



Partial Differential Equations — *New classes of entire solutions for semilinear elliptic problems in \mathbb{R}^n* , by ANDREA MALCHIODI.

Dedicated to the memory of Renato Caccioppoli

ABSTRACT. — The goal of this paper is to describe some new results concerning entire solutions of semilinear elliptic equations in \mathbb{R}^n with non trivial asymptotic behavior at infinity. We describe in particular the (focusing, subcritical) nonlinear Schrödinger equation and the Allen-Cahn equation, which enjoy some common features but also present rather different aspects.

KEY WORDS: Semilinear elliptic equations, entire solutions, Lyapunov-Schmidt reduction, weighted spaces.

AMS SUBJECT CLASSIFICATION: 35B40, 35J10, 35J60, 58C15.

1. INTRODUCTION

In this paper we discuss some recent existence theorems for semilinear elliptic equations in \mathbb{R}^n , with the feature that the solutions considered here have a non trivial behavior at infinity. We are going to focus in particular on a couple of well studied equations, the elliptic Nonlinear Schrödinger and the Allen-Cahn.

We begin by considering the following equation

$$(E_p) \quad -\Delta u + u = u^p \quad \text{in } \mathbb{R}^n,$$

where p is greater than 1 and subcritical with respect to the Sobolev embedding of $H^1(\mathbb{R}^n)$ into $L^{p+1}(\mathbb{R}^n)$, namely $p \in (1, \frac{n+2}{n-2})$, and with no upper bound for $n = 2$. Equation (E_p) arises when considering the (focusing) Nonlinear Schrödinger equation

$$i\hbar \frac{\partial \tilde{\psi}}{\partial t} = -\hbar^2 \Delta \tilde{\psi} - |\tilde{\psi}|^{p-1} \tilde{\psi} \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+.$$

The latter equation admits a special class of solutions called *standing waves*, which are complex-valued functions $\psi(x, t)$ of the form $\psi(x, t) = e^{-i\omega t} u(x)$, where ω is a real constant and $u : \mathbb{R}^n \rightarrow \mathbb{R}$ a real-valued function satisfying

$$(1) \quad -\hbar^2 \Delta u + (V(x) + \omega)u = u^p \quad \text{in } \mathbb{R}^n,$$

which corresponds to (E_p) in the case when $V + \omega \equiv 1$ and $\hbar = 1$. Apart from this particular case, (E_p) is useful to understand the properties of (1) through concentration-compactness arguments and especially the limit profiles of solutions, as we will see in more detail, in the semiclassical limit.

Other motivations for considering (E_p) arise in the study of models from biology: for example, the *Gierer-Meinhardt* system

$$(GM) \quad \begin{cases} \mathcal{U}_t = d_1 \Delta \mathcal{U} - \mathcal{U} + \frac{\mathcal{U}^p}{\mathcal{V}^q} & \text{in } \Omega \times (0, +\infty), \\ \mathcal{V}_t = d_2 \Delta \mathcal{V} - \mathcal{V} + \frac{\mathcal{U}^r}{\mathcal{V}^s} & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial \mathcal{U}}{\partial \nu} = \frac{\partial \mathcal{V}}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, +\infty). \end{cases}$$

The functions \mathcal{U} , \mathcal{V} represent the densities of a slowly-diffusing chemical activator and of a fast-diffusing inhibitor respectively. Under suitable assumptions on the numbers p , q , r , s , in the limits $d_1 = \varepsilon \rightarrow 0$ and $d_2 \rightarrow +\infty$, the function \mathcal{V} is close to a constant (in the stationary case) and \mathcal{U} is, with a good approximation, a solution of

$$(P_\varepsilon) \quad \begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \\ u > 0 & \text{in } \Omega, \end{cases}$$

which is nothing but (E_p) in Ω with Neumann boundary conditions.

Equation (E_p) is well known to possess a radial solution U which decays to zero exponentially fast, see [5], [29]. This *ground state soliton* gives rise, via a scaling in *all* variables, to a family of solutions of both (1) and the Neumann problem corresponding to (GM) which are called *spike layers* and which have been intensively studied in the literature, see for example [1], [27] and references therein. Typically, for (1), as $\hbar \rightarrow 0$, solutions tend to concentrate near stationary points of the potential, while for (P_ε) they concentrate at the boundary of the domain or in the interior, respectively at critical points of the mean curvature or at singular points of the distance from $\partial \Omega$. These kinds of solutions are produced either via variational and penalization techniques, or via finite-dimensional Lyapunov-Schmidt reductions.

Recently, a different kind of solutions (whose existence has been conjectured for some time, see [27]) has been shown to exist, either for (1) or for (P_ε) . These have a different profile and scale only in one direction (or, more generally, in k directions, with $k \in \{1, \dots, n-1\}$), corresponding to solutions of the equation in (E_p) which are independent of some of the variables, see [2], [3], [4], [8], [11], [18], [20], [21], [26]. Apart from some special cases, when some symmetry is present, these results asserts that concentration occurs provided we choose suitable values of ε tending to zero. The reason is that these solutions have an increasing Morse index, and therefore resonance occurs. As a consequence, if one wishes to employ local inversion arguments, it is necessary to avoid some values of the parameter ε , so that the linearized equation is invertible. This resonance is also underlying when considering the model problem (E_p) . For example, one can start

from entire (decaying) solutions in lower dimension, say in \mathbb{R}^{n-1} , and extend them (with obvious notation) to the whole \mathbb{R}^n by setting $\tilde{U}(x_1, x') = U_{n-1}(x')$. Restricting ourselves to the strip $D_L := \{-\frac{L}{2} \leq x_1 \leq \frac{L}{2}\}$, and imposing Neumann conditions at the boundary: by iterative reflections, each solution can be extended to all of \mathbb{R}^n . However, it is possible to prove that the Morse index of solutions in D_L diverges as $L \rightarrow +\infty$. In [7], N. Dancer showed that whenever the Morse index changes bifurcation of non-cylindrical solutions from \tilde{U} occurs, and these new solutions are periodic in x_1 .

A similar method has been previously used by R. Schoen to prove multiplicity of solutions for the Yamabe problem, see [28]. Indeed, other geometric problems exhibit this kind of phenomenon, like that of finding surfaces in \mathbb{R}^3 with constant mean curvature (shortly, CMC). Considering for example axially-symmetric objects, it turns out that from the cylinder bifurcates a family of surfaces, the *Delaunay unduloids*, which are periodic along the axis of the cylinder.

Delaunay unduloids have been used as *building blocks* to produce complete surfaces in \mathbb{R}^3 with constant mean curvature which are union of a compact set and a finite number of *ends*, namely subsets with the topology of the cylinder which are asymptotically close to Delaunay surfaces. We refer for example to [13], [14], [15], [22], [23] and [24] for details. Analogous constructions can be done with Yamabe metrics which are defined on domains of \mathbb{R}^n with a finite number of points removed, and which are singular at these points, see e.g. [16], [25].

In [19] it was shown that a similar structure is present for solutions to (E_p) : to our knowledge there were no previous examples which raised in a pure PDE context, see also the comments below. Denoting points of \mathbb{R}^n by couples $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$, we consider first a family of solutions u_L to (E_p) which are periodic in the x_1 variable and which decay to zero at an exponential rate away from $x' = 0$, counterparts of the Delaunay surfaces. We focus on the case of large period L , which allows to construct the solutions of [7] using perturbative methods. In fact, setting $\mathbf{z}_i = (iL, 0, \dots, 0)$ and considering the function $u_{0,L} = \sum_{i \in \mathbb{N}} U(\cdot - \mathbf{z}_i)$, this satisfies the Neumann boundary conditions on ∂D_L and is an approximate solution of (E_p) for L large. Using the implicit function theorem, one can add a correction \bar{w}_L to $u_{0,L}$ so that $u_L = u_{0,L} + \bar{w}_L$ solves (E_p) exactly. To state the next result, we need to introduce some extra notation: set $\Pi = \{(z_1, z_2, 0, \dots, 0) : (z_1, z_2) \in \mathbb{R}^2\} \subseteq \mathbb{R}^n$ and also, given $\theta \in S^{n-1} (\subseteq \mathbb{R}^n) \cap \Pi$, we define the ray $l_\theta = \{t\theta : t \geq 0\}$. We also let R_θ denote the rotation in the plane Π (extended naturally to all of \mathbb{R}^n) of the angle θ . The distance function between two points (or between two sets) of \mathbb{R}^n is denoted by $dist(\cdot, \cdot)$. In the statement of Theorem 1.1 u_L stands for the solution of (E_p) periodic in x_1 just described.

THEOREM 1.1 ([19]). *Problem (E_p) admits a three-dimensional (up to rotations and translations) family of solutions which decay exponentially away from three rays originating from the origin, and which have an asymptotic periodic profile along the rays. More precisely, there exist a positive constant C , a neighborhood \mathcal{U} of 0 in \mathbb{R}^3 , smooth functions $\theta_1, \theta_2, \theta_3 : \mathcal{U} \rightarrow S^{n-1} \cap \Pi$, $L_1, L_2, L_3 : \mathcal{U} \rightarrow \mathbb{R}$,*

$y_1, y_2, y_3 : \mathcal{U} \rightarrow \Pi$ and a map from \mathcal{U} into $L^\infty(\mathbb{R}^n)$, $\zeta \in \mathcal{U} \mapsto u_\zeta$, such that the following properties hold

- (i) u_ζ is a positive solution of (E_p) ;
- (ii) if $l_{\theta_1}, l_{\theta_2}, l_{\theta_3}$ are the rays corresponding to the directions θ_1, θ_2 and θ_3 respectively, then

$$u_\zeta(x) \leq Ce^{-(1/C)\text{dist}(x, (l_{\theta_1} \cup l_{\theta_2} \cup l_{\theta_3}))} \quad \text{for every } x \in \mathbb{R}^n;$$

- (iii) for any $t_i \rightarrow +\infty$, given any compact set K of \mathbb{R}^n one has

$$\|u(\cdot - t_i\theta_a) - u_{L_a}(R_{\theta_a}(\cdot - y_a))\|_{C^2(K)} \leq C_K e^{(1/C)|t_i|}, \quad \text{for } a = 1, 2, 3.$$

It is possible to characterize with more precision these solutions in terms of their asymptotic behavior at infinity. In the above construction the values of the numbers L_a , $a = 1, 2, 3$, can be chosen arbitrarily large, but the differences $|L_a - L_b|$, with $a \neq b$, stay uniformly bounded. Also, we have $\theta_a \angle \theta_b > \frac{\pi}{3}$ for every $a \neq b$, where $\theta_a \angle \theta_b$ stands for the angle between the two versors θ_a and θ_b . Let us now consider the following function, which can be proven to be positive and monotone decreasing in L ($L \gg 1$)

$$G(L) := \frac{1}{4} \int_{\partial D_L} (|\nabla u_L|^2 + u_L^2) d\sigma - \frac{1}{2(p+1)} \int_{\partial D_L} |u_L|^{p+1} d\sigma,$$

and that it determines uniquely the asymptotic period and profile of the functions u_L . In analogy with a *balance condition* for the CMC surfaces or the singular Yamabe metrics one has the following result.

THEOREM 1.2 ([19]). *Let u be a function satisfying the properties (i)–(iii) in Theorem 1.1, and let θ_a, L_a , $a = 1, 2, 3$, be the corresponding quantities. Assume that the angles $\theta_a \angle \theta_b$ between any two different θ 's are greater than $\frac{\pi}{3}$. Then $\sum_{a=1,2,3} \theta_a G(L_a) = 0$.*

Existence of solutions of semilinear elliptic equations with infinitely-many bumps has been considered in other works, but in a different spirit. For example, in [6], similar equations in the presence of a slowly-oscillating potential have been considered. While in that paper it is the potential which mainly determines the locations of the *bumps*, here are precisely their mutual interactions which allow us to perform the construction of Theorem 1.1. We will see in the next section that these interactions are governed by a *discrete Toda system*, which plays a role in the study of particles distributed over lattices.

Concerning the Neumann problem mentioned above, we believe that the functions constructed in Theorem 1.1, scaled in ε , might lead to the existence of solutions concentrating at a singular set in Ω , with a triple point. This would be a new type of phenomenon, since so far concentration at sets of dimension greater than zero has been proved for smooth curves or manifolds only.

A result somehow related to Theorem 1.1 was proven in [9], where some other solutions to (E_p) were constructed, with a different profile. In fact, the limit shape of these solutions is nearly cylindrical and more precisely it is given by the solutions constructed by Dancer in [7]. To state the result, we need to introduce some more notation. We use a parameter δ (real, close to zero) to describe the bifurcation branch of solutions close to the cylindrical ones. It is possible to prove that the following formula holds, in cylindrical coordinates (x, z)

$$u_\delta(x, z) = u_0(x) + \delta Z(x) \cos(\lambda_1 z) + O(\delta^2)e^{-|x|},$$

where λ_1 is the square root of the inverse of the limit period of the solutions, and $Z(x)$ is an exponentially decaying axially symmetric function.

The solutions produced in [9] are defined for $n = 2$ and are of the form

$$u(x, z) \simeq \sum_{j=0}^k u_{\delta_j}(x - f_j(z), z),$$

for some small numbers δ_j and some even functions $f_1(z) \ll f_2(z) \ll \dots \ll f_k(z)$. The authors showed that the f_i 's should approximately satisfy a second order system of differential equations, the *Toda system*, given by

$$(2) \quad c_0 f_j'' = e^{f_{j-1} - f_j} - e^{f_j - f_{j+1}}; \quad i = 1, \dots, k,$$

for some positive constant c_0 and with the conventions $f_0 = -\infty$ and $f_{k+1} = +\infty$. It can be shown that the f_j 's are asymptotically affine near plus or minus infinity, see the last section for more details.

THEOREM 1.3 ([9]). *Assume that $n = 2$ and that $p > 2$. Given $k \geq 2$ there exist numbers β_i , $i = 1, \dots, k$, with $|\beta_i - \beta_j|$ large for $i \neq j$ and solutions of (2) with $f_j(0) = \beta_j$ and $f_j'(0) = 0$ such that (E_p) has solutions of the form*

$$(3) \quad u(x, z) = (1 + o(1)) \sum_{j=0}^k u_{\delta_j}(x - f_j(z), z),$$

where $o(1), \delta_j \rightarrow 0$ as $|\beta_i - \beta_j|$ tend to $+\infty$.

A similar kind of solutions was found in [10] for the Allen-Cahn equation

$$(4) \quad \Delta u = u^3 - u \quad \text{in } \mathbb{R}^n,$$

which has been the subject of several works, mostly for its role in the theory of phase transitions. For reasons of brevity we do not recall here the main features of the equations, and the contributions which can be found in the literature: we simply refer the interested reader to the introductions of [10] and [12].

In one dimension equation (4) possesses a heteroclinic solution H such that

$$H(t) \rightarrow \pm 1 \quad \text{as } t \rightarrow \pm\infty,$$

and which converge to ± 1 exponentially fast. The function H has indeed the explicit expression $\tanh((t - t_0)/\sqrt{2})$, where t_0 is real and arbitrary. The function H , in the singularly perturbed version of (4), gives the profile of the transition layers from one state of the physical system to another.

The solutions to (4) constructed in [10] exhibit multiple transitions, which are still related to a Toda system of the form (2) but with some difference in the coefficients, precisely

$$(5) \quad c_1 f_j'' = e^{\sqrt{2}(f_{j-1}-f_j)} - e^{\sqrt{2}(f_j-f_{j+1})}; \quad i = 1, \dots, k,$$

where c_1 is another positive constant and where we kept the same notation for f_0, f_{k+1} as in (2). The difference in the coefficients with respect to the case of equation (E_p) is due to the factor $\sqrt{2}$ in the expression of H . In [10] the following result was proved.

THEOREM 1.4 ([10]). *Assume that $n = 2$: given $k \geq 2$ there exist numbers β_i , $i = 1, \dots, k$, with $|\beta_i - \beta_j|$ large for $i \neq j$ and solutions of (5) with $f_j(0) = \beta_j$ and $f_j'(0) = 0$ such that (4) has solutions of the form*

$$u(x, z) = \sum_{j=1}^k H(x - f_j(z)) + \sigma_k + \varphi(x, z); \quad \sigma_k = -\frac{1}{2}(1 + (-1)^k),$$

where $\varphi \rightarrow 0$ uniformly in \mathbb{R}^2 as $|\beta_i - \beta_j|$ tend to $+\infty$.

We remark that also for the solutions found in Theorems 1.3 and 1.4 (for $k = 2$), there are some counterparts for the CMC problem, see the above mentioned references regarding this topic. It is also worth noticing that the Toda system, but in its discrete form, appears also in the proof of Theorem 1.1. It would indeed be interesting to see whether one could possibly relate the solutions constructed in Theorem 1.1 and in Theorem 1.3, showing for example that they both belong to a continuum of solutions.

The proofs of the above theorems (except for Theorem 1.2, which uses mostly integration by parts) rely on infinite-dimensional Lyapunov-Schmidt reductions, combined with analysis in weighted spaces. While Theorem 1.1 uses a *discrete* reduction, connected to the location of an infinite sequence of points, Theorems 1.3 and 1.4 use a *continuous* reduction, related to variations of the profiles normal to a given surface. With the latter techniques, in [11] a long-standing conjecture by De Giorgi was proved, regarding entire solutions of (4) which are monotone in some direction. This is the main result in [12].

THEOREM 1.5 ([12]). *Suppose $n \geq 9$: then there is a minimal x_n -graph F in \mathbb{R}^n which is not an affine function and such that for all $\gamma > 0$ sufficiently small there*

exists a bounded solution u_γ of (4) such that $u_\gamma(0) = 0$, such that $\partial_{x_n} u_\gamma > 0$ everywhere and such that $|u_\gamma(x)| \rightarrow 1$ as $\text{dist}(x, \Gamma_\gamma) \rightarrow +\infty$. Here Γ_γ is the graph of the function

$$F_\gamma(x') := \gamma^{-1} F(\gamma x'); \quad x' \in \mathbb{R}^{n-1}.$$

In Section 2 we describe the main arguments in the proof of Theorem 1.1, which relies on a discrete Lyapunov-Schmidt reduction. In Section 3 instead we sketch the proofs of Theorems 1.3–1.5, which instead are based on continuous Lyapunov-Schmidt reductions.

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2. DISCRETE INFINITE-DIMENSIONAL L-S REDUCTIONS

To begin, we recall some basic properties of the ground state solution U to (E_p) : its asymptotic behavior is the following

$$(6) \quad \lim_{r \rightarrow \infty} e^r r^{(n-1)/2} U(r) = \alpha_{n,p}; \quad \lim_{r \rightarrow \infty} \frac{U'(r)}{U(r)} = -1 \quad (r = |x|).$$

Moreover, the kernel of the operator $L_0 v := -\Delta v + v - pU^{p-1}v$ (the linearization of (E_p) at U) is spanned by $\frac{\partial U}{\partial x_1}, \dots, \frac{\partial U}{\partial x_n}$. We will work within the space of functions which are rotationally invariant in the last $n-2$ variables, so under this condition the elements of $\ker(L_0)$ will be linear combinations of $\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}$.

Our strategy consists in starting with approximate solutions which have the desired behavior at infinity, and to use then a Lyapunov-Schmidt reduction to fully solve the equation.

We introduce the three half-spaces $V_a = \{x \in \mathbb{R}^n : \langle x, \theta_a \rangle \geq \frac{L}{2} - 1\}$, and also $\psi_a(x) = \psi(d(x, V_a))$, where ψ is a fixed smooth cutoff function defined on \mathbb{R} with values into $[0, 1]$ such that $\psi(t) = 1$ for $t \leq 0$, $\psi(t) = 0$ for $t \geq 1$. Let $\theta_1, \theta_2, \theta_3$ be three unit vectors in $\Pi := \{(x_1, x_2, 0, \dots, 0)\} \subseteq \mathbb{R}^n$ which satisfy

$$(7) \quad \theta_1 \angle \theta_2 \geq \frac{\pi}{3} + \theta_0; \quad \theta_2 \angle \theta_3 \geq \frac{\pi}{3} + \theta_0; \quad \theta_1 \angle \theta_3 \geq \frac{\pi}{3} + \theta_0$$

for some $\theta_0 > 0$, and for any $a = 1, 2, 3$ recall that R_{θ_a} stands for the rotation in Π by the angle θ_a . If \bar{w}_L is as above, we define next the function \bar{w}_{L_a, θ_a} as $R_{\theta_a} \bar{w}_L$. Next we choose three large numbers L_1, L_2, L_3 (with $|L_a - L_b| + |L_a - L|$ uniformly bounded by a fixed constant C), points y_a , $a = 1, 2, 3$ and $(P_{a,i})_{a,i}$ such that

$$(8) \quad |y_a| \leq c_{\theta_0}, \quad |P_{a,i} - i\theta_i L_a - y_a| \leq C_{\theta_0} e^{-\tau|P_i|}, \quad a = 1, 2, 3, i = 1, 2, \dots$$

for some constants c_{θ_0} , C_{θ_0} and τ (uniformly bounded in L). We set for simplicity $\{P_I\} = \{0\} \cup_{a,i} \{P_{a,i}\}$, $X = \bigcup_I \{P_I\}_I$, $Y = (y_1, y_2, y_3)$ and $U_I(\cdot) = (\cdot - P_I)$ for any index I . We finally define

$$(9) \quad u_{X,Y}(x) = \sum_I U_I(x) + \sum_{a=1}^3 \psi_a(x) \bar{w}_{L_a, \theta_a}(x - y_a).$$

By our choice, this function is exponentially close to a rotation of u_{L_a} along each direction θ_a : indeed it is possible to prove the following quantitative estimate on $u_{X,Y}$.

LEMMA 2.1. *Let $S_0(u_{X,Y}) = -\Delta u_{X,Y} + u_{X,Y} - u_{X,Y}^p$. Then, if $(y_a)_a, (P_I)_I$ satisfy (8) and $(\theta_a)_a$ (7), for any $\gamma \in (0, 1)$ we have the following estimate on $S_0(u_{X,Y})$*

$$(10) \quad \|S_0(u_{X,Y})\|_{C^\gamma(B_1(x))} \leq C e^{-(1+\xi)(L/2)} e^{-\sigma d(x, \bigcup_i x_i)} [e^{-\eta|x|} + C_{\theta_0} e^{-\tau|x|}]; \quad x \in \mathbb{R}^n,$$

where ξ , σ and η are positive constants depending only on n , p and θ_0 , but not on L , and where C is a fixed constant (depending only on n , p , γ and θ_0) also independent of L .

The Lyapunov-Schmidt reduction consists in transferring problem (E_p) into determining the appropriate location of the points $\{P_I\}_I$. For doing this we can exploit the linear properties of L_0 , and as a first step solve the equation up to, basically, a sequence of Lagrange multipliers in the kernel of $L_I := -\Delta + 1 - pU_I^{p-1}$. Precisely, one can prove the following result.

PROPOSITION 2.2. *Suppose $(y_a)_a, (P_I)_I$ satisfy (8), and let $u_{X,Y}$ be as in (9). Then, for L sufficiently large, there exists a function $w_{X,Y}$ and a sequence $(\alpha^I)_I$ of elements of \mathbb{R}^2 , $\alpha^I = (\alpha^{I,j})_j$, $j = 1, 2$, which satisfy the following two properties*

$$(a) \quad -\Delta(u_{X,Y} + w_{X,Y}) + (u_{X,Y} + w_{X,Y}) - (u_{X,Y} + w_{X,Y})^p = \sum_{I,j} \alpha^{I,j} U_I^{p-1} \partial_j U_I;$$

$$(b) \quad \int_{\mathbb{R}^n} w_{X,Y} U_I^{p-1} \frac{\partial U_I}{\partial z_j} = 0 \text{ for every } I \text{ and for every } j = 1, 2.$$

While this method is rather standard when dealing with a *finite* number of solitons, some technical difficulties arise when dealing with infinitely-many ones: the proof uses crucially weighted spaces and Toeplitz type operators. The final step of the proof consists in adjusting the positions of the points $(P_I)_I$ in order to make all the coefficients $(\alpha^I)_I$ vanish. First of all, using Lemma 2.1 with $C_{\theta_0} = 0$, one can estimate the α^I 's corresponding to the function $u_{X(Y),Y}$ where $X(Y)$ denotes the special configuration of points satisfying

$$(11) \quad P_{a,i} = y_a + i\theta_a L_a \quad \text{for every } a = 1, 2, 3 \text{ and every } i \in \mathbb{N},$$

and where the symbol Y stands for the triple (y_1, y_2, y_3) .

LEMMA 2.3. For X and Y satisfying (8) and (11) we have the following estimates

$$\alpha_{X(Y), Y}^0 = - \sum_{a=1,2,3} F_1(|P_{a,1}|) \frac{P_{a,1}}{|P_{a,1}|} + O(e^{-(1+\xi)L});$$

$$\alpha_{X(Y), Y}^I = \left[F_1(|P_{a,1}|) \frac{P_{a,1}}{|P_{a,1}|} + F_1(|P_{a,1} - P_{a,2}|) \frac{P_{a,1} - P_{a,2}}{|P_{a,1} - P_{a,2}|} \right] + O(e^{-(1+\xi)L}),$$

if $P_i = P_{a,1}$, $a = 1, 2, 3$;

$$|\alpha_{X(Y), Y}^I| \leq C e^{-(1+\xi)L} e^{-\eta|P_I|} + CC_{\theta_0} e^{-\tau(|P_{a,h-1}|)} F_0(L),$$

if $P_I = P_{a,h}$ for $a = 1, 2, 3$, and $h > 1$,

where F_1 satisfies $F_1(t) = (1 + o_t(1))F_0(t)$, and where C, η, ξ are constants depending on n and p .

From energetic expansions, one can think of the solitons U_I as attracting each other via a force which is proportional to e^{-d} , where d is the distance between two neighboring ones, this factor is due to the decay rate in (6). The coefficients α^I represent the total forces acting on each soliton U_I .

Next, we study the variation of the α^I 's depending on the points $(P_I)_I$ and $(y_a)_a$. To understand this, looking at the expansions in Lemma 2.3, one can imagine α^I to behave like

$$\alpha^I \simeq - \sum_{S \neq I} \frac{P_S - P_I}{|P_S - P_I|} e^{-|P_S - P_I|}.$$

By the presence of the exponential term, the main contribution to the above expression will be given by the points closest to P_I : three when $P_I = 0$ and two for $P_I \neq 0$ (here condition (7) is also used). In particular, along each l_{θ_a} when the configuration of points $P_{a,i}$ is nearly periodic the linearization looks like a *Toda operator* which, in matrix form with respect to the index i , qualitatively looks like

$$\begin{pmatrix} \ddots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \vdots \\ \vdots & 0 & -1 & 2 & -1 & 0 & \cdots & \cdots & \vdots \\ \vdots & \cdots & 0 & -1 & 2 & -1 & 0 & \cdots & \vdots \\ \vdots & \cdots & \cdots & 0 & -1 & 2 & -1 & 0 & \vdots \\ \cdots & \cdots \end{pmatrix}.$$

The latter operator can be viewed as a discretization of the Laplacian in one dimension, and it is indeed possible to invert it via convolution with a kernel which is piecewise affine in the index i . If ξ, η are given by Lemmas 2.1 and 2.3, using the above invertibility, one finds the following result.

PROPOSITION 2.4. *Suppose $\theta_1, \theta_2, \theta_3$ satisfy (7), and L_1, L_2, L_3 $|L_a - L_b| + |L_a - L| \leq C$ for C fixed and L sufficiently large. Then, if we choose $\tau < \min\{\xi, \frac{\eta}{2}\}$ there exist $(y_a)_a$ and $(P_I)_I = (P_{a,i})_{a,i}$ such that (8) hold true for some uniformly bounded (in L) $c_{\theta_0}, C_{\theta_0}$, and with $\alpha^I = 0$ for all $I \neq 0$.*

Notice that we have a six-dimensional family of configurations satisfying the properties of Proposition 2.4. The final step consists in choosing the L_a 's and the θ_a 's so that also α^0 vanishes, which leaves four parameters free: taking the quotient with respect to rotations in Π , we obtain a genuine three-dimensional family of solutions.

3. CONTINUOUS INFINITE-DIMENSIONAL L-S REDUCTIONS

In this section we give a brief sketch of the proofs of Theorems 1.3, 1.4 and 1.5, which rely on continuous infinite-dimensional Lyapunov-Schmidt reductions. For simplicity, we describe in some detail only the arguments needed for Theorem 1.4: concerning the other results, we only give few ideas.

Also in this case, one starts by constructing approximate solutions, which one can defined by the formula

$$(12) \quad \tilde{u}(x, z) = \sum_{j=1}^k H(x - f_j(z)) + \sigma_k; \quad \sigma_k = -\frac{1}{2}(1 + (-1)^k),$$

where the functions f_j satisfy the system (5), with initial data $f_j(0) = \beta_j$ and $f_j'(0) = 0$. More specifically, it is possible to show that there exist families of solutions with these asymptotics ($\alpha > 0$ small)

$$(13) \quad f_j(z) = \tilde{f}_j(\alpha z) + \sqrt{2} \left(j - \frac{k+1}{2} \right) \log \frac{1}{\alpha}, \quad j = 1, \dots, k,$$

where

$$(14) \quad \tilde{f}_j(z) = a_j |z| + b_j + \hat{f}_j(z), \quad |z| \gg 1,$$

where

$$a_j - a_{j-1} > \varrho > 0 \quad \text{and} \quad \|\hat{f}_j e^{\varrho|z|}\|_{C^4(\mathbb{R})} < C,$$

for some large constants $C, \varrho > 0$.

One has then the following result (here in the definition of the approximate solutions we are not completely precise for reasons of brevity, however the function \tilde{u} gives an idea of how the approximate solutions in [9] look like).

PROPOSITION 3.1. *Suppose \tilde{u} is defined as in (12), with f_j as in (13) satisfying (5). Let also*

$$S(\tilde{u}) = \Delta \tilde{u} + \tilde{u} - \tilde{u}^3.$$

Then there exists positive constants $C > 0$ and $\sigma \in (0, \frac{\sqrt{2}-1}{\sqrt{2}})$ such that

$$\|S(\tilde{u})\|_{\sigma, \varrho, \alpha} \leq C\alpha^{2(1-\sigma)},$$

where the latter norm is defined as

$$\|\varphi\|_{\sigma, \varrho, \alpha} = \left\| \left(\sum_{j=1}^k e^{-\sigma|x-f_j(z)|-\varrho\alpha|z|} \right)^{-1} \varphi \right\|_{\infty},$$

where ϱ is as in (14).

We can give a heuristic explanation of why this configuration is a good candidate for being an approximate solution. From energetic expansions similar to those as for Lemma 2.3 one finds that two distant transition layers attract each other with an *intensity* proportional to $e^{-\sqrt{2}d}$, where d stands for the distance between the two layers. If the layers stay parallel, they would tend to collapse one onto each other but if they are bent, closer points would feel stronger forces. On the other hand, a curved transition layer would tend to move according to its curvature vector, since this motion would tend to decrease locally the length of the layer, and hence it would be energetically favorable. The above choice of the functions f_j gives a balance between the forces due to the curvature and the mutual interactions. For this reason we see both the second derivatives and the exponential terms appearing in formula (5).

To continue in the proof of the theorem one uses the following result, which reduces the problem to determining k real functions only (also here we are not completely precise, for reasons of brevity).

PROPOSITION 3.2. *There exist small numbers α_0, σ_0 such that for $\alpha \in (0, \alpha_0)$ and $\delta \in (0, \delta_0)$, and any function satisfying (13) there exists a correction w to \tilde{u} for which*

$$(15) \quad S(\tilde{u} + w) = \sum_{j=1}^k c_j(z)H'(x - f_j(z)),$$

and the orthogonality condition

$$(16) \quad \int_{\mathbb{R}} H'(x - f_j(z))w(x, z) dx = 0; \quad j = 1, \dots, k.$$

Similarly to Proposition 2.2, to prove this result one uses the fact that the heteroclinic H is a non degenerate solution of (4) in one dimension, except for the generators of translations in \mathbb{R} . The functions c_j in Proposition 3.2 play the role of Lagrange multipliers once the constraint equation 16 is imposed. A similar procedure has also been used in [17], [18] (or in other papers regarding the Allen-Cahn equations), but one of the difficulties here is that the functions f_j are defined

on the whole real line, which requires introducing weighted norms in the variable z as well.

To conclude the proof, one is then reduced to find functions f_j for which all the c_j 's in (15) vanish. In this step, one shows that the changes of the c_j 's with respect to a variation of the f_j 's are ruled at main order by the linearization of the system (5). Since within a certain family of functions (which enjoy some symmetry properties) this linearization is invertible, one is then able to make all the functions c_j vanish, and hence to prove Theorem 1.4. The proof of Theorem 1.3 is similar in spirit but slightly more difficult, because of the resonance phenomenon described before. To tackle this difficulty as well, one uses a modification of the profile as (3), to control also the component of the equation along this extra degenerate direction.

The proof of Theorem 1.5 uses a similar procedure, but the main issue there is to prove that the sets Γ_γ are non-degenerate except for the actions of translations, rotations and dilations. To show this the authors derive (from a clever choice of conformal coordinates) a refined asymptotic expansion of the functions F_γ . Those functions were constructed in 1969 by Bombieri, De Griogi and Giusti, and in [12] their asymptotic behavior is improved passing from a polynomial control, to an estimate of order $o(1)$ near infinity. Once this is done, the Jacobi fields for the minimal surface operator are classified, and one can use a reduction method as in Proposition 3.2.

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