rewrite $\mathcal{F}_d^\gamma(q)$ as follows.

$$\mathcal{F}_d^\gamma(q) = \sum_{k; k|d} \sum_{\substack{(\vec{\mu}, \vec{\nu}, \vec{\lambda}); \\ r\text{-set of} \\ \text{degree } d/k; \\ \text{gcd}(\vec{\mu}, \vec{\nu}, \vec{\lambda})=1}} \frac{1}{z_k \vec{\lambda} z_k \vec{\mu} z_k \vec{\nu}} \sum_{W \in \text{Comb}_\gamma^\circ(\vec{\mu}, \vec{\nu}, \vec{\lambda})} \mathcal{H}(W_{(k)}).$$

Then $G_d^\gamma(q)$ is equal to

$$\begin{aligned} G_d^\gamma(q) &= \sum_{k; k|d} \sum_{\substack{(\vec{\mu}, \vec{\nu}, \vec{\lambda}); \\ r\text{-set of} \\ \text{degree } d/k; \\ \text{gcd}(\vec{\mu}, \vec{\nu}, \vec{\lambda})=1}} \sum_{k'; k'|k} \frac{k'}{k} \mu\left(\frac{k}{k'}\right) \frac{1}{z_{k'} \vec{\lambda} z_{k'} \vec{\mu} z_{k'} \vec{\nu}} \\ &\quad \times \sum_{W \in \text{Comb}_\gamma^\circ(\vec{\mu}, \vec{\nu}, \vec{\lambda})} \mathcal{H}(W_{(k')})|_{q \rightarrow q^{k/k'}} \\ &= \sum_{k; k|d} \sum_{\substack{(\vec{\mu}, \vec{\nu}, \vec{\lambda}); \\ r\text{-set of} \\ \text{degree } d/k; \\ \text{gcd}(\vec{\mu}, \vec{\nu}, \vec{\lambda})=1}} \frac{1}{k \# \text{aut}(\vec{\mu}) \# \text{aut}(\vec{\nu}) \# \text{aut}(\vec{\lambda})} \sum_{W \in \text{Comb}_\gamma^\circ(\vec{\mu}, \vec{\nu}, \vec{\lambda})} \mathcal{G}_k(W), \end{aligned}$$

where

$$\mathcal{G}_k(W) = \sum_{k':k'|k} \mu\left(\frac{k}{k'}\right) k'^{-l(\vec{\mu})-l(\vec{\nu})-l(\vec{\lambda})+1} \mathcal{H}(W_{(k')}) \Big|_{q \rightarrow q^{k/k'}}.$$

Therefore if we show

$$(13) \quad t \cdot \mathcal{G}_k(W) \in \mathbb{Q}[t]$$

where $t = [1]^2$, then the proof of the theorem will be finished. This is the subject of the next section.

Since the entire proof is quite long, let us briefly explain the main idea. The amplitude $\mathcal{H}(W)$ turns out to be a polynomial in t if W has the cycle rank > 0 and clearly (13) holds. In the case the cycle rank is zero, $\mathcal{H}(W)$ has poles. However, the poles of $\mathcal{H}(W)$ and $\mathcal{H}(W_{(k)})$ are related in some way and almost all of them cancel in $\mathcal{G}_k(W)$, leaving at most a pole at $t = 0$. Let us describe the situation in the simplest case, when W does not contain a certain type of VEV trees ($V_{III}(W) = \emptyset$ in the notation of the next section), and k is odd. Then

$$\mathcal{H}(W_{(k)}) = \frac{g_W \cdot k^{(\vec{\mu})+l(\vec{\nu})+l(\vec{\lambda})-1}}{t_k} + \text{polynomial in } t.$$

Here $t_k = [k]^2$ and g_W is a constant determined by W . Therefore, if k is odd,

$$\mathcal{G}_k(W) = \frac{g_W}{t_k} \sum_{k':k'|k} \mu\left(\frac{k}{k'}\right) + \text{polynomial in } t.$$

(13) follows from the formula of Möbius function

$$(14) \quad \sum_{k':k'|k} \mu\left(\frac{k}{k'}\right) = \begin{cases} 1 & (k = 1) \\ 0 & (k > 1, k \in \mathbb{N}). \end{cases}$$

So the factor $1/t_k$ in $\mathcal{G}_k(W)$ vanishes unless $k = 1$. The proofs in other cases are more complicated, but similar.

§6. Pole Structure

In this section, we first study poles of the amplitude $\mathcal{B}(T)$ and then move to the study of $\mathcal{H}(W)$. In Subsection 6.1, we see that $\mathcal{B}(T)$ is actually a function of $t = [1]^2$ and study its poles. Then we study $\mathcal{H}(W)$ in Subsection 6.2. The proof of the main theorem is completed at the last of this section when we give a proof of (13).

§6.1. $\mathcal{B}(T)$

In this subsection, we study poles of the amplitudes $\mathcal{B}(T)$ of a VEV tree T . We first state the main proposition and check some examples. Next we collect some lemmas necessary for the proof and prove the proposition.

6.1.1. Main results

We set

$$t = [1]^2, \quad t_k = [k]^2 \quad (k \in \mathbb{N}).$$

It is known that t_k is a polynomial in t with integer coefficients [BP]

$$t_k = \sum_{j=1}^k \frac{k}{j} \binom{j+k-1}{2j-1} t^j \in \mathbb{Z}[t].$$

We define $m(T)$ of a VEV tree T by

$$m(T) = \gcd(\mu, \nu) \quad \text{if } T \in \overline{\text{Graph}}_a^\circ(\mu, \nu).$$

It turns out that $\mathcal{B}(T)$ has poles at $t_{m(T)} = 0$.

Proposition 6.1. *Let T be a VEV tree in $\overline{\text{Graph}}_a^\circ(\mu, \nu)$.*

1. *There exists $g_T \in \mathbb{Z}$ and $f_T(t) \in \mathbb{Z}[t]$ such that*

$$\mathcal{B}(T) = \begin{cases} \frac{g_T}{t_{m(T)}} + f_T(t_{m(T)}) & (m(T) \text{ odd or } n_{\text{root}}/m(T) \text{ even}) \\ \frac{g_T}{t_{m(T)}} \left(1 + \frac{t_{m(T)/2}}{2}\right) + f_T(t_{m(T)}) & (m(T) \text{ even and } n_{\text{root}}/m(T) \text{ odd}) \end{cases}$$

We remark that $n_{\text{root}} = ad$ where $d = |\mu| = |\nu|$.

2. *Moreover,*

$$g_T = g_{T_{(0)}} \cdot m(T)^{l(\mu)+l(\nu)-1}.$$

Here $T_{(0)}$ is the VEV tree which is the same as T except all the vertex-labels is multiplied by $1/m(T)$.

We will give a proof for the case $m(T) = 1$ first, then the case $m(T) > 1$.

Before giving a proof, let us check simple examples. Consider the VEV forest $T \in \overline{\text{Graph}}_a^\circ((d), (d))$ ($a \neq 0$) below. T has $m(T) = d$.



If $a = 1$ and $d = 3$,

$$\mathcal{B}(T) = \frac{[9]}{[3]^3} = \frac{1}{[3]^2}(q^3 + 1 + q^{-3}) = \frac{t_3 + 3}{t_3}, \quad g_T = 3.$$

$$\mathcal{B}(T_{(0)}) = \frac{[1]}{[1]^3} = \frac{1}{t}, \quad g_{T_{(0)}} = 1.$$

Here we have used the relation

$$q^k + q^{-k} = t_k + 2.$$

If $a = 3$ and $d = 2$,

$$\mathcal{B}(T) = \frac{[12]}{[6][2]^2} = \frac{1}{[2]^2}(q^3 + q^{-3}) = \frac{t_3 + 2}{t_2} = \frac{t + 2}{t_2} + t + 2, \quad g_T = 2.$$

$$\mathcal{B}(T_{(0)}) = \frac{[3]}{[3][1]^2} = \frac{1}{t}, \quad g_{T_{(0)}} = 1.$$

Thus the proposition holds in these examples.

6.1.2. Lemmas

We collect some lemmas on q -numbers necessary for the proof of the Proposition 6.1. Here is the list indicating where these lemmas are needed.

In the proof of the $k = 1$ case, Lemma 6.3.4 plays the guiding role. Lemmas 6.4 and 6.5 are used to show that the third assumption of Lemma 6.3.4. In the proof of the $k > 1$ case, we use Lemmas 6.4.2 and 6.6 together with the result of the $k = 1$ case. Lemmas 6.2 and 6.3.1,2,3 are the preliminary for the other lemmas. We remark that the proof of Lemma 6.6 is very similar to the $k = 1$ case.

Let us start from some notations. For a (dummy) variable x , the ring of polynomials with integer (rational) coefficients is denoted by $\mathbb{Z}[x]$ ($\mathbb{Q}[x]$), the ring of rational functions with rational coefficients by $\mathbb{Q}(x)$. The ring of Laurent polynomials with integer coefficients is denoted by $\mathbb{Z}[x, x^{-1}]$. We define

$$\mathbb{Z}^+[x, x^{-1}] = \{f(x) \in \mathbb{Z}[x, x^{-1}] \mid f(x) = f(x^{-1})\}.$$

This is a subring of $\mathbb{Z}[x, x^{-1}]$.

We also define

$$\mathcal{L}[x] = \left\{ \frac{f_2(x)}{f_1(x)} \mid f_1(x), f_2(x) \in \mathbb{Z}[x], f_1(x) : \text{monic} \right\}.$$

It is not difficult to see that this is a subring of $\mathbb{Q}(x)$ (see [P]).

We set

$$t = [1]^2, \quad t_k = [k]^2, \quad y = [1/2]^2.$$

Note that $t = y(y + 4)$. We define the relation $<_y$ in $\mathcal{L}[y]$ by

$$f_1(y) <_y f_2(y) \Leftrightarrow \exists f_3(y) \in \mathbb{Z}[y], f_3(0) = 1, \quad f_1(y) = f_2(y)f_3(y).$$

$f_1(y) <_y f_2(y)$ implies that $f_1(y)$ has poles at most where $f_2(y)$ does.

Lemma 6.2.

$$\mathbb{Z}^+[q, q^{-1}] \cong \mathbb{Z}[t], \quad \mathbb{Z}^+[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \cong \mathbb{Z}[y].$$

Proof. We only show the first isomorphism. The proof of the second is the same if q is replaced by $q^{\frac{1}{2}}$.

\subseteq : An element $f(q)$ in $\mathbb{Z}^+[q, q^{-1}]$ is written as follows.

$$f(q) = b_0 + \sum_{i=1}^m b_j(q^i + q^{-i}) = b_0 + \sum_{i=1}^m b_j(t_j + 2) \quad (b_j \in \mathbb{Z}),$$

where we have used the relation $t_j = q^j + q^{-j} - 2$. Since $t_j \in \mathbb{Z}[t]$, $f(q) \in \mathbb{Z}[t]$.

\supseteq : An element $f(t)$ in $\mathbb{Z}[t]$ is written as follows.

$$f(t) = \sum_{i=0}^m b_i t^i = \sum_{i=0}^m b_i (q + q^{-1} - 2)^i \quad (b_j \in \mathbb{Z}).$$

Thus $f(t) \in \mathbb{Z}^+[q, q^{-1}]$. □

Lemma 6.3. *Let $a_1, \dots, a_m, b_1, \dots, b_n$ be integers.*

1. $\prod_{i=1}^m [a_i] \in \mathbb{Z}[y] \Leftrightarrow m$ is even.
2. $\prod_{i=1}^m [a_i] \in \mathbb{Z}[t] \Leftrightarrow$ both m and $\sum_{i=1}^m a_i$ are even.
3. $\prod_{i=1}^m [a_i] / \prod_{i=1}^n [b_i] \in \mathcal{L}[y] \Leftrightarrow m + n$ is even.
4. If both $m + n$ and $\sum_{i=1}^m a_i + \sum_{i=1}^n b_i$ are even and $\prod_{i=1}^m [a_i] / \prod_{i=1}^n [b_i] \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, then $\prod_{i=1}^m [a_i] / \prod_{i=1}^n [b_i] \in \mathbb{Z}[t]$.

Proof. 1 and 2 follow immediately from the previous lemma.

3. If $m + n$ is odd, $\prod [a_i] / \prod [b_j]$ is anti-symmetric under $q \rightarrow q^{-1}$. Thus it can not be written with y . If both m and n are even, then $\prod [a_i], \prod [b_j] \in \mathbb{Z}[y]$. If both m and n are odd, then $[1] \prod [a_i], [1] \prod [b_j] \in \mathbb{Z}[y]$. Thus 3 is proved.

4. The condition that $m + n$ is even means that $\prod [a_i] / \prod [b_j] \in \mathcal{L}[y]$. The assumption that it is a polynomial implies that $l = \sum |a_i| - \sum |b_j| \geq 0$ and that it is written as follows.

$$\frac{\prod [a_i]}{\prod [b_j]} = \sum_{i=0}^l c_i q^{\frac{l}{2}-i} \quad (c_i \in \mathbb{Z}, \quad c_i = c_{l-i}).$$

The condition that l is even implies that $\prod [a_i] / \prod [b_j] \in \mathbb{Z}^+[q, q^{-1}]$. Therefore 4 follows from the previous lemma. \square

Recall that $[\lambda] = \prod_{i=1}^{l(\lambda)} [\lambda_i]$ where λ is a partition.

Lemma 6.4. *Let $k \in \mathbb{N}$ be a positive integer.*

1. *Let $a \in \mathbb{N}$ be a positive integer. Then $[ka]/[a] \in \mathbb{Z}[y]$. Moreover if k is odd or a is even, $[ka]/[a] \in \mathbb{Z}[t]$ and the constant term is k . If k is even and a is odd, the remainder of $[ka]/[a]$ by $y(y + 4)$ is $k(1 + y/2)$.*
2. *Let λ be a partition. Then $[k\lambda]/[\lambda]$ is written as follows.*

$$\frac{[k\lambda]}{[\lambda]} = \begin{cases} k^{l(\lambda)} + t \times u_{k,\lambda}(t) & (k \text{ odd or } |\lambda| \text{ even}) \\ k^{l(\lambda)} \left(1 + \frac{y}{2}\right) + t \times u_{k,\lambda}(y) & (k \text{ even and } |\lambda| \text{ odd}). \end{cases}$$

Here, if k is odd or $|\lambda|$ is even, $u_{k,\lambda}(t) \in \mathbb{Z}[t]$ is a polynomial of degree $(k - 1)|\lambda|/2 - 1$, if k is even and $|\lambda|$ is odd, $u_{k,\lambda}(y) \in \mathbb{Z}[y]$ is a polynomial of degree $(k - 1)|\lambda| - 2$.

Weaker statements of the lemma are written with $<_y$ in the following simple form:

$$\frac{[ka]}{[a]} <_y k, \quad \frac{[k\lambda]}{[\lambda]} <_y k^{l(\lambda)}.$$

Proof. 1. Recall the formula $(x^k - 1)/(x - 1) = 1 + x + \dots + x^{k-1}$. Then it is easily calculated that

$$\frac{[ka]}{[a]} = \sum_{i=0}^{k-1} q^{\binom{k-1}{2}a - i}.$$

If $(k - 1)a$ is even, $[ka]/[a] \in \mathbb{Z}^+[q, q^{-1}]$. Thus it is in $\mathbb{Z}[t]$. Since $t = 0$ is equivalent to $q = 1$, the value at $t = 0$ is the value at $q = 1$, which is k .

If $(k-1)a$ is odd, $[ka]/[a] \in \mathbb{Z}[y]$. To compute the remainder by $y(y+4)$, let us write it as $b_1 + b_2y$. We compare the values at $y = 0$ and $y = -4$, or equivalently at $q^{\frac{1}{2}} = 1$ and $q^{\frac{1}{4}} = \sqrt{-1}$:

$$b_1 = k, \quad b_1 - 4b_2 = -k.$$

Therefore the first part is proved.

2. By the result of 1, $[k\lambda]/[\lambda]$ is in $\mathbb{Z}[y]$ with constant term $k^{l(\lambda)}$.

If k is odd or $|\lambda|$ is even, all the three conditions of Lemma 6.3.4 are satisfied and $[k\lambda]/[\lambda] \in \mathbb{Z}[t]$.

If k is even and $|\lambda|$ is odd, there are odd number of odd parts in λ . Take one odd part, say λ_1 , and denote by λ' by the partition $\lambda \setminus \{\lambda_1\}$. Then $[k\lambda']/[\lambda'] \in \mathbb{Z}[t]$ with constant term $k^{l(\lambda)-1}$. On the other hand, $[k\lambda_1]/[\lambda_1] \in \mathbb{Z}[y]$ and the remainder by $t = y(y+4)$ is $k(1+y/2)$. Therefore the second part of the lemma follows. \square

Lemma 6.5. *Let $a, b, c \in \mathbb{N}$ be positive integers satisfying $\gcd(a, b, c) = 1$. Then*

$$\frac{[\text{lcm}(a, b, c)][\gcd(a, b)][\gcd(b, c)][\gcd(c, a)]}{[a][b][c][1]} \in \mathbb{Z}[t],$$

and the constant term is 1.

With $<_y$, (a weaker statement of) the lemma are written as follows:

$$\frac{[\text{lcm}(a, b, c)]}{[a][b][c]} <_y \frac{[1]}{[\gcd(a, b)][\gcd(b, c)][\gcd(c, a)]}.$$

Proof. We check the three conditions to apply Lemma 6.3.4. The first condition is clearly satisfied.

We write a, b, c as follows:

$$a = pra', \quad b = psb', \quad c = qsc',$$

where $p = \gcd(a, b)$, $s = \gcd(b, c)$, $r = \gcd(a, c)$. Since $\gcd(a, b, c) = 1$, any pair of p, s, r, a', b', c' are prime to each other.

Then

$$\begin{aligned} l &:= psra'b'c' + p + s + r - pra' - psb' - src' - 1 \\ &= p(ra' - 1)(sb' - 1) + s(rc' - 1)(sb' - 1) - (p - 1)(sb' - 1) \equiv 0 \pmod{2}. \end{aligned}$$

Thus the second condition is also satisfied.

To check the third condition, we write:

$$\frac{[\text{lcm}(a, b, c)][\text{gcd}(a, b)][\text{gcd}(b, c)][\text{gcd}(c, a)]}{[a][b][c][1]} = \frac{[psra'b'c'][p][r][s]}{[pra'][psb'][src'][1]}$$

$$= q^{-l} \underbrace{\frac{(q^{psra'b'c'} - 1)(q^p - 1)(q^s - 1)(q^r - 1)}{(q^{pra'} - 1)(q^{psb'} - 1)(q^{src'} - 1)(q - 1)}}_{\star}.$$

The numerator of (\star) has a zero of 4-th order at $q = 1$, double zeros at $q = e^{2\pi\sqrt{-1}j}$ ($j \in I_1$) and simple zeros at $q = e^{2\pi\sqrt{-1}j}$ ($j \in I_2$) where

$$I_1 = \left\{ \frac{1}{p}, \dots, \frac{p-1}{p}, \frac{1}{s}, \dots, \frac{s-1}{s}, \frac{1}{r}, \dots, \frac{r-1}{r} \right\},$$

$$I_2 = \left\{ \frac{1}{psra'b'c'}, \dots, \frac{psra'b'c' - 1}{psra'b'c} \right\} \setminus I_1.$$

On the other hand, the denominator has a zero of 4-th order at $q = 1$, double zeros at I_1 and simple zeros at $q = e^{2\pi\sqrt{-1}j}$ ($j \in I_3$) where

$$I_3 = \left\{ \frac{1}{pra'}, \dots, \frac{pra' - 1}{pra'}, \frac{1}{psb'}, \dots, \frac{psb' - 1}{psb'}, \frac{1}{src'}, \dots, \frac{src' - 1}{src'} \right\} \setminus I_1.$$

Since $I_1 \subseteq I_3$, (\star) is a polynomial in q . Moreover, since both the numerator and the denominator are elements of $\mathbb{Z}[q]$ and monic, it is a polynomial with integer coefficients. Therefore the third condition is satisfied. By Lemma 6.3.4, $[psra'b'c'][p][s][r]/[pra'][psb'][src'][1] \in \mathbb{Z}[t]$.

The constant term is equal to, by Lemma 6.4.1,

$$\frac{[psra'b'c']}{[pra']} \Big|_{q=1} \cdot \frac{[p]}{[psb']} \Big|_{q=1} \cdot \frac{[s]}{[src']} \Big|_{q=1} \cdot \frac{[r]}{[1]} \Big|_{q=1} = sb'c' \cdot \frac{1}{sb'} \cdot \frac{1}{rc'} \cdot r = 1.$$

□

Lemma 6.6. *Let λ be a partition and $k \in \mathbb{N}$. Assume they satisfy the following conditions. 1. $l(\lambda) \geq 2$. 2. $\text{gcd}(k, \lambda \setminus \lambda_i) = 1 (1 \leq \forall i \leq l(\lambda))$. 3. k is odd or $|\lambda|$ is even. Then there exists $w_{k,\lambda}(t) \in \mathbb{Z}[t]$ and*

$$\frac{[k\lambda]}{[k]^2[\lambda]} = \frac{k^{l(\lambda)-2}}{t} + w_{k,\lambda}(t).$$

Proof. Define $p_i = \text{gcd}(k, \lambda_1, \dots, \lambda_i)$ ($1 \leq i \leq l(\lambda)$), $p_0 = k$, $k_i = \text{gcd}(k, \lambda_{i+1}, \dots, \lambda_{l(\lambda)})$ ($0 \leq i \leq l(\lambda)$), $k_{l(\lambda)} = k$. Note that $\text{gcd}(\lambda_i, p_{i-1}) = p_i$, $\text{gcd}(\lambda_i, k_i) = k_{i-1}$ and $\text{gcd}(p_{i-1}, k_i) = p_{l(\lambda)} = k_0 = 1$ because of the second condition.

We write the LHS as follows.

$$\frac{[k\lambda]}{[k]^2[\lambda]} = \frac{1}{[k]^2} \prod_{i=1}^{l(\lambda)} \frac{[k\lambda_i]}{[\text{lcm}(\lambda_i, p_{i-1}, k_i)]} \prod_{i=1}^{l(\lambda)} \frac{[\text{lcm}(\lambda_i, p_{i-1}, k_i)]}{[\lambda_i][p_{i-1}][k_i]} \prod_{i=1}^{l(\lambda)} [p_{i-1}][k_i].$$

By Lemmas 6.5 and 6.4.1,

$$\begin{aligned} (RHS) &<_y \frac{1}{[k]^2} \prod_{i=1}^{l(\lambda)} \frac{1}{[p_i][k_{i-1}]} \prod_{i=1}^{l(\lambda)} [p_{i-1}][k_i] \cdot \prod_{i=1}^{l(\lambda)} \frac{k\lambda_i}{\text{lcm}(\lambda_i, p_{i-1}, k_i)} \\ &= \frac{1}{[1]^2} \prod_{i=1}^{l(\lambda)} \frac{k\lambda_i}{\text{lcm}(\lambda_i, p_{i-1}, k_i)} = \frac{1}{[1]^2} \times (\text{positive integer}). \end{aligned}$$

Therefore $[k\lambda][1]^2/[k]^2[\lambda]$ is in $\mathbb{Z}[y]$. Applying Lemma 6.3.4, we find that $[k\lambda][1]^2/[k]^2[\lambda] \in \mathbb{Z}[t]$. Its constant term is $[k\lambda]/[\lambda]|_{y=0} \cdot [1]^2/[k]^2|_{y=0} = k^{l(\lambda)-2}$. \square

6.1.3. Proof of Proposition 6.1: Case $m(T) = 1$

We first show the case $m(T) = 1$.

The statement actually holds for a more general VEV tree. Let $\vec{c} = (c_1, \dots, c_l)$, $\vec{n} = (n_1, \dots, n_l)$ be the pair satisfying the assumptions of the Subsection 3.2 and $\gcd(\vec{c}, \vec{n}) = 1$. We extend the definition of $\mathcal{B}(T)$ to $T \in \overline{\text{Graph}}^\circ(\vec{c}, \vec{n})$ as follows:

$$\mathcal{B}(T) = \frac{\mathcal{A}(T)}{\prod_{i=1}^l [m_i]}$$

where $m_i = \gcd(c_i, n_i)$ ($1 \leq i \leq l$). We will show $t \cdot \mathcal{B}(T) \in \mathbb{Z}[t]$.

We give a proof of the case $|\vec{n}| \neq 0$ and omit the case $|\vec{n}| = 0$ since the proof is almost the same.

Our strategy of the proof is Lemma 6.3.4. Since $[1]^2\mathcal{B}(T)$ is written by the products of q -numbers:

$$[1]^2\mathcal{B}(T) = \frac{[1]^2 \prod_{v \in V_2(T)} [\zeta_v]}{[|\vec{n}|] \prod_{i=1}^l [m_i]},$$

it is sufficient to show the followings:

1. $\#V_2(T) - l - 1 \equiv 0 \pmod{2}$;
2. $\sum_{v \in V_2(T)} \zeta_v - |\vec{n}| - \sum_{i=1}^l m_i \equiv 0 \pmod{2}$;
3. $[1]^2\mathcal{B}(T) \in \mathbb{Z}[y]$

1. The first condition is immediately checked since $\#V_2(T) = l - 1$.
2. Since $c_v = c_{L(v)} + c_{R(v)}$ and $n_v = n_{L(v)} + n_{R(v)}$,

$$\zeta_v = c_{L(v)}n_{L(v)} + c_{R(v)}n_{R(v)} - c_v n_v + 2c_{L(v)}n_{R(v)}.$$

Then

$$\begin{aligned} & \sum_{v \in V_2(F)} \zeta_v - |\vec{n}| - \sum_{i=1}^l m_i \pmod 2 \\ \equiv & \sum_{i=1}^l c_i n_i - |\vec{n}| - \sum_{i=1}^l m_i - |\vec{c}| \pmod 2 \quad (\cdot \cdot | \vec{c} | = 0) \\ \equiv & \sum_{i=1}^l [m_i^2(c'_i - 1)(n'_i - 1) + m_i(m_i - 1)(c'_i + n'_i) - m_i(m_i - 1)] \pmod 2 \\ \equiv & 0 \pmod 2 \end{aligned}$$

where c'_i, n'_i are defined by $c_i = m_i c'_i$ and $n_i = m_i n'_i$ for $1 \leq i \leq l$.

3. Let us introduce some notations. For a vertex v ,

$$\begin{aligned} \vec{c}_v &= \{c_i \in \vec{c} \mid \text{the } i\text{-th leaf is a descendent of } v\}, \\ \vec{n}_v &= \{n_i \in \vec{n} \mid \text{the } i\text{-th leaf is a descendent of } v\}, \\ p_v &= \gcd(\vec{c}_v, \vec{n}_v), \quad k_v = \gcd(\vec{c} \setminus \vec{c}_v, \vec{n} \setminus \vec{n}_v, |\vec{n}|). \end{aligned}$$

p_v, k_v 's satisfy the following relations. (For $a, b \in \mathbb{N}$, $a|b$ means b is divisible by a .)

$$\begin{aligned} p_v, k_v &| \gcd(c_v, n_v), \quad p_{L(v)}, p_{R(v)}, k_v | \zeta_v, \\ \gcd(p_{R(v)}, k_v) &= k_{L(v)}, \quad \gcd(p_{L(v)}, k_v) = k_{R(v)}, \\ p_{i\text{-th leaf}} &= m_i, \quad k_{\text{root}} = |\vec{n}|, \quad \gcd(p_v, k_v) = k_{\text{leaf}} = p_{\text{root}} = 1. \end{aligned}$$

Note that the last identities are the consequence of the assumption $\gcd(\vec{c}, \vec{v}) = 1$.

Now we decompose the amplitude $\mathcal{B}(T)$ as follows.

$$\begin{aligned} \mathcal{B}(T) &= \underbrace{\prod_{v \in V_2(T)} \frac{[\zeta_v]}{[\text{lcm}(k_v, p_{L(v)}, p_{R(v)})]}}_{(a)} \underbrace{\prod_{v \in V_2(T)} \frac{[\text{lcm}(k_v, p_{L(v)}, p_{R(v)})]}{[k_v][p_{L(v)}][p_{R(v)}]}}_{(b)} \\ &\times \underbrace{\prod_{v \in V_2(T)} [k_v][p_{L(v)}][p_{R(v)}]}_{(c)} \cdot \underbrace{\frac{1}{[|\vec{n}|] \prod_{i=1}^l [m_i]}}_{(d)}. \end{aligned}$$

Since $k_v, p_{L(v)}, p_{R(v)}$ divide ζ_v , so does $\text{lcm}(k_v, p_{L(v)}, p_{R(v)})$. By Lemma 6.4.1,

$$(a) <_y \prod_{v \in V_2(T)} \frac{\zeta_v}{\text{lcm}(k_v, p_{L(v)}, p_{R(v)})} =: g_T \in \mathbb{Z}.$$

We apply Lemma 6.5 to (b). Note that the assumption is satisfied since $\text{gcd}(k_v, p_{L(v)}, p_{R(v)}) = 1$.

$$\begin{aligned} (b) &<_y \prod_{v \in V_2(T)} \frac{[1]}{[k_{L(v)}][k_{R(v)}][p_v]} \\ &= [1]^{\#V_2(T)} \cdot \frac{1}{[p_{\text{root}}]} \cdot \prod_{v:\text{leaf}} \frac{1}{[k_v]} \cdot \prod_{\substack{v \in V_2(T) \\ v \neq \text{root}}} \frac{1}{[p_v][k_v]} \\ &= \frac{1}{[1]^2} \prod_{\substack{v \in V_2(T) \\ v \neq \text{root}}} \frac{1}{[p_v][k_v]}. \end{aligned}$$

We have used $[k_{\text{leaf}}] = [p_{\text{root}}] = [1]$ and $\#V_2(T) = \#(\text{leaves}) - 1$.

The factor (c) is rewritten as follows:

$$(c) = [k_{\text{root}}] \cdot \prod_{v:\text{leaf}} [p_v] \cdot \prod_{\substack{v \in V_2(T) \\ v \neq \text{root}}} [p_v][k_v] = [|\vec{n}|] \cdot \prod_{i=1}^l [m_i] \prod_{\substack{v \in V_2(T) \\ v \neq \text{root}}} [p_v][k_v].$$

Thus all the factors except $1/[1]^2$ cancel and we obtain

$$\mathcal{B}(T) <_y \frac{g_T}{[1]^2}.$$

Thus we find that $[1]^2 \mathcal{B}(T) \in \mathbb{Z}[y]$. All the assumptions of Lemma 6.3.4 are checked and the $m(T) = 1$ case is proved.

6.1.4. Proof of Proposition 6.1: case $m(T) > 1$

Next we give a proof in the case $m(T) > 1$. Here T is a VEV tree in $\text{Graph}_a^\circ(\mu, \nu)$ with $|\mu| = |\nu| = d$. For simplicity of the writing, let us set $k = m(T) = \text{gcd}(\mu, \nu)$.

$\mathcal{B}(T)$ is written in terms of $\mathcal{B}(T_{(0)})$ as follows.

$$\mathcal{B}(T) = \underbrace{\frac{[k\mu'][k\nu']}{[k]^2[\mu'][\nu']}}_{(a')} \cdot \underbrace{\frac{[kad']}{[ad']}}_{(b')} \times \underbrace{\left([1]^2 \mathcal{B}(T_{(0)}) \right)}_{(c')} \Big|_{q \rightarrow q^{k^2}}$$

where μ', ν' and d' are defined by $\mu = k\mu', \nu = k\nu'$ and $d = kd'$.

We will apply Lemma 6.6 to the factor (a') by substituting $\lambda' \cup \mu'$ into λ . Let us check the assumptions. The first and the third are clearly satisfied since $l(\mu') + l(\nu') \geq 2$ and $|\mu'| + |\nu'| = 2d'$. The second assumption is also satisfied since if we assume otherwise, then it contradicts with $|\mu'| = |\nu'|$. Therefore by the lemma, there exists $w_{k,\mu' \cup \nu'}(t) \in \mathbb{Z}[t]$ such that

$$(a') = \frac{k^{l(\mu)+l(\nu)-2}}{t_k} + w_{k,\mu' \cup \nu'}(t_k).$$

By Lemma 6.4, the factor (b') is written as follows.

$$(b') = \frac{[kad']}{[ad']} \Big|_{q \rightarrow q^k} = \begin{cases} k + t_k \times u_{k,(d')}(t_k) & (k \text{ odd or } ad' \text{ even}) \\ k \left(1 + \frac{t_{k/2}}{2}\right) + t_k \times u_{k,(d')}(t_{k/2}) & (k \text{ even and } ad' \text{ odd}) \end{cases}$$

where, in both cases, $u_{k,(d')}$ is a polynomial in one variable with integer coefficients.

From the result of the $m(T) = 1$ case, $(c') \in \mathbb{Z}[t_{k^2}] \subset \mathbb{Z}[t_k]$ with the constant term $g_{T_{(0)}}$. Thus the $m(T) > 1$ case is proved. This completes the proof of the proposition.

§6.2. Poles of combined amplitudes

We study the pole of the combined amplitude and finish the proof of the main theorem by showing (13): $t \mathcal{G}_k(W) \in \mathbb{Q}[t]$.

As it turns out, the pole structure of $\mathcal{H}(W)$ of $W \in \text{Comb}_\gamma^\circ(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ depends on the cycle rank $\beta(W)$ and $k = \text{gcd}(\vec{\mu}, \vec{\nu}, \vec{\lambda})$. We first see that if W has the cycle rank > 0 , then $\mathcal{H}(W)$ is a polynomial in t and if W has the cycle rank zero, $\mathcal{H}(W)$ has a simple pole at $t_k = 0$. Then we prove the more detailed pole structure in the latter case. We also prove (13).

6.2.1.

Proposition 6.7. *Let $(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ be an r -set and $k = \text{gcd}(\vec{\mu}, \vec{\nu}, \vec{\lambda})$. Let $W \in \text{Comb}_\gamma^\circ(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ be a connected combined forest.*

1. *If $\beta(W) = 0$, then $t_k \mathcal{H}(W) \in \mathbb{Z}[t]$.*
2. *If $\beta(W) > 1$, then $\mathcal{H}(W) \in \mathbb{Z}[t]$.*

Proof. We write the combined amplitude as follows.

$$\mathcal{H}(W) = (-1)^{L_1(W)+L_2(W)} \prod_{T: \text{VEV tree in } W} \mathcal{B}(T) \cdot \prod_{b: \text{bridges in } W} t_{h(b)}.$$

Since every $\mathcal{B}(T)$ has poles at $t_{m(T)} = 0$, the first factor has poles at

$$\{t_{m(T)} = 0 \mid T: \text{VEV tree in } W\}.$$

On the other hand the second factor has zeros at

$$\{t_{h(b)} = 0 \mid b: \text{bridges in } W\}.$$

Do the poles cancel with the zeros? Note that if a bridge b is incident to a VEV tree T , then the zeros of $t_{h(b)}$ cancel the poles of $t_{m(T)}$ because $m(T)$ is a divisor of $h(b)$. Therefore the question is whether it is possible to couple every VEV tree in W with at least one of its incident bridges.

The answer is yes if $\beta(W) > 0$. Therefore $\mathcal{H}(W)$ is a polynomial in t . As an example, consider the first graph in Figure 1. The cycle rank is two in this case. Let T_1, T_2, T_3 denote the VEV trees and b_1, b_2, b_3 the bridges as depicted below. (For simplicity, we contract each VEV tree to a vertex.)



If we take the coupling (T_3, b_3) , (T_2, b_1, b_2) , (T_1, b_4) , we could easily see that the pole of $\mathcal{B}(T_3)$ is canceled by the zeros of $t_{h(b_3)}$ etc.

If the cycle rank $\beta(W)$ is zero, the answer to the question is no. However, if we pick one VEV tree T , then it is possible to couple every other VEV tree with one of its incident bridges. As an example, consider the combined forest in the Figure 2. For simplicity, we contract VEV trees to a vertex as before (see below).



If we pick T_4 and choose the pairing (T_1, b_3) , (T_2, b_1) , (T_3, b_2) , then we could easily see that the pole of $\mathcal{B}(T_1)$ cancel with the zeros of $t_{h(b_3)}$, etc, and that only the pole of $\mathcal{B}(T_4)$ remains. Moreover, the choice of a VEV tree T is arbitrary. Therefore $\mathcal{H}(W)$ has poles at most

$$\begin{aligned} \bigcap_{T: \text{VEV tree in } W} \{t_{m(T)} = 0\} &= \{t_{\text{gcd}(m(T), T: \text{VEV tree in } W)} = 0\} \\ &= \{t_{(\text{gcd}(\vec{\mu}, \vec{\nu}, \vec{\lambda}))} = 0\} = \{t_k = 0\}. \end{aligned}$$

Therefore $t_k \mathcal{H}(W)$ is a polynomial in t .

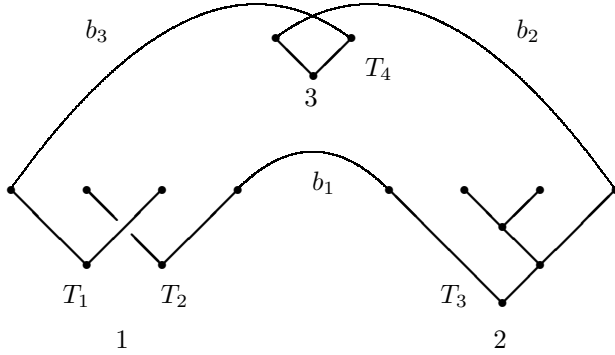


Figure 2. A combined forest in $\text{Comb}_\gamma^\bullet(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ where $r = 3$, $\gamma_i + 2 \neq 0$ and $\lambda^1 = (1) = \lambda^3 = \lambda^2 = (1)$, $\mu^1 = (1) = \mu^2 = \nu^1 = \nu^2, \mu^3 = \nu^3 = \emptyset$. The cycle rank is zero.

The integrality of the coefficients follows from Lemma 6.3.4. □

6.2.2.

We study the property of the case $\beta(W) = 0$ in more details. For convenience, we define the following two relations.

- The relation $\sim_{\mathbb{Z}}$ in the ring $\mathcal{L}[t]$ is defined by

$$u_1(t) \sim_{\mathbb{Z}} u_2(t) \Leftrightarrow u_1(t) - u_2(t) \in \mathbb{Z}[t].$$

- The relation $\sim_{\mathbb{Q}}$ in the ring $\mathbb{Q}(t)$ is defined by

$$u_1(t) \sim_{\mathbb{Q}} u_2(t) \Leftrightarrow u_1(t) - u_2(t) \in \mathbb{Q}[t].$$

Both $u_1(t) \sim_{\mathbb{Z}} u_2(t)$ and $u_1(t) \sim_{\mathbb{Q}} u_2(t)$ imply that $u_1(t)$ and $u_2(t)$ has the same poles.

Proposition 6.8. *Let $(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ be an r -set such that $\text{gcd}(\vec{\mu}, \vec{\nu}, \vec{\lambda}) = 1$. Let $W \in \text{Comb}_\gamma^\circ(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ be a connected combined forest with the cycle rank $\beta(W) = 0$.*

1. *If k is an odd positive integer,*

$$\mathcal{H}(W_{(k)}) \sim_{\mathbb{Z}} k^{l(\vec{\mu})+l(\vec{\nu})+l(\vec{\lambda})-1} \mathcal{H}(W)|_{t \rightarrow t_k}.$$

2. If k is an even positive integer,

$$\mathcal{H}(W_{(k)}) \sim_{\mathbb{Z}} (-1)^{L_2(W)} k^{l(\vec{\mu})+l(\vec{\nu})+l(\vec{\lambda})-1} \mathcal{H}(W)|_{t \rightarrow t_k} \prod_{T \in VT_I(W)} \left(1 + \frac{t_{m(T)k/2}}{2}\right).$$

$VT_I(W)$ denotes the set of VEV trees in W such that both $m(T)$ and n_{root} are odd.

3.

$$\mathcal{G}_k(W) \sim_{\mathbb{Q}} \begin{cases} 0 & (k > 2) \\ \text{const.}/t & (k = 1, 2). \end{cases}$$

As we have seen in Section 5, the main Theorem 1.3 follows directly from the third statement.

Before giving a proof, let us check the proposition in simple examples. Consider the combined forest in Figure 2 (and also see (16)) with $\gamma_1 = \gamma_2 = \gamma_3 = -1$. In this case $l(\vec{\mu}) + l(\vec{\nu}) + l(\vec{\lambda}) = 7$. VEV trees of type I are T_1, T_2, T_4 . All VEV tree have $m(T_i) = 1$. The amplitude of W is equal to

$$\mathcal{H}(W) = \frac{1}{[1]^2} = \frac{1}{t}.$$

As a check for the odd k case, take $k = 3$. The amplitude of $W_{(3)}$ is

$$\frac{3^6}{t_3} + 3^7 + 3^5 11t_3 + 3^3 5 \cdot 13t_3^2 + 3^3 5^2 t_3^3 + 3^2 17t_3^4 + 19t_3^5 + t_3.$$

As an example for the even k , we take $k = 2$. The amplitude of $W_{(2)}$ is

$$\frac{2^6(1 + \frac{t}{2})^2}{t_2} + 32 + 96t + 86t^2 + 41t^3 + 10t^4 + t^5.$$

Therefore the first and the second of the proposition hold. It is easy to see that the third statement also holds in these cases.

Proof. 1 and 2. Since the pole structure of $\mathcal{B}(T)$ depends on $m(T)$ and n_{root} (Proposition 6.1), we separate VEV trees in W into three types:

$$\begin{aligned} VT_I(W) &= \{T : \text{VEV tree} | m(T) \text{ odd and } n_{\text{root}}/m(T) \text{ odd}\}, \\ VT_{II}(W) &= \{T : \text{VEV tree} | n_{\text{root}}/m(T) \text{ even}\}, \\ VT_{III}(W) &= \{T : \text{VEV tree} | m(T) \text{ even and } n_{\text{root}}/m(T) \text{ odd}\}. \end{aligned}$$

The combined amplitude $\mathcal{H}(W)$ is written as follows.

$$(17) \quad \mathcal{H}(W) \sim_{\mathbb{Z}} (-1)^{L_1(W)+L_2(W)} \prod_{b:\text{bridge}} t_{h(b)} \\ \times \prod_{T:\text{VEV tree}} \frac{g_T}{t_m(T)} \cdot \prod_{T \in V_{III}(W)} \left(1 + \frac{t_m(T)/2}{2}\right).$$

Consider the amplitude of $W_{(k)}$. If k is odd,

$$VT_I(W_{(k)}) \cong VT_I(W), \quad VT_{II}(W_{(k)}) \cong VT_{II}(W), \\ VT_{III}(W_{(k)}) \cong VT_{III}(W).$$

If k is even,

$$VT_I(W_{(k)}) = \emptyset, \quad VT_{II}(W_{(k)}) \cong VT_{II}(W), \\ VT_{III}(W_{(k)}) \cong VT_I(W) \cup VT_{III}(W).$$

Thus we have

$$(18) \quad \mathcal{H}(W_{(k)}) \sim_{\mathbb{Z}} (-1)^{L_1(W)+kL_2(W)} k^{\#(\text{leaves})-\#(\text{trees})} \\ \times \prod_{b:\text{bridge in } W} t_{kh(b)} \cdot \prod_{T:\text{VEV tree in } W} \frac{g_T}{t_{km}(T)} \\ \times \begin{cases} \prod_{T \in V_{III}(W)} \left(1 + \frac{t_{km}(T)/2}{2}\right) & (k:\text{odd}) \\ \prod_{T \in V_I(W) \cup V_{III}(W)} \left(1 + \frac{t_{km}(T)/2}{2}\right) & (k:\text{even}) \end{cases}$$

Given that $\beta(W) = \#(\text{bridges}) - \#(\text{trees}) + 1 = 0$, $\#(\text{bridges}) = l(\vec{\lambda})$ and that $\#(\text{leaves}) = l(\vec{\mu}) + l(\vec{\nu}) + 2l(\vec{\lambda})$,

$$k^{\#(\text{leaves})-\#(\text{trees})} = k^{l(\vec{\mu})+l(\vec{\nu})+l(\vec{\lambda})-1}.$$

Comparing (17) and (18), we obtain the first and the second statements.

3. In the proof of the third statement, the key formula is the formula for the Möbius function (14).

(i) k is odd. Since the divisors of k are all odd,

$$\mathcal{G}_k(W) \sim_{\mathbb{Q}} \mathcal{H}(W)|_{q \rightarrow q^k} \sum_{k':k'|k} \mu\left(\frac{k}{k'}\right) = \begin{cases} \mathcal{H}(W) & (k = 1) \\ 0 & (k > 1) \end{cases}$$

Therefore this case follows from the previous proposition.

(ii) k is divisible by 4. If we write $k = 4k_0$ ($k_0 \geq 1$), every divisor k' of k is written as $4k'', 2k'', k''$ with a divisor k'' of k_0 . For $k' = k''$, the Möbius

function $\mu(k/k')$ is zero since k/k' contains 2^2 . For $k' = 4k'', 2k''$, we could write the result of Proposition 6.8.2 as

$$\mathcal{H}(W_{(k')}) \sim_{\mathbb{Q}} \mathcal{H}(W_{(2)})|_{q \rightarrow q^{k'/2}} \left(\frac{k'}{2}\right)^{l(\bar{\mu})+l(\bar{\nu})+l(\bar{\lambda})-1}$$

Therefore

$$\mathcal{G}_k(W) \sim_{\mathbb{Q}} (2^{-l(\bar{\mu})-l(\bar{\nu})-l(\bar{\lambda})+1} \mathcal{H}(W_{(2)})|_{q \rightarrow q^{k/2}} \sum_{k':k'|k} \mu\left(\frac{k}{k'}\right) = 0.$$

This proves the case $4|k$.

(iii) $k = 2k_0$ where $k_0 > 1$ is an odd positive integer. Note that any divisor of k is written as either k' or $2k'$ with a divisor k' of k_0 . Therefore

$$\mathcal{G}_k(W) \sim_{\mathbb{Q}} (2^{-l(\bar{\mu})-l(\bar{\nu})-l(\bar{\lambda})+1} \mathcal{H}(W_{(2)})|_{q \rightarrow q^{k/2}} - \mathcal{H}(W)|_{q \rightarrow q^k}) \sum_{k':k'|k_0} \mu\left(\frac{k}{k'}\right) = 0.$$

(iv) $k = 2$.

$$\mathcal{G}_2(W) \sim_{\mathbb{Q}} -\mathcal{H}(W)|_{t \rightarrow t(t+4)} \left(1 + (-1)^{L_2(W)} \prod_{T \in VT_I(W)} \left(1 + \frac{t_{m(T)}}{2}\right)\right)$$

Since $\#VT_I(W) \equiv \sum_T n_{\text{root}} \equiv L_2(W) \pmod{2}$, the $k = 2$ case follows from the next lemma.

Lemma 6.9. *Let m_1, m_2, \dots, m_l be positive odd integers. Then*

$$\frac{(-1)^{l+1} + \prod_{i=1}^l \left(1 + \frac{t_{m_i}}{2}\right)}{t + 4} \in \mathbb{Q}[t].$$

Proof. To show the lemma, we only have to show that the denominator has zero at $t = -4$. At $t = -4$, or equivalently at $q^{\frac{1}{2}} = \sqrt{-1}$, $t_{m_i} = (q^{\frac{1}{2}})^{2m_i} + (q^{\frac{1}{2}})^{-2m_i} - 2 = -4$. Thus the denominator is zero at $t = -4$ and the lemma is proved. □

This completes the proof of the Proposition 6.8. □

Appendix A. Exponential Formula

In this appendix, we formulate the exponential formula used in Section 4. First, we present the conditions for a graph set.

Definition A.1. A G -set is a set of (labeled) graphs \mathbb{G}^\bullet together with a family of subsets, $\{\mathbb{G}_d^\bullet\}_{d \geq 1}$, satisfying the following conditions.

1. $\mathbb{G}^\bullet = \coprod_{d \geq 1} \mathbb{G}_d^\bullet$.
2. If $G \in \mathbb{G}_d^\bullet$ and $G' \in \mathbb{G}_{d'}^\bullet$, then $G \cup G' \in \mathbb{G}_{d+d'}^\bullet$.
3. Every connected component G_i of $G \in \mathbb{G}^\bullet$ is also an element of \mathbb{G}^\bullet .
4. If G and G' are two elements in \mathbb{G}^\bullet , then G and G' are not isomorphic.
5. Every \mathbb{G}_d^\bullet ($d \geq 1$) is a finite set.

If $G \in \mathbb{G}_d^\bullet$, the *degree* of G is d and is denoted by $d(G)$.

Remark A.2. The conditions 2 and 3 together with the condition 4 imply that the decomposition of a graph into connected components:

$$G = G_1 \cup \dots \cup G_l \mapsto \{G_1, \dots, G_l\}$$

gives a one-to-one correspondence

$$\mathbb{G}_{(l)}^\bullet \xrightarrow{\sim} SP^l \mathbb{G}^\circ,$$

where $\mathbb{G}_{(l)}^\bullet \subset \mathbb{G}^\bullet$ is the subset consisting of graphs with l connected components and $SP^l \mathbb{G}^\circ$ is the l -th symmetric product of \mathbb{G}° , the subset consisting of all connected graphs.

Next we set the condition for an amplitude. We recall the definition of a graded ring. A graded ring is a commutative associative ring R together with a family $(R_d)_{d \geq 0}$ of subgroups of the additive group of R , such that $R = \bigoplus_{d \geq 0} R_d$ and $R_d \cdot R_{d'} \subseteq R_{d+d'}$ for all $d, d' \geq 0$.

Definition A.3. Let $(\mathbb{G}^\bullet, R, \Psi)$ be a triple of a G -set, a graded ring and a grading preserving map $\Psi : \mathbb{G} \rightarrow R$. It is a *triple of graph amplitude* (GA triple) if the map Ψ is multiplicative with respect to the graph union:

$$\Psi(G \cup G') = \Psi(G) \cdot \Psi(G') \quad (\forall G, G' \in \mathbb{G}^\bullet).$$

Let $(\mathbb{G}^\bullet, R, \Psi)$ be a GA triple and $\mathbb{G}_d^\circ \subset \mathbb{G}_d^\bullet$ be the subset of connected graphs. By the finiteness assumption 5 of the G -set, the following sums over \mathbb{G}_d^\bullet and \mathbb{G}_d° are well-defined elements of R :

$$\sum_{G \in \mathbb{G}_d^\bullet} \frac{1}{\#\text{aut}(G)} \Psi(G), \quad \sum_{G \in \mathbb{G}_d^\circ} \frac{1}{\#\text{aut}(G)} \Psi(G).$$

These are related by the relation:

Proposition A.4.

$$1 + \sum_{d \geq 1} \sum_{G \in \mathbb{G}_d^\bullet} \frac{1}{\#\text{aut}(G)} \Psi(G)x^d = \exp \left[\sum_{d \geq 1} \sum_{G \in \mathbb{G}_d^\circ} \frac{1}{\#\text{aut}(G)} \Psi(G)x^d \right]$$

as formal power series in x .

We call this the *exponential formula* since it is a variation of the exponential formula ([S], chapter 5).

Proof. Expanding the RHS, we obtain

$$1 + \sum_{d \geq 1} x^d \left(\sum_{l \geq 1} \sum_{\substack{\{(G_i, n_i)\}_{1 \leq i \leq l}; \\ G_i \in \mathbb{G}^\circ, n_i \in \mathbb{N}; \\ \sum_{i=1}^l n_i d(G_i) = d}} \underbrace{\frac{1}{n_1! \dots n_l!} \prod_{i=1}^l \left(\frac{\Psi(G_i)}{\#\text{aut}(G_i)} \right)^{n_i}}_{*} \right).$$

As we mentioned in Remark A.2, each $\{(G_1, n_1), \dots, (G_l, n_l)\}$ uniquely corresponds to the graph

$$G = \bigcup_{i=1}^l \underbrace{G_i \cup \dots \cup G_i}_{n_i \text{ times}}$$

and (*) is exactly the same as the contribution of G to the LHS, as we now explain. The number of automorphisms of such G is

$$\#\text{aut}(G) = \prod_{i=1}^l n_i! (\#\text{aut}(G_i))^{n_i}.$$

The amplitude $\Psi(G)$ is the product of $\Psi(G_i)^{n_i}$ ($1 \leq i \leq l$) because of the multiplicativity of Ψ . Therefore the contribution of G to the LHS and (*) are the same. □

Appendix B. Proof of Free Energy

We give a proof of Proposition 4.1 by using the exponential formula. All we have to do is to define a suitable GA triple. We first construct a G-set.

For a combined forest W , we define \overline{W} to be the graph obtained by forgetting all leaf indices. Two combined forests W and W' are *equivalent* if $\overline{W} \cong \overline{W}'$. The set of equivalence classes in $\text{Comb}^\bullet(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ and $\text{Comb}^\circ(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ are denoted

by $\overline{\text{Comb}}^\bullet(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ and $\overline{\text{Comb}}^\circ(\vec{\mu}, \vec{\nu}, \vec{\lambda})$. Let us define

$$\begin{aligned} \overline{\text{Comb}}_\gamma^\bullet(d) &= \prod_{|\vec{d}|=d} \prod_{\substack{(\vec{\mu}, \vec{\nu}, \vec{\lambda}) \\ r\text{-set of} \\ \text{degree } \vec{d}}} \overline{\text{Comb}}_\gamma^\bullet(\vec{\mu}; \vec{\nu}; \vec{\lambda}) \quad (d \geq 1), \\ \overline{\text{Comb}}_\gamma^\bullet &= \prod_{d \geq 1} \overline{\text{Comb}}_\gamma^\bullet(d). \end{aligned}$$

The set $\overline{\text{Comb}}_\gamma^\bullet$ clearly satisfies the conditions 1,3,4,5 of the G-set. It also satisfies the second condition, because

$$\begin{aligned} \overline{\text{Comb}}_\gamma^\bullet(\vec{\mu}, \vec{\nu}, \vec{\lambda}) &= \prod_{k \geq 1} \prod_{\{(\vec{\mu}^j, \vec{\nu}^j, \vec{\lambda}^j)\}_{1 \leq j \leq k} \in D_k(\vec{\mu}, \vec{\nu}, \vec{\lambda})} \\ &\overline{\text{Comb}}_\gamma^\circ(\vec{\mu}^1, \vec{\nu}^1, \vec{\lambda}^1) \times \dots \times \overline{\text{Comb}}_\gamma^\circ(\vec{\mu}^k, \vec{\nu}^k, \vec{\lambda}^k) \end{aligned}$$

where D_k is the set of all partitions of $(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ into k parts $\{(\vec{\mu}^j, \vec{\nu}^j, \vec{\lambda}^j)\}_{1 \leq j \leq k}$ such that every part is an r -set.

Secondly, we define the amplitude Ψ of $G \in \overline{\text{Comb}}_\gamma^\bullet(\vec{\mu}, \vec{\nu}, \vec{\lambda})$ so that it satisfies

$$\frac{1}{\#\text{aut}(G)} \Psi(G) = \sum_{\substack{W: \text{combined forest} \\ \overline{W} = G}} \frac{1}{z_{\vec{\mu}} z_{\vec{\nu}} z_{\vec{\lambda}}} \mathcal{H}(W) \vec{Q}^{\vec{d}}$$

where \vec{d} is the degree of the r -set $(\vec{\mu}, \vec{\nu}, \vec{\lambda})$. Solving this, we have

$$\Psi(G) = \#\text{aut}(G) \cdot N(W) \frac{1}{z_{\vec{\mu}} z_{\vec{\nu}} z_{\vec{\lambda}}} \mathcal{H}(W) \vec{Q}^{\vec{d}} \quad (W \text{ such that } \overline{W} = G)$$

where $N(W)$ is the number of combined forests equivalent to W .

It is easy to see that Ψ is a grade preserving map to the ring $R = \mathbb{Q}(t)[[\vec{Q}]]$. To check the multiplicativity of Ψ , we need to know $N(W)$. Since $N(W)$ is equal to the number of ways to assign leaf indices to \overline{W} , it consists of two factors. The one is the number of ways to distribute the leaf indices to VEV trees in \overline{W} ; we define the partitions $\lambda(T, T'), \mu(T), \nu(T)$ for a VEV tree T, T' in W by:

$$\begin{aligned} \lambda(T, T') &= \{h(b) \mid b \text{ is a bridge joining } T, T'\}, \\ \mu(T) &= \{c_v \mid v \text{ is a left leaf of } T, \text{ not incident to a bridge}\}, \\ \nu(T) &= \{c_v \mid v \text{ is a right leaf of } T, \text{ not incident to a bridge}\}; \end{aligned}$$

then there are

$$\frac{\#\text{aut}(\vec{\mu}) \#\text{aut}(\vec{\nu}) \#\text{aut}(\vec{\lambda})}{|\text{aut}(\overline{W})| \prod_{T \neq T'} \#\text{aut}(\lambda(T, T')) \prod_T \#\text{aut}(\mu(T)) \#\text{aut}(\nu(T))}$$

ways to distribute the leaf indices to VEV trees. The other factor is the number of ways to assign the leaf indices in each VEV tree. It is the same as the number of (connected) combined forests equivalent to T which we write as $N'(T)$.

Therefore

$$N(W) = \frac{\#\text{aut}(\vec{\mu}) \#\text{aut}(\vec{\nu}) \#\text{aut}(\vec{\lambda})}{|\text{aut}(\overline{W})|} \times \underbrace{\frac{1}{\prod_{T \neq T'} \#\text{aut}(\lambda(T, T'))} \prod_T \frac{N'(T)}{\#\text{aut}(\mu(T)) \#\text{aut}(\nu(T))}}_{\star}$$

The factor (\star) is multiplicative. Non-multiplicative factors cancel with those of $z_{\vec{\mu}}, z_{\vec{\nu}}, z_{\vec{\lambda}}$ and $\#\text{aut}(\overline{W})$:

$$\frac{\#\text{aut}(G) \cdot N(W)}{z_{\vec{\mu}} z_{\vec{\nu}} z_{\vec{\lambda}}} = \prod_{i=1}^r \left(\prod_{j=1}^{l(\mu^i)} \mu_j^i \cdot \prod_{j=1}^{l(\nu^i)} \nu_j^i \cdot \prod_{j=1}^{l(\lambda^i)} \lambda_j^i \right)^{-1} \times (\star).$$

This is multiplicative and so are $\mathcal{H}(W)$ and $\vec{Q}^{\vec{d}}$. Thus we finished the check of the multiplicativity of Ψ and $(\overline{\text{Comb}}_{\gamma}^{\bullet}, R, \Psi)$ is a GA triple.

With the GA triple, the partition function is expressed as

$$\mathcal{Z}^{\gamma}(q; \vec{Q}) = 1 + \sum_{G \in \overline{\text{Comb}}_{\gamma}^{\bullet}} \frac{1}{\#\text{aut}(G)} \Psi(G).$$

Therefore we apply the exponential formula and obtain the free energy:

$$\begin{aligned} \mathcal{F}^{\gamma}(q; \vec{Q}) &= \sum_{G \in \overline{\text{Comb}}_{\gamma}^{\bullet}} \frac{1}{\#\text{aut}(G)} \Psi(G) \\ &= \sum_{\vec{d} \neq \vec{0}} Q^{\vec{d}} \sum_{\substack{(\vec{\mu}, \vec{\nu}, \vec{\lambda}); \\ r\text{-set of} \\ \text{degree } \vec{d}}} \frac{1}{z_{\vec{\lambda}} z_{\vec{\mu}} z_{\vec{\nu}}} \sum_{W \in \text{Comb}_{\gamma}^{\circ}(\vec{\mu}, \vec{\nu}, \vec{\lambda})} \mathcal{H}(W). \end{aligned}$$

Since $\mathcal{F}_d^{\gamma}(q)$ is the coefficient of $\vec{Q}^{\vec{d}}$, this implies the Proposition 4.1.

Appendix C.

In the proof of Proposition 6.8, we have claimed the existence of a certain coupling between VEV trees and bridges in a combined forest W . If we make the simpler graph \hat{W} by contracting every VEV tree in W to a vertex, then the coupling is the same as that of vertices and edges in \hat{W} (see, for example, Figure 2 and (16)). Then the existence of such coupling follows if we apply the next lemma to \hat{W} .

Lemma C.1. *Let Γ be a connected graph without self-loops (a self loop is an edge that joins a single vertex to itself).*

1. *If the cycle rank $\beta(\Gamma)$ is zero, then for all $v \in V(\Gamma)$, there exists a map $\varphi_v : E(\Gamma) \rightarrow V(\Gamma)$ satisfying the following conditions; $\varphi_v(e)$ is either of the incident vertices; $|\varphi_v^{-1}(v')| = 1$ for $v' \neq v$; $|\varphi_v^{-1}(v)| = 0$. We call φ_v an edge-map of v .*
2. *If the cycle rank $\beta(\Gamma) > 1$, then there exists $\varphi : E(\Gamma) \rightarrow V(\Gamma)$ satisfying the following two conditions; $\varphi(e)$ is either of the incident vertices; $|\varphi^{-1}(v)| \geq 1$ for all $v \in V(\Gamma)$. We call φ an edge-map of Γ .*

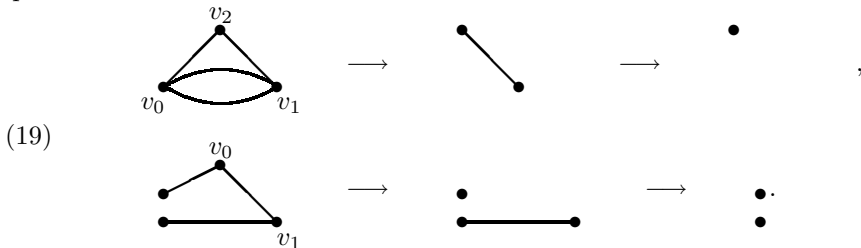
Proof. Firstly, we make a (finite) sequence $(\Gamma_i)_{i \geq 0}$ as follows.

1. $\Gamma_0 = \Gamma$.
2. Assume Γ_i is given. Then pick one vertex $v_i \in V(\Gamma_i)$. (This choice is completely arbitrary.) Γ_{i+1} is the graph obtained by deleting v_i and all of its incident edges.
3. Repeat the step 2 until we obtain the graph consisting only of vertices. Note that this process ends after at most $\#V(\Gamma)$ steps.

We write the sequence as follows.

$$\Gamma = \Gamma_0 \longrightarrow \Gamma_1 \longrightarrow \dots \longrightarrow \Gamma_n = (\text{vertices})$$

As examples, consider the figures in (15)(16). We can take the following sequences:



Now, by induction, we will construct a map

$$\varphi_i : E(\Gamma_i) \rightarrow V(\Gamma_i)$$

such that $\varphi_i(e)$ is one of the endpoints of an edge e .

1. We define φ_n to be the empty map since there is no edges in Γ_n .
2. Next assume that φ_{i+1} is given. If an edge $e \in E(\Gamma_i)$ is not incident to v_i , it has the corresponding edge e' in Γ_{i+1} . We define

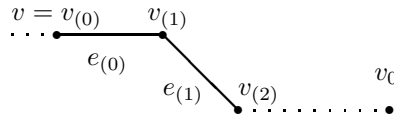
$$\varphi_i(e) = \varphi_{i+1}(e')$$

If an edge $e \in E(\Gamma_i)$ is incident to v_i , we define

$$\begin{cases} \varphi_i(e) = v_i & \text{if } \varphi_{i+1}^{-1}(v) \neq \emptyset \\ \varphi_i(e) = v & \text{if } \varphi_{i+1}^{-1}(v) = \emptyset \end{cases}$$

where v is the other endpoint of e .

Thus we obtain $\varphi_0 : E(\Gamma) \rightarrow V(\Gamma)$. If Γ has the cycle rank > 0 , φ_0 is its edge-map. If the cycle rank is zero, φ_0 is an edge-map of v_0 . We make an edge-map of other vertex $v \in V(\Gamma)$ as follows. We take the path $v = v_{(0)} \rightarrow v_{(1)} \rightarrow \dots$ where $e_{(i)} = \varphi_{v_i}^{-1}(v_{(i)})$ and $v_{(i+1)}$ is the other endpoint of $e_{(i)}$:



Since the cycle-rank is zero, this path arrives at v_0 at some time. Define

$$\varphi_v(e) = \begin{cases} \varphi_0(e) & \text{if } e \text{ is not on the path} \\ v_{(i+1)} & \text{if } e = e_{(i)}. \end{cases}$$

Then φ_v is an edge-map of v . This completes the proof. □

The examples of coupling given after (15) and (16) were constructed in this way from (19).

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