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Low Rank Cohomology of the Classifying Spaces of Gauge Groups over 3-manifolds

By

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Abstract

The purpose of this paper is to calculate the cohomology of the function space $\mathrm{Map}(M, BG)$ for degree less than or equal to 3, where G is a simply connected compact Lie group and M is a closed orientable 3-manifold. The calculation enables us to obtain a simple proof and an improvement of the result [4, Theorem 1.2].

§1. Introduction

Let G be a simply connected compact Lie group and M a closed orientable 3-manifold. Since BG is 3-connected, any principal G-bundles over M are trivial. Then we call the gauge group of the trivial G-bundle over M the gauge group over M and denote it by $\mathcal{G}$. It is well-known that

$$B\mathcal{G} \simeq \mathrm{Map}(M, BG)$$

([2]). The cohomology of $B\mathcal{G}$ in low dimensions is considered in [4] by making use of the Eilenberg-Moore spectral sequence.

Theorem 1.1 [4, Theorem 1.2]. *Suppose that* $\mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}/2, R) = 0$. *Let* G *be a simply-connected compact Lie group such that the integral cohomology of* BG *is torsion free and let* M *be a closed orientable* 3*-manifold. We denote* $H^i(\mathrm{Map}(M, BG); R)$ *by* H^i. *Then there exists a short exact sequence*

$$0 \to H_1(M; R)^{\oplus r} \oplus R^{\oplus s} \oplus H^1 \otimes H^2 \xrightarrow{\alpha} H^3 \to (R/2R)^{\oplus r} \to 0,$$

where $\alpha|_{H^1 \otimes H^2}$ *is the cup product,* $r = \mathrm{rank} H^4(BG)$ *and* $s = \mathrm{rank} H^6(BG)$. *Moreover* H^1 *is a free* R*-module for any* R*, and* H^2 *is also free if* R *is a PID.*

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The purpose of this paper is to refine Theorem 1.1 and to give a simple proof. We determine the integral cohomology of $H^i(\mathrm{Map}(M, BG))(i \le 3)$ without the assumption that $H^*(BG)$ is torsion free. It is known that G is the direct product of simply connected compact simple Lie groups ([6]). Then we reduce Theorem 1.1 to the case that G is a simply connected compact simple Lie group and obtain

Theorem 1.2. *Let G be a simply connected compact simple Lie group and M a closed orientable 3-manifold. We denote $H^i(\mathrm{Map}(M, BG))$ by H^i. Then we have*

$$H^i \cong \begin{cases} \mathbb{Z} & i = 0 \\ H^1(\Omega^2 G) & i = 1 \\ H_2(M) & i = 2 \\ H^1(\Omega^2 G) \otimes H_2(M) \oplus H_1(M) \oplus H^3(\Omega^2 G) & i = 3. \end{cases}$$

Moreover, the cup product $H^1 \otimes H^2 \to H^3$ maps $H^1 \otimes H^2$ isomorphically onto the direct summand $H^1(\Omega^2 G) \otimes H_2(M) \subset H^3$.

Remark. Let G be a simply connected compact simple Lie group. We here describe the integral cohomology $H^*(\Omega^2 G)$ for $* \le 3$.

Table 1. $H^*(\Omega^2 G)$

$H^i(\Omega^2 G)$	type of G $A_l (l \ge 2)$	$C_l (l \ge 1)$	otherwise
$i = 1$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$i = 2$	0	0	0
$i = 3$	$\mathbb{Z}$	$\mathbb{Z}/2$	0

Remark. Since the inclusion of the 1-skeleton $\bigvee^{g-1} S^1 \to \overbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}^{g}$ induces an isomorphism on mod p cohomology for each odd prime p, Theorem 1.1 in [4] is easily shown without the assumption that $H^*(G)$ is p-torsion free by [5, Proposition 4.2] and [3, Ch. VI, Proposition 7.1].

§2. Approximation of G by Infinite Loop Spaces

Let G be a simply connected compact Lie group. In this section we approximate G by an infinite loop space in low dimensions.

It is known that G is the direct product of simply connected compact simple Lie groups. Since simply connected compact simple Lie groups are classified by their Lie algebras as $A_l, B_l, C_l, D_l (l \geq 1), E_l (l = 6, 7, 8), F_4, G_2$, we give an approximation to each type.

Proposition 2.1. *Let G be a simply connected compact simple Lie group. Then there exist an infinite loop space $\mathbf{B}$ and a 7-equivalence $t : BG \to \mathbf{B}$.*

Proof. First we note that there are the following correspondence in low ranks ([6]):

$$A_1 = B_1 = C_1,\ B_2 = C_2,\ D_2 = A_1 \times A_1,\ D_3 = A_3.$$

Therefore we have only to consider the types $A_l(l \geq 2), B_l(l \geq 3), C_l(l \geq 1), D_l(l \geq 4), E_l(l = 6, 7, 8), F_4$ and G_2. In the case where G is of type $A_l(l \geq 2)$, there exists a 7-equivalence $BG \to BSU$ induced by the inclusion $G \to SU$. Similarly in the case where G is of type $C_l(l \geq 1)$, there exists a 7-equivalence $BG \to BSp$. For G otherwise, we have $\pi_i(G) = 0$ $(i = 1, 2, 4, 5)$ ([6]). Then a representative of a generator of $H^4(BG; \mathbb{Z}) \cong \mathbb{Z}$ is a 7-equivalence $BG \to K(\mathbb{Z}, 4)$. $\square$

Corollary 2.1. *Let M be a 3-dimensional complex. Then we have $H^i(\mathrm{Map}(M, \mathbf{B})) \cong H^i(\mathrm{Map}(M, BG))$ $(i \leq 3)$, where $\mathbf{B}$ is as in Proposition 2.1.*

Proof. Let $\mathrm{Map}_*(X, Y)$ denote the space of basepoint preserving maps from X to Y, where X, Y are based spaces.

We consider the following commutative diagram

$$\begin{array}{ccc} \pi_k(\mathrm{Map}_*(M, BG)) & \xrightarrow{\pi_*(\mathrm{Map}_*(Id,t))} & \pi_k(\mathrm{Map}_*(M, \mathbf{B})) \\ \cong\downarrow & & \cong\downarrow \\ [S^k \wedge M, BG] & \xrightarrow{t_*} & [S^k \wedge M, \mathbf{B}]. \end{array}$$

Since the second row is an isomorphism for $k \leq 3$ and a surjection for $k = 4$ by J.H.C. Whitehead theorem, we see that $\mathrm{Map}_*(Id, t) : \mathrm{Map}_*(M, BG) \to$

$\mathrm{Map}_*(M, \mathbf{B})$ is a 4-equivalence. Consider the following commutative diagram of evaluation fibrations

$$\begin{array}{ccccc} \mathrm{Map}_*(M, BG) & \longrightarrow & \mathrm{Map}(M, BG) & \longrightarrow & BG \\ {\scriptstyle \mathrm{Map}_*(Id,t)}\big\downarrow & & {\scriptstyle \mathrm{Map}(Id,t)}\big\downarrow & & {\scriptstyle t}\big\downarrow \\ \mathrm{Map}_*(M, \mathbf{B}) & \longrightarrow & \mathrm{Map}(M, \mathbf{B}) & \longrightarrow & \mathbf{B} \end{array}$$

Since $\mathrm{Map}_*(Id, t)$ is a 4-equivalence and t is a 7-equivalence, it follows that $\mathrm{Map}(Id, t) : \mathrm{Map}(M, BG) \to \mathrm{Map}(M, \mathbf{B})$ is a 4-equivalence. □

§3. Proof of Theorem 1.2

Let G be a simply connected compact simple Lie group. By Proposition 2.1 there exists an infinite loop space $\mathbf{B}$ and a 7-equivalence $t : BG \to \mathbf{B}$. Since $\mathbf{B}$ is a homotopy group, we have a homotopy equivalence

$$\begin{aligned} \mathrm{Map}(M, \mathbf{B}) &\simeq \mathrm{Map}_*(M, \mathbf{B}) \times \mathbf{B} \\ f &\mapsto (f \cdot f(*)^{-1}, f(*)), \end{aligned}$$

where $*$ denotes the basepoint of M. Since the infinite loop space $\mathbf{B}$ is 3-connected, the inclusion map $\mathrm{Map}_*(M, B) \to \mathrm{Map}(M, B)$ induces the isomorphism on homology for degree less than or equal to 3. Then we compute $H^*(\mathrm{Map}_*(M, \mathbf{B}))$ to determine $H^*(\mathrm{Map}(M, \mathbf{B}))$.

Proposition 3.1. *We have*

$$H^k(\mathrm{Map}_*(M, \mathbf{B})) \cong \bigoplus_{i+j=k} H^i(\mathrm{Map}_*(M^2, \mathbf{B})) \otimes H^j(\Omega^3\mathbf{B})$$

for $k \leq 3$, where M^2 is the 2-skeleton of M.

Proof. Since a closed orientable 3-manifold is parallelizable, the top cell of M is split off stably ([1]). Actually by Freudenthal suspension theorem the top cell of M is split off after double suspension. Then we have

$$\begin{aligned} \mathrm{Map}_*(M, \mathbf{B}) &\simeq \mathrm{Map}_*(M, \Omega^2 B^2\mathbf{B})) \\ &\simeq \mathrm{Map}_*(\Sigma^2 M, B^2\mathbf{B}) \\ &\simeq \mathrm{Map}_*(\Sigma^2 M^2 \vee S^5, B^2\mathbf{B}) \\ &\simeq \mathrm{Map}_*(M^2 \vee S^3, \mathbf{B}) \\ &\simeq \mathrm{Map}_*(M^2, \mathbf{B}) \times \Omega^3\mathbf{B}. \end{aligned}$$

Since $H^k(\Omega^3\mathbf{B})$ $(k<3)$ is either 0 or $\mathbb{Z}$, the proof is completed by Künneth Theorem. □

To compute $H^i(\mathrm{Map}_*(M^2,\mathbf{B}))$ $(i\le 3)$ we need the following technical lemma. Let X,Y,Z be based spaces and $f:X\to Y$ be a based map. We denote by $f^\#$ the induced map $\mathrm{Map}_*(f,Id):\mathrm{Map}_*(Y,Z)\to\mathrm{Map}_*(X,Z)$.

Lemma 3.1. *Let X be a based space such that there is a $(p+q+1)$-equivalence $g:X\to K(\mathbb{Z},p+q)$ and let $f:\bigvee^l S^p\to\bigvee^m S^p$ be a based map. Suppose $(f^\#)^*:H^q(\Pi^l\Omega^pX)\to H^q(\Pi^m\Omega^pX)$ is represented by a matrix A for a certain basis. Then $f_*:H_p(\bigvee^l S^p)\to H_p(\bigvee^m S^p)$ is also represented by A for a suitable basis.*

Proof. Since $g_*:[S^{p+q},X]\to[S^{p+q},K(\mathbb{Z},p+q)]$ is an isomorphism, we have the following commutative diagram

$$\begin{array}{ccccccc}
H_q(\Pi^m\Omega^pX) & \xrightarrow[hur]{\cong} & \pi_q(\Pi^m\Omega^pX) & = & [\bigvee^m S^{p+q},X] & \xrightarrow{\cong} & H^{p+q}(\bigvee^m S^{p+q}) \\
\downarrow{\scriptstyle (f^\#)_*} & & \downarrow{\scriptstyle (f^\#)_*} & & \downarrow & & \downarrow{\scriptstyle (\Sigma^q f)^*} \\
H_q(\Pi^l\Omega^pX) & \xrightarrow[hur]{\cong} & \pi_q(\Pi^l\Omega^pX) & = & [\bigvee^l S^{p+q},X] & \xrightarrow{\cong} & H^{p+q}(\bigvee^l S^{p+q}),
\end{array}$$

where hur is the Hurewicz homomorphism. Since $\Omega^pX\to K(\mathbb{Z},q)$ is a $(q+1)$-equivalence, the proof is completed by taking the dual. □

Proposition 3.2. $$H^i(\mathrm{Map}_*(M^2,\mathbf{B}))\cong\begin{cases}0 & i=1\\ H_2(M) & i=2\\ H_1(M) & i=3\end{cases}$$

Proof. We have the following cofibration sequence $\bigvee^l S^1\xrightarrow{f}\bigvee^m S^1\xrightarrow{i} M^2\to\bigvee^l S^2$, where f is the attaching map of 2-cells of M and i is the inclusion. Then we have the fibration

$$\Pi^l\Omega^2\mathbf{B}\to\mathrm{Map}_*(M^2,\mathbf{B})\xrightarrow{i^\#}\Pi^m\Omega\mathbf{B}.$$

We consider the Leray-Serre spectral sequence (E_r,d_r) of the fibration above. Since $E_2^{p,q}\cong H^p(\Pi^m\Omega\mathbf{B})\otimes H^q(\Pi^l\Omega^2\mathbf{B})$, $\mathbf{B}$ is 3-connected and $H^4(\mathbf{B})\cong\mathbb{Z}$, the non-trivial differential $d_r:E_r^{p,q}\to E_r^{p+r,q-r+1}$ $(p+q\le 4)$ occurs only when $r=3$ and $(p,q)=(0,2)$. Then we obtain $H^1(\mathrm{Map}_*(M^2,\mathbf{B}))=0$. Next we determine $d_3:E_3^{0,2}\to E_3^{3,0}$ to compute $H^i(\mathrm{Map}_*(M^2,\mathbf{B}))$ $(i=2,3)$. We consider the commutative diagram

$$\begin{array}{ccccccc} \bigvee^l S^1 & \xrightarrow{f} & \bigvee^m S^1 & \xrightarrow{i} & M^2 & \longrightarrow & \bigvee^l S^2 \\ {\scriptstyle f}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow & & {\scriptstyle \Sigma f}\downarrow \\ \bigvee^m S^1 & \xrightarrow{1} & \bigvee^m S^1 & \xrightarrow{j} & \bigvee^m D^2 & \longrightarrow & \bigvee^m S^2, \end{array}$$

where j is the inclusion. Applying $\mathrm{Map}_*(\ ,\mathbf{B})$ to above, we have the following commutative diagram

$$\begin{array}{ccccc} \Pi^m\Omega\mathbf{B} & \xleftarrow{i^\#} & \mathrm{Map}_*(M^2,\mathbf{B}) & \longleftarrow & \Pi^l\Omega^2\mathbf{B} \\ {\scriptstyle 1}\uparrow & & \uparrow & & {\scriptstyle (\Sigma f)^\#}\uparrow \\ \Pi^m\Omega\mathbf{B} & \xleftarrow{j^\#} & \Pi^m P(\Omega\mathbf{B}) & \longleftarrow & \Pi^m\Omega^2\mathbf{B}, \end{array}$$

where the second row is the product of the path space fibrations of $\Omega\mathbf{B}$. Comparing the Leray-Serre spectral sequence of fibrations above, we obtain $d_3 = \tau(\Sigma f)^{\#*} : E_3^{0,2} \to E_3^{3,0}$, where $\tau : H^2(\Pi^m\Omega^2\mathbf{B}) \xrightarrow{\cong} H^3(\Pi^m\Omega\mathbf{B})$ is the transgression.

Let A be a matrix which represents $((\Sigma f)^\#)^* : H^2(\Pi^l\Omega^2\mathbf{B}) \to H^2(\Pi^m\Omega^2\mathbf{B})$. By Lemma 3.1, $(\Sigma f)_* : H_2(\bigvee^l S^2) \to H_2(\bigvee^m S^2)$ is represented by A and so is $f_* : H_1(\bigvee^l S^1) \to H_1(\bigvee^m S^1)$. Then we have the exact sequence

$$0 \to H_2(M^2) \to H_1\left(\bigvee^l S^1\right) \xrightarrow{A} H_1\left(\bigvee^m S^1\right) \to H_1(M^2) \to 0.$$

Since $H_i(M^2) \cong H_i(M)$ $(i \le 2)$, we have

$$\begin{aligned} H^2(\mathrm{Map}_*(M^2,\mathbf{B})) &\cong \mathrm{Ker}\{d_3 : E_3^{0,2} \to E_3^{3,0}\} \cong \mathrm{Ker}A \cong H_2(M), \\ H^3(\mathrm{Map}_*(M^2,\mathbf{B})) &\cong \mathrm{Coker}\{d_3 : E_3^{0,2} \to E_3^{3,0}\} \cong \mathrm{Coker}A \cong H_1(M). \end{aligned}$$

□

Proof of Theorem 1.2. By Corollary 2.1, Proposition 3.1 and Proposition 3.2, Theorem 1.2 is proved. □

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