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Infinite Dimensionality of the Middle L^2-cohomology on Non-compact Kähler Hyperbolic Manifolds

By

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Abstract

We prove that the space of L^2 harmonic forms of middle degree is infinite dimensional on any non-compact Kähler hyperbolic manifold.

§1. Introduction

Let (M, ω) be a complete Kähler manifold of dimension n and let $\mathcal{H}_2^{p,q}(M)$ denote the space of L^2-harmonic $(p,q)-$forms. In a groundbreaking paper, Donnelly and Fefferman [8] discovered a new L^2-estimate which implies the vanishing of $\mathcal{H}_2^{p,q}(M)$ outside of middle degree for those manifolds which have a complete Kähler metric $\omega = \partial\bar{\partial}\rho$ with the global potential satisfying $\sup_M |d\rho|_\omega < \infty$. Inspired by their work, Ohsawa and Takegoshi [18] proved the remarkable L^2-extension theorem. Ohsawa also found several interesting applications of the Donnelly-Fefferman estimate, for instance, to the Hodge theory on singular complex spaces and to the study of the Bergman metric (cf. [16], [5] etc). The main result in [8] is $\mathcal{H}_2^{p,q}(M) = 0$ for $p + q \neq n$ and $\dim \mathcal{H}_2^{p,q}(M) = \infty$ for $p+q = n$, associated to the Bergman metric on bounded strongly pesudoconvex domains. A different approach of infinite dimensionality was proposed by Ohsawa [17].

In a more geometric direction, Gromov [10] introduced a new notion of hyperbolicity as follows. A Kähler manifold M is called Kähler hyperbolic if

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there is a complete Kähler metric Ω which is $d-$bounded, i.e., $\omega = d\eta$ for some $1-$form η with $\|\eta\|_{L^\infty} < \infty$. Since such a metric cannot exist on compact Kähler manifolds, Gromov called a compact Kähler manifold (M, ω) Kähler hyperbolic if the lift of ω to the universal covering of M is $d-$bounded. Examples of non-compact Kähler hyperbolic manifolds include all hyperconvex manifolds (i.e., there is a bounded C^∞ strictly plurisubharmonic exhaustion function) and those $D\backslash S$ where D is a bounded hyperconvex domain and S is a complex submanifold defined in a neighborhood of $\bar{D}$. Gromov proved that the L^2-cohomology vanishes outside the middle degree for non-compact Kähler hyperbolic manifolds and the existence of L^2-harmonic forms of middle degree on the universal covering of every compact Kähler hyperbolic manifold, associated to the lifting metric.

Using the idea of Gromov, Donnelly [6] gave a more transparent proof of the results from [8]. He also discovered some new examples of Kähler hyperbolic domains with respect to the Bergman metric, for instance, bounded pseudoconvex domains of finite type in $\mathbf{C}^2$ or convex domains of finite type in $\mathbf{C}^n$ (cf. [7]). The L^2-cohomology with respect to the Bergman metric is of independent interest, simply because the latter is a canonical invariant metric. Recently, the author [4] showed that the Teichmüller space with the Bergman metric is Kähler hyperbolic (Earlier, McMullen [14] constructed a $d-$bounded Kähler metric by using the Weil-Petersson metric and hyperbolic length functions).

Comparing to the rather strong vanishing theorems in [8], [10], the conditions for non-vanishing results seem to be less transparent. In this spirit, we will show

Theorem 1. *Let (M, ω) be a complete $n-$dimensional Kähler manifold such that ω is $d-$bounded. Then we have*

$$\dim \mathcal{H}_2^{p,q}(M) = \infty, \quad p + q = n.$$

From 0.1.B. in [10], we obtain

Corollary 1. *If (M, ω) is a complete simply connected Kähler manifold with sectional curvature bounded above by a negative constant, then*

$$\dim \mathcal{H}_2^{p,q}(M) = \infty, \quad p + q = n.$$

Let $\mathcal{T}_{g,n}$ denote the Teichmüller space of a Riemann surface of genus g and with n punctures. It is a complex manifold of dimension $3g - 3 + n$. Since the Bergman metric on $\mathcal{T}_{g,n}$ is $d-$bounded (cf. [4]), one has

Corollary 2. *With respect to the Bergman metric,*

$$\dim \mathcal{H}_2^{p,q}(\mathcal{T}_{g,n}) = \infty, \quad p+q = 3g-3+n.$$

Assume M is a holomorphic family of Riemann surfaces of genus g and n–punctures over the unit polydisc Δ^m in $\mathbf{C}^m$. According to the celebrated Bers simultaneous uniformization, the universal covering $\tilde{M}$ of M is a holomorphic family of conformal discs over Δ^m (in particular, $\tilde{M}$ is a domain in $\mathbf{C}^{m+1}$), and there is a holomorphic map $f : \Delta^m \to \mathcal{T}_{g,n}$, which naturally induces a holomorphic map $\hat{f} : \tilde{M} \to \mathcal{F}_{g,n}$, where $\mathcal{F}_{g,n}$ is the Bers fiber space over $\mathcal{T}_{g,n}$, which maps fibers to fibers. Set $\tilde{\omega} = ds^2_{\Delta^m} + \hat{f}^*(ds^2_{\mathcal{F}_{g,n}})$, where $ds^2_{\Delta^m}$, $ds^2_{\mathcal{F}_{g,n}}$ denote the Bergman metrics on Δ^m, $\mathcal{F}_{g,n}$ respectively.

Corollary 3. *With respect to $\tilde{\omega}$,*

$$\dim \mathcal{H}_2^{p,q}(\tilde{M}) = \infty, \quad p+q = m+1.$$

Proof. Since $\mathcal{F}_{g,n}$ is biholomorphic to $\mathcal{T}_{g,n+1}$ (cf. [2]), by a similar argument as in [4], we can show that $ds^2_{\mathcal{F}_{g,n}}$ is also d–bounded, which implies $\rho = -e^{-\epsilon \log K_{\mathcal{F}_{g,n}}}$ is a negative strictly plurisubharmonic exhaustion function for sufficiently small $\epsilon > 0$, where $K_{\mathcal{F}_{g,n}}$ denotes the Bergman kernel of $\mathcal{F}_{g,n}$. Since

$$\begin{aligned} \hat{f}^*(ds^2_{\mathcal{F}_{g,n}}) &= \partial\bar{\partial} \log(K_{\mathcal{F}_{g,n}} \circ \hat{f}) \\ &= \partial\bar{\partial}\left(-\frac{1}{\epsilon}\log(-\rho\circ\hat{f})\right) \\ &\geq \epsilon\partial\left(-\frac{1}{\epsilon}\log(-\rho\circ\hat{f})\right)\bar{\partial}\left(-\frac{1}{\epsilon}\log(-\rho\circ\hat{f})\right), \end{aligned}$$

we conclude that $\tilde{\omega}$ is a d–bounded complete Kähler metric on $\tilde{M}$. □

The proof of Theorem 1 is a modification of the original argument of Donnelly-Fefferman, which turns out to be quite simple since we only use the vanishing theorem, while in [8] the existence of L^2–harmonic forms in the unit ball and asymptotic behavior of the Bergman metric on strongly pseudoconvex domains (cf. [9], [13]) play an essential role, even in the special case of the ball one has to use some deep theorems such as Atiyah's L^2–index theorem [1] and the Hirzebruch proportionality principle [11].

Vanishing theorems in [8], [10] have been extended to certain "non-elliptic" cases in [3], [12], [15]. A typical example of those results is $\mathbf{C}^n$ equipped with the Euclidean metric. Clearly, one cannot expect the existence of L^2–harmonic forms.

§2. Proof of Theorem 1

Let (M,ω) be a complete Kähler manifold of dimension n. Let $L_2^{p,q}(M)$ denote the Hilbert space of $(p,q)-$forms with respect to the norm defined by

$$\|\psi\|_2 = \left(\int_M \psi \wedge \bar{*}\psi\right)^{1/2}$$

where $\bar{*}$ is the conjugate of the Hodge star operator $*$ associated to ω. Let $\bar{\partial}^*$ denote the adjoint of $\bar{\partial}$. The space of L^2-harmonic forms is

$$\mathcal{H}_2^{p,q}(M) = \{\psi \in L_2^{p,q}(M) : \bar{\partial}\psi = 0,\ \bar{\partial}^*\psi = 0\}.$$

We need the following important observation of Gromov:

Theorem (cf. [10]). *Let (M,ω) be a complete Kähler manifold of dimension n and $\omega = d\eta$ where η is a bounded 1-form on M. Then every L^2-form ψ of degree $p+q \neq n$ satisfies the inequality*

$$\|\bar{\partial}\psi\|_2^2 + \|\bar{\partial}^*\psi\|_2^2 \geq \lambda_0^2\|\psi\|_2^2 \tag{1}$$

when the left hand side of the inequality exists, where λ_0 is a strictly positive constant which depends only on $n = \dim M$ and the bound on η,

$$\lambda_0 \geq \mathrm{const}_n \|\eta\|_{L_\infty}^{-1}.$$

Furthermore, inequality (1) *is satisfied by the L^2-forms of middle degree which are orthogonal to the harmonic forms.*

The idea of following key lemma comes from [8]:

Lemma 2. *Let (M,ω) be a complete Kähler manifold of dimension n and ω is $d-$bounded. Let (N,g) be another complete Kähler manifold of dimension n such that $\mathcal{H}_2^{p,q}(N) \neq 0$ for $p+q=n$. Suppose that for any $r>0$, there exist two sequences of mutually disjoint geodesic balls $B(x_j,r) \subset M$ and $B(y_j,r) \subset N$ such that the metric ω and its first derivatives are asymptotic on $B(x_j,r)$ to those of g on $B(y_j,r)$ as $j \to \infty$ by some diffeomorphisms. Then*

$$\dim \mathcal{H}_2^{p,q}(M) = \infty, \quad p+q=n.$$

Proof. Since $\mathcal{H}_2^{p,q}(N) \neq 0$ for $p+q=n$, for any $\epsilon > 0$ there exists a $\psi \in L_2^{p,q}(N)$ such that

$$\|\bar{\partial}\psi\|_2^2 + \|\bar{\partial}^*\psi\|_2^2 < \epsilon\|\psi\|_2^2.$$

For sufficiently large r and for all j, we have such ψ_j whose support is contained in the geodesic ball $B(y_j, r)$ of (N, g). Therefore, for every sufficiently large j we may transplant ψ_j to get a copy $\varphi_j \in C_0^{p,q}(B(x_j, r))$ such that

$$\|\bar{\partial}\varphi_j\|_2^2 + \|\bar{\partial}^*\varphi_j\|_2^2 < 2\epsilon\|\varphi_j\|_2^2,$$

where the L^2−norms are associated to (M, ω). Now assume $\dim \mathcal{H}_2^{p,q}(M) < \infty$. Then there exist $\varphi_{j_1}, \varphi_{j_2}, \ldots, \varphi_{j_m}$ where $j_k >> 1$ and constants $c_1, \ldots, c_m$ with at least one non-vanishing such that

$$\varphi = \sum_{k=1}^{m} c_k \varphi_{j_k} \in (\mathcal{H}_2^{p,q}(M))^{\perp}.$$

Applying Gromov's theorem we obtain

$$\lambda_0^2\|\varphi\|_2^2 \leq \|\bar{\partial}\varphi\|_2^2 + \|\bar{\partial}^*\varphi\|_2^2 < 2\epsilon\|\varphi\|_2^2$$

since the supports of φ_{j_k} are disjoint. If we take $\epsilon < \frac{\lambda_0^2}{2}$, then $\varphi = 0$, which is absurd.

Proof of Theorem 1. We start from the unit polydisc Δ^n with the standard metric

$$\omega_0 = \sum_{j=1}^{n} \partial\bar{\partial}(-\log(1 - |z_j|^2)).$$

For any $p + q = n$, it is not difficult to verify

$$\alpha = dz_1 \wedge \cdots \wedge dz_p \wedge d\bar{z}_{p+1} \wedge \cdots \wedge d\bar{z}_n \in \mathcal{H}_2^{p,q}(\Delta^n),$$

hence $\mathcal{H}_2^{p,q}(\Delta^n) \neq 0$. Set

$$\omega_1 = \epsilon\, \partial\bar{\partial}\{\chi(|z|)(-\log\log 1/|z|)\} + \omega_0$$

where χ is a cut-off function satisfying $\chi|_{(-\infty,1/4)} = 1$ and $\chi|_{(1/2,\infty)} = 0$. Clearly ω_1 gives a d−bounded complete Kähler metric on the punctured polydisc $\Delta^n\backslash\{0\}$ provided $\epsilon > 0$ small enough. Note that $\omega_1|_{\Delta^n\backslash B^n_{1/2}} = \omega_0$, and for any $r > 0$, $B(x, r) \subset \Delta^n\backslash B^n_{1/2}$ as $x \to \partial\Delta^n$, where $B^n_{1/2}$ denotes the Euclidean ball of radius $1/2$ and $B(x, r)$ are geodesic balls associated to ω_1. By Lemma 2, we obtain

$$\dim \mathcal{H}_2^{p,q}(\Delta^n\backslash\{0\}) = \infty, \quad p + q = n$$

with respect to ω_1. Now let (M, ω) be the Kähler manifold in Theorem 1. Fix a point $p \in M$ and take a coordinate polydisc Δ^n at p. Since ω is d−bounded, we can define a d−bounded complete Kähler metric on $M\backslash\{p\}$ by

$$\omega_2 = \epsilon\, \partial\bar{\partial}\{\chi(|z|)(-\log\log 1/|z|)\} + \omega$$

if ϵ is sufficiently small. Fix such a ϵ. Since the eigenvalues of $\partial\bar{\partial}(-\log\log 1/|z|)$ with respect to the Euclidean metric, say $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, are bounded below by

$$|z|^{-2}(-\log|z|)^{-2}.$$

It follows that for any $r > 0$, the metric ω_2 and its first derivatives are asymptotic (via normal coordinate comparison) to those of ω_1 on geodesic balls (w.r.t. ω_2) $B(x,r) \subset M\backslash\{p\} \cap \Delta^n\backslash\{0\}$ as $x \to p$. Hence the middle L^2-cohomology is non-vanishing for $(M\backslash\{p\}, \omega_2)$. Finally, since ω coincides with ω_2 outside a neighborhood of p, a similar argument as above shows the infinite dimensionality of the middle L^2-cohomology for (M, ω).

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