Radon Transforms of Constructible Functions on Grassmann Manifolds

By

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Abstract

P. Schapira studied Radon transforms of constructible functions and obtained a formula related to an inversion formula. We generalize this formula to more complicated cases including Radon transformations between any Grassmann manifolds. In particular, we give an inversion formula for the Radon transformation and characterize images of Radon transforms of characteristic functions of Schubert cells.

§1.   Introduction

A constructible function ϕ on a real analytic or complex manifold X is a $\mathbb{Z}$-valued function which is constant along a stratification. We can choose a stratification according to the problem under consideration, so we work with subanalytic stratifications here.

In [11], P. Schapira defined Radon transforms of constructible functions. This is a kind of integral transformations. We consider the following diagram:

$$
\begin{array}{ccc}
 & S & \\
 f \swarrow & & \searrow g \\
X & & Y.
\end{array}
$$

Here X and Y are real analytic or complex manifolds, S is a locally closed subanalytic subset of $X \times Y$, and f and g are real or complex analytic maps, respectively. Then we can define the Radon transform $\mathcal{R}_S(\phi)$ of a constructible function ϕ on X by

$$
\mathcal{R}_S(\phi) = \int_g f^* \phi.
$$

In [11], P. Schapira obtained a formula for $\mathcal{R}_S$ in general situation. This formula gives an inversion formula for Radon transforms of constructible functions from a real projective space to its dual in the case where the whole dimension is odd. Here inversion means left inverse. We can, that is, reconstruct a constructible function ϕ on the projective space from its Radon transform $\mathcal{R}_S(\phi)$. As a result, we can reconstruct the original subanalytic set K from the knowledge of the topological Euler numbers $\chi(K \cap H)$ for all affine hyperplanes H.

Many mathematicians have been working on topological Radon transformations. Topological Radon transforms of constructible sheaves were dealt with in [2] and topological Radon transforms of constructible functions were dealt with in [3], [4], [13]. They considered Radon transformations from a projective space to a Grassmannian and obtained many results.

In this paper, we study these topological Radon transforms of constructible functions from $X = F_{n+1}(p)$ to $Y = F_{n+1}(q)$. We denote it by $\mathcal{R}_{(n+1;p,q)}$. Here $F_{n+1}(p)$ is the Grassmann manifold, that is, the set of all p dimensional subspaces in an $n+1$ dimensional vector space. Moreover, we consider not only real cases but also complex cases. We obtain the following main results.

First we prove an inversion formula for the Radon transformation $\mathcal{R}_{(n+1;p,q)}$ (Theorem 3.1 in this paper).

Theorem 1.1. *We consider the case where $p < q$. We obtain an inversion formula for $\mathcal{R}_{(n+1;p,q)}$ if either one of the following conditions are satisfied;*

(i) *$p + q \leq n + 1$ under the complex Grassmann case,*

(ii) *$p + q \leq n + 1$ and $q - p$ is even under the real Grassmann case.*

We concretely construct an inversion transformation $\mathcal{R}^{-1}$ as a left inverse in Section 3.2. We remark that the assumption (2.4) of Schapira's formula is not satisfied in the case where p is not equal to 1. So our situation is more general. Moreover we prove the following inverse theorem (Theorem 4.1):

Theorem 1.2. *In the case where $p + q = n + 1$, an inversion formula obtained in Theorem 1.1 is the inverse formula for $\mathcal{R}_{(n+1;p,q)}$.*

Namely, we show that $\mathcal{R}^{-1}$ is also right inverse.

Second we characterize the images of Radon transforms of characteristic functions of Schubert cells of the Grassmannian (Theorem 5.1).

Theorem 1.3. *Let $\alpha \in \Lambda_{p,n-p}$.*

(i) *In the complex case, we have*

$$
\mathcal{R}_{(n+1;p,q)}(\mathbf{1}_{\Omega_\alpha^\circ}) =
\begin{cases}
\displaystyle\sum_{\hat{\alpha} \subset \hat{\beta}} \mathbf{1}_{\Omega_\beta^\circ} & \text{for } p \leq q, \\[2em]
\displaystyle\sum_{\hat{\alpha} \supset \hat{\beta}} \mathbf{1}_{\Omega_\beta^\circ} & \text{for } p \geq q,
\end{cases}
$$

where $\hat{\beta}$ ranges through sequences in $\Lambda_{q,n}$ containing (or contained by) $\hat{\alpha}$.

(ii) *In the real case, we have*

$$
\mathcal{R}_{(n+1;p,q)}(\mathbf{1}_{\Omega_\alpha^\circ}) =
\begin{cases}
\displaystyle\sum_{\hat{\alpha} \subset \hat{\beta}} (-1)^{c_{\hat{\alpha},\hat{\beta}}} \mathbf{1}_{\Omega_\beta^\circ} & \text{for } p \leq q, \\[2em]
\displaystyle\sum_{\hat{\alpha} \supset \hat{\beta}} (-1)^{c_{\hat{\beta},\hat{\alpha}}} \mathbf{1}_{\Omega_\beta^\circ} & \text{for } p \geq q,
\end{cases}
$$

where $\hat{\beta}$ ranges through sequences in $\Lambda_{q,n}$ containing (or contained by) $\hat{\alpha}$.

Here $\Lambda_{p,q}$ is the set of the complement of Young diagrams and Ω_α° is the Schubert cell corresponding to $\alpha \in \Lambda_{p,q}$. For precise definitions of the sequence $\hat{\alpha}$ and the constant $c_{\hat{\alpha},\hat{\beta}}$, see Section 5; the sequence $\hat{\alpha}$ depends on $\alpha \in \Lambda_{p,q}$ and the constant $c_{\hat{\alpha},\hat{\beta}}$ only depends on $\alpha, \beta \in \Lambda_{p,q}$.

The plan of this paper is as follows.

We first recall basic properties of Grassmann manifolds, constructible functions and Schapira's formula in general case in Section 2.

In Section 3, we construct an inversion formula for $\mathcal{R}_{(n+1;p,q)}$ in each Grassmann case. In Section 3.1, we modify Schapira's formula in general case under the almost same assumptions as Schapira's. This gives an inversion formula for the Radon transformation $\mathcal{R}_S$ (Proposition 3.2). We can apply this formula to the Radon transformation $\mathcal{R}_{(n+1;1,q)}$ (Proposition 3.3). Moreover, in Section 3.2 we concretely construct an inversion transformation $\mathcal{R}^{-1}$ for $\mathcal{R}_{(n+1;p,q)}$ ($p \neq 1$) by modifying the kernel function of this inversion transformation under suitable conditions of p and q (Theorem 3.1).

In Section 4, we prove that an inversion transformation $\mathcal{R}^{-1}$ constructed as a left inverse in Section 3.2 is right inverse in the case where $p + q = n + 1$. This show that the Radon transformation $\mathcal{R}_{(n+1;p,n+1-p)}$ is the non-trivial isomorphism between $CF(F_{n+1}(p))$ and its dual $CF(F_{n+1}(n+1-p))$ (Theorem 4.1). Here $CF(X)$ is the set of constructible functions on X.

In Section 5, we calculate the images of Radon transforms of characteristic functions of Schubert cells. We characterize these images by Young diagrams.

We have some remarks on this topics.

The meaning of our integration is not usual one but topological one based on the Euler-Poincaré indices of slices. On the other hand, in [8], T. Kakehi constructed an inversion formula for Radon transforms of C^∞-functions on $F_{n+1}(p)$. In spite of the difference of definition of Radon transforms, the sufficient condition under which we obtain an inversion formula in both cases coincide with each other; namely in the real Grassmann case only when $q - p$ is even, we obtain both inversion formulas. It would be interesting to investigate the reason why the condition coincides. Moreover, in [7], recently, E. Grinberg and B. Rudin constructed an inversion formula of Radon transforms of C^∞-functions on $F_{n+1}(p)$ for any p, q. We might construct an inversion formula for our topological Radon transformation in the real Grassmann case where $q - p$ is odd.

Moreover, in [12], we have succeeded in proving of Hergason's support theorem for Radon transforms of constructible functions .

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§2. Preliminaries

§2.1. A cell decomposition of Grassmann manifolds

We recall the notation and well-known results on a cell decomposition of the Grassmann manifold, that is Schubert decomposition. For more details, we refer to [1], [5], [6].

Definition 2.1.   Let E be an $n-$dimensional vector space over $k = \mathbb{R}$ or $\mathbb{C}$, and p, q integers satisfying $1 \leqq p \leqq q \leqq n$. We set

 (i) $F_n(p) = \{x \mid x$ is a linear subspace of E, whose dimension is $p.\}$,

 (ii) $F_n(p, q) = \{(x, y) \in F_n(p) \times F_n(q) \mid x \subset y \ \}$,

 (iii) $F_n(q, p) = \{(y, x) \in F_n(q) \times F_n(p) \mid y \supset x \ \}$,

 (iv) $\mu_n(p) = \chi(F_n(p))$: the topological Euler-Poincaré index of $F_n(p)$.

We calculate $\mu_n(p)$ concretely in A.

We fix a basis $e_1, e_2, \ldots, e_n$ of E. We set $V_i = \text{span}[e_1, e_2, \ldots, e_i]$ for $i = 1, 2, \ldots, n$. Then we have a complete flag of vector spaces;

$$V_1 \subset V_2 \subset \cdots \subset V_n, \quad \dim V_i = i.$$

We recall a cell decomposition of $F_n(p)$, which is called Schubert decomposition.

Definition 2.2.

(i) Let $\lambda = (a_1, a_2, \ldots, a_p)$ be a sequence of integers such that

$$n - p \geq a_1 \geq a_2 \geq \cdots \geq a_p \geq 0.$$

This sequence corresponds to what is called a Young diagram with at most p rows and $n-p$ columns. We identify this sequence with a Young diagram.

(ii) For a Young diagram $\lambda = (a_1, a_2, \ldots, a_p)$, we define its complement $\lambda^c = (b_1, b_2, \ldots, b_p)$ by

$$b_j = n - p - a_j \quad \text{for } j = 1, 2, \ldots, p.$$

(iii) For a sequence $\lambda = (a_1, a_2, \ldots, a_p)$, we set

$$|\lambda| = \sum_{k=1}^{p} a_k.$$

Definition 2.3. Let λ be a Young diagram, and $\lambda^c = (b_1, b_2, \ldots, b_p)$ its complement. Then we define the Schubert cell corresponding to λ by

$$\Omega_\lambda^\circ = \left\{ x \in F_n(p) \ \middle| \ \begin{array}{l} \dim(x \cap V_{b_i+i}) = i, \\ \dim(x \cap V_{b_i+i-1}) = i - 1 \end{array} \ (1 \leq i \leq p) \right\}.$$

Proposition 2.1. Let λ be a Young diagram with p rows and $n - p$ columns. Then we have

(i) $\Omega_\lambda^\circ \simeq k^{|\lambda^c|} = k^{p(n-p)-|\lambda|}$,   (ii) $F_n(p) = \coprod_\lambda \Omega_\lambda^\circ$.

Definition 2.4. Let λ be a Young diagram. We define the Schubert variety for λ by

$$\Omega_\lambda = \{ x \in F_n(p) \ | \ \dim(x \cap V_{b_i+i}) \geq i, \ (1 \leq i \leq p) \},$$

where $\lambda^c = (b_1, b_2, \ldots, b_p)$. Note that Ω_λ is an analytic submanifold of $F_n(p)$.

Proposition 2.2.   *Let λ be a Young diagram λ. Then we have*

$$\Omega_\lambda = \coprod_{\lambda \subset \mu} \Omega_\mu^\circ,$$

where μ ranges through Young diagrams containing λ as a subset.

§2.2.   Constructible functions

We recall the notation and results on constructible functions without proofs. For more details, we refer to [9].

Let X be a real analytic manifold.

Definition 2.5.   A function $\phi : X \to \mathbb{Z}$ is set to be constructible if:

(i) For any $m \in \mathbb{Z}$, $\phi^{-1}(m)$ is subanalytic,

(ii) the family $\{\phi^{-1}(m)\}_{m \in \mathbb{Z}}$ is locally finite in X.

We denote by $CF(X)$ the abelian group of all the constructible functions on X, and by $\mathscr{CF}_X$ the sheaf $U \mapsto CF(U)$ on X.

It follows from the Hardt triangulation theorem that ϕ is constructible if and only if there exists a locally finite family of compact subanalytic contractible subsets $\{K_i\}_i$ of X such that

$$\phi = \sum_i c_i \mathbf{1}_{K_i}.$$

Here $c_i \in \mathbb{Z}$ and $\mathbf{1}_A$ is the characteristic function of the subset A.

Example 1.   Let $\mathbf{D}_{\mathbb{R}-c}^b(X)$ be the derived category of the category of complexes of $\mathbb{R}$-constructible sheaves and $F \in Ob(\mathbf{D}_{\mathbb{R}-c}^b(X))$ (the base ring is a field k with characteristic zero). Then its local Euler-Poincaré index

$$\chi(F)(x) = \sum_j (-1)^j \dim H^j(F)_x$$

is a constructible function.

From now on, χ denotes the local Euler-Poincaré index.

Proposition 2.3.

(i) *Let F, $G \in Ob(\mathbf{D}^b_{\mathbb{R}-c}(X))$. Then we have*

    (a) $\chi(F \oplus G) = \chi(F) + \chi(G)$,   (b) $\chi(F \otimes G) = \chi(F) \cdot \chi(G)$.

(ii) *Let $F' \to F \to F'' \xrightarrow{+1}$ be a distinguished triangle in $\mathbf{D}^b_{\mathbb{R}-c}(X)$. Then we have*

$$\chi(F) = \chi(F') + \chi(F'').$$

We denote by $\mathbf{K}_{\mathbb{R}-c}(X)$ the Grothendieck group of $\mathbf{D}^b_{\mathbb{R}-c}(X)$. This group is obtained as the quotient group of the free abelian group generated by $Ob(\mathbf{D}^b_{\mathbb{R}-c}(X))$ under the following equivalence relations: $F = F' + F''$ if there exists a distinguished triangle $F' \to F \to F'' \xrightarrow{+1}$.

Theorem 2.1 ([9, Theorem 9.7.11])**.**   *The group homomorphism induced by the local Euler-Poincaré index χ*

$$\chi : \mathbf{K}_{\mathbb{R}-c}(X) \to CF(X)$$

is an isomorphism.

Next, we recall operations on constructible functions [9]. These operations are induced by operations of $\mathbf{K}_{\mathbb{R}-c}(X)$ through the Euler-Poincaré index χ.

Definition 2.6.   Let X and Y be two real analytic manifolds, and $f \colon Y \to X$ a real analytic map.

(i) The inverse image: Let $\phi \in CF(X)$. We set

$$f^*\phi(y) = \phi(f(y)).$$

Note that if $\phi = \chi(F)$, then $f^*\phi = \chi(f^{-1}F)$.

(ii) The integral: Let $\phi \in CF(X)$. Assume that ϕ is represented as $\phi = \chi(F) = \sum_i c_i \mathbf{1}_{K_i}$. Here $F \in Ob(\mathbf{D}^b_{\mathbb{R}-c}(X))$, and $\{K_i\}$ is a locally finite family of compact subanalytic contractible subsets. Assume moreover that ϕ has compact support. Then we set

$$\int_X \phi = \sum_i c_i = \chi(R\Gamma(X; F)).$$

(iii) The direct image: Let $\psi \in CF(Y)$. Assume that $f\colon \mathrm{supp}(\psi) \to X$ is proper. Here $\mathrm{supp}(\psi)$ denotes a support of ψ. We set

$$\left(\int_f \psi \right)(x) = \int_Y (\psi \cdot \mathbf{1}_{f^{-1}(x)}).$$

Note that if $\psi = \chi(G)$ and f is proper on $\mathrm{supp}(G)$, then $\int_f \psi = \chi(Rf_! G)$.

Remark 1.   Let A be a locally closet subset of a manifold X. Then the integral $\int_X \mathbf{1}_A$ is not the usual integral, but a kind of topological integrals. By Theorem 2.1 and the definition, we have the following equalities:

$$\int_X \mathbf{1}_A = \chi(R\Gamma(X; k_A)) = \chi(R\Gamma(X; i_! i^{-1} k_X)) = \chi(R\Gamma_c(A; k_A)) = \chi_c(A).$$

Here k is $\mathbb{R}$ or $\mathbb{C}$, $i : A \to X$ is an inclusion morphism and χ_c is the topological Euler-Poincaré index with compact supports.

Let A_1, A_2 be two locally closed subsets of a manifold X such that $A_2 \subset A_1$. Then we have distinguished triangles

$$\mathbb{C}_{A_1 \setminus A_2} \to \mathbb{C}_{A_1} \to \mathbb{C}_{A_2} \xrightarrow{+1},$$
$$R\Gamma_c(X; \mathbb{C}_{A_1 \setminus A_2}) \to R\Gamma_c(X; \mathbb{C}_{A_1}) \to R\Gamma_c(X; \mathbb{C}_{A_2}) \xrightarrow{+1}.$$

Therefore we have the additivity of χ_c;

$$\chi_c(A_1) = \chi_c(A_1 \setminus A_2) + \chi_c(A_2).$$

By this additivity of χ_c, we have some examples;

$$\int_{\mathbb{R}} \mathbf{1}_{[0,1]} = 1, \quad \int_{\mathbb{R}} \mathbf{1}_{[0,1)} = 0, \quad \int_{\mathbb{R}} \mathbf{1}_{(0,1)} = -1.$$

Proposition 2.4.

(i) *The following operations are well-defined morphisms of sheaves*;

(a) $f^* : f^{-1}\mathscr{CF}_X \to \mathscr{CF}_Y$,   (b) $\displaystyle\int_f : f_! \mathscr{CF}_Y \to \mathscr{CF}_X.$

(ii) *Inverse and direct images have functorial properties. Precisely, if $f : Y \to X$ and $g : Z \to Y$ are real analytic maps, then we have*;

(c) $g^* \circ f^* = (f \circ g)^*$,   (d) $\displaystyle\int_{f \circ g} = \int_f \circ \int_g .$

(iii) *Consider a Cartesian diagram of morphisms of real analytic manifolds*:

$$
\begin{array}{ccc}
Y' & \xrightarrow{\ f'\ } & X' \\
{\scriptstyle h}\downarrow & \square & \downarrow{\scriptstyle g} \\
Y & \xrightarrow{\ f\ } & X.
\end{array}
$$

Let $\psi \in CF(Y)$. *Suppose that f is proper on* suppψ. *Then we have*

$$
g^* \int_f \psi = \int_{f'} (h^*\psi).
$$

§2.3.   Radon transforms of constructible functions and Schapira's formula

We recall the definition of Radon transforms of constructible functions and Schapira's formula ([11]).

Let X and Y be two real analytic manifolds, and S a locally closed sub-analytic subset of $X \times Y$. Denote by p_1 and p_2 the first and second projections defined on $X \times Y$, and by f and g the restrictions of p_1 and p_2 to S respectively:

(2.1)

$$
\begin{array}{ccc}
 & X \times Y & \\
 & \cup & \\
{}^{p_1}\diagup & S & \diagdown{}^{p_2} \\
{}_{f}\diagdown & & \diagup{}_{g} \\
X & & Y.
\end{array}
$$

We assume;

(2.2) p_2 is proper on $\bar{S}$ (the closure of S in $X \times Y$).

Definition 2.7.   For a $\phi \in CF(X)$, we set

$$
\mathcal{R}_S(\phi) = \int_g f^*\phi = \int_{p_2} \mathbf{1}_S(p_1^*\phi).
$$

We call $\mathcal{R}_S(\phi)$ the Radon transform of ϕ.

Let $S' \subset Y \times X$ be another locally closed subanalytic subset. We denote again by p_2 and p_1 the first and second projections defined on $Y \times X$, by f' and g' the restrictions of p_1 and p_2 to S', and by r the projection $S \underset{Y}{\times} S' \to X \times X$.

Then Schapira posed the following assumptions:

(2.3) $\qquad\qquad\qquad p_1$ is proper on $\bar{S}'$ (the closure of S' in $Y \times X$),

(2.4) $\qquad \exists \lambda, \mu \in \mathbb{Z}$ s.t. $\lambda \neq \mu$ and $\chi(r^{-1}(x, x')) = \begin{cases} \lambda \ (x \neq x'), \\ \mu \ (x = x'), \end{cases}$

where χ is the topological Euler-Poincaré index. We use the same symbol χ as the local Euler-Poincaré index.

In this paper, we refer to the assumption (2.4) as Schapira's condition.

Under the notation above, Schapira's formula is stated as follows.

Theorem 2.2 ([11, Theorem 3.1]). *Assume (2.2), (2.3) and (2.4). Then, for any $\phi \in CF(X)$, we have*

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) = (\mu - \lambda)\phi + \left(\int_X \lambda\phi \right) \mathbf{1}_X.$$

Proof. For the convenience of readers, we recall the proof of this theorem. Denote by h and h' the projections from $S \underset{Y}{\times} S'$ to S and S' respectively.

Consider the following diagram:
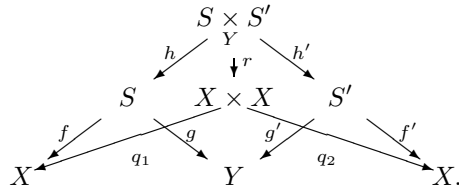
Since the square

is of Cartesian, we have

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) = \int_{f'} (g'^* \int_g (f^*\phi)) = \int_{f' \circ h'} ((f \circ h)^*\phi) = \int_{q_2} \int_r r^* q_1^* \phi$$
$$= \int_{q_2} k(x, x') q_1^* \phi.$$

Here we have

$$k(x, x') = \int_r r^* \mathbf{1}_{X \times X} = \int_r \mathbf{1}_{S \underset{Y}{\times} S'}.$$

By Schapira's condition (2.4), we have

$$\int_r \mathbf{1}_{S \times S'} \underset{Y}{} = \mu \mathbf{1}_{\Delta_X} + \lambda \mathbf{1}_{X \times X \setminus \Delta_X} = (\mu - \lambda) \mathbf{1}_{\Delta_X} + \lambda \mathbf{1}_{X \times X},$$

where Δ_X is the diagonal of $X \times X$.

Since $\int_{q_2} \mathbf{1}_{\Delta_X} q_1^* \phi = \phi$ and $\int_{q_2} \mathbf{1}_{X \times X} q_1^* \phi = \int_X \phi$, we obtain the result. $\square$

In [11], Schapira applied this formula to correspondences of real flag manifolds; that is, we consider the following diagram called the correspondence;

(2.5)
$$
\begin{array}{ccc}
 & F_{n+1}(1, q) & \\
 {}^f \swarrow & & \searrow {}^g \\
F_{n+1}(1) & & F_{n+1}(q),
\end{array}
$$

where f and g are projections.

We set $\mathcal{R}_{(n+1;1,q)} = \mathcal{R}_S$ and $\mathcal{R}_{(n+1;q,1)} = \mathcal{R}_{S'}$, where $S = F_{n+1}(1, q)$ and $S' = F_{n+1}(q, 1)$. Then this situation satisfies the assumptions of Schapira's formula, because we have

$$r^{-1}(x, x') \simeq \begin{cases} F_{n-1}(q-2) \ (x \neq x'), \\ F_n(q-1) \quad (x = x'). \end{cases}$$

Therefore we can apply Theorem 2.2 to this case.

Proposition 2.5 ([11, Proposition 4.1]). *Consider the correspondence* (2.5). *For any* $\phi \in CF(F_{n+1}(1))$, *we have*

$$\mathcal{R}_{(n+1;q,1)} \circ \mathcal{R}_{(n+1;1,q)}(\phi)$$

$$= (\mu_n(q-1) - \mu_{n-1}(q-2))\phi + \mu_{n-1}(q-2) \left(\int_{F_{n+1}(1)} \phi \right) \mathbf{1}_{F_{n+1}(1)}.$$

In particular, if n is odd, we obtain an inversion formula for $\mathcal{R}_{(n+1;1,n)}$.

§3. Inversion of Radon Transforms of Constructible Functions

We generalize (2.5); that is, we consider the following diagram:

(3.1)
$$
\begin{array}{ccc}
 & F_{n+1}(p) \times F_{n+1}(q) & \\
 & \cup & \\
 {}^{p_1} \swarrow \quad F_{n+1}(p, q) \quad \searrow {}^{p_2} & & \\
 {}_f \swarrow & & \searrow {}_g \\
F_{n+1}(p) & & F_{n+1}(q).
\end{array}
$$

We set $X = F_{n+1}(p)$, $Y = F_{n+1}(q)$ and $S = F_{n+1}(p, q)$.

We consider the following problems;

YUTAKA MATSUI

(i) an inversion formula for $\mathcal{R}_{(n+1;1,q)}$ in the case where n is even or $q \neq n$,

(ii) an inversion formula for $\mathcal{R}_{(n+1;p,q)}$ in the case where $1 < p$ and $1 < q$.

Namely, we consider the reconstruction of ϕ from $\mathcal{R}_S(\phi)$ on Grassmann manifolds.

We remark that Schapira already considered this diagram (3.1) in [11], but he could not obtain results for these problems.

§3.1.   A minor modification of Schapira's formula

We modify Schapira's formula. We inherit the notation from Section 2.3.

Definition 3.1.   For a $\psi \in CF(Y)$, we set

$$\mathcal{R}_0(\psi) = \int_{p_1} \mathbf{1}_{Y \times X} (p_2^* \psi) = \int_{p_1} (p_2^* \psi) = \left(\int_Y \psi \right) \mathbf{1}_X.$$

Definition 3.2.   We define the transposed set of S by

$${}^t S = \{ (y, \ x) \in Y \times X \mid (x, \ y) \in S \ \}.$$

In this section, we assume Schapira's assumptions (2.2), (2.3), (2.4) and the following assumption:

(3.2) $${}^t S = S'.$$

Proposition 3.1.   *Let $\phi \in CF(X)$. Then we have*

$$\mathcal{R}_0 \circ \mathcal{R}_S(\phi) = \int_X (\mu \phi) \, \mathbf{1}_X.$$

Proof.   A constructible function ϕ is represented by $\phi = \sum_i c_i \mathbf{1}_{K_i}$, where $\{K_i\}$ is a locally finite family of compact subanalytic contractible subsets. By the linearity of transformations, it is enough to show this formula only for a characteristic function $\mathbf{1}_K$ of a compact subanalytic contractible subset K.

Since the square

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\ p_2\ } & Y \\
{\scriptstyle p_1} \downarrow & \square & \downarrow {\scriptstyle a_Y} \\
X & \xrightarrow{\ a_X\ } & \{pt\}
\end{array}
$$

is of Cartesian, we have

$$\mathcal{R}_0 \circ \mathcal{R}_S(\mathbf{1}_K) = \int_{p_1} p_2^* \int_{p_2} (\mathbf{1}_S \cdot p_1^* \mathbf{1}_K) = a_X^* \int_{a_Y} \int_{p_2} (\mathbf{1}_S \cdot p_1^* \mathbf{1}_K)$$

$$= a_X^* \int_{a_X} \int_{p_1} (\mathbf{1}_S \cdot p_1^* \mathbf{1}_K).$$

For any $\phi \in CF(X)$, we have

$$\left(a_X^* \int_{a_X} \phi \right)(x) = \left(\int_{a_X} \phi \right)(\{pt\}) = \int_X \phi(x') \mathbf{1}_{a_X^{-1}(\{pt\})}(x') = \left(\int_X \phi \right) \mathbf{1}_X(x).$$

Since the Euler-Poincaré index of

$$\{x\} \times Y \cap S \simeq \{y \in Y \mid (x, y) \in S\}$$

is μ defined in (2.4), we have

$$\left(\int_{p_1} \mathbf{1}_S \cdot p_1^* \mathbf{1}_K \right)(x) = \int_{X \times Y} \mathbf{1}_{((\{x\} \cap K) \times Y) \cap S}(x', y') = \mu \mathbf{1}_K.$$

Therefore we obtain the desired result. $\qquad\square$

Definition 3.3. For a $\psi \in CF(Y)$, we set

$$\mathcal{R}^{-1}(\psi) = \int_{p_1} (\mu \mathbf{1}_{S'} - \lambda \mathbf{1}_{Y \times X})(p_2^* \psi) = \mu \mathcal{R}_{S'}(\psi) - \lambda \mathcal{R}_0(\psi).$$

Proposition 3.2. *Let $\phi \in CF(X)$. Then we have*

$$\mathcal{R}^{-1} \circ \mathcal{R}_S(\phi) = \mu(\mu - \lambda)\phi.$$

In particular, if $\mu(\mu - \lambda)$ is not zero, we can reconstruct the original constructible function ϕ from its Radon transform $\mathcal{R}_S(\phi)$ by dividing the last term by this constant $\mu(\mu - \lambda)$.

Proof. By Theorem 2.2 and Proposition 3.1, we have

$$\mathcal{R}^{-1} \circ \mathcal{R}_S(\phi) = \mu \mathcal{R}_{S'} \circ \mathcal{R}_S(\phi) - \lambda \mathcal{R}_0 \circ \mathcal{R}_S(\phi) = \mu(\mu - \lambda)\phi.$$

$\qquad\square$

We apply this result to the complex or real Grassmann manifolds. We recall the Euler-Poincaré index of the Grassmann manifold. For more detail calculation, see A.

In the complex case, we have

$$(3.3) \qquad\qquad \mu_n(p) = \binom{n}{p}.$$

In the real case, we have

$$(3.4) \qquad \mu_n(p) = \begin{cases} 0 & \text{(if } p(n-p) \text{ is odd)}, \\[2mm] \dbinom{E\left(\frac{n}{2}\right)}{E\left(\frac{p}{2}\right)} & \text{(if } p(n-p) \text{ is even)}. \end{cases}$$

Here $E\left(\frac{n}{2}\right)$ denotes the integral part of $\frac{n}{2}$, $\binom{a}{b}$ is the binomial coefficient.

We consider the correspondence (2.5). Then the assumptions (2.2), (2.3), (2.4) and (3.2) are satisfied. We remark that

$$\mu = \mu_n(q-1), \qquad \lambda = \mu_{n-1}(q-2).$$

We consider the conditions of q that $\mu(\mu - \lambda) \neq 0$ from (3.3) and (3.4). Therefore we can apply Proposition 3.2 to the Grassmann cases.

Proposition 3.3. *We have $\mu(\mu - \lambda) \neq 0$ if either one of the following conditions are satisfied;*

(i) *$q > 1$ under the complex Grassmann case,*

(ii) *q is odd and $1 < q < n + 1$ under the real Grassmann case.*

In particular, then we obtain an inversion formula for $\mathcal{R}_{(n+1;1,q)}$.

§3.2.   Inversion formulas on Grassmann manifolds

For $p < q$, we consider the diagram (3.1).

We remark that Schapira's condition (2.4) is not satisfied if $1 < p$. This is because we consider $(p+1)$ cases according to $\dim(x_1 \cap x_2)$ to study $r^{-1}(x_1, x_2)$.

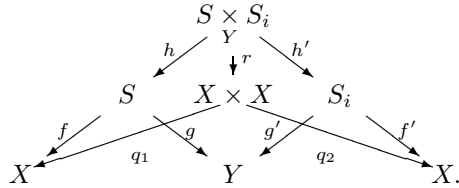We introduce new sets in order to construct to an inversion transformation for $\mathcal{R}_{(n+1;p,q)}$.

Definition 3.4.   We set

(i) $S_i = \{(y,x) \in Y \times X \mid \dim(y \cap x) = i\,\}$ for $i = 0, 1, \ldots, p$,

(ii) $Z_j = \{(x_1, x_2) \in X \times X \mid \dim(x_1 \cap x_2) = j\,\}$ for $j = 0, 1, \ldots, p$.

Remark 2.   We have

$$X \times X = \coprod_{j=0}^{p} Z_j.$$

Consider the following diagram:

$$
\begin{array}{ccc}
& S \underset{Y}{\times} S_i & \\
{}^{h}\swarrow \quad \downarrow{}^{r} \quad \searrow{}^{h'} & & \\
S \quad X \times X \quad S_i & & \\
\end{array}
$$

Denote by h and h' the projections from $S \underset{Y}{\times} S_i$ to S and S' respectively.

Note that $Z_p = \{(x_1, x_2) \in X \times X \mid x_1 = x_2\}$ and we have $\int_{q_2} \mathbf{1}_{Z_p} q_1^* \phi = \phi$. In order to apply the same argument as in the proof of Theorem 2.2 and Section 3.1, we modify the kernel such that $\int_r (\text{kernel})$ is equal to $\mathbf{1}_{Z_p}$.

We calculate $\int_r \mathbf{1}_{S \underset{Y}{\times} S_i}$;

$$
\left(\int_r \mathbf{1}_{S \underset{Y}{\times} S_i} \right)(x_1, x_2) = \sum_{j=0}^{p} \left(\int_{S \underset{Y}{\times} S_i} \mathbf{1}_{r^{-1}(x_1, x_2) \cap r^{-1}(Z_j)} \right) \cdot \mathbf{1}_{Z_j}.
$$

We consider $r^{-1}(x_1, x_2) \cap S \underset{Y}{\times} S_i$ when we fix $(x_1, x_2) \in Z_j$. We have

$$
r^{-1}(x_1, x_2) \cap S \underset{Y}{\times} S_i = \left\{ y \in F_{n+1}(q) \,\middle|\, \begin{array}{l} x_1 \subset y \\ \dim(x_2 \cap y) = i \end{array} \right\} \quad (\dim(x_1 \cap x_2) = j).
$$

Here we consider conditions in the quotient space E/x_1. Then we have

$$
r^{-1}(x_1, x_2) \cap S \underset{Y}{\times} S_i \simeq
\begin{cases}
\emptyset & (i < j), \\
\left\{ y \in F_{n+1-p}(q-p) \,\middle|\, \dim(x \cap y) = i - j \right\} & (i \geq j) \\
\quad (\dim x = p - j)
\end{cases}
$$

$$
=
\begin{cases}
\emptyset & (i < j), \\
\Omega_{i,j} \setminus \Omega_{i+1,j} & (i \geq j),
\end{cases}
$$

where we set

(3.5) $\quad \Omega_{i,j} = \{y \in F_{n+1-p}(q-p) \mid \dim(x \cap y) \geq i - j\} \quad (\dim x = p - j)$.

This set is nothing but a Schubert variety of $F_{n+1-p}(q-p)$.

Therefore we calculate the Euler-Poincaré index $\chi_c(\Omega_{i,j} \setminus \Omega_{i+1,j})$ with compact supports. It is enough to calculate $\chi_c(\Omega_{i,j})$, because we have the additivity

$$\chi_c(\Omega_{i,j} \setminus \Omega_{i+1,j}) = \chi_c(\Omega_{i,j}) - \chi_c(\Omega_{i+1,j}).$$

We can calculate $\chi_c(\Omega_{i,j})$ by using a cell decomposition of $\Omega_{i,j}$ and the Young diagram corresponding to $\Omega_{i,j}$. For more details, see A.

(i) If we consider complex Grassmannians, i.e. $E = \mathbb{C}^{n+1}$, then we obtain

$$\chi_c(r^{-1}(x_1, x_2) \cap r^{-1}(Z_j) \cap S \underset{Y}{\times} S_i) = \begin{cases} 0 & (i < j), \\ \binom{p-j}{i-j}\binom{n+1-2p+j}{q-p-i+j} & (i \geq j) \end{cases}$$
$$=: c_{ij}.$$

(ii) If we consider real Grassmannians, i.e. $E = \mathbb{R}^{n+1}$, then we obtain

$$(-1)^{(q-p)(n+1-q)}\chi_c(r^{-1}(x_1, x_2) \cap r^{-1}(Z_j) \cap S \underset{Y}{\times} S_i)$$

$$= \begin{cases} 0, & (i < j) \\ \left| \sum_{l=0}^{p-i}(-1)^l \mu_{l+i-j-1}(l)\mu_{n+1-p-i+j-l}(q-p-i+j) \right| \\ \quad - \left| \sum_{l=0}^{p-i-1}(-1)^l \mu_{l+i-j}(l)\mu_{n-p-i+j-l}(q-p-i+j-1) \right|, & (i > j) \\ \mu_{n+1-2p+j}(q-p) & (i = j) \end{cases}$$
$$=: (-1)^{(q-p)(n+1-q)}c_{ij}.$$

We can unify these two cases. Note that c_{ij} is independent of the choice of (x_1, x_2) in Z_j. Therefore we have

$$\left(\int_r \mathbf{1}_{S \underset{Y}{\times} S_i} \right)(x_1, x_2) = \sum_{j=0}^{p} c_{ij}\mathbf{1}_{Z_j}.$$

Here, we denote by $C^{p,q}$ the square matrix $(c_{ij})_{0 \leq i,j \leq p}$ of size $(p+1)$. Since this is the lower triangular matrix, we have

$$(3.6) \qquad |\det C^{p,q}| = \prod_{j=0}^{p} \mu_{n+1-2p+j}(q-p)$$

in both cases. In particular it is $\mathbb{Z}$-valued.

In the argument here after, we consider the case where $\det C^{p,q} \neq 0$. We derive the following conditions for $\det C^{p,q} \neq 0$ from (3.3) and (3.4):

(i) $p + q \leq n + 1$ in the complex Grassmann case,

(ii) $p + q \leq n + 1$ and $q - p$ is even in the real Grassmann case.

Under the preliminaries above, we define the kernel function of an inversion formula for $\mathcal{R}_{(n+1;p,q)}$.

We obtain the equation

$$
C^{p,q} \begin{pmatrix} \mathbf{1}_{Z_0} \\ \mathbf{1}_{Z_1} \\ \vdots \\ \mathbf{1}_{Z_p} \end{pmatrix} = \begin{pmatrix} \int_r \mathbf{1}_{S \times S_0 \atop Y} \\ \int_r \mathbf{1}_{S \times S_1 \atop Y} \\ \vdots \\ \int_r \mathbf{1}_{S \times S_p \atop Y} \end{pmatrix}.
$$

When $\det C^{p,q} \neq 0$, we can solve this equation with respect to $\mathbf{1}_{Z_p}$ by Cramer's formula:

$$
\det C^{p,q} \cdot \mathbf{1}_{Z_p} = \det \begin{pmatrix} c_{00} & 0 & \cdots & 0 & \int_r \mathbf{1}_{S \times S_0 \atop Y} \\ c_{10} & c_{11} & \ddots & \vdots & \int_r \mathbf{1}_{S \times S_1 \atop Y} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ c_{p-1,0} & c_{p-1,1} & \cdots & c_{p-1,p-1} & \int_r \mathbf{1}_{S \times S_{p-1} \atop Y} \\ c_{p,0} & c_{p,1} & \cdots & c_{p,p-1} & \int_r \mathbf{1}_{S \times S_p \atop Y} \end{pmatrix}.
$$

Definition 3.5. If $\det C^{p,q} \neq 0$, we set

$$
K_{p,q} = \det \begin{pmatrix} c_{00} & 0 & \cdots & 0 & \mathbf{1}_{S_0} \\ c_{10} & c_{11} & \ddots & \vdots & \mathbf{1}_{S_1} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ c_{p-1,0} & c_{p-1,1} & \cdots & c_{p-1,p-1} & \mathbf{1}_{S_{p-1}} \\ c_{p,0} & c_{p,1} & \cdots & c_{p,p-1} & \mathbf{1}_{S_p} \end{pmatrix}.
$$

Then we can define $\mathcal{R}^{-1}(\psi)$ for a $\psi \in CF(F_{n+1}(q))$ by

$$
\mathcal{R}^{-1}(\psi) = \int_{p_1} K_{p,q} \cdot (p_2^* \psi).
$$

The main result in this paper is:

Theorem 3.1. *Consider the diagram* (3.1). *If* $\det C^{p,q} \neq 0$, *then for any* $\phi \in CF(F_{n+1}(p))$ *we have*

$$
\mathcal{R}^{-1} \circ \mathcal{R}_{(n+1;p,q)}(\phi) = \det C^{p,q} \cdot \phi.
$$

This means that we can reconstruct the original constructible function ϕ from its Radon transform $\mathcal{R}_{(n+1;p,q)}(\phi)$ by dividing the last term by the constant $\det C^{p,q}$. In particular, we obtain an inversion formula for $\mathcal{R}_{(n+1;p,q)}$ if either one of the following conditions are satisfied;

(i) $p + q \leq n + 1$ under the complex Grassmann case,

(ii) $p + q \leq n + 1$ and $q - p$ is even under the real Grassmann case.

Proof. In the same way as in the proof of Theorem 2.2, we have

$$
\mathcal{R}^{-1} \circ \mathcal{R}_{(n+1;p,q)}(\phi) = \int_{p_1} K_{p,q} \cdot \left(p_2^* \int_{p_2} 1_S \cdot p_1^* \phi \right)
$$

$$
= \int_{q_2} \left\{ \det \begin{pmatrix} c_{00} & 0 & \dots & 0 & \int_r \mathbf{1}_{S \times S_0 \atop Y} \\ c_{10} & c_{11} & \ddots & \vdots & \int_r \mathbf{1}_{S \times S_1 \atop Y} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ c_{p-1,0} & c_{p-1,1} & \dots & c_{p-1,p-1} & \int_r \mathbf{1}_{S \times S_{p-1} \atop Y} \\ c_{p,0} & c_{p,1} & \dots & c_{p,p-1} & \int_r \mathbf{1}_{S \times S_p \atop Y} \end{pmatrix} \right\} q_1^* \phi
$$

$$
= \int_{q_2} \det C^{p,q} \mathbf{1}_{Z_p} q_1^* \phi
$$

$$
= \det C^{p,q} \cdot \phi.
$$

Note that $\int_{q_2} \mathbf{1}_{Z_p} q_1^* \phi = \phi$.   □

Remark 3.   In the argument above, we consider only the case where $p < q$. However, we obtain results in other cases.

When $p = q$, the inversion formula is trivial because $\mathcal{R}_{(n+1;p,q)} = id_{CF(F_{n+1}(p))}$.

When $p > q$, we have only to consider the result in the case of $p < q$ by the dualities of Grassmann manifolds. Namely, we obtain an inversion formula for $\mathcal{R}_{(n+1;p,q)}$ if $p + q \geq n + 1$ in the complex case or if $p + q \geq n + 1$ and $p - q$ is even in the real case.

§4.   The Inverse Radon Transformation

In this section, we show the following theorem

Theorem 4.1.   *Let $p + q = n + 1$ hold. The inversion transformation $\mathcal{R}^{-1}$ defined in Definition 3.5 gives the inverse transformation for the Radon*

transformation $\mathcal{R}_{(n+1;p,q)}$. *Namely, the Radon transformation* $\mathcal{R}_{(n+1:p,q)}$ *is the non-trivial isomorphism from* $CF(F_{n+1}(p))$ *to* $CF(F_{n+1}(q))$ *up to constant if either one of the following conditions are satisfied;*

(i) *the complex case,*

(ii) $p + q = n + 1$ *and* $q - p$ *is even in the real case.*

Moreover, through the Euler-Poincaré index χ, *the Radon transformation gives the non-trivial isomorphism between Grothendieck groups.*

First, we remark that it is enough to show that $\mathcal{R}^{-1}$ gives a right inverse transformation of $\mathcal{R}_{(n+1:p,q)}$. Moreover, in the same argument as Section 3.2, it is enough to consider only when $p < q$ and $p + q = n + 1$.

Before going into the proof of this theorem, we need some preliminaries.

We consider the diagram (3.1). We introduce the following sets similarly to Section 3.2.

Definition 4.1. We set

(i) $S_i = \{(y, x) \in Y \times X \mid \dim(y \cap x) = i\}$ for $i = 0, 1, \ldots, p$,

(ii) $Z'_j = \{(y_1, y_2) \in Y \times Y \mid \dim(y_1 \cap y_2) = j + (q - p)\}$ for $j = 0, 1, \ldots, p$.

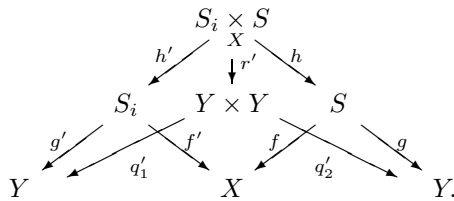*Remark* 4. We have an inequality

$$q - p \leq \dim(y_1 \cap y_2) \leq q.$$

Moreover we have that

$$Y \times Y = \coprod_{j=0}^{p} Z'_j.$$

We calculate $\mathcal{R}_S \circ \mathcal{R}^{-1}$ similarly to Section 3.2.

Consider the following diagram:

$$
\begin{array}{ccccc}
 & & S_i \underset{X}{\times} S & & \\
 & \overset{h'}{\swarrow} & \downarrow r' & \overset{h}{\searrow} & \\
 & S_i & Y \times Y & S & \\
\overset{g'}{\swarrow} & & \underset{f'}{\swarrow} \quad \underset{f}{\searrow} & & \overset{g}{\searrow} \\
Y & & q'_1 \quad X \quad q'_2 & & Y.
\end{array}
$$

In the same way as in the proof of Theorem 2.2, we calculate $\int_{r'} \mathbf{1}_{S_i \underset{X}{\times} S}$;

Proposition 4.1. *We have*

$$\left(\int_{r'} \mathbf{1}_{S_i \underset{X}{\times} S} \right) = \sum_{j=0}^{p} c_{ij} \cdot \mathbf{1}_{Z'_j},$$

where $C^{p,q} = (c_{ij})_{0 \leq i,j \leq p}$ *is the coefficient matrix defined in Section* 3.2.

Proof.

$$\left(\int_{r'} \mathbf{1}_{S_i \underset{X}{\times} S} \right)(y_1, y_2) = \sum_{j=0}^{p} \left(\int_{S_i \underset{X}{\times} S} \mathbf{1}_{r'^{-1}(y_1,y_2) \cap r'^{-1}(Z'_j)} \right) \cdot \mathbf{1}_{Z'_j}.$$

First, we consider $r'^{-1}(y_1, y_2) \cap S_i \underset{X}{\times} S$ for $(y_1, y_2) \in Z'_j$.

We have

$$r'^{-1}(y_1, y_2) \cap S_i \underset{X}{\times} S = \left\{ x \in F_{n+1}(p) \ \middle| \ \begin{array}{l} x \subset y_1 \\ \dim(x \cap y_2) = i \end{array} \right\}$$

$$= \{ x \in F_q(p) \mid \dim(x \cap y_1 \cap y_2) = i \}$$

$$= \Omega'_{i,j} \setminus \Omega'_{i+1,j},$$

where we denote by

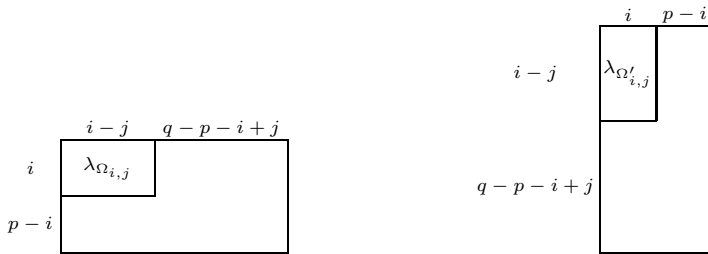$$\Omega'_{i,j} = \{ x \in F_q(p) \mid \dim(x \cap y) \geq i \} \quad (\dim(y) = j + q - p).$$

This set is a Schubert variety of $F_q(p)$.

Similarly to in Section 3.2, it is enough to calculate $\chi_c(\Omega'_{i,j})$.

Here we consider the Young diagram corresponding to $\Omega_{i,j}$ in (3.5) and that of $\Omega'_{i,j}$.

These Young diagrams have the following shapes;

(Figure 1)

This implies that

$$\chi_c(\Omega_{i,j}) = \chi_c(\Omega'_{i,j}).$$

Therefore we have

$$\chi_c(r'^{-1}(y_1, y_2) \cap r'^{-1}(Z'_j) \cap S_i \underset{X}{\times} S) = c_{ij}.$$

□

Finally, we show Theorem 4.1. For any $\psi \in CF(F_{n+1}(q))$, we have

$$\mathcal{R}_{(n+1;p,q)} \circ \mathcal{R}^{-1}(\psi) = \int_{p_2} 1_S \cdot \left(p_1^* \int_{p_1} K_{p,q} \cdot p_2^* \psi \right)$$

$$= \int_{q'_2} \left\{ \det \begin{pmatrix} c_{00} & 0 & \cdots & 0 & \int_r \mathbf{1}_{S_0 \underset{X}{\times} S} \\ c_{10} & c_{11} & \ddots & \vdots & \int_r \mathbf{1}_{S_1 \underset{X}{\times} S} \\ \vdots & \vdots & \ddots & 0 & \vdots \\ c_{p-1,0} & c_{p-1,1} & \cdots & c_{p-1,p-1} & \int_r \mathbf{1}_{S_{p-1} \underset{X}{\times} S} \\ c_{p,0} & c_{p,1} & \cdots & c_{p,p-1} & \int_r \mathbf{1}_{S_p \underset{X}{\times} S} \end{pmatrix} \right\} q_1'^* \psi$$

$$= \int_{q'_2} \det C^{p,q} \mathbf{1}_{Z'_p} q_1'^* \psi$$

$$= \det C^{p,q} \cdot \psi.$$

The third identity is due to the result of Section 3.2 and we remark that $\int_{q'_2} \mathbf{1}_{Z'_p} q_1'^* \psi = \psi$.

§5. The Image of Radon Transform of the Characteristic Function on a Schubert Cell

In this section, we characterize the image of the Radon transform of the characteristic function of a Schubert cell. We consider the correspondence (3.1). Moreover, we take account of the complement λ^c of a Young diagram λ when we study the Schubert cell in this section. We set

$$\Lambda_{p,n-p} = \{(a_1, a_2, \ldots, a_p) \mid 0 \leq a_1 \leq a_2 \leq \cdots \leq a_p \leq n - p.\}.$$

This is the set of increasing sequences like λ^c. We denote by Ω_α° the Schubert cell corresponding to λ for $\lambda^c = \alpha \in \Lambda_{p,n-p}$.

Definition 5.1.

(i) For an $\alpha = (a_1, a_2, \ldots, a_p) \in \Lambda_{p,n-p}$, we define a new sequence $\hat{\alpha} \in \Lambda_{p,n}$ by

$$\hat{\alpha} = (a_1 + 1, a_2 + 2, \ldots, a_p + p).$$

(ii) Let $\alpha = (a_1, a_2, \ldots, a_p) \in \Lambda_{p,n}$ and $\beta = (b_1, b_2, \ldots, b_q) \in \Lambda_{q,m}$ for $p < q$. Then we define the relation $\alpha \subset \beta$ if for each i $(1 \le i \le p)$ there exists j $(1 \le j \le q)$ such that $a_i = b_j$. We denote by $\sigma_{\alpha,\beta}$ this correspondence of numbers, that is, $\sigma_{\alpha,\beta}(i) = j$.

Definition 5.2. Let $\alpha = (a_1, a_2, \ldots, a_p) \in \Lambda_{p,n-p}$, $\beta = (b_1, b_2, \ldots, b_q) \in \Lambda_{q,n-q}$ such that $\alpha \subset \beta$. Then we set

$$c_{\alpha,\beta} = \sum_{k=1}^{p} \sigma_{\alpha,\beta}(k) - k.$$

We characterize the image of the Radon transform of $\mathbf{1}_{\Omega_\alpha^\circ}$.

Theorem 5.1. Let $\alpha \in \Lambda_{p,n-p}$.

(i) *In the complex case, we have*

$$\mathcal{R}_{(n+1;p,q)}(\mathbf{1}_{\Omega_\alpha^\circ}) = \begin{cases} \displaystyle\sum_{\hat\alpha \subset \hat\beta} \mathbf{1}_{\Omega_{\hat\beta}^\circ} & \text{for } p \le q, \\ \displaystyle\sum_{\hat\alpha \supset \hat\beta} \mathbf{1}_{\Omega_{\hat\beta}^\circ} & \text{for } p \ge q, \end{cases}$$

where $\hat\beta$ ranges through sequences in $\Lambda_{q,n}$ containing (or contained by) $\hat\alpha$.

(ii) *In the real case, we have*

$$\mathcal{R}_{(n+1;p,q)}(\mathbf{1}_{\Omega_\alpha^\circ}) = \begin{cases} \displaystyle\sum_{\hat\alpha \subset \hat\beta} (-1)^{c_{\hat\alpha,\hat\beta}} \mathbf{1}_{\Omega_{\hat\beta}^\circ} & \text{for } p \le q, \\ \displaystyle\sum_{\hat\alpha \supset \hat\beta} (-1)^{c_{\hat\beta,\hat\alpha}} \mathbf{1}_{\Omega_{\hat\beta}^\circ} & \text{for } p \ge q, \end{cases}$$

where $\hat\beta$ ranges through sequences in $\Lambda_{q,n}$ containing (or contained by) $\hat\alpha$.

Proof. It is enough to consider the case where $p < q$ and to calculate

$$\mathcal{R}_S(\mathbf{1}_{\Omega_\alpha^\circ})(y) = \int_X \mathbf{1}_{g^{-1}(y) \cap S \cap f^{-1}(\Omega_\alpha^\circ)}.$$

We have

$$g^{-1}(y) \cap S \cap f^{-1}(\Omega_\alpha^\circ)$$
$$\simeq \left\{ x \in F_{n+1}(p) \;\middle|\; \begin{array}{l} x \subset y, \\ \dim(x \cap V_{a_i+i}) = i, \\ \dim(x \cap V_{a_i+i-1}) = i-1 \end{array} \;(i = 1, 2, \ldots, p) \right\}.$$

Here we fix any $x \in \Omega_\alpha$. If $x \subset y$, then y have the same gaps of dimensions of intersection with the complete flag of E as ones of x. Therefore $\hat{\beta}$ satisfies $\hat{\beta} \supset \hat{\alpha}$.

Let $x \in \Omega_\alpha$, $y \in \Omega_\beta$ where $\hat{\alpha} \subset \hat{\beta}$. Then we can choose the basis of the whole space E to contain the basis of y without changing the original complete flag. y has the complete flag which is a subflag of the complete flag of E. Namely, we define the complete flag of y by

$$V_{b_1+1} \subset V_{b_2+2} \subset \cdots \subset V_{b_q+q}$$
$$\| \quad\quad \| \quad \cdots \quad \|$$
$$V_1' \subset V_2' \subset \cdots \subset V_q' = y.$$

By considering in y, the fiber above is a Schubert cell of $F_q(p)$;

$$g^{-1}(y) \cap S \cap f^{-1}(\Omega_\alpha^\circ)$$
$$\simeq \left\{ x \in F_q(p) \;\middle|\; \begin{array}{l} \dim(x \cap V_{a_i+i}) = i, \\ \dim(x \cap V_{a_i+i-1}) = i-1 \end{array} \; (i=1,2,\ldots,p) \right\}$$
$$\simeq \left\{ x \in F_q(p) \;\middle|\; \begin{array}{l} \dim(x \cap V'_{\sigma_{\hat{\alpha},\hat{\beta}}(i)}) = i, \\ \dim(x \cap V'_{\sigma_{\hat{\alpha},\hat{\beta}}(i)-1}) = i-1 \end{array} \; (i=1,2,\ldots,p) \right\}.$$

Therefore we calculate the Euler-Poincaré index with compact supports of this Schubert cell. In the complex case, it is equal to 1. In the real case, it is equal to $(-1)^{c_{\hat{\alpha},\hat{\beta}}}$. So we obtain the desired results. $\square$

We can represent this formula by Young diagrams when $p=1$, $q=n$.

Definition 5.3. We denote by λ_k the Young diagram (k) with at most one row and n columns $(0 \leq k \leq n)$. For $\lambda = \lambda_k$, we define its dual with at most n rows and one column;

$$\lambda^* = (1,1,\ldots,1, \overset{n-k}{1}, 0,0,\ldots,0).$$

Definition 5.4. Let λ be a Young diagram with at most one row and n columns. For a Young diagram μ with at most n rows and one column, we set

$$\tau_\lambda(\mu) = \begin{cases} n - |\lambda| & (\text{for } n-|\lambda| < |\mu|), \\ n - |\lambda| - 1 & (\text{for } n-|\lambda| \geq |\mu|). \end{cases}$$

Proposition 5.1. *Let λ be a Young diagram with one row and n columns.*

(i) *In the complex case, we have*

$$\text{(5.1)} \qquad \mathcal{R}_{(n+1;1,n)}(\mathbf{1}_{\Omega_\lambda^\circ}) = \sum_{\mu \neq \lambda^*} \mathbf{1}_{\Omega_\mu^\circ},$$

where μ ranges through Young diagrams with at most n rows and one column which are not equal to λ^*. We can rewrite (5.1) as

$$\mathcal{R}_{(n+1;1,n)} \begin{pmatrix} \mathbf{1}_{\Omega_{\lambda_0}} \\ \mathbf{1}_{\Omega_{\lambda_1}} \\ \vdots \\ \mathbf{1}_{\Omega_{\lambda_n}} \end{pmatrix} = \begin{pmatrix} 0\,1\,1\cdots 1 \\ 1\,0\,1\cdots 1 \\ 1\,1\,0\cdots 1 \\ \vdots\,\vdots\quad\ddots\,\vdots \\ 1\,1\,1\ldots 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_{\Omega_{\lambda_0^*}} \\ \mathbf{1}_{\Omega_{\lambda_1^*}} \\ \vdots \\ \mathbf{1}_{\Omega_{\lambda_n^*}} \end{pmatrix}.$$

(ii) *In the real case, we have*

$$\text{(5.2)} \qquad \mathcal{R}_{(n+1;1,n)}(\mathbf{1}_{\Omega_\lambda^\circ}) = \sum_{\mu \neq \lambda^*} (-1)^{\tau_\lambda(\mu)} \mathbf{1}_{\Omega_\mu^\circ},$$

where μ ranges through Young diagrams with at most n rows and one column which are not equal to λ^*. We can rewrite (5.2) as

$$\mathcal{R}_{(n+1;1,n)} \begin{pmatrix} \mathbf{1}_{\Omega_{\lambda_0}} \\ \mathbf{1}_{\Omega_{\lambda_1}} \\ \vdots \\ \mathbf{1}_{\Omega_{\lambda_n}} \end{pmatrix} = \begin{pmatrix} 0 & (-1)^{n-1} & (-1)^{n-1} & \cdots & (-1)^{n-1} \\ (-1)^{n-1} & 0 & (-1)^{n-2} & \cdots & (-1)^{n-2} \\ (-1)^{n-2} & (-1)^{n-2} & 0 & & (-1)^{n-3} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_{\Omega_{\lambda_0^*}} \\ \mathbf{1}_{\Omega_{\lambda_1^*}} \\ \vdots \\ \mathbf{1}_{\Omega_{\lambda_n^*}} \end{pmatrix}.$$

A.   The Calculation of Euler-Poincaré Indices of Schubert Varieties

In this appendix, we calculate the Euler-Poincaré indices with compact supports of Schubert varieties straightforwardly. These calculations are elementary, but we write for the completeness of this paper. For the other more technical calculations with characteristic classes, we refer to [6], [10].

First, we extend the definition of $\binom{n}{p}$ by

$$\binom{n}{p} = \begin{cases} \binom{n}{p} & (n \geq p \geq 0), \\ 0 & \text{(otherwise)}. \end{cases}$$

Let x be an m-dimensional subspace of E. We calculate the Euler-Poincaré index with compact supports of the following Schubert variety:

$$\Omega^{m,k} := \{y \in F_n(p) \mid \dim(x \cap y) \geq k\},$$

which corresponds to the Young diagram $\lambda = (a_1, a_2, \ldots, a_p)$ with

$$a_j = \begin{cases} n - p - m + k \ (1 \leq j \leq k), \\ 0 \qquad\qquad\quad (k+1 \leq j \leq p). \end{cases}$$

We denote the Euler-Poincaré index with compact supports by χ_c.

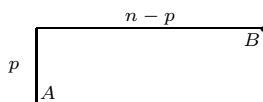First we consider complex Grassmann manifolds, i.e. $E = \mathbb{C}^{n+1}$.

Proposition A.1. *We have*

$$\mu_n(p) = \chi(F_n(p)) = \binom{n}{p}.$$

Proof. We calculate the Euler-Poincaré index with compact supports of a Schubert cell. Since $\mathbb{C}$ is a real 2-dimensional vector space, we have

$$\chi_c(\Omega_\mu^\circ) = \chi_c(\mathbb{C}^{p(n-p)-|\mu|}) = \chi(\mathbb{C}^{p(n-p)-|\mu|}) = 1.$$

By Proposition 2.1, we count the number of the shortest ways which connect from A to B (see Figure 2).
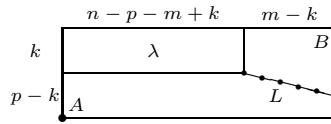

(Figure 2)

□

Proposition A.2. *We have*

$$\chi_c\left(\Omega^{m,k}\right) = \sum_{l=0}^{m-k} \binom{n-m}{p-k-l}\binom{m}{k+l}.$$

Proof. By the additivity of χ_c, we have for a Schubert variety

$$\chi_c(\Omega^{m,k}) = \chi_c\left(\coprod_{\lambda \subset \mu} \Omega_\mu^\circ\right) = \sum_{\lambda \subset \mu} \chi_c(\Omega_\mu^\circ) = \sum_{\lambda \subset \mu} 1 = \sharp\{\mu \mid \lambda \subset \mu\}.$$

We count the number of the shortest ways which connect from A to B through each point on the L (see Figure 3).

(Figure 3)

Next, we consider real Grassmann manifolds, i.e. $E = \mathbb{R}^{n+1}$.

First, we calculate the Euler-Poincaré index with compact supports of a Schubert cell. By the Poincaré duality we have

$$\chi_c(\Omega_\mu^\circ) = \chi_c(\mathbb{R}^{p(n-p)-|\mu|}) = (-1)^{p(n-p)-|\mu|}.$$

By the additivity of χ_c, we have for a Schubert variety

$$\chi_c(\Omega_\lambda) = \chi_c\left(\coprod_{\lambda \subset \mu} \Omega_\mu^\circ\right) = \sum_{\lambda \subset \mu} \chi_c(\Omega_\mu^\circ) = \sum_{\lambda \subset \mu} (-1)^{p(n-p)-|\mu|}$$
$$= (-1)^{p(n-p)} \sum_{\lambda \subset \mu} (-1)^{|\mu|}.$$

So we count the numbers of Young diagrams; the number of Young diagrams containing λ with at most p rows $n - p$ columns.

Definition A.1.   We set

(i) $e_n(p) = \sharp \left\{ \mu \ \middle| \ \begin{array}{l} \mu \text{ is a Young diagram with at most } p \text{ rows} \\ \text{and } n - p \text{ columns. } |\mu| \text{ is even.} \end{array} \right\}$,

(ii) $o_n(p) = \sharp \left\{ \mu \ \middle| \ \begin{array}{l} \mu \text{ is a Young diagram with at most } p \text{ rows} \\ \text{and } n - p \text{ columns. } |\mu| \text{ is odd.} \end{array} \right\}$.

Proposition A.3.   *We have*

(i) $e_n(p) = \dfrac{1}{2} \left\{ \dbinom{n}{p} + \mu_n(p) \right\},$ (ii) $o_n(p) = \dfrac{1}{2} \left\{ \dbinom{n}{p} - \mu_n(p) \right\},$

where $\mu_n(p)$ is the Euler index of the real Grassmann manifold $F_n(p)$.

Proof.   By Definition A.1, we have

$$e_n(p) - o_n(p) = \mu_n(p), \quad e_n(p) + o_n(p) = \binom{n}{p}.$$

Proposition A.4. *We have*

$$(A.1) \qquad \mu_n(p) = \chi(F_n(p)) = \begin{cases} 0 & (\text{if } p(n-p) \text{ is odd}), \\ \begin{pmatrix} E\left(\dfrac{n}{2}\right) \\ E\left(\dfrac{p}{2}\right) \end{pmatrix} & (\text{if } p(n-p) \text{ is even}). \end{cases}$$

Here $E\left(\frac{n}{2}\right)$ *denotes the integral part of* $\frac{n}{2}$*,* $\binom{a}{b}$ *is the binomial coefficient.*

Proof. We show this proposition by induction on p.

In the case where $p = 1$, we have (A.1) for all $n \geq 1$ from

$$o_n(1) = E\left(\frac{n}{2}\right), \qquad e_n(1) = E\left(\frac{n+1}{2}\right).$$

In the cases where $p > 1$, we consider the following four cases.

(i) p is odd, n is even.

For each Young diagram λ, we consider λ^c. Since the number of total boxes with at most p rows $n - p$ columns is odd, we have the equality $o_n(p) = e_n(p)$, i.e. $\mu_n(p) = 0$. This proves (A.1) in the case of (i).

In the following cases, we count the number of boxes in a Young diagram by dividing the cases according to the number of boxes in the first row.

(ii) p is odd, n is odd. Since $n - p$ is even, we have

$$o_n(p) = o_{n-1}(p-1) + e_{n-2}(p-1) + o_{n-3}(p-1) + \cdots + e_p(p-1),$$
$$e_n(p) = e_{n-1}(p-1) + o_{n-2}(p-1) + e_{n-3}(p-1) + \cdots + o_p(p-1) + 1.$$

Therefore we have

$$\mu_n(p) = \mu_{n-1}(p-1) - \mu_{n-2}(p-1) + \mu_{n-3}(p-1) - \cdots - \mu_p(p-1) + 1$$

$$= \sum_{k=p-1}^{n-1} (-1)^k \begin{pmatrix} E\left(\dfrac{k}{2}\right) \\ E\left(\dfrac{p-1}{2}\right) \end{pmatrix} = \begin{pmatrix} E\left(\dfrac{n-1}{2}\right) \\ E\left(\dfrac{p-1}{2}\right) \end{pmatrix} = \begin{pmatrix} E\left(\dfrac{n}{2}\right) \\ E\left(\dfrac{p}{2}\right) \end{pmatrix}.$$

(iii) p is even $(= 2q)$, n is even $(= 2m)$. Since $n - p$ is even, we have

$$o_n(p) = o_{n-1}(p-1) + e_{n-2}(p-1) + o_{n-3}(p-1) + \cdots + e_p(p-1),$$
$$e_n(p) = e_{n-1}(p-1) + o_{n-2}(p-1) + e_{n-3}(p-1) + \cdots + o_p(p-1) + 1.$$

Therefore we have

$$
\begin{aligned}
\mu_n(p) &= \mu_{n-1}(p-1) - \mu_{n-2}(p-1) + \mu_{n-3}(p-1) - \cdots - \mu_p(p-1) + 1 \\
&= \begin{pmatrix} E\left(\dfrac{n-1}{2}\right) \\[4pt] E\left(\dfrac{p-1}{2}\right) \end{pmatrix} - 0 + \begin{pmatrix} E\left(\dfrac{n-3}{2}\right) \\[4pt] E\left(\dfrac{p-1}{2}\right) \end{pmatrix} + \cdots - 0 + 1 \\
&= \sum_{k=q-1}^{m-1} \binom{k}{q-1} = \binom{m}{q} = \begin{pmatrix} E\left(\dfrac{n}{2}\right) \\[4pt] E\left(\dfrac{p}{2}\right) \end{pmatrix}.
\end{aligned}
$$

(iv) p is even $(= 2q)$, n is odd $(= 2m+1)$. Since $n - p$ is odd, we have

$$
\begin{aligned}
o_n(p) &= e_{n-1}(p-1) + o_{n-2}(p-1) + e_{n-3}(p-1) + \cdots + e_p(p-1), \\
e_n(p) &= o_{n-1}(p-1) + e_{n-2}(p-1) + o_{n-3}(p-1) + \cdots + o_p(p-1) + 1.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\mu_n(p) &= -\mu_{n-1}(p-1) + \mu_{n-2}(p-1) - \mu_{n-3}(p-1) - \cdots - \mu_p(p-1) + 1 \\
&= -0 + \begin{pmatrix} E\left(\dfrac{n-2}{2}\right) \\[4pt] E\left(\dfrac{p-1}{2}\right) \end{pmatrix} - 0 + \cdots - 0 + 1 \\
&= \sum_{k=q-1}^{m-1} \binom{k}{q-1} = \binom{m}{q} = \begin{pmatrix} E\left(\dfrac{n}{2}\right) \\[4pt] E\left(\dfrac{p}{2}\right) \end{pmatrix}.
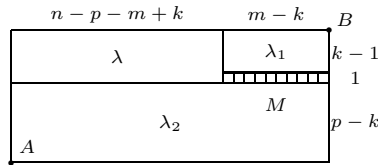\end{aligned}
$$

$\square$

Proposition A.5.   *We have*

$$
\chi_c\left(\Omega^{m,k}\right) = \begin{cases} (-1)^{p(n-p)} \left| \displaystyle\sum_{l=0}^{m-k} (-1)^l \mu_{l+k-1}(l)\mu_{n-k-l}(p-k) \right| & (k \geq 1), \\[12pt] (-1)^{p(n-p)} \mu_n(p) & (k = 0). \end{cases}
$$

Proof.   We set $eo_n^j(p) = \dfrac{1}{2}\left\{ \binom{n}{p} + (-1)^j \mu_n(p) \right\}$.

Then we have

$$
(-1)^{p(n-p)} \cdot \chi_c\left(\Omega^{m,k}\right)
$$

$$
= \begin{cases} \left| \begin{array}{c} \sum\limits_{l=0}^{m-k} eo_{l+k-1}^{l-1}(l)o_{n-k-l}(p-k) + eo_{l+k-1}^{l}(l)e_{n-k-l}(p-k) \\ -eo_{l+k-1}^{l}(l)o_{n-k-l}(p-k) - eo_{l+k-1}^{l-1}(l)e_{n-k-l}(p-k) \end{array} \right| & (k \geq 1), \\ \mu_n(p) & (k = 0) \end{cases}
$$

$$
= \begin{cases} \left| \sum\limits_{l=0}^{m-k}(-1)^l \mu_{l+k-1}(l)\mu_{n-k-l}(p-k) \right| & (k \geq 1), \\ \mu_n(p) & (k = 0) \ . \end{cases}
$$

(Figure 4)

Here, λ_1 is a Young diagram with k rows and $m - k$ columns. Further λ_2 is a Young diagram with $p - k$ rows and $n - p$ columns.

For example, if $k(m - k)$ is even, we count the number of Young diagrams which have even boxes in λ_1 and odd boxes in λ_2. We divide our problem into cases where diagrams have $m - k - j$ boxes at M from the left ($0 \leq j \leq m - k$). If j is even, the number of Young diagrams that we count is $e_{k-1+j}(k-1) \times o_{n-k-j}(p-k)$. If j is odd, the number of Young diagrams that we should count is $o_{k-1+j}(k-1) \times o_{n-k-j}(p-k)$ (see Figure 4). □

At the end of this appendix, we calculate $\chi_c(\Omega^{m,0}) - \chi_c(\Omega^{m,1})$ in another way, which plays an important role in Section 3.2.
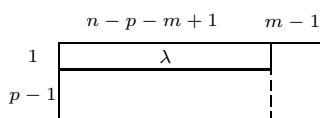
Proposition A.6. *We have*

$$
\chi_c(\Omega^{m,0}) - \chi_c(\Omega^{m,1}) = (-1)^{p(n-p)}\mu_{n-m}(p).
$$

Proof. We denote by $e_p^\lambda(n)$ (resp. $o_p^\lambda(n)$) the number of Young diagrams containing λ with p rows and $n - p$ columns whose number of boxes is even (resp. odd).

Then for the Young diagram $\lambda = (n - p - m + 1, 0, \ldots, 0)$, we have

$$
\begin{aligned}
(-1)^{p(n-p)}\{\chi_c(\Omega^{m,0}) - \chi_c(\Omega^{m,1})\} &= \mu_n(p) - (e_p^\lambda(n) - o_p^\lambda(n)) \\
&= (e_p(n) - e_p^\lambda(n)) - (o_p(n) - o_p^\lambda(n)) \\
&= e_p^{\lambda'}(n) - o_p^{\lambda'}(n) \\
&= \mu_{n-m}(p),
\end{aligned}
$$

where $\lambda' = (n - p - m, 0, \ldots, 0)$ (see Figure 5).

(Figure 5)

$\square$

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