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Another Proof of Gustafson's C_n-type Summation Formula via 'Elementary' Symmetric Polynomials

By

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Abstract

We introduce new symmetric polynomials which induce a q-difference equation associated with a basic hypergeometric sum of type C_n investigated by Gustafson. Using them we give another proof for Gustafson's C_n-type summation formula.

§1. Introduction

In a series of the papers [5, 6, 7, 8], Gustafson established some basic hypergeometric sums associated with Lie algebras. They are natural generalization of Milne's basic hypergeometric sums [15]. The summation formulas investigated by Gustafson are some multiple generalizations of Bailey's ${}_6\psi_6$ summation formula [3]. Using terminology of Jackson integrals, one of his formulas is rewritten as follows:

(1)
$$\int_0^{\xi\infty}\Phi_{\mathrm{G}}(z)\Delta(z)\frac{d_qz_1}{z_1}\cdots\frac{d_qz_n}{z_n}$$
$$=(1-q)^n(q)_\infty^n\frac{\prod_{1\le\mu<\nu\le 2n+2}(qa_\mu^{-1}a_\nu^{-1})_\infty}{(qa_1^{-1}a_2^{-1}\dots a_{2n+2}^{-1})_\infty}$$
$$\times\prod_{i=1}^n\frac{\xi_i^{i-\alpha_1-\alpha_2-\cdots-\alpha_{2n+2}}\theta(\xi_i^2)}{\prod_{m=1}^{2n+2}\theta(a_m\xi_i)}\prod_{1\le j<k\le n}\theta(\xi_j/\xi_k)\theta(\xi_j\xi_k)$$

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where $\theta(x) := (x)_\infty(q/x)_\infty$ and

$$\Phi_{\mathrm{G}}(z) := \prod_{i=1}^{n}\prod_{m=1}^{2n+2} z_i^{1/2-\alpha_m}\frac{(qa_m^{-1}z_i)_\infty}{(a_m z_i)_\infty}, \quad q^{\alpha_m} = a_m,$$

$$\Delta(z) := \prod_{i=1}^{n}\frac{1-z_i^2}{z_i}\prod_{1\le j<k\le n}\frac{(1-z_j/z_k)(1-z_jz_k)}{z_j}$$

(For more details, see the definition of the Jackson integral in Section 3). We call it *Gustafson's C_n-type summation formula.* This formula implies that the basic hypergeometric sum represented by the left-hand side of (1) can be expressed as a product of q-gamma functions.

On the other hand, Gustafson's formula is very closely related to the following formula:

$$\begin{aligned} &\int_0^{\xi\infty}\Phi_{\mathrm{vD}}(z)\Delta(z)\frac{d_qz_1}{z_1}\cdots\frac{d_qz_n}{z_n} \\ &= (1-q)^n(q)_\infty^n\prod_{i=1}^{n}\frac{(qt^{-i})_\infty}{(qt^{-1})_\infty}\frac{\prod_{1\le\mu<\nu\le 4}(qt^{-(n-i)}a_\mu^{-1}a_\nu^{-1})_\infty}{(qt^{-(n+i-2)}a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})_\infty} \\ &\quad\times\prod_{i=1}^{n}\frac{\xi_i\,\theta(\xi_i^2)}{\prod_{m=1}^{4}\xi_i^{\alpha_m}\theta(a_m\xi_i)}\prod_{1\le j<k\le n}\frac{\theta(\xi_j/\xi_k)\theta(\xi_j\xi_k)}{\xi_j^{2\tau}\theta(t\xi_j/\xi_k)\theta(t\xi_j\xi_k)} \end{aligned} \tag{2}$$

where $q^{\alpha_m} = a_m, q^\tau = t$ and

$$\begin{aligned} \Phi_{\mathrm{vD}}(z) := &\prod_{i=1}^{n}\prod_{m=1}^{4} z_i^{1/2-\alpha_m}\frac{(qa_m^{-1}z_i)_\infty}{(a_mz_i)_\infty} \\ &\times\prod_{1\le j<k\le n} z_j^{1-2\tau}\frac{(qt^{-1}z_j/z_k)_\infty}{(tz_j/z_k)_\infty}\frac{(qt^{-1}z_jz_k)_\infty}{(tz_jz_k)_\infty}. \end{aligned}$$

The formula (2) was proved by van Diejen [4], who showed it to calculate a certain multiple Jackson integral in two ways using the formula (1), following Gustafson's method [6].

In [13], we introduced some new symmetric polynomials $e_i(z)$, $0 \le i \le n$, which we call the *'elementary' symmetric polynomials associated with the weight function* $\Phi_{\mathrm{vD}}(z)$. (See [13] for the reason we call them 'elementary'.) We found the following relation between $e_i(z)$ and $e_{i-1}(z)$:

$$\begin{aligned} &\int_0^{\xi\infty} e_i(z)\Phi_{\mathrm{vD}}(z)\Delta(z)\varpi_q \\ &= -\frac{t^{i-1}(1-t^{n-i+1})\prod_{k=2}^{4}(1-a_ka_1t^{n-i})}{t^{n-i}(1-t^i)a_1(1-a_1a_2a_3a_4t^{2n-i-1})}\int_0^{\xi\infty} e_{i-1}(z)\Phi_{\mathrm{vD}}(z)\Delta(z)\varpi_q, \end{aligned}$$

where ϖ_q is the abbreviation for the symbol $\frac{d_qz_1}{z_1}\cdots\frac{d_qz_n}{z_n}$. From repeated use of this relation, for the sum in the left-hand side of (2), we can immediately deduce its q-difference equations with respect to parameters. As a consequence, we gave another proof of the product formula (2) as if we construct the gamma product expression of the beta function from its difference equations and its asymptotic behavior at infinity of parameters.

In this paper, for Gustafson's formula case, we also found other new symmetric polynomials $e'(z)$, $0 \le i \le n$, similar to those associated with $\Phi_{\mathrm{vD}}(z)$. (See Section 4 for the definition of $e'_i(z)$.) We can also obtain q-difference equations with respect to parameters from the following theorem:

Theorem 1.1. *The following relation holds for the polynomials $e'_0(z)$ and $e'_n(z)$:*

$$\int_0^{\xi\infty} e'_n(z)\Phi_{\mathrm{G}}(z)\Delta(z)\varpi_q = (-a_1)^{-n}\frac{\prod_{k=2}^{2n+2}(1-a_1a_k)}{1-a_1a_2\dots a_{2n+2}}\int_0^{\xi\infty} e'_0(z)\Phi_{\mathrm{G}}(z)\Delta(z)\varpi_q.$$

The aim of this paper is to introduce these symmetric polynomials $e'_i(z)$ associated with $\Phi_{\mathrm{G}}(z)$ and to give another proof of the formula (1) using the Theorem 1.1. From the previous result by the author (Lemma 3.1), the problem is to determine the constant C in (9). To this aim the Theorem 1.1 is used.

Throughout this paper, we use the notation $(x)_\infty := \prod_{i=0}^{\infty}(1-q^ix)$ and $(x)_N := (x)_\infty/(q^Nx)_\infty$ where $0<q<1$.

§2. Symplectic Schur Functions $\chi_\lambda(z)$

Let W_{C_n} be the Weyl group of type C_n, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathcal{S}_n$ where $\mathcal{S}_n$ is the symmetric group of nth order. W_{C_n} is generated by the following transformations of the coordinates $(z_1,z_2,\dots,z_n)\in(\mathbb{C}^*)^n$:

$$(z_1,z_2,\dots,z_n)\to(z_1^{-1},z_2,\dots,z_n),$$
$$(z_1,z_2,\dots,z_n)\to(z_{\sigma(1)},z_{\sigma(2)},\dots,z_{\sigma(n)})\quad \sigma\in\mathcal{S}_n.$$

For a function $f(z)$ of $z\in(\mathbb{C}^*)^n$, the Weyl group action is defined by

$$wf(z) := f(w^{-1}(z)) \quad \text{for} \quad w\in W_{C_n},$$

and we denote by $\mathcal{A}f(z)$ the alternating sum over W_{C_n} defined by

$$\mathcal{A}f(z) := \sum_{w\in W_{C_n}} (\operatorname{sgn} w)\, wf(z).$$

Let P be the set of partitions defined by

$$P := \{(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n\,;\, \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0\}.$$

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in P$, we set

$$\mathcal{A}_\lambda(z) := \mathcal{A}(z_1^{\lambda_1} z_2^{\lambda_2} \dots, z_n^{\lambda_n}).$$

The following holds for $\rho := (n, n-1, \dots, 2, 1) \in P$,

$$\mathcal{A}_\rho(z) = \prod_{i=1}^{n} (z_i - z_i^{-1}) \prod_{1 \le j < k \le n} \frac{(z_k - z_j)(1 - z_j z_k)}{z_j z_k} \tag{3}$$

which is called Weyl's denominator formula. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in P$, we define the *symplectic Schur function*

$$\chi_\lambda(z) := \frac{\mathcal{A}_{\lambda+\rho}(z)}{\mathcal{A}_\rho(z)} = \frac{\mathcal{A}_{(\lambda_1+n, \lambda_2+n-1, \dots, \lambda_{n-1}+2, \lambda_n+1)}(z)}{\mathcal{A}_{(n,n-1,\dots,2,1)}(z)},$$

which occurs in Weyl's character formula. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in P$, if we denote by m_i the multiplicity of i in λ, i.e., $m_i = \#\{j\,;\, \lambda_j = i\}$, it is convenient to use the symbols $\lambda = (1^{m_1} 2^{m_2} \dots r^{m_r} \dots)$ and $\chi_\lambda(z) = \chi_{(1^{m_1} 2^{m_2} \dots r^{m_r} \dots)}(z)$ as used in the example $\chi_{(2,1,1,0)}(z_1, z_2, z_3, z_4) = \chi_{(1^2 2)}(z_1, z_2, z_3, z_4)$.

We state two lemmas which will be used technically when we prove a property of the 'elementary' symmetric polynomials in Section 4. Note in passing that the number of variables for $\chi_{(1^{i-j})}$ and $\chi_{(j)}$ are different in the following:

Lemma 2.1. *The following holds for $i = 0, 1, 2, \dots, n$:*

$$\sum_{j=0}^{i} (-1)^j \chi_{(1^{i-j})}(z_1, z_2, \dots, z_n) \chi_{(j)}(z_1, z_2, \dots, z_{n-i+1}) = \begin{cases} 0 \; (i \ne 0), \\ 1 \; (i = 0). \end{cases}$$

Proof. See [13]. □

Lemma 2.2. *The following holds for $\chi_\lambda(z)$ and $\mathcal{A}_\rho(z)$:*

$$\sum_{j=0}^{n} (-1)^j \chi_{(1^{n-j})}(z_1, z_2, \dots, z_n) \chi_{(j)}(z_{n+1}) = \frac{\mathcal{A}_{(n+1,n,\dots,1)}(z_1, z_2, \dots, z_n, z_{n+1})}{\mathcal{A}_{(n,n-1,\dots,1)}(z_1, \dots, z_n) \mathcal{A}_{(1)}(z_{n+1})}.$$

Proof. See [13]. □

§3. Jackson Integral of Gustafson's C_n-type

For $z = (z_1, z_2, \dots, z_n) \in (\mathbb{C}^*)^n$, we set

$$\Phi_{\mathrm{G}}(z) := \prod_{i=1}^{n} \prod_{m=1}^{2n+2} z_i^{1/2-\alpha_m} \frac{(qa_m^{-1}z_i)_\infty}{(a_m z_i)_\infty},$$

$$\Delta(z) := \prod_{i=1}^{n} \frac{1-z_i^2}{z_i} \prod_{1\le j<k\le n} \frac{(1-z_j/z_k)(1-z_jz_k)}{z_j}$$

where $q^{\alpha_m} = a_m$. Weyl's denominator formula (3) says

$$\Delta(z) = (-1)^n \mathcal{A}_\rho(z) \quad \text{where} \quad \rho = (n, n-1, \dots, 2, 1) \in P. \tag{4}$$

For an arbitrary $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in (\mathbb{C}^*)^n$, we define the q-shift $\xi \to q^\nu \xi$ by a lattice point $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{Z}^n$, where

$$q^\nu \xi := (q^{\nu_1}\xi_1, q^{\nu_2}\xi_2, \dots, q^{\nu_n}\xi_n) \in (\mathbb{C}^*)^n.$$

For $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in (\mathbb{C}^*)^n$ and a function $h(z)$ of $z \in (\mathbb{C}^*)^n$, we define the sum over the lattice $\mathbb{Z}^n$ by

$$\int_0^{\xi_1\infty} \!\!\cdots\! \int_0^{\xi_n\infty} h(z)\, \frac{d_qz_1}{z_1} \cdots \frac{d_qz_n}{z_n} := (1-q)^n \sum_{\nu\in\mathbb{Z}^n} h(q^\nu\xi), \tag{5}$$

which we call the *Jackson integral* if it converges. We abbreviate the LHS of (5) to $\int_0^{\xi\infty} h(z)\,\varpi_q$. We now define the Jackson integral whose integrand is $\Phi_{\mathrm{G}}(z)\Delta(z)$ as follows:

$$J_{\mathrm{G}}(\xi) := \int_0^{\xi\infty} \Phi_{\mathrm{G}}(z)\Delta(z)\varpi_q, \tag{6}$$

which converges if

$$|a_1a_2 \dots a_{2n+2}| > q$$

and

$$a_m\xi_i \notin \{q^l\,;\, l \in \mathbb{Z}\} \quad \text{for} \quad 1 \le m \le 2n+2,\ 1 \le i \le n.$$

(See [11] for the convergence condition.) We call the sum $J_{\mathrm{G}}(\xi)$ the *Jackson integral of Gustafson's C_n-type*. The sum $J_{\mathrm{G}}(\xi)$ is invariant under the shifts $\xi \to q^\nu\xi$ for $\nu \in \mathbb{Z}^n$.

Since $(q^{1+k})_\infty = 0$ if k is a negative integer, for the special point

$$\zeta := (a_1, a_2, \dots, a_{n-1}, a_n) \in (\mathbb{C}^*)^n,$$

it follows that

$$\Phi_{\mathrm{G}}(q^\nu\zeta) = 0 \quad \text{if} \quad \nu \notin D$$

where D forms the cone in the lattice $\mathbb{Z}^n$ defined by

$$D := \{\nu \in \mathbb{Z}^n\,;\, \nu_1 \geq 0,\ \nu_2 \geq 0, \ldots,\ \nu_{n-1} \geq 0\ \text{ and }\ \nu_n \geq 0\}.$$

This implies that $J(\zeta)$ is written as the sum over the cone D as follows:

$$J_{\mathrm{G}}(\zeta) = (1-q)^n \sum_{\nu \in D} \Phi_{\mathrm{G}}(q^\nu\zeta)\Delta(q^\nu\zeta) \tag{7}$$

We call its Jackson integral summed over D *truncated*. We just write

$$J_{\mathrm{G}}(\zeta) = \int_0^\zeta \Phi_{\mathrm{G}}(z)\Delta(z)\varpi_q$$

omitting ∞ from the integral area only if $\xi = \zeta$.

Let $\Theta_{\mathrm{G}}(\xi)$ be the function defined by

$$\Theta_{\mathrm{G}}(\xi) := \prod_{i=1}^n \frac{\xi_i^{i-\alpha_1-\alpha_2-\cdots-\alpha_{2n+2}}\theta(\xi_i^2)}{\prod_{m=1}^{2n+2}\theta(a_m\xi_i)} \prod_{1\leq j<k\leq n} \theta(\xi_j/\xi_k)\theta(\xi_j\xi_k) \tag{8}$$

where $\theta(x) := (x)_\infty(q/x)_\infty$. We state a lemma for subsequent section.

Lemma 3.1. *The Jackson integral $J_{\mathrm{G}}(\xi)$ is expressed as*

$$J_{\mathrm{G}}(\xi) = C\,\Theta_{\mathrm{G}}(\xi) \tag{9}$$

where C is a constant not depending on $\xi \in (\mathbb{C}^*)^n$

Proof. See [10]. □

We will discuss the constant C later in Section 6.

§4. 'Elementary' Symmetric Polynomials $e'_i(z)$

For $i = 0, 1, 2, 3, \ldots, n$, we define the following symmetric polynomials in terms of $\chi_\lambda(z)$:

$$e'_i(z) := \sum_{j=0}^i (-1)^j \chi_{(1^{i-j})}(\underbrace{z_1, z_2, \ldots, z_n}_{n})\,\chi_{(j)}(\underbrace{a_1, a_2, \ldots, a_{n-i+1}}_{n-i+1}), \tag{10}$$

which we call the *ith 'elementary' symmetric polynomials associated with the weight function* $\Phi_{\mathrm{G}}(z)$. The reason we call it 'elementary' is mentioned in [13].

Lemma 4.1. *The product expression of the* n*th 'elementary' symmetric polynomial* $e'_n(z)$ *is the following:*

$$e'_n(z) = \prod_{i=1}^{n} \frac{(a_1 - z_i)(1 - a_1 z_i)}{a_1 z_i}. \tag{11}$$

Proof. Using Weyl's denominator formula (3), we have

$$\prod_{i=1}^{n} \frac{(a_1 - z_i)(1 - a_1 z_i)}{a_1 z_i} = \frac{\mathcal{A}_{(n+1,n,\dots,1)}(z_1, z_2, \dots, z_n, a_1)}{\mathcal{A}_{(n,n-1,\dots,1)}(z_1, \dots, z_n)\mathcal{A}_{(1)}(a_1)} \tag{12}$$

Taking $z_{n+1} = a_1$ at Lemma 2.2, we have

$$\begin{aligned} &\frac{\mathcal{A}_{(n+1,n,\dots,1)}(z_1, z_2, \dots, z_n, a_1)}{\mathcal{A}_{(n,n-1,\dots,1)}(z_1, \dots, z_n)\mathcal{A}_{(1)}(a_1)} \\ &= \sum_{j=0}^{n} (-1)^j \chi_{(1^{n-j})}(z_1, z_2, \dots, z_n)\chi_{(j)}(a_1) \\ &= e'_n(z). \end{aligned} \tag{13}$$

From (12) and (13), we have (11). □

Let x be an arbitrary real number satisfying $x > 0$. For $i = 1, 2, \dots, n+1$, we set

$$\zeta_i = (\zeta_{i1}, \zeta_{i2}, \dots, \zeta_{in}) \in (\mathbb{C}^*)^n, \tag{14}$$

where

$$\zeta_{ij} := \begin{cases} x^{i-j} & \text{if} \quad 1 \le j < i, \\ a_{n-j+1} & \text{if} \quad i \le j \le n. \end{cases}$$

The explicit expression of ζ_i is the following:

$$\begin{aligned} \zeta_1 &= (a_n, a_{n-1}, \dots, a_2, a_1), \\ \zeta_2 &= (x, a_{n-1}, a_{n-2}, \dots, a_2, a_1), \\ \zeta_3 &= (x^2, x, a_{n-2}, a_{n-3}, \dots, a_2, a_1), \\ &\ \vdots \\ \zeta_n &= (x^{n-1}, \dots, x^2, x, a_1), \\ \zeta_{n+1} &= (x^n, x^{n-1}, \dots, x^2, x). \end{aligned}$$

Lemma 4.2. *If $1 \le j \le i \le n$, then*

$$e'_i(z_1, z_2, \dots, z_{j-1}, a_{n-j+1}, a_{n-j}, \dots, a_2, a_1) = 0.$$

Proof. Since $\chi_\lambda(z)$ is symmetric, by definition (10), we have

$$\begin{aligned}
&e'_i(z_1, z_2, \dots, z_{j-1}, a_{n-j+1}, a_{n-j}, \dots, a_2, a_1) \\
&= \sum_{k=0}^{i} (-1)^k \chi_{(1^{i-k})}(z_1, z_2, \dots, z_{j-1}, a_{n-j+1}, a_{n-j}, \dots, a_2, a_1) \\
&\qquad\qquad \times \chi_{(k)}(a_1, a_2, \dots, a_{n-i+1}) \\
&= \sum_{k=0}^{i} (-1)^k \chi_{(1^{i-k})}(a_1, a_2, \dots, a_{n-i+1}, a_{n-i+2}, \dots, a_{n-j+1}, z_1, z_2, \dots, z_{j-1}) \\
&\qquad\qquad \times \chi_{(k)}(a_1, a_2, \dots, a_{n-i+1}).
\end{aligned}$$

Applying Lemma 2.1, the right-hand side of the above equation is equal to 0. This completes the proof. □

The explicit expression of Lemma 4.2 is the following:

$$\begin{aligned}
e'_1(a_n, a_{n-1}, \dots, a_2, a_1) &= 0, \\
e'_2(z_1, a_{n-1}, \dots, a_2, a_1) &= 0, \\
&\vdots \\
e'_n(z_1, z_2, \dots, z_{n-1}, a_1) &= 0.
\end{aligned}$$

In particular,

Corollary 4.1. *If $1 \le j \le i \le n$, then $e'_i(\zeta_j) = 0$.*

Proof. It is straightforward from the definition (14) of ζ_i and Lemma 4.2. □

§5. Main Theorem

In this section, to specify the number of variables n, we simply use $e_i^{(n)}(z)$ and $\mathcal{A}^{(n)}(z)$ instead of the 'elementary' symmetric polynomials $e'_i(z)$ and Weyl's denominator $\mathcal{A}_\rho(z)$ respectively. The symbol (n) on the right shoulder of e_i or $\mathcal{A}$ indicates the number of variables of $z = (z_1, z_2, \dots, z_n)$ for $e'_i(z)$ or $\mathcal{A}_\rho(z)$.

Let τ_1 and σ_i be reflections of the coordinates $z=(z_1,z_2,\ldots,z_n)$ defined as follows:

$$\begin{aligned}&\tau_1: z_1 \longleftrightarrow z_1^{-1},\\ &\sigma_i: z_1 \longleftrightarrow z_i \quad \text{for} \quad i=2,3,\ldots,n.\end{aligned}$$

Since the Weyl group W_{C_n} of type C_n is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathcal{S}_n$, we write

$$W_{C_n}=\langle \tau_1,\sigma_2,\sigma_3,\ldots,\sigma_n\rangle, \tag{15}$$

which means W_{C_n} is generated by τ_1 and σ_i, $i=2,3,\ldots,n$.

For $w\in W_{C_n}$, if we set

$$U_w(z):=\frac{w\Phi_{\mathrm{G}}(z)}{\Phi_{\mathrm{G}}(z)}, \tag{16}$$

then the function $U_w(z)$ satisfies the following property:

$$U_{w_1w_2}(z)=U_{w_1}(z)\times w_1U_{w_2}(z). \tag{17}$$

For the generators $\tau_1,\sigma_2,\sigma_3,\ldots,\sigma_n$ of W_{C_n}, from the definition of $\Phi_{\mathrm{G}}(z)$ and (16) we have the following explicitly:

$$\begin{aligned}&U_{\tau_1}(z)=\prod_{m=1}^{2n+2}\frac{z_1^{2\alpha_m-1}\theta(a_mz_1)}{\theta(qa_m^{-1}z_1)},\\ &U_{\sigma_i}(z)=1 \quad \text{for} \quad i=2,3,\ldots,n.\end{aligned}$$

Since the function $\theta(x)$ satisfies $\theta(qx)=-\theta(x)/x$, the functions $U_{\tau_1}(z)$ and $U_{\sigma_i}(z)$ are invariant under the q-shifts $\xi\to q^\nu\xi$ for $\nu\in\mathbb{Z}^n$. Moreover, from repeated use of the property (17), for $w\in W_{C_n}$ the function $U_w(z)$ is also invariant under the q-shifts $\xi\to q^\nu\xi$ for $\nu\in\mathbb{Z}^n$.

Let T_{z_1} be the q-shift of variable z_1 such that $T_{z_1}: z_1\to qz_1$. Set

$$\nabla\varphi(z):=\varphi(z)-\frac{T_{z_1}\Phi_{\mathrm{G}}(z)}{\Phi_{\mathrm{G}}(z)}\,T_{z_1}\varphi(z), \tag{18}$$

where $T_{z_1}\Phi_{\mathrm{G}}(z)/\Phi_{\mathrm{G}}(z)$ is written as follows by definition:

$$\frac{T_{z_1}\Phi_{\mathrm{G}}(z)}{\Phi_{\mathrm{G}}(z)}=q^{n+1}\prod_{k=1}^{2n+2}\frac{(1-a_kz_1)}{(a_k-qz_1)}$$

Lemma 5.1. *Let $\varphi(z)$ be an arbitrary function such that $\int_0^{\xi\infty}\varphi(z)\,\Phi_{\mathrm{G}}(z)\varpi_q$ converges. The following holds for $\varphi(z)$:*

$$\int_0^{\xi\infty}\Phi_{\mathrm{G}}(z)\,\nabla\varphi(z)\,\varpi_q=0. \tag{19}$$

In particular,

$$\int_0^{\xi\infty}\Phi_{\mathrm{G}}(z)\,\mathcal{A}\nabla\varphi(z)\,\varpi_q=0. \tag{20}$$

Proof. Since a Jackson integral is invariant under q-shift, it follows that

$$\int_0^{\xi\infty}\varphi(z)\Phi_{\mathrm{G}}(z)\varpi_q=\int_0^{\xi\infty}T_{z_1}\varphi(z)\,T_{z_1}\Phi_{\mathrm{G}}(z)\varpi_q,$$

so that

$$\int_0^{\xi\infty}\varphi(z)\Phi_{\mathrm{G}}(z)\varpi_q-\int_0^{\xi\infty}T_{z_1}\varphi(z)\,\frac{T_{z_1}\Phi_{\mathrm{G}}(z)}{\Phi_{\mathrm{G}}(z)}\Phi_{\mathrm{G}}(z)\varpi_q=0.$$

This implies (19) from the definition (18) of ∇. For $w\in W_{C_n}$, we have

$$\begin{aligned}
&w\int_0^{\xi\infty}\Phi_{\mathrm{G}}(z)\,\nabla\varphi(z)\,\varpi_q=\int_0^{(w^{-1}\xi)\infty}\Phi_{\mathrm{G}}(z)\,\nabla\varphi(z)\,\varpi_q\\
&=\int_0^{\xi\infty}w\Phi_{\mathrm{G}}(z)\,w\nabla\varphi(z)\,\varpi_q=\int_0^{\xi\infty}U_w(z)\Phi_{\mathrm{G}}(z)\,w\nabla\varphi(z)\,\varpi_q\\
&=U_w(\xi)\int_0^{\xi\infty}\Phi_{\mathrm{G}}(z)\,w\nabla\varphi(z)\,\varpi_q
\end{aligned}$$

because $U_w(z)$ is invariant under the q-shift $z\to q^\nu z$. If (19) holds, then

$$\int_0^{\xi\infty}\Phi_{\mathrm{G}}(z)\,w\nabla\varphi(z)\,\varpi_q=\frac{1}{U_w(\xi)}\;w\int_0^{\xi\infty}\Phi_{\mathrm{G}}(z)\,\nabla\varphi(z)\,\varpi_q=0\quad\text{for}\quad w\in W_{C_n}.$$

Thus, for the function $\mathcal{A}\nabla\varphi(z)=\sum_{w\in W_{C_n}}(\operatorname{sgn}w)\;w\nabla\varphi(z)$, we have

$$\int_0^{\xi\infty}\Phi_{\mathrm{G}}(z)\,\mathcal{A}\nabla\varphi(z)\,\varpi_q=\sum_{w\in W_{C_n}}(\operatorname{sgn}w)\int_0^{\xi\infty}\Phi_{\mathrm{G}}(z)\,w\nabla\varphi(z)\varpi_q=0,$$

which completes the proof. □

Let $f(z)$ and $g(z)$ be functions defined as follows:

$$f(z) := \prod_{m=1}^{2n+2}(a_m - z_1), \quad g(z) := \prod_{m=1}^{2n+2}(1 - a_m z_1).$$

For $i = 2, 3, \ldots, n$, we set

$$f_i(z) := \sigma_i f(z), \quad g_i(z) := \sigma_i g(z) \tag{21}$$

and simply $f_1(z) := f(z)$, $g_1(z) := g(z)$. For $i = 1, 2, \ldots, n$, the explicit forms of $f_i(z)$ and $g_i(z)$ are the following:

$$f_i(z) = \prod_{m=1}^{2n+2}(a_m - z_i), \quad g_i(z) = \prod_{m=1}^{2n+2}(1 - a_m z_i). \tag{22}$$

By definition, we have

$$\tau_1\left(\frac{f_1(z)}{z_1^{n+1}}\right) = \frac{g_1(z)}{z_1^{n+1}}. \tag{23}$$

Let $\overline{\varphi}_i(z)$, $1 \le i \le n$, be the function defined by

$$\overline{\varphi}_i(z) := \frac{\mathcal{A}\nabla\varphi_i(z)}{2}$$

where

$$\varphi_i(z) := \frac{f(z)}{z_1^{n+1}}\, z_2^{n-1} z_3^{n-2} \ldots z_n\, e_{i-1}^{(n-1)}(z_2, z_3, \ldots, z_n). \tag{24}$$

Proposition 5.1. *The functions $\overline{\varphi}_i(z)$ are expressed as*

$$\overline{\varphi}_i(z) = \sum_{k=1}^{n}(-1)^{k+1}\frac{f_k(z) - g_k(z)}{z_k^{n+1}} e_{i-1}^{(n-1)}(\widehat{z}_k)\mathcal{A}^{(n-1)}(\widehat{z}_k) \tag{25}$$

where $\widehat{z}_k := (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n)$.

Proof. By definition (18) of ∇ and (24), we have

$$\nabla\varphi_i(z) = \frac{f(z) - g(z)}{z_1^{n+1}} z_2^{n-1} z_3^{n-2} \ldots z_n\, e_{i-1}^{(n-1)}(\widehat{z}_1). \tag{26}$$

Then, from (15) and (23), it follows that

$$\begin{aligned} \overline{\varphi}_i(z) &= \mathcal{A}\nabla\varphi_i(z)/2 \\ &= \frac{f_1(z) - g_1(z)}{z_1^{n+1}} e_{i-1}^{(n-1)}(\widehat{z}_1)\mathcal{A}^{(n-1)}(\widehat{z}_1) \\ &\quad + \sum_{k=2}^{n}(\operatorname{sgn}\sigma_k)\ \sigma_k\Big[\frac{f_1(z) - g_1(z)}{z_1^{n+1}} e_{i-1}^{(n-1)}(\widehat{z}_1)\,\mathcal{A}^{(n-1)}(\widehat{z}_1)\Big]. \end{aligned} \tag{27}$$

Thus, we obtain the expression (25) by substituting (21) and the following for (27):

$$\operatorname{sgn}\sigma_k = -1, \quad \sigma_k e_{i-1}^{(n-1)}(\widehat{z}_1) = e_{i-1}^{(n-1)}(\widehat{z}_k), \quad \sigma_k \mathcal{A}^{(n-1)}(\widehat{z}_1) = (-1)^k \mathcal{A}^{(n-1)}(\widehat{z}_k).$$

This completes the proof. □

On the other hand, $\overline{\varphi}_i(z)$ are expanded as a linear combination of the functions $e_j^{(n)}(z)\mathcal{A}^{(n)}(z)$, $0 \le j \le i$, as we will see in Proposition 5.2 later. For proving Proposition 5.2, we show three lemmas.

Lemma 5.2. *For $(\ell_1, \ell_2, \dots, \ell_n) \in \mathbb{Z}^n$, the alternating sum $\mathcal{A}(z_1^{\ell_1} z_2^{\ell_2} \dots z_n^{\ell_n})$ has the following properties*:

$$\mathcal{A}(z_1^{\ell_1} \dots z_i^{\ell_i} \dots z_n^{\ell_n}) = -\ \mathcal{A}(z_1^{\ell_1} \dots z_i^{-\ell_i} \dots z_n^{\ell_n}) \tag{28}$$

$$\mathcal{A}(z_1^{\ell_1} \dots z_i^{\ell_i} \dots z_j^{\ell_j} \dots z_n^{\ell_n}) = -\ \mathcal{A}(z_1^{\ell_1} \dots z_i^{\ell_j} \dots z_j^{\ell_i} \dots z_n^{\ell_n}). \tag{29}$$

Moreover

$$\mathcal{A}(z_1^{\ell_1} \dots z_i^{0} \dots z_n^{\ell_n}) = 0, \tag{30}$$

$$\mathcal{A}(z_1^{\ell_1} \dots z_i^{\ell} \dots z_j^{\ell} \dots z_n^{\ell_n}) = 0. \tag{31}$$

Proof. Eq. (28) and (29) are straightforward from the definition of $\mathcal{A}(z_1^{\ell_1} z_2^{\ell_2} \dots z_n^{\ell_n})$. Eq. (30) and (31) are the consequences of (28) and (29) respectively. □

Lemma 5.3. *Let $\{r_2, r_3, \dots, r_n\}$ be a sequence consisting of $-1, 0$ or 1. For $\{r_2, r_3, \dots, r_n\}$, set $\delta := \#\{i\,;\, 2 \le i \le n,\ r_i = -1,\ r_{i+1} = 1\}$. Let ℓ be an integer satisfying $1 \le \ell \le n$. If $\mathcal{A}\Big(z_1^{\ell}(z_2^{n-1} z_3^{n-2} \dots z_n) z_2^{r_2} z_3^{r_3} \dots z_n^{r_n}\Big) \neq 0$, then $\{r_2, r_3, \dots, r_n\}$ satisfies the following conditions*:

(i) $r_2 = r_3 = \dots = r_{n-\ell+1} = 1$.

(ii) *If there exists i such that $n - \ell + 2 \le i < n$ and $r_i = -1$, then $r_{i+1} = 1$.*

(iii) *If there exists i such that $n - \ell + 2 < i \le n$ and $r_i = 1$, then $r_{i-1} = -1$.*

Conversely if $\{r_2, r_3, \dots, r_n\}$ satisfies the conditions (i), (ii) *and* (iii), *then*

$$\mathcal{A}\Big(z_1^{\ell}(z_2^{n-1} z_3^{n-2} \dots z_n) z_2^{r_2} z_3^{r_3} \dots z_n^{r_n}\Big) = (-1)^{n-\ell+\delta} \mathcal{A}_\rho(z). \tag{32}$$

Moreover, if $\mathcal{A}\Big(z_1^{n+1}(z_2^{n-1} z_3^{n-2} \dots z_n) z_2^{r_2} z_3^{r_3} \dots z_n^{r_n}\Big) \neq 0$, then there exists $j \in \{1, 2, \dots, n\}$ such that $\{r_2, r_3, \dots, r_n\}$ satisfies the following conditions:

(iv) $r_2 = r_3 = \cdots = r_j = 1$.

(v) *If there exists i such that $j+1 \le i < n$ and $r_i = -1$, then $r_{i+1} = 1$.*

(vi) *If there exists i such that $j+1 < i \le n$ and $r_i = 1$, then $r_{i-1} = -1$.*

Conversely if $\{r_2, r_3, \ldots, r_n\}$ satisfies the conditions (iv), (v) *and* (vi), *then*

$$\begin{aligned}\mathcal{A}\Big(z_1^{n+1}(z_2^{n-1}z_3^{n-2}\ldots z_n)z_2^{r_2}z_3^{r_3}\ldots z_n^{r_n}\Big) &= (-1)^{\delta}\mathcal{A}_{(1^j)+\rho}(z) \\ &= (-1)^{\delta}\chi_{(1^j)}(z)\mathcal{A}_{\rho}(z).\end{aligned}$$

Proof. We abbreviate the left-hand side of (32) to A. First we prove the following two claims:

(C1) For $2 \le i < j < n$, if $A \neq 0$ and $r_i = r_{i+1} = \cdots = r_j = -1$, then $r_{j+1} = -1$.

(C2) For $2 < i < j \le n$, if $A \neq 0$ and $r_i = r_{i+1} = \cdots = r_j = 1$, then $r_{i-1} = 1$.

For (C1), if $r_{j+1} = 0$ or 1, using (31), we have $A = \mathcal{A}(\ldots z_j^{n-j}z_{j+1}^{n-j}\ldots) = 0$ or $A = \mathcal{A}(\ldots z_{j-1}^{n-j+1}\ldots z_{j+1}^{n-j+1}\ldots) = 0$ respectively. For (C2), if $r_{i-1} = 0$ or -1, using (31), we also have $A = \mathcal{A}(\ldots z_{i-1}^{n-i+2}z_i^{n-i+2}\ldots) = 0$ or $A = \mathcal{A}(\ldots z_{i-1}^{n-i+1}\ldots z_{i+1}^{n-i+1}\ldots) = 0$ respectively.

We prove the former part of Lemma 5.3. Suppose that $A \neq 0$.

If $r_{n-\ell+1} = 0$, then $A = \mathcal{A}(z_1^{\ell}\ldots z_{n-\ell+1}^{\ell}\ldots) = 0$ by using (31). Thus we have $r_{n-\ell+1} \neq 0$. Assume $r_{n-\ell+1} = -1$. If $r_{n-\ell+2} = 1$ or 0, then $A = \mathcal{A}(z_1^{\ell}\ldots z_{n-\ell+2}^{\ell}\ldots) = 0$ or $A = \mathcal{A}(\ldots z_{n-\ell+1}^{\ell-1}z_{n-\ell+2}^{\ell-1}\ldots) = 0$ respectively. If $r_{n-\ell+2} = -1$, using (C1) repeatedly, we have $r_n = -1$. Then $A = \mathcal{A}(\ \ldots\ z_n^0) = 0$ by (30). Thus we have $r_{n-\ell+1} \neq -1$.

From the above, we obtain $r_{n-\ell+1} = 1$ if $A \neq 0$. If $r_{n-\ell} = -1$ or 0, then $A = \mathcal{A}(z_1^{\ell}\ldots z_{n-\ell}^{\ell}\ldots) = 0$ or $A = \mathcal{A}(\ldots z_{n-\ell}^{\ell+1}z_{n-\ell+1}^{\ell+1}\ldots) = 0$ respectively, Thus we have $r_{n-\ell} = 1$. By repeated use of (C2), it follows that $r_2 = r_3 = \cdots = r_{n-\ell+1} = 1$, which is the condition (i).

Next we prove the condition (ii). Suppose $r_i = -1$ for some i with $n-\ell+2 \le i < n$. If $r_{i+1} = 0$ then $A = \mathcal{A}(\ldots z_i^{n-i}z_{i+1}^{n-i}\ldots) = 0$. If $r_{i+1} = -1$, using (C1) repeatedly, we have $r_n = -1$. Then $A = \mathcal{A}(\ \ldots\ z_n^0) = 0$. Thus we obtain $r_{i+1} = 1$. This is the condition (ii).

We prove the condition (iii). Suppose $r_i = 1$ for some i with $n-\ell+2 < i \le n$. If $r_{i-1} = 0$ then $A = \mathcal{A}(\ldots z_{i-1}^{n-i+2}z_i^{n-i+2}\ldots) = 0$. If $r_{i-1} = 1$, using (C2) repeatedly, we have $r_{n-\ell+2} = 1$. Then $A = \mathcal{A}(z_1^{\ell}\ldots z_{n-\ell+2}^{\ell}\ldots) = 0$. Thus we obtain $r_{i-1} = -1$. This is the condition (iii).

Conversely if $\{r_2, r_3, \ldots, r_n\}$ satisfies the conditions (i), (ii) and (iii), using (29), it follows that

$$\begin{aligned} A &= \mathcal{A}(z_1^{\ell} z_2^{n} z_3^{n-1} \ldots z_{n-\ell+1}^{\ell+1} z_{n-\ell+2}^{\ell-1+r_{n-\ell+2}} \ldots z_n^{1+r_n}) \\ &= (-1)^{n-\ell}\mathcal{A}(z_1^{n} z_2^{n-1} \ldots z_{n-\ell+1}^{\ell} z_{n-\ell+2}^{\ell-1+r_{n-\ell+2}} \ldots z_n^{1+r_n}) \\ &= (-1)^{n-\ell+\delta}\mathcal{A}(z_1^{n} z_2^{n-1} \ldots z_n^{1}) \;=\; (-1)^{n-\ell+\delta}\mathcal{A}_\rho(z). \end{aligned}$$

Thus we complete the proof of the former part. Almost the same argument as above is valid for the latter part of Lemma 5.3, and it is left to readers. $\square$

For $z = (z_1, z_2, \ldots, z_n) \in (\mathbb{C}^*)^n$ and $0 \le k \le n$, we define

$$m_k^{(n)}(z) := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} (z_{i_1} + z_{i_1}^{-1})(z_{i_2} + z_{i_2}^{-1}) \cdots (z_{i_k} + z_{i_k}^{-1}). \tag{33}$$

By definition (10), the polynomials $e_i^{(n)}(z)$ are expanded as a linear combination of the functions $\chi_{(1^j)}(z)$, $0 \le j \le i$. The polynomials $\chi_{(1^i)}(z)$ are expanded as a linear combination of the functions $m_j^{(n)}(z)$, $0 \le j \le i$. Thus the polynomials $e_i^{(n)}(z)$ are written in the form

$$e_i^{(n)}(z) = \sum_{j=0}^{i} E_{ij}\ m_j^{(n)}(z)$$

where $E_{ii} = 1$. For $\widehat{z}_1 = (z_2, z_3, \ldots, z_n) \in (\mathbb{C}^*)^{n-1}$ the polynomials $e_{i-1}^{(n-1)}(\widehat{z}_1)$ are also written as

$$e_{i-1}^{(n-1)}(\widehat{z}_1) = \sum_{j=1}^{i} E'_{ij}\ m_{j-1}^{(n-1)}(\widehat{z}_1). \tag{34}$$

Lemma 5.4. *If $1 \le \ell \le n$, then*

$$\mathcal{A}\Big(z_1^{\ell}(z_2^{n-1} z_3^{n-2} \ldots z_n)\, m_{j-1}^{(n-1)}(\widehat{z}_1)\Big) = c'_{\ell j}\mathcal{A}_\rho(z),$$

where $c'_{\ell j}$'s are integers satisfying $c'_{\ell j} = 0$ if $j \le n-\ell$, and $c'_{\ell, n-\ell+1} = (-1)^{n-\ell}$. Moreover

$$\mathcal{A}\Big(z_1^{n+1}(z_2^{n-1} z_3^{n-2} \ldots z_n)\, m_{j-1}^{(n-1)}(\widehat{z}_1)\Big) = \sum_{k=1}^{j} c''_{jk}\, \chi_{(1^k)}(z)\mathcal{A}_\rho(z),$$

where $c''_{\ell j}$'s are some integers and $c''_{jj} = 1$.

Proof. From the explicit expression (33) of $m_j^{(n)}(z)$, we have

$$\mathcal{A}\Big(z_1^\ell(z_2^{n-1}z_3^{n-2}\ldots z_n)\, m_{j-1}^{(n-1)}(\widehat{z}_1)\Big) = \mathcal{A}\Big(z_1^\ell(z_2^{n-1}z_3^{n-2}\ldots z_n)(z_2z_3\ldots z_j+\cdots)\Big).$$

Using Lemma 5.3 we obtain Lemma 5.4. □

We now state the other expression of the function $\overline{\varphi}_i(z)$.

Proposition 5.2. *The functions $\overline{\varphi}_i(z)$ are expanded as a linear combination of the functions $e_j^{(n)}(z)\mathcal{A}^{(n)}(z)$, $0 \le j \le i$, as follows*:

$$\overline{\varphi}_i(z) = \sum_{j=0}^{i} c_{ij}\, e_j^{(n)}(z)\mathcal{A}^{(n)}(z). \tag{35}$$

Remark 5.2.1. As we will see in Lemma 5.8 later, the coefficients c_{ij}'s except two of them vanish.

Proof. If we set

$$s_k = \sum_{1\le i_1<\cdots<i_k\le 2n+2} a_{i_1}a_{i_2}\ldots a_{i_k},$$

from (22), the function $f(z)-g(z)$ is expanded as

$$\frac{f(z)-g(z)}{z_1^{n+1}} = \sum_{\ell=1}^{n+1} d_\ell\,(z_1^\ell - z_1^{-\ell})$$

where $d_\ell = s_{n+1-\ell} - s_{n+1+\ell}$. Using this and (26), we have

$$\begin{aligned}\mathcal{A}\nabla\varphi_i(z) &= \mathcal{A}\bigg(\frac{f(z)-g(z)}{z_1^{n+1}} z_2^{n-1}z_3^{n-2}\ldots z_n\, e_{i-1}^{(n-1)}(\widehat{z}_1)\bigg)\\ &= \sum_{\ell=1}^{n+1} d_\ell\, \mathcal{A}\Big((z_1^\ell - z_1^{-\ell})z_2^{n-1}z_3^{n-2}\ldots z_n\, e_{i-1}^{(n-1)}(\widehat{z}_1)\Big)\\ &= 2\sum_{\ell=1}^{n+1} d_\ell\, \mathcal{A}\Big(z_1^\ell z_2^{n-1}z_3^{n-2}\ldots z_n\, e_{i-1}^{(n-1)}(\widehat{z}_1)\Big).\end{aligned} \tag{36}$$

From (34) and (36), it follows that

$$\begin{aligned}\overline{\varphi}_i(z) &= \mathcal{A}\nabla\varphi_i(z)/2\\ &= \sum_{\ell=1}^{n+1}\sum_{j=1}^{i} d_\ell E'_{ij}\,\mathcal{A}\Big(z_1^\ell z_2^{n-1} z_3^{n-2} \cdots z_n\, m_{j-1}^{(n-1)}(\widehat{z}_1)\Big)\\ &= \sum_{\ell=1}^{n}\sum_{j=1}^{i} d_\ell E'_{ij}\,\mathcal{A}\Big(z_1^\ell z_2^{n-1} z_3^{n-2} \cdots z_n\, m_{j-1}^{(n-1)}(\widehat{z}_1)\Big)\\ &\quad + \sum_{j=1}^{i} d_{n+1} E'_{ij}\,\mathcal{A}\Big(z_1^{n+1} z_2^{n-1} z_3^{n-2} \cdots z_n\, m_{j-1}^{(n-1)}(\widehat{z}_1)\Big).\end{aligned}$$

Using Lemma 5.4, we have

$$\begin{aligned}\overline{\varphi}_i(z) &= \Big(\sum_{j=1}^{i}\sum_{\ell=1}^{n} d_\ell E'_{ij} c'_{\ell j}\Big)\mathcal{A}_\rho(z) + \sum_{j=1}^{i}\sum_{k=1}^{j} d_{n+1} E'_{ij} c''_{jk}\,\chi_{(1^k)}(z)\mathcal{A}_\rho(z)\\ &= \Big(\sum_{j=1}^{i}\sum_{\ell=1}^{n} d_\ell E'_{ij} c'_{\ell j}\Big)\mathcal{A}_\rho(z) + \sum_{k=1}^{i}\Big(\sum_{j=k}^{i} d_{n+1} E'_{ij} c''_{jk}\Big)\chi_{(1^k)}(z)\mathcal{A}_\rho(z).\end{aligned}$$

This indicates that the functions $\overline{\varphi}_i(z)$ are expanded as a linear combination of the functions $\chi_{(1^j)}(z)\mathcal{A}^{(n)}(z)$, $0 \le j \le i$. From the definition (10), the polynomials $\chi_{(1^i)}(z)$ are also expanded as a linear combination of the polynomials $e_j^{(n)}(z)$, $0 \le j \le i$. Therefore we have the expression (35). This completes the proof. □

Lemma 5.5. *The following holds for $f_k(z)$, $g_k(z)$ and $\zeta_j \in (\mathbb{C}^*)^n$:*

$$\begin{aligned} f_k(\zeta_j) &= 0 \quad \text{if} \quad j \le k \le n,\\ \lim_{x\to 0} f_k(\zeta_j) &= \textstyle\prod_{m=1}^{2n+2} a_m \quad \text{if} \quad k < j \le n+1,\\ g_k(\zeta_j) &= \textstyle\prod_{m=1}^{2n+2}(1 - a_m a_{n-k+1}) \quad \text{if} \quad j \le k \le n,\\ \lim_{x\to 0} g_k(\zeta_j) &= 1 \quad \text{if} \quad k < j \le n+1.\end{aligned}$$

Proof. It is straightforward from (22) and definition (14) of ζ_j. □

Lemma 5.6. *The following holds for $1 \le j \le i+1$:*

$$\lim_{x\to 0}\Big[\Big(\prod_{l=1}^{j-1} z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\Big]_{z=\zeta_j} = (-1)^{j-1} c_{i,j-1}\mathcal{A}^{(n-j+1)}(a_{n-j+1},\ldots,a_1). \tag{37}$$

Proof. From (35), it follows that

$$\Big(\prod_{l=1}^{j-1} z_l^{n-l+2}\Big)\overline{\varphi}_i(z)=\sum_{k=0}^{i} c_{ik}\Big(z_1z_2\dots z_{j-1}\, e_k^{(n)}(z)\Big)\Big(z_1^n z_2^{n-1}\dots z_{j-1}^{n-j+2}\,\mathcal{A}^{(n)}(z)\Big).$$

Put

$$z=\zeta_j=(\underbrace{x^{j-1},x^{j-2},\dots,x}_{j-1},\underbrace{a_{n-j+1},a_{n-j},\dots,a_2,a_1}_{n-j+1}). \tag{38}$$

Since $e_k^{(n)}(\zeta_j)=0$ if $j\le k$ by Corollary 4.1, we have

$$\begin{aligned}\Big[\Big(\prod_{l=1}^{j-1} z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\Big]_{z=\zeta_j}\\ =\sum_{k=0}^{j-1} c_{ik}\Big[\Big(z_1z_2\dots z_{j-1}\, e_k^{(n)}(z)\Big)\Big(z_1^n z_2^{n-1}\dots z_{j-1}^{n-j+2}\,\mathcal{A}^{(n)}(z)\Big)\Big]_{z=\zeta_j}.\end{aligned} \tag{39}$$

From the definition (10) of $e_k^{(n)}(z)$ and the explicit expression (38) of ζ_j, we have

$$\lim_{x\to 0}\Big[\Big(z_1z_2\dots z_{j-1}\, e_k^{(n)}(z)\Big)\Big]_{z=\zeta_j}=\begin{cases}0 & \text{if } \ k<j-1,\\ 1 & \text{if } \ k=j-1.\end{cases} \tag{40}$$

From Weyl's denominator formula (3) and the expression (38) of ζ_j, it follows

$$\lim_{x\to 0}\Big[\Big(z_1^n z_2^{n-1}\dots z_{j-1}^{n-j+2}\,\mathcal{A}^{(n)}(z)\Big)\Big]_{z=\zeta_j}=(-1)^{j-1}\mathcal{A}^{(n-j+1)}(a_{n-j+1},\dots,a_1). \tag{41}$$

Taking the limit $x\to 0$ in both sides of (39) and using (40) and (41), we obtain (37). This completes the proof. □

Lemma 5.7. *The following holds for the point $\zeta_j\in(\mathbb{C}^*)^n$, $j\le i$:*

$$\Big[e_{i-1}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_j}=0 \quad \textit{if } \ 1\le k<i.$$

Proof. It is straightforward from (14) and Lemma 4.2. □

Lemma 5.8. *The coefficient c_{ij} in* (35) *vanishes if $1\le j\le i-1$. In more details, $\overline{\varphi}_i(z)$ is expanded as*

$$\overline{\varphi}_i(z)=\Big(c_{ii}e_i^{(n)}(z)+c_{i0}e_0^{(n)}(z)\Big)\mathcal{A}^{(n)}(z).$$

In particular, if $i=n$, the constant c_{n0} is evaluated as

$$c_{n0}=(-1)^{n-1}a_1^{-n}\textstyle\prod_{m=2}^{2n+2}(1-a_1a_m). \tag{42}$$

Proof. From Lemma 5.6, in order to prove $c_{ij}=0$ for $1\le j\le i-1$, it is sufficient to show that

$$\lim_{x\to 0}\Big[\Big(\prod_{l=1}^{j-1} z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\Big]_{z=\zeta_j}=0 \quad \text{if} \quad 2\le j\le i. \tag{43}$$

We now suppose $1\le j\le i$. By Lemma 5.7, if $j\le i$, then

$$\Big[e_{i-1}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_j}=0 \quad \text{if} \quad 1\le k<i.$$

Since the summand of $\overline{\varphi}_i(z)$ in the form (25) has the factor $e_{i-1}^{(n-1)}(\widehat{z}_k)$, if we put $z=\zeta_j$, then we have

$$\overline{\varphi}_i(\zeta_j)=\Big[\sum_{k=i}^{n}(-1)^{k+1}\frac{f_k(z)-g_k(z)}{z_k^{n+1}}e_{i-1}^{(n-1)}(\widehat{z}_k)\mathcal{A}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_j}.$$

Since $f_k(\zeta_j)=0$ if $1\le j\le k\le n$ by Lemma 5.5, we have

$$\overline{\varphi}_i(\zeta_j)=\Big[\sum_{k=i}^{n}(-1)^{k}\frac{g_k(z)}{z_k^{n+1}}e_{i-1}^{(n-1)}(\widehat{z}_k)\mathcal{A}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_j}. \tag{44}$$

Thus, if $1\le j\le i$, we have

$$\begin{aligned}
&\Big[\Big(\prod_{l=1}^{j-1} z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\Big]_{z=\zeta_j}\\
&\quad=\sum_{k=i}^{n}(-1)^k\Big[\frac{g_k(z)}{z_k^{n+1}}\Big(z_1^2z_2^2\dots z_{j-1}^2e_{i-1}^{(n-1)}(\widehat{z}_k)\Big)\\
&\qquad\qquad\times\Big(z_1^{n-1}z_2^{n-2}\dots z_{j-1}^{n-j+1}\mathcal{A}^{(n-1)}(\widehat{z}_k)\Big)\Big]_{z=\zeta_j}.
\end{aligned} \tag{45}$$

Since $g_k(\zeta_j)$ is constant if $j\le k\le n$ by Lemma 5.5, for $i\le k\le n$,

$$\lim_{x\to 0}\Big[\frac{g_k(z)}{z_k^{n+1}}\Big]_{z=\zeta_j} \tag{46}$$

is also constant. For $i\le k\le n$,

$$\lim_{x\to 0}\Big[z_1^2z_2^2\dots z_{j-1}^2e_{i-1}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_j}=0 \quad \text{if} \quad 2\le j\le i. \tag{47}$$

If $i\le k\le n$, from (38) and Weyl's denominator formula (3), it follows that

$$\begin{aligned}(48)\quad &\lim_{x\to 0}\Big[z_1^{n-1}z_2^{n-2}\dots z_{j-1}^{n-j+1}\,\mathcal{A}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_j}\\ &= (-1)^{j-1}\mathcal{A}^{(n-j)}(a_{n-j+1},\dots,a_{n-k+2},a_{n-k},\dots,a_1)\quad \text{if}\quad 1\le j\le i.\end{aligned}$$

Combining (45), (46), (47) and (48), we obtain (43).

Next we evaluate the coefficient c_{n0}. Putting $i=n$ and $j=1$ in (44), we have

$$\overline{\varphi}_n(\zeta_1) = (-1)^n\Big[\frac{g_n(z)}{z_n^{n+1}}e_{n-1}^{(n-1)}(\widehat{z}_n)\mathcal{A}^{(n-1)}(\widehat{z}_n)\Big]_{z=\zeta_1}. \tag{49}$$

From (13), it follows that

$$e_{n-1}^{(n-1)}(\widehat{z}_n)\mathcal{A}^{(n-1)}(\widehat{z}_n) = \frac{\mathcal{A}^{(n)}(z_1,z_2,\dots,z_{n-1},a_1)}{\mathcal{A}^{(1)}(a_1)},$$

so that

$$\Big[e_{n-1}^{(n-1)}(\widehat{z}_n)\mathcal{A}^{(n-1)}(\widehat{z}_n)\Big]_{z=\zeta_1} = \frac{\mathcal{A}^{(n)}(\zeta_1)}{\mathcal{A}^{(1)}(a_1)}.$$

Thus we have

$$\overline{\varphi}_n(\zeta_1) = (-1)^n\frac{g_n(\zeta_1)}{a_1^{n+1}}\frac{\mathcal{A}^{(n)}(\zeta_1)}{\mathcal{A}^{(1)}(a_1)}. \tag{50}$$

On the other hand, from Lemma 5.6 we have

$$\overline{\varphi}_n(\zeta_1) = c_{n0}\,\mathcal{A}^{(n)}(\zeta_1). \tag{51}$$

Comparing (50) and (51) we obtain

$$c_{n0} = \frac{(-1)^n g_n(\zeta_1)}{a_1^{n+1}\mathcal{A}^{(1)}(a_1)}.$$

Since $g_n(\zeta_1)=\prod_{m=1}^{2n+2}(1-a_1a_m)$ by Lemma 5.5, using $\mathcal{A}^{(1)}(a_1) = -a_1^{-1}(1-a_1^2)$ we obtain (42). This completes the proof. □

Lemma 5.9. *The coefficient c_{ii} in* (35) *is evaluated as*

$$c_{ii} = 1 - a_1a_2\dots a_{2n+2}.$$

Proof. If we put

$$z=\zeta_{i+1} = (\underbrace{x^i,x^{i-1},\dots,x}_{i},\underbrace{a_{n-i},a_{n-i-1},\dots,a_2,a_1}_{n-i}) \tag{52}$$

in (25), then

$$\overline{\varphi}_i(\zeta_{i+1}) = \Big[\sum_{k=1}^{n}(-1)^{k+1}\frac{f_k(z)-g_k(z)}{z_k^{n+1}}e_{i-1}^{(n-1)}(\widehat{z}_k)\mathcal{A}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_{i+1}}.$$

Thus it follows that

$$\Big[\Big(\prod_{l=1}^{i} z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\Big]_{z=\zeta_{i+1}} = S_1(\zeta_{i+1}) + S_2(\zeta_{i+1})$$

where $S_1(z)$ and $S_2(z)$ are functions defined by the following:

$$\begin{aligned} S_1(z) := & \sum_{k=1}^{i}(-1)^{k+1}\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}\Big(f_k(z)-g_k(z)\Big) \\ & \times\Big(z_1 z_2\ldots z_{k-1}\, z_{k+1}\ldots z_i\, e_{i-1}^{(n-1)}(\widehat{z}_k)\Big) \\ & \times\Big(z_1^{n-1}z_2^{n-2}\ldots z_{k-1}^{n-k+1}z_{k+1}^{n-k}\ldots z_i^{n-i+1}\,\mathcal{A}^{(n-1)}(\widehat{z}_k)\Big), \end{aligned} \tag{53}$$

$$\begin{aligned} S_2(z) := & \sum_{k=i+1}^{n}(-1)^{k+1}\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{i-1}}{z_k}\Big(\frac{z_i}{z_k}\Big)^2\Big(\frac{f_k(z)-g_k(z)}{z_k^{n-i}}\Big) \\ & \times\Big(z_1 z_2\ldots z_{i-1}\, e_{i-1}^{(n-1)}(\widehat{z}_k)\Big) \\ & \times\Big(z_1^{n-1}z_2^{n-2}\ldots z_i^{n-i}\,\mathcal{A}^{(n-1)}(\widehat{z}_k)\Big). \end{aligned} \tag{54}$$

We show $\lim_{x\to 0} S_2(\zeta_{i+1}) = 0$ first. If $i < k \le n$, from Lemma 5.5,

$$\Big[\frac{f_k(z)-g_k(z)}{z_k^{n-i}}\Big]_{z=\zeta_{i+1}} \tag{55}$$

is constant. If $i < k \le n$, from the explicit form (52) of ζ_{i+1} and the definition (10) of $e_i^{(n)}(z)$, we have

$$\lim_{x\to 0}\Big[\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{i-1}}{z_k}\Big(\frac{z_i}{z_k}\Big)^2\Big]_{z=\zeta_{i+1}} = 0 \tag{56}$$

and

$$\lim_{x\to 0}\Big[z_1 z_2\ldots z_{i-1}\, e_{i-1}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_{i+1}} = 1. \tag{57}$$

If $i < k \le n$, we also have

$$\begin{aligned} & \lim_{x\to 0}\Big[z_1^{n-1}z_2^{n-2}\ldots z_i^{n-i}\,\mathcal{A}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_{i+1}} \\ & = (-1)^i\mathcal{A}^{(n-i-1)}(a_{n-i},\ldots,a_{n-k+2},a_{n-k},\ldots,a_1) \end{aligned} \tag{58}$$

by using (52) and Weyl's denominator formula (3). From (54), (55), (56), (57) and (58), we obtain $\lim_{x\to 0} S_2(\zeta_{i+1})=0$.

Next, we evaluate $\lim_{x\to 0} S_1(\zeta_{i+1})$. If $1\le k\le i$, from the explicit form (52) of ζ_{i+1} and the definition (10) of $e_i^{(n)}(z)$, we have

$$\lim_{x\to 0}\Big[\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}\Big]_{z=\zeta_{i+1}}=\begin{cases}1 & \text{if} \quad k=1\\ 0 & \text{if} \quad 2\le k\le i\end{cases}. \tag{59}$$

and

$$\lim_{x\to 0}\Big[z_1z_2\ldots z_{k-1}\,z_{k+1}\ldots z_i\,e_{i-1}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_{i+1}}=1. \tag{60}$$

If $1\le k\le i$, we also have

$$\begin{aligned}(61)\qquad &\lim_{x\to 0}\Big[z_1^{n-1}z_2^{n-2}\ldots z_{k-1}^{n-k+1}z_{k+1}^{n-k}\ldots z_i^{n-i+1}\,\mathcal{A}^{(n-1)}(\widehat{z}_k)\Big]_{z=\zeta_{i+1}}\\ &=\ (-1)^{i-1}\mathcal{A}^{(n-i)}(a_{n-i},\ldots,a_2,a_1)\end{aligned}$$

by using (52) and Weyl's denominator formula (3). Thus, from (53), (59), (60) and (61), we have

$$\begin{aligned}(62)\qquad \lim_{x\to 0}\Big[\Big(\prod_{l=1}^{i}z_l^{n-l+2}\Big)\overline{\varphi}_i(z)\Big]_{z=\zeta_{i+1}} &= \lim_{x\to 0}S_1(\zeta_{i+1})\\ =(-1)^{i-1}\mathcal{A}^{(n-i)}(a_{n-i},\ldots,a_2,a_1)&\lim_{x\to 0}\Big[f_1(z)-g_1(z)\Big]_{z=\zeta_{i+1}}.\end{aligned}$$

Comparing (37) with (62) and using Lemma 5.5, we obtain

$$c_{ii}=-\lim_{x\to 0}\Big[f_1(z)-g_1(z)\Big]_{z=\zeta_{i+1}}=1-a_1a_2\ldots a_{2n+2},$$

which completes the proof. □

Theorem 5.1. *For $1\le i\le n$, the following relation holds between $e_i^{(n)}(z)$ and $e_0^{(n)}(z)$:*

$$\int_0^{\xi\infty}e_i^{(n)}(z)\Phi_{\mathrm{G}}(z)\mathcal{A}^{(n)}(z)\varpi_q=-\frac{c_{i0}}{c_{ii}}\int_0^{\xi\infty}e_0^{(n)}(z)\Phi_{\mathrm{G}}(z)\mathcal{A}^{(n)}(z)\varpi_q, \tag{63}$$

In particular, if i=n, then

$$-\frac{c_{n0}}{c_{nn}}=(-a_1)^{-n}\frac{\prod_{k=2}^{2n+2}(1-a_1a_k)}{1-a_1a_2\ldots a_{2n+2}}.$$

Remark 5.1.1. In other words, from the definition (4) Theorem 5.1 is equivalent to Theorem 1.1.

Proof. Since $\int_0^{\xi\infty} \Phi_{\mathrm{G}}(z)\,\overline{\varphi}_i(z)\,\varpi_q = 0$ by (20) in Lemma 5.1, from Lemma 5.8, it follows that

$$\int_0^{\xi\infty} \Phi_{\mathrm{G}}(z)\Big(c_{ii}\, e_i^{(n)}(z) + c_{i0}\, e_0^{(n)}(z)\Big)\mathcal{A}^{(n)}(z)\varpi_q = 0.$$

We therefore obtain (63). The evaluation of the coefficient $-c_{n0}/c_{nn}$ is given by Lemma 5.8 and 5.9. The proof is now complete. □

§6. Product Formula

The aim of this section is to prove Gustafson's C_n-type formula (1).

Theorem 6.1 (Gustafson)**.** *The constant C in the expression* (9) *is the following*:

$$C = (1-q)^n (q)_\infty^n \frac{\prod_{1\le i<j\le 2n+2}(qa_i^{-1}a_j^{-1})_\infty}{(qa_1^{-1}a_2^{-1}\dots a_{2n+2}^{-1})_\infty}.$$

Before proving Theorem 6.1, we have to establish q-difference equations and asymptotic behavior for the Jackson integral of Gustafson's C_n-type.

§6.1. q-difference Equations

First we deduce a recurrence relation of $J_{\mathrm{G}}(\xi)$ from Theorem 1.1.

Corollary 6.1. *Let T_{a_1} be the q-shift of parameter a_1 such that T_{a_1} : $a_1 \to qa_1$. Then*

$$T_{a_1} J_{\mathrm{G}}(\xi) = (-a_1)^{-n} \frac{\prod_{k=2}^{2n+2}(1-a_1a_k)}{1-a_1a_2\dots a_{2n+2}} J_{\mathrm{G}}(\xi).$$

Remark 6.1.1. The parameters $a_1, a_2, \dots, a_{2n+2}$ can be replaced symmetrically in the above equation.

Remark 6.1.2. The shift of the parameter $a_j \to qa_j$ is equivalent to that of $\alpha_j \to \alpha_j + 1$.

Proof. The function $T_{a_1} J_{\mathrm{G}}(\xi)$ is written

$$T_{a_1} J_{\mathrm{G}}(\xi) = \int_0^{\xi\infty} \frac{T_{a_1}\Phi_{\mathrm{G}}(z)}{\Phi_{\mathrm{G}}(z)} \Phi_{\mathrm{G}}(z)\Delta(z)\varpi_q = \int_0^{\xi\infty} e_n'(z)\Phi_{\mathrm{G}}(z)\Delta(z)\varpi_q,$$

because the following holds for $\Phi_{\mathrm{G}}(z)$ by Lemma 4.1:

$$\frac{T_{a_1}\Phi_{\mathrm{G}}(z)}{\Phi_{\mathrm{G}}(z)}=\prod_{i=1}^{n}\frac{(a_1-z_i)(1-a_1z_i)}{a_1z_i}=e'_n(z).$$

Moreover, from Theorem 1.1 we have

$$\int_0^{\xi\infty}e'_n(z)\Phi_{\mathrm{G}}(z)\Delta(z)\varpi_q=(-a_1)^{-n}\prod_{j=1}^{n}\frac{\prod_{k=2}^{2n+2}(1-a_1a_k)}{1-a_1a_2\dots a_{2n+2}}J_{\mathrm{G}}(\xi).$$

This completes the proof. □

Let T^N be the q-shift of parameters for the special direction defined by

$$T^N:\quad {}_n\begin{cases}a_1\to a_1q^{(n+1)N},\\ a_2\to a_2q^{(n+1)N},\\ \quad\vdots\\ a_n\to a_nq^{(n+1)N},\end{cases}\qquad {}_{n+2}\begin{cases}a_{n+1}\to a_{n+1}q^{-nN},\\ a_{n+2}\to a_{n+2}q^{-nN},\\ \quad\vdots\\ a_{2n+2}\to a_{2n+2}q^{-nN}.\end{cases}$$

Lemma 6.1. *The following holds for the shift T^N:*

$$\begin{aligned}J_{\mathrm{G}}(\xi)=T^NJ_{\mathrm{G}}(\xi)\;&\frac{(a_1a_2\dots a_n)^{n^2N}(a_{n+1}a_{n+2}\dots a_{2n+2})^{n(n+1)N}}{q^{n(n+1)(1+n+2nN)N/2}\;(qa_1^{-1}a_2^{-1}\cdots a_{2n+2}^{-1})_{nN}}\\ &\times\frac{\prod_{n+1\le\mu<\nu\le 2n+2}(qa_\mu^{-1}a_\nu^{-1})_{2nN}}{\prod_{1\le j<k\le n}(a_ja_k)_{2(n+1)N}\;\prod_{i=1}^{n}\prod_{m=n+1}^{2n+2}(a_ia_m)_N}.\end{aligned}$$

Proof. Applying Corollary 6.1 to $J_{\mathrm{G}}(\xi)$ repeatedly, we obtain the above relation between $J_{\mathrm{G}}(\xi)$ and $T^NJ_{\mathrm{G}}(\xi)$. □

§6.2. Asymptotic Behavior of Truncated Jackson Integral

Next we consider an asymptotic behavior of $J_{\mathrm{G}}(\zeta)$.

Lemma 6.2. *The asymptotic behavior of the truncated Jackson integral $T^NJ_{\mathrm{G}}(\zeta)$ at $N\to+\infty$ is the following:*

$$\begin{aligned}T^NJ_{\mathrm{G}}(\zeta)\sim&\frac{(1-q)^n(q)_\infty^n\,q^{n(n+1)(1+n+2nN)N/2}}{(a_1a_2\dots a_n)^{n^2N}(a_{n+1}a_{n+2}\dots a_{2n+2})^{n(n+1)N}}\\ &\times\prod_{i=1}^{n}a_i^{i-\alpha_1-\alpha_2-\cdots-\alpha_{2n+2}}\prod_{1\le j<k\le n}\theta(a_j/a_k).\end{aligned}$$

Proof. We divide $\Phi_{\mathrm{G}}(z)\Delta(z)$ into the following three parts:

$$\Phi_{\mathrm{G}}(z)\Delta(z)=I_1(z)\,I_2(z)\,I_3(z)$$

where

$$\begin{aligned}
I_1(z)&=\prod_{i=1}^{n} z_i^{1-\alpha_1-\alpha_2-\cdots-\alpha_{2n+2}}\prod_{1\le j<k\le n}(z_k-z_j),\\
I_2(z)&=\prod_{i=1}^{n}\prod_{m=1}^{n}(qa_m^{-1}z_i)_\infty,\\
I_3(z)&=\prod_{i=1}^{n}(1-z_i^2)\bigg(\prod_{\ell=1}^{n}\frac{1}{(a_\ell z_i)_\infty}\bigg)\bigg(\prod_{m=n+1}^{2n+2}\frac{(qa_m^{-1}z_i)_\infty}{(a_m z_i)_\infty}\bigg)\\
&\quad\times\prod_{1\le j<k\le n}(1-z_jz_k).
\end{aligned}$$

$T^N J_{\mathrm{G}}(\zeta)$ is expressed as

$$\begin{aligned}
T^N J_{\mathrm{G}}(\zeta)&=(1-q)^n\sum_{\nu\in D}T^N\Big(\Phi_{\mathrm{G}}(q^\nu\zeta)\Delta(q^\nu\zeta)\Big)\\
&=(1-q)^n\sum_{\nu\in D}T^N I_1(q^\nu\zeta)\,T^N I_2(q^\nu\zeta)\,T^N I_3(q^\nu\zeta)
\end{aligned}\tag{64}$$

where

$$\begin{aligned}
T^N I_1(q^\nu\zeta)&=\prod_{i=1}^{n}(a_iq^{\nu_i+(n+1)N})^{1-\alpha_1-\alpha_2-\cdots-\alpha_{2n+2}+nN}\\
&\quad\times\prod_{1\le j<k\le n}(a_kq^{\nu_k+(n+1)N}-a_jq^{\nu_j+(n+1)N}),
\end{aligned}$$

$$\begin{aligned}
T^N I_2(q^\nu\zeta)&=\prod_{i=1}^{n}\prod_{m=1}^{n}(qa_m^{-1}a_iq^{\nu_i})_\infty,\\
T^N I_3(q^\nu\zeta)&=\Bigg[\prod_{i=1}^{n}(1-a_i^2q^{2\nu_i+2(n+1)N})\bigg(\prod_{\ell=1}^{n}\frac{1}{(a_\ell a_iq^{\nu_i+2(n+1)N})_\infty}\bigg)\\
&\quad\times\bigg(\prod_{m=n+1}^{2n+2}\frac{(qa_m^{-1}a_iq^{\nu_i+(2n+1)N})_\infty}{(a_ma_iq^{\nu_i+N})_\infty}\bigg)\Bigg]\\
&\quad\times\prod_{1\le j<k\le n}(1-a_ja_kq^{\nu_j+\nu_k+2(n+1)N}).
\end{aligned}$$

Eq. (64) indicates that the summand $T^N\Big(\Phi_{\rm G}(q^\nu\zeta)\Delta(q^\nu\zeta)\Big)$ of $T^N J_{\rm G}(\zeta)$ corresponding to $\nu=(0,0,\dots,0)\in D$ gives the principal term of the asymptotic behavior of $T^N J_{\rm G}(\zeta)$ at $N\to+\infty$ because the point $(0,0,\dots,0)\in D$ is the vertex of the cone D. Hence we have

$$T^N J_{\rm G}(\zeta)\sim(1-q)^n T^N I_1(\zeta)\,T^N I_2(\zeta)\,T^N I_3(\zeta). \tag{65}$$

Moreover the asymptotic behavior of each $T^N I_i(\zeta)$ at $N\to+\infty$ is the following:

$$\begin{aligned}
T^N I_1(\zeta) &= \prod_{i=1}^n (a_i q^{(n+1)N})^{1-\alpha_1-\alpha_2-\cdots-\alpha_{2n+2}+nN}\\
&\quad\times\prod_{1\le j<k\le n}(a_k q^{(n+1)N}-a_j q^{(n+1)N}),\\
&=\frac{(a_1a_2\dots a_n)^{1-\alpha_1-\alpha_2-\cdots-\alpha_{2n+2}}q^{n(n+1)(1+nN)N}}{(a_1a_2\dots a_n)^{n^2N}(a_{n+1}a_{n+2}\dots a_{2n+2})^{n(n+1)N}}\\
&\quad\times q^{(n-1)n(n+1)N/2}\prod_{1\le j<k\le n}(a_k-a_j)\\
&=\prod_{i=1}^n a_i^{i-\alpha_1-\alpha_2-\cdots-\alpha_{2n+2}}\prod_{1\le j<k\le n}(1-a_j/a_k)\\
&\quad\times\frac{q^{n(n+1)(1+nN)N+(n-1)n(n+1)N/2}}{(a_1a_2\dots a_n)^{n^2N}(a_{n+1}a_{n+2}\dots a_{2n+2})^{n(n+1)N}}\\
&=\prod_{i=1}^n a_i^{i-\alpha_1-\alpha_2-\cdots-\alpha_{2n+2}}\prod_{1\le j<k\le n}(1-a_j/a_k)\\
&\quad\times\frac{q^{n(n+1)(1+n+2nN)N/2}}{(a_1a_2\dots a_n)^{n^2N}(a_{n+1}a_{n+2}\dots a_{2n+2})^{n(n+1)N}},
\end{aligned}\tag{66}$$

$$\begin{aligned}
T^N I_2(\zeta) &= \prod_{i=1}^n\prod_{m=1}^n (qa_m^{-1}a_i)_\infty=(q)_\infty^n\prod_{1\le j<k\le n}(qa_k^{-1}a_j)_\infty(qa_j^{-1}a_k)_\infty\\
&=(q)_\infty^n\prod_{1\le j<k\le n}\frac{\theta(a_j/a_k)}{1-a_j/a_k},
\end{aligned}\tag{67}$$

$$\begin{aligned}
T^N I_3(\zeta) &= \left[\prod_{i=1}^n(1-a_i^2q^{2(n+1)N})\left(\prod_{\ell=1}^n\frac{1}{(a_\ell a_i q^{2(n+1)N})_\infty}\right)\right.\\
&\qquad\left.\times\left(\prod_{m=n+1}^{2n+2}\frac{(a_m^{-1}a_iq^{1+(2n+1)N})_\infty}{(a_ma_iq^N)_\infty}\right)\right]\\
&\quad\times\prod_{1\le j<k\le n}(1-a_ja_kq^{2(n+1)N})\\
&\quad\sim 1\quad(N\to+\infty).
\end{aligned}\tag{68}$$

Combining (65), (66), (67) and (68), we obtain Lemma 6.2. □

§6.3. Proof of Theorem 6.1

Theorem 6.2. *The truncated Jackson integral $J_{\mathrm{G}}(\zeta)$ is evaluated as*

$$J_{\mathrm{G}}(\zeta) = (1-q)^n (q)_\infty^n \times \frac{\prod_{n+1\le\mu<\nu\le 2n+2}(qa_\mu^{-1}a_\nu^{-1})_\infty}{(qa_1^{-1}a_2^{-1}\cdots a_{2n+2}^{-1})_\infty} \prod_{i=1}^{n} \frac{a_i^{i-\alpha_1-\cdots-\alpha_{2n+2}}}{\prod_{m=n+1}^{2n+2}(a_i a_m)_\infty} \prod_{1\le j<k\le n} \frac{\theta(a_j/a_k)}{(a_j a_k)_\infty}.$$

Proof. It is straightforward from Lemma 6.1 and Lemma 6.2. □

As a consequence of Theorem 6.2, we deduce Theorem 6.1.

Proof of Theorem 6.1. The constant C is written $C = J_{\mathrm{G}}(\xi)/\Theta_{\mathrm{G}}(\xi)$ by virtue of Lemma 3.1. In particular, putting $\xi = \zeta$, from Theorem 6.2 we obtain

$$C = \frac{J_{\mathrm{G}}(\zeta)}{\Theta_{\mathrm{G}}(\zeta)} = (1-q)^n (q)_\infty^n \frac{\prod_{1\le i<j\le 2n+2}(qa_i^{-1}a_j^{-1})_\infty}{(qa_1^{-1}a_2^{-1}\cdots a_{2n+2}^{-1})_\infty},$$

because $\Theta_{\mathrm{G}}(\xi)$ in (8) is evaluated at $\xi = \zeta$ as

$$\Theta_{\mathrm{G}}(\zeta) = \prod_{i=1}^{n} \frac{a_i^{i-\alpha_1-\alpha_2-\cdots-\alpha_{2n+2}}}{\prod_{m=n+1}^{2n+2}\theta(a_m a_i)} \prod_{1\le j<k\le n} \frac{\theta(a_j/a_k)}{\theta(a_j a_k)}.$$

The proof of Theorem 6.1 is now complete. □

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