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On the Occupation Time on the Half Line of Pinned Diffusion Processes

By

Yuko Yano*

Abstract

The aim of the present paper is to generalize Lévy's result of the occupation time on the half line of pinned Brownian motion for pinned diffusion processes. An asymptotic behavior of the distribution function at the origin of the occupation time $\Gamma_+(t)$ and limit theorem for the law of the fraction $\Gamma_+(t)/t$ when $t \to \infty$ are studied. An expression of the distribution function by the Riemann–Liouville fractional integral for pinned skew Bessel diffusion processes is also obtained. Krein's spectral theory and Tauberian theorem play important roles in the proofs.

§0. Introduction

P. Lévy's arc-sine law is a well-known result for a Brownian motion: Let $B = \{B_t, P_x\}$ be a standard Brownian motion on $\mathbb{R}^1$ and $\Gamma_+(t) = \int_0^t 1_{[0,\infty)}(B_s)ds$, i.e., the occupation time on $[0, \infty)$. Then, for each $t > 0$, we have

$$P_0\Big(\frac{1}{t}\Gamma_+(t) \le x\Big) = P_0(\Gamma_+(1) \le x) = \frac{2}{\pi} \arcsin \sqrt{x}, \quad 0 \le x \le 1 \tag{0.1}$$

and

$$P_0\Big(\frac{1}{t}\Gamma_+(t) \le x \,\Big|\, B_t = 0\Big) = P_0(\Gamma_+(1) \le x | B_1 = 0) = x, \quad 0 \le x \le 1. \tag{0.2}$$

Many authors have been interested in these results and tried to extend them for more general stochastic processes. In the present paper we shall confine

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*Department of Information Sciences, Ochanomizu University, 2-1-1 Otsuka Bunkyo-ku Tokyo 112-8610, Japan.
e-mail: `yano@is.ocha.ac.jp`

ourselves to the case of one-dimensional diffusion processes and we are interested in the following two results: The first one is due to S. Watanabe [10]. He proved that all possible limiting laws of the fraction $\Gamma_+(t)/t$ as $t \to \infty$ are Lamperti's laws (3.1), which can be identified with the laws of the the time spent on the positive side by skew Bessel diffusion processes. The second one is due to Y. Kasahara and the author [5], which studies the relationship between the asymptotic behavior of the distribution function of $\Gamma_+(t)$ and that of the speed measure at $x = 0$.

However, we notice that these results are only concerned with (0.1) and it might be natural to study similar problems for (0.2). Thus the aim of this article is to study the same problems as above for pinned diffusion processes. First, we shall determine the asymptotic behavior of the distribution function at $x = 0$, which explains how it depends on the positive and negative sides of the speed measure (Theorem 2.1). Secondly, we shall obtain an expression of the distribution function by the Riemann–Liouville fractional integral for pinned skew Bessel diffusion processes (Theorem 3.1). Finally, we shall study a limit theorem for pinned diffusion processes (Theorem 3.2).

The idea of the proofs for the pinned diffusion processes are essentially the same as for non-pinned cases. The key to our proofs is the double Laplace transform formula (2.4) and Tauberian theorems for Laplace transform. However, it should be emphasized here that the proofs cannot be carried out completely in parallel. The difficulty is that we cannot apply the continuity theorem directly, because, unlike the non-pinned case, the index of the regular variation of the functions appearing in the double Laplace transform (2.4) is out of the range.

In Section 1, we shall introduce some notations and a brief review of Krein's correspondence and the generalized diffusion processes. Although they might be cumbersome to readers who are familiar to these materials, they are necessary to state and to prove our results. We shall state the asymptotic result with the proof in Section 3, where the double Laplace transform formula is also proved. Section 4 is devoted to our limit theorem. In Appendix, we shall give a simple version of the inversion formula for the (generalized) Stieltjes transform of arbitrary order.

§1. Preliminaries

We adopt the same notation as in [5] but we state it here for the completeness of the paper. See [6] and [4] for details.

Let $m : [0, l) \to [0, \infty)$ be a right-continuous, nondecreasing function where $0 \le l \le \infty$. We put $m(0-) = 0$ and $m(x) = \infty$ for $x \ge l$ when $l < \infty$ so that the

Borel measure dm is defined on $[0, l)$. Such dm is referred to as an *inextensible measure*. Let $\mathcal{M}$ be the class of all such functions m. $m \in \mathcal{M}$ is sometimes called a *string*. To a given $m \in \mathcal{M}$, we assign a function $h(\lambda)$ defined on $(0, \infty)$ in the following way: If $m(x) \equiv \infty$, then $h(\lambda) \equiv 0$. Otherwise, for $\lambda > 0$, we consider the following integral equations:

$$\phi(x, \lambda) = 1 + \lambda \int_0^x d\xi \int_{0-}^{\xi+} \phi(u, \lambda) dm(u),$$
$$\psi(x, \lambda) = x + \lambda \int_0^x d\xi \int_{0-}^{\xi+} \psi(u, \lambda) dm(u)$$

on the interval $[0, l)$. The equations have unique continuous solutions $\phi(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ on $[0, l)$ for each $\lambda > 0$. We then define

$$h(\lambda) = \lim_{x \uparrow l} \frac{\psi(x, \lambda)}{\phi(x, \lambda)} = \int_0^l \frac{dx}{\phi(x, \lambda)^2}.$$

We note that, if $m(x) = 0$ for $0 \leq x < l$, then $h(\lambda) \equiv l$. The correspondence $m \in \mathcal{M} \mapsto h(\lambda)$ is called *Krein's correspondence*. $h(\lambda)$ is called the *spectral characteristic function of the string* m.

The spectral characteristic function $h(\lambda)$, $h \not\equiv \infty$, has a unique representation

$$h(\lambda) = c + \int_{[0,\infty)} \frac{d\sigma(\xi)}{\lambda + \xi} \tag{1.1}$$

for some $0 \leq c < \infty$ and nonnegative Radon measure $d\sigma$ on $[0, \infty)$ such that

$$\int_{[0,\infty)} \frac{d\sigma(\xi)}{1 + \xi} < \infty.$$

The measure $d\sigma$ is called the *spectral measure* and the function $\sigma(t) = \int_{[0,t]} d\sigma(\xi)$ is called the *spectral function associated with the string* m. In fact, it holds that

$$\lim_{\lambda \to \infty} h(\lambda) = c = \inf\{x \geq 0; m(x) > 0\} = \inf \operatorname{Supp}(dm). \tag{1.2}$$

Let $\mathcal{H}$ be the class of all functions h of the form (1.1) and $h \equiv \infty$. An important result is that Krein's correspondence

$$m \in \mathcal{M} \mapsto h \in \mathcal{H}$$

is one-to-one and onto and defines a homeomorphism if suitable topologies are introduced on $\mathcal{M}$ and $\mathcal{H}$.

Let $m^* := m^{-1} \in \mathcal{M}$ be the right-continuous inverse of $m \in \mathcal{M}$. If $m \longleftrightarrow h$ is Krein's correspondence, then

$$m^*(x) \longleftrightarrow \frac{1}{\lambda h(\lambda)} =: h^*(\lambda)\Big(= c^* + \int_{[0,\infty)} \frac{d\sigma^*(\xi)}{\lambda+\xi}\Big) \tag{1.3}$$

is also Krein's correspondence. The function m^* is called the *dual string* of m.

Put $\psi = 1/h$. The function ψ is called the *spectral characteristic exponent of the string* m. We remark that $\psi(\lambda)$, $\lambda > 0$, is known to have the form

$$\psi(\lambda) = c_0 + c_1\lambda + \int_0^\infty (1-e^{-\lambda u})n(u)du \tag{1.4}$$

where $c_0 = 1/l$, $c_1 = c^*$ and

$$n(u) = \int_{(0,\infty)} e^{-\xi u}\xi d\sigma^*(\xi).$$

Now let $m_+, m_- \in \mathcal{M}$ such that

$$m_\pm : [0, l_\pm) \to [0,\infty)$$

and $m_-(0) = 0$. We define a Radon measure $dm(x)$ on $(-l_-, l_+)$ by

$$dm(x) = \begin{cases} dm_+(x) & on \quad [0,l_+), \\ d\check{m}_-(x) & on \quad (-l_-,0) \end{cases}$$

where $d\check{m}_-(x)$ is the image measure of dm_- under the mapping $x \mapsto -x$. Then a strong Markov process $X = \{X_t, P_x\}$ on $E_m = (\mathrm{Supp}(dm) \cup \{-l_-\} \cup \{l_+\}) \cap (-\infty,\infty)$ associated with the Feller generator $A = \frac{d}{dm(x)}\frac{d}{dx}$ can be constructed from the Brownian motion B on $\mathbb{R}^1$ by the time change. Here the boundaries l_+ and $-l_-$ are traps for the Markov process X. This process is called the *generalized diffusion process corresponding to the pair of strings* $\{m_+, m_-\}$. We denote the transition probability density of X with respect to dm as $p(t,x,y)$, i.e.,

$$P_x(X_t \in dy) = p(t,x,y)dm(y).$$

We remark that

$$\int_0^\infty e^{-\lambda t}p(t,0,0)dt = \frac{1}{\psi_+(\lambda)+\psi_-(\lambda)}, \quad \lambda > 0. \tag{1.5}$$

Let $X = \{X_t, P_x\}$ be a generalized diffusion process on $(-l_-, l_+)$ corresponding to the pair $\{m_+, m_-\}$ so that $m_\pm \in \mathcal{M}$ with $m_-(0) = 0$ and let $h_\pm$

be the spectral characteristic functions and $\psi_\pm$ be the spectral characteristic exponents associated with strings $m_\pm$, respectively. We call a process X the *skew Bessel diffusion process of dimension* $2-2\alpha$, $0<\alpha<1$, *with the skew parameter* p, $0\le p\le 1$, and denote it as $SKEWBES(2-2\alpha,p)$ if it corresponds to the pair $\{m_+,m_-\}$ given by

$$\begin{aligned} m_+(x) &= p^{1/\alpha}x^{1/\alpha-1}, \quad l_+=\infty, \\ m_-(x) &= (1-p)^{1/\alpha}x^{1/\alpha-1}, \quad l_-=\infty. \end{aligned}$$

We remark that, if $p=0$ or $p=1$, the process X is the Bessel process in its canonical scale which is reflecting at 0. The conditions for $m_\pm$ is equivalent to the following:

$$h_+(\lambda)=D_\alpha p^{-1}\lambda^{-\alpha}, \quad h_-(\lambda)=D_\alpha(1-p)^{-1}\lambda^{-\alpha}$$

and

$$\psi_+(\lambda)=pD_\alpha^{-1}\lambda^\alpha, \quad \psi_-(\lambda)=(1-p)D_\alpha^{-1}\lambda^\alpha$$

where $D_\alpha=\{\alpha(1-\alpha)\}^{-\alpha}\Gamma(1+\alpha)/\Gamma(1-\alpha)$. When $p=\alpha=1/2$, it is a Brownian motion with a multiplicative constant.

§2. Asymptotic Behavior of the Occupation Time Distribution Function

Our main result of this section is the following:

Theorem 2.1. *Let $X=\{X_t,P_x\}$ be a generalized diffusion process on $(-\infty,\infty)$ corresponding to $\{m_+,m_-\}$ and let $\Gamma_+(t)=\int_0^t 1_{[0,\infty)}(X_s)ds$. Let $\varphi(x)$ vary regularly at* 0 *with exponent* $1/\alpha$, $0<\alpha<1$. *If*

$$m_+(x)\sim\frac{\varphi(x)}{x}, \quad x\to 0+, \tag{2.1}$$

then

$$P_0(\Gamma_+(t)\le x|X_t=0)\sim\frac{D_\alpha^2}{\Gamma(1+2\alpha)}\cdot\frac{-\frac{d}{dt}g_-^*(t)}{p(t,0,0)}\{\varphi^{-1}(x)\}^2, \quad x\to 0+ \tag{2.2}$$

where

$$g_-^*(t)=\int_{[0,\infty)}e^{-t\xi}d\sigma_-^*(\xi). \tag{2.3}$$

Remark. If $\varphi(x)$ is a regularly varying function at 0 with exponent $1/\alpha$, then the asymptotic inverse $\varphi^{-1}(y)$ is defined and varies regularly with exponent α.

Remark. $g_-^*(t)$ is equal to the transition density $p_-^*(t,0,0)$ of a diffusion process on $(-\infty,0]$ corresponding to $\{0,m_-^*\}$ (0 is a reflecting barrier).

To prove Theorem 2.1, we prepare the following formula:

Theorem 2.2. *Let $X=\{X_t,P_x\}$ be a generalized diffusion process corresponding to $\{m_+,m_-\}$ on $(-l_-,l_+)$ and $\Gamma_+(t)=\int_0^t 1_{[0,\infty)}(X_s)ds$. Let $\psi_\pm$ be the characteristic exponents of $m_\pm$, respectively. Then, for $\lambda>0$, $\mu>0$,*

$$\int_0^\infty e^{-\mu t}E_0[e^{-\lambda\Gamma_+(t)}|X_t=0]p(t,0,0)dt=\frac{1}{\psi_+(\lambda+\mu)+\psi_-(\mu)}. \tag{2.4}$$

Proof. Put

$$z_\varepsilon(x)=E_x\Big[\int_0^\infty \frac{1_{[0,\varepsilon)}(X_t)}{m[0,\varepsilon)}e^{-\mu t-\lambda\Gamma_+(t)}dt\Big]$$

for $\lambda>0$, $\mu>0$. Then, according to Feynman–Kac formula, the function $z_\varepsilon(x)$ satisfies the equation

$$\Big(\mu+\lambda\cdot 1_{[0,\infty)}-\frac{d}{dm(x)}\frac{d}{dx}\Big)z_\varepsilon(x)=\frac{1_{[0,\varepsilon)}(x)}{m[0,\varepsilon)}.$$

The value at $x=0$ of the unique bounded solution of the preceding equation is

$$z_\varepsilon(0)=\frac{1}{m[0,\varepsilon)}\frac{1}{\psi_+(\lambda+\mu)+\psi_-(\mu)}\int_{[0,\varepsilon)}u_2(y,\lambda+\mu)dm(y)$$

where u_2 is the positive non-increasing solution of $(\mu+\lambda-\frac{d}{dm(x)}\frac{d}{dx})u(x)=0$ with $u(0)=1$. Then, letting $\varepsilon\to 0$, we have

$$z_\varepsilon(0)\to\frac{1}{\psi_+(\lambda+\mu)+\psi_-(\mu)}.$$

On the other hand,

(2.5)

$$\begin{aligned}z_\varepsilon(0)&=E_0\Big[\int_0^\infty \frac{1_{[0,\varepsilon)}(X_t)}{m[0,\varepsilon)}e^{-\mu t-\lambda\Gamma_+(t)}dt\Big]\\&=\int_0^\infty e^{-\mu t}E_0\Big[\frac{1_{[0,\varepsilon)}(X_t)}{m[0,\varepsilon)}e^{-\lambda\Gamma_+(t)}\Big]dt\\&=\int_0^\infty e^{-\mu t}\frac{1}{m[0,\varepsilon)}\int_{[0,\varepsilon)}E_0\big[e^{-\lambda\Gamma_+(t)}|X_t=y\big]p(t,0,y)dm(y).\end{aligned}$$

Now let

$$\tilde{p}(t,x,y) := E_x\big[e^{-\lambda\Gamma_+(t)}|X_t = y\big]p(t,x,y).$$

Then $\tilde{p}(t,x,y)$ is the transition density of the diffusion process with generator $\tilde{A} = \frac{d}{dm(x)}\frac{d}{dx} - \lambda \cdot 1_{[0,\infty)}$. It is shown by McKean [8] that $\tilde{p}(t,x,y)$, as a function of y, belongs to the domain of $\tilde{A}$, in particular, it is continuous in y. Hence

$$\text{the RHS of (2.5)} \to \int_0^\infty e^{-\mu t}E_0[e^{-\lambda\Gamma_+(t)}|X_t = 0]p(t,0,0)dt, \quad \varepsilon \to 0+.$$

Therefore we obtain

$$\int_0^\infty e^{-\mu t}E_0[e^{-\lambda\Gamma_+(t)}|X_t = 0]p(t,0,0)dt = \frac{1}{\psi_+(\lambda+\mu)+\psi_-(\mu)}.$$

□

Now we give the proof to our theorem.

Proof. By Theorem 2.2, we see

$$\int_0^\infty e^{-\mu t}E_0[e^{-\lambda\Gamma_+(t)}|X_t = 0]p(t,0,0)dt = \frac{1}{\psi_+(\lambda+\mu)+\psi_-(\mu)}$$

for $\lambda > 0, \mu > 0$. Differentiating the both sides with respect to μ, we have

$$\int_0^\infty e^{-\mu t}E_0[e^{-\lambda\Gamma_+(t)}|X_t = 0]p(t,0,0)tdt = \frac{\frac{\partial}{\partial\mu}\big(\psi_+(\lambda+\mu)+\psi_-(\mu)\big)}{\big(\psi_+(\lambda+\mu)+\psi_-(\mu)\big)^2}. \tag{2.6}$$

We note by [4] that (2.1) is equivalent to

$$h_+(\lambda) = \frac{1}{\psi_+(\lambda)} \sim D_\alpha\varphi^{-1}\Big(\frac{1}{\lambda}\Big), \quad \lambda \to \infty. \tag{2.7}$$

Thus, $\psi_+(\lambda)$ varies regularly at ∞ with exponent $0 < \alpha < 1$ and hence $\frac{\partial}{\partial\mu}\psi_+(\lambda+\mu) \to 0$ as $\lambda \to \infty$. Therefore, as $\lambda \to \infty$,

$$\text{the RHS of (2.6)} \sim \frac{d}{d\mu}\psi_-(\mu) \Big/ \psi_+^2(\lambda)$$

and we obtain

$$\psi_+^2(\lambda)\int_0^\infty e^{-\mu t}E_0[e^{-\lambda\Gamma_+(t)}|X_t = 0]p(t,0,0)tdt \to \frac{d}{d\mu}\psi_-(\mu), \quad \lambda \to \infty. \tag{2.8}$$

On the other hand, by (1.3), $1/(\mu h_-(\mu)) = \psi_-(\mu)/\mu$ is the characteristic function of the dual string m_-^* and hence there exists a nonnegative Radon measure $d\sigma_-^*$ on $[0,\infty)$ such that $\int_0^\infty d\sigma_-^*(\xi)/(1+\xi) < \infty$ and

$$\frac{\psi_-(\mu)}{\mu} = \frac{1}{\mu h_-(\mu)} = \int_0^\infty \frac{d\sigma_-^*(\xi)}{\mu+\xi} = \int_0^\infty e^{-\mu t}dt \int_0^\infty e^{-t\xi} d\sigma_-^*(\xi)$$
$$= \int_0^\infty e^{-\mu t} g_-^*(t)dt.$$

We note that the constant $c_-^* = 0$ by $m_-(0) = 0$ and (1.2). Therefore, we have

$$\frac{d}{d\mu}\psi_-(\mu) = \int_0^\infty e^{-\mu t}\Big\{-\frac{d}{dt}g_-^*(t)\Big\}tdt.$$

Since $\tilde{p}(t,0,0) = E_0[e^{-\lambda\Gamma_+(t)}|X_t = 0]p(t,0,0)$ is monotone in t, (2.8) implies

$$\psi_+^2(\lambda)E_0[e^{-\lambda\Gamma_+(t)}|X_t = 0]p(t,0,0) \to -\frac{d}{dt}g_-^*(t), \quad \lambda \to \infty$$

by the continuity lemma (see Lemma 2 of [5]). Therefore

$$E_0[e^{-\lambda\Gamma_+(t)}|X_t = 0] \sim \frac{1}{\psi_+^2(\lambda)} \cdot \frac{-\frac{d}{dt}g_-^*(t)}{p(t,0,0)}, \quad \lambda \to \infty.$$

By (2.7) and Karamata's Tauberian theorem, we have

$$P_0(\Gamma_+(t) \le x|X_t = 0) \sim \frac{D_\alpha^2}{\Gamma(1+2\alpha)} \cdot \frac{-\frac{d}{dt}g_-^*(t)}{p(t,0,0)}\{\varphi^{-1}(x)\}^2, \quad x \to 0+$$

which completes the proof of the theorem. □

Example 1. When $X = \{X_t, P_x\}$ is $SKEWBES(2-2\alpha, p)$, $0<\alpha<1$, $0<p<1$, we put

$$G_{\alpha,p}(x) := P_0(\Gamma_+(1) \le x|X_1 = 0) = P_0\Big(\frac{1}{t}\Gamma_+(t) \le x \;\Big|\; X_t = 0\Big). \tag{2.9}$$

As is shown in the proof of Theorem 3.1 below, $G_{\alpha,p}(x)$ is characterized by

$$\int_0^1 \frac{dG_{p,\alpha}(x)}{(\xi+x)^\alpha} = \frac{1}{p(1+\xi)^\alpha + (1-p)\xi^\alpha}. \tag{2.10}$$

By inverting this, we obtain an expression (3.2) by the Riemann–Liouville fractional integral as given below. When $\alpha = 1/2$, that is, the case of skew Brownian motions, we have a more explicit formula: $G_{\frac{1}{2},p}(x)$ has the density $g_{\frac{1}{2},p}(x)$ given by

$$g_{\frac{1}{2},p}(x) = \frac{p(1-p)}{2}\{(1-2p)x+p^2\}^{-\frac{3}{2}}, \quad 0<x<1. \tag{2.11}$$

It is readily verified by an elementary calculus that this satisfies (2.10). In particular, $G_{\frac{1}{2},\frac{1}{2}}(x) = x$, which is just Lévy's result (0.2).

Theorem 2.1 implies that the asymptotic behavior of $G_{\alpha,p}(x)$ is as follows:

$$G_{\alpha,p}(x) \sim \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)\Gamma(1-\alpha)} \cdot \frac{1-p}{p^2} x^{2\alpha}, \quad x \to 0+.$$

This can be obtained from the expression (3.2) given as below.

§3. Limit Theorem

In the case of non-pinned $SKEWBES(2-2\alpha, p)$, $0 < \alpha < 1$, $0 < p < 1$, the density formula of the fraction of the occupation time is given by the form

(3.1)
$$f_{\alpha,p}(x) = \frac{\sin \alpha\pi}{\pi} \frac{p(1-p)x^{\alpha-1}(1-x)^{\alpha-1}}{p^2(1-x)^{2\alpha} + (1-p)^2 x^{2\alpha} + 2p(1-p)x^\alpha(1-x)^\alpha \cos \alpha\pi}$$

for $0 < x < 1$ (see [7] and [10]). This is Lamperti's density formula. In the pinned $SKEWBES(2-2\alpha, p)$ case, we can determine the distribution function of the fraction $\Gamma_+(t)/t$, i.e., $G_{\alpha,p}(x)$, $0 < \alpha < 1$, $0 < p < 1$, by inverting double Laplace transform formula (2.10).

Theorem 3.1. *For* $0 \le x \le 1$,

(3.2)
$$G_{\alpha,p}(x) = \frac{\sin \alpha\pi}{\pi} \int_0^x \frac{(1-p)(x-s)^{\alpha-1} s^\alpha ds}{p^2(1-s)^{2\alpha} + (1-p)^2 s^{2\alpha} + 2p(1-p)s^\alpha(1-s)^\alpha \cos \alpha\pi}.$$

Proof. Applying Theorem 2.2, we have

$$\int_0^\infty e^{-\mu t} E_0[e^{-\lambda \Gamma_+(t)} | X_t = 0] \frac{D_\alpha}{\Gamma(\alpha)} t^{\alpha-1} dt = \frac{D_\alpha}{p(\lambda+\mu)^\alpha + (1-p)\mu^\alpha} \tag{3.3}$$

for $\lambda > 0, \mu > 0$ in the $SKEWBES(2-2\alpha, p)$ case. By the self-similarity of the Bessel process, i.e., $(\Gamma_+(t), X_t) \overset{d}{=} (t\Gamma_+(1), t^\alpha X_1)$, we have

$$\begin{aligned}
\text{the LHS of (3.3)} &= \int_0^\infty e^{-\mu t} E_0[e^{-\lambda t \Gamma_+(1)} | X_1 = 0] \frac{D_\alpha}{\Gamma(\alpha)} t^{\alpha-1} dt \\
&= \frac{D_\alpha}{\Gamma(\alpha)} E_0\Big[\int_0^\infty e^{-\{\mu+\lambda\Gamma_+(1)\}t} t^{\alpha-1} dt \,\Big|\, X_1 = 0\Big] \\
&= D_\alpha E_0\Big[\frac{1}{\{\mu+\lambda\Gamma_+(1)\}^\alpha} \,\Big|\, X_1 = 0\Big] \\
&= D_\alpha \int_0^1 \frac{dG_{p,\alpha}(x)}{(\mu+\lambda x)^\alpha}.
\end{aligned}$$

Setting $\xi = \mu/\lambda (> 0)$, we obtain

$$\int_0^1 \frac{dG_{p,\alpha}(x)}{(\xi + x)^\alpha} = \frac{1}{p(1+\xi)^\alpha + (1-p)\xi^\alpha}. \tag{2.10}$$

By the inversion formula for the Stieltjes transform (see Appendix or [9]), we obtain the equality (3.2) and complete the proof. □

Now we introduce a class of random variables $\{\xi_{\alpha,p}\}$, $0 < \alpha \leq 1$, $0 \leq p \leq 1$, as follows: Each $\xi_{\alpha,p}$ is a $[0,1]$-valued random variable with the Stieltjes transform of order α given by

$$E\Big[\frac{1}{(\lambda + \xi_{\alpha,p})^\alpha}\Big] = \frac{1}{p(1+\lambda)^\alpha + (1-p)\lambda^\alpha}, \quad \lambda > 0. \tag{3.4}$$

When $0 < \alpha < 1$, the distribution function of $\xi_{\alpha,p}$ is given by $G_{\alpha,p}(x)$. If $\alpha = 1$, $\xi_{1,p}$ is the constant random variable such that $P[\xi_{1,p} = p] = 1$. If $p = 0$ or $p = 1$, $\xi_{\alpha,0} = 0$ a.s. and $\xi_{\alpha,1} = 1$ a.s.

We now state the limit theorem for the law of $\Gamma_+(t)/t$ as $t \to \infty$ for the pinned diffusion processes. Let $X = \{X_t\}$ be a generalized diffusion process corresponding to $\{m_+, m_-\}$. We always assume that $X_0 = 0$, that is, we consider the probability law P_0 unless otherwise stated.

Theorem 3.2. *Let* $\Gamma_+(t) = \int_0^t 1_{[0,\infty)}(X_s)ds$. *If*

$$m_\pm(x) \sim x^{\frac{1}{\alpha}-1} K_\pm(x), \quad x \to \infty \tag{3.5}$$

for $0 < \alpha \leq 1$ *where* $K_\pm(x)$ *are slowly varying functions at* ∞ *with*

$$\lim_{x\to\infty} \frac{K_+(x)}{K_-(x)} = \frac{p^{\frac{1}{\alpha}}}{(1-p)^{\frac{1}{\alpha}}}, \tag{3.6}$$

then the distribution of $\Gamma_+(t)/t$ *conditional on* $X_t = 0$ *converges in law to that of* $\xi_{\alpha,p}$ *as* $t \to \infty$ *i.e.,*

$$P\Big(\frac{1}{t}\Gamma_+(t) \leq x \,\Big|\, X_t = 0\Big) \to P(\xi_{\alpha,p} \leq x), \quad t \to \infty.$$

Remark. The condition (3.5) with (3.6) is equivalent to the following (see [4]):

$$h_\pm(\lambda) = \frac{1}{\psi_\pm} \sim D_\alpha \lambda^{-\alpha} L_\pm\Big(\frac{1}{\lambda}\Big), \quad \lambda \to 0+ \tag{3.7}$$

where $L_\pm(\lambda)$ are slowly varying functions at ∞ with

$$\lim_{\lambda\to\infty}\frac{L_+(\lambda)}{L_-(\lambda)}=\frac{p}{1-p} \tag{3.8}$$

and

$$D_\alpha=\begin{cases}1, & \alpha=1,\\ \{\alpha(1-\alpha)\}^{-\alpha}\Gamma(1+\alpha)/\Gamma(1-\alpha), & 0<\alpha<1.\end{cases}$$

We prove Theorem 3.2. By (1.5), for $\mu>0$ and $\beta>0$, we have

$$\begin{aligned}\int_0^\infty e^{-\frac{\mu}{\beta}t}p(t,0,0)dt &= \beta\int_0^\infty e^{-\mu t}p(t\beta,0,0)dt\\ &= \frac{1}{\psi_+(\frac{\mu}{\beta})+\psi_-(\frac{\mu}{\beta})}\\ &= \frac{1}{\psi_-(\frac{\mu}{\beta})}\cdot\frac{1}{\psi_+(\frac{\mu}{\beta})/\psi_-(\frac{\mu}{\beta})+1}\\ &\sim \frac{1}{\psi_-(\frac{\mu}{\beta})}\cdot(1-p),\quad \beta\to\infty\\ &\sim \frac{1-p}{\psi_-(\frac{1}{\beta})\mu^\alpha},\quad \beta\to\infty\end{aligned}$$

by the assumption that $\psi_\pm$ vary regularly with α, $0<\alpha\le 1$. Hence we have

$$\frac{1}{1-p}\cdot\beta\psi_-\Big(\frac{1}{\beta}\Big)\int_0^\infty e^{-\mu t}p(t\beta,0,0)dt\to\frac{1}{\mu^\alpha}=\int_0^\infty e^{-\mu t}\frac{t^{\alpha-1}}{\Gamma(\alpha)}dt,\quad \beta\to\infty.$$

Then we obtain

$$\frac{1}{1-p}\cdot\beta\psi_-\Big(\frac{1}{\beta}\Big)\cdot p(t\beta,0,0)\to\frac{t^{\alpha-1}}{\Gamma(\alpha)},\quad \beta\to\infty. \tag{3.9}$$

By Theorem 2.2 and (3.7), we have

$$\begin{aligned}&\frac{1}{1-p}\cdot\beta\psi_-\Big(\frac{1}{\beta}\Big)\int_0^\infty e^{-\mu t}E\big[e^{-\frac{\lambda}{\beta}\Gamma_+(t\beta)}\mid X_{t\beta}=0\big]p(t\beta,0,0)dt\\ &\quad=\frac{1}{1-p}\cdot\psi_-\Big(\frac{1}{\beta}\Big)\int_0^\infty e^{-\frac{\mu}{\beta}t}E\big[e^{-\frac{\lambda}{\beta}\Gamma_+(t)}\mid X_t=0\big]p(t,0,0)dt\\ &\quad=\frac{1}{1-p}\cdot\psi_-\Big(\frac{1}{\beta}\Big)\cdot\frac{1}{\psi_+(\frac{\lambda+\mu}{\beta})+\psi_-(\frac{\mu}{\beta})}\\ &\quad=\frac{1}{1-p}\cdot\frac{\psi_-(\frac{1}{\beta})}{\psi_-(\frac{\mu}{\beta})}\cdot\frac{1}{\psi_+(\frac{\lambda+\mu}{\beta})/\psi_-(\frac{\mu}{\beta})+1}\\ &\quad\to\frac{1}{1-p}\cdot\frac{1}{\mu^\alpha}\cdot\frac{1}{p(\lambda+\mu)^\alpha/(1-p)\mu^\alpha+1}=\frac{1}{p(\lambda+\mu)^\alpha+(1-p)\mu^\alpha}\end{aligned}$$

as $\beta \to \infty$. By (3.3),

$$\frac{1}{p(\lambda+\mu)^\alpha + (1-p)\mu^\alpha} = \int_0^\infty e^{-\mu t} E[e^{-\lambda t \xi_{\alpha,p}}] \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt.$$

Combining these with (3.9), we obtain that

$$E\big[e^{-\frac{\lambda}{\beta}\Gamma_+(t\beta)} \mid X_{t\beta} = 0\big] \to E[e^{-\lambda t \xi_{\alpha,p}}], \quad \beta \to \infty$$

and hence

$$P\Big(\frac{1}{\beta}\Gamma_+(t\beta) \le x \;\Big|\; X_{t\beta} = 0\Big) \to P(t\xi_{\alpha,p} \le x), \quad \beta \to \infty$$

for every $t > 0$. This clearly implies that

$$P\Big(\frac{1}{t}\Gamma_+(t) \le x \;\Big|\; X_t = 0\Big) \to P(\xi_{\alpha,p} \le x) = G_{\alpha,p}(x), \quad t \to \infty,$$

which completes the proof.

§4. Appendix: An Inversion Formula for the Stieltjes Transform

First we recall the Riemann–Liouville fractional integral of order $\rho > 0$:

$$\begin{aligned}\mathcal{I}^\rho[f](x) :=& \frac{1}{\Gamma(\rho)} \int_{0+}^{x-} (x-\xi)^{\rho-1} f(\xi) d\xi \\ =& \frac{1}{\Gamma(\rho)} \int 1_{(0,x)}(\xi)(x-\xi)^{\rho-1} f(\xi) d\xi.\end{aligned}$$

Obviously, when $\rho \ge 1$, the integral $\mathcal{I}^\rho[f](x)$ is well-defined and is a continuous function on $[0,\infty)$.

Lemma 4.1. *Let $0 < \rho < 1$ and let $f(x)$ be a Borel measurable function on $[0,\infty)$ which is bounded on each compact subset of $[0,\infty)$. Then the Riemann–Liouville fractional integral of order ρ of the function f is well-defined and is a continuous function on $(0,\infty)$.*

Proof. It is obvious noting the following: Let $\varepsilon > 0$ such that $\varepsilon < \rho$. Then there exists a constant C such that

$$|x^{\rho-1} - y^{\rho-1}| \le C|x-y|^\varepsilon \{x^{\rho-\varepsilon-1} + y^{\rho-\varepsilon-1}\}$$

for any $x, y > 0$. □

It is easy to check the following:

Lemma 4.2. *Let $\rho_1, \rho_2 > 0$ and let $f(x)$ be a Borel measurable function on $[0, \infty)$ which is bounded on each compact subset of $[0, \infty)$. Then it holds that*

$$\mathcal{I}^{\rho_1} \circ \mathcal{I}^{\rho_2}[f](x) = \mathcal{I}^{\rho_1 + \rho_2}[f](x)$$

for any $x \in (0, \infty)$.

Lemma 4.3. *Let $\rho > 1$ and let $f(x)$ be a Borel measurable function on $[0, \infty)$ which is bounded on each compact subset of $[0, \infty)$. Then $\mathcal{I}^\rho[f](x)$ has a continuous derivative given by*

$$\frac{d}{dx}\mathcal{I}^\rho[f](x) = \mathcal{I}^{\rho-1}f(x)$$

for any $x \in (0, \infty)$.

Let $F(x)$ be a non-negative increasing function on $[0, \infty)$. For $0 < \rho < 1$ the Stieltjes transform of the Stieltjes measure dF of order ρ is defined by

$$\mathcal{S}_\rho[dF](x) = \int_{[0,\infty)} \frac{dF(\xi)}{(x+\xi)^\rho}.$$

If the Riemann–Stieltjes integral of the right hand side converges at a point $x \in (0, \infty)$, then it converges uniformly on any compact subset of $\mathbb{C} \setminus (-\infty, 0]$ and hence the function $\mathcal{S}_\rho[dF]$ is a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$.

Theorem 4.1. *Suppose that the Stieltjes transform $\mathcal{S}_\rho[dF]$ converge at a point $x \in (0, \infty)$. Then the limit*

$$\Phi(x) := \lim_{\eta \to 0+} \frac{1}{2\pi i} \int_{0+}^{x-} d\sigma \{\mathcal{S}_\rho[dF](-\sigma - i\eta) - \mathcal{S}_\rho[dF](-\sigma + i\eta)\}$$

exists for any $x \in (0, \infty)$ and the equality

$$\Phi(x) = \frac{1}{\Gamma(\rho)}\mathcal{I}^{1-\rho}[F - F(0)](x)$$

holds for any $x \in (0, \infty)$. That is, the function $x \mapsto \Phi(x)$ is a continuous function in x such that

$$\int_{0+}^{x-} \{F(\xi) - F(0)\}d\xi = \Gamma(\rho)\mathcal{I}^\rho[\Phi](x)$$

for any $x \in (0, \infty)$. Moreover, suppose the limit

$$\phi(\sigma) := \lim_{\eta \to 0+} \frac{1}{2\pi i}\{\mathcal{S}_\rho[dF](-\sigma - i\eta) - \mathcal{S}_\rho[dF](-\sigma + i\eta)\}$$

exist for any $\sigma \in (0,\infty)$ *and there exist a function* $\tilde{\phi}$ *such that*

$$|\mathcal{S}_\rho[dF](-\sigma - i\eta) - \mathcal{S}_\rho[dF](-\sigma + i\eta)| \le \tilde{\phi}(\sigma)$$

and

$$\int_{0+}^{x-} \tilde{\phi}(\sigma) d\sigma < \infty$$

for any $x \in (0,\infty)$. *Then it follows that* $F(x)$ *is continuous and*

$$F(x) - F(0) = \Gamma(\rho)\mathcal{I}^\rho[\phi](x)$$

for any $x \in (0,\infty)$.

Proof. Let $x > 0$ be fixed.

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{0+}^{x-} d\sigma\{\mathcal{S}_\rho[dF](-\sigma - i\eta) - \mathcal{S}_\rho[dF](-\sigma + i\eta)\} \\
&\quad = \frac{1}{2\pi i}\int_{0+}^{x-} d\sigma \int_{[0,\infty)} \{(\xi - \sigma - i\eta)^{-\rho} - (\xi - \sigma + i\eta)^{-\rho}\} dF(\xi) \\
&\quad = \frac{1}{2\pi i}\int_{0+}^{x-} d\sigma \int_{[0,x)} \{(\xi - \sigma - i\eta)^{-\rho} - (\xi - \sigma + i\eta)^{-\rho}\} dF(\xi) \\
&\qquad + \frac{1}{2\pi i}\int_{0+}^{x-} d\sigma \int_{[x,2x)} \{(\xi - \sigma - i\eta)^{-\rho} - (\xi - \sigma + i\eta)^{-\rho}\} dF(\xi) \\
&\qquad + \frac{1}{2\pi i}\int_{0+}^{x-} d\sigma \int_{[2x,\infty)} \{(\xi - \sigma - i\eta)^{-\rho} - (\xi - \sigma + i\eta)^{-\rho}\} dF(\xi) \\
&\quad =: S_1 + S_2 + S_3.
\end{aligned}$$

Then we show that

$$\lim_{\eta\to 0+} S_1 = \frac{1}{\Gamma(\rho)}\mathcal{I}^{1-\rho}[F](x) \qquad \text{and} \qquad \lim_{\eta\to 0+} S_2 = \lim_{\eta\to 0+} S_3 = 0.$$

First, we obtain that

$$\begin{aligned}
S_1 =& \frac{1}{2\pi i}\int_{[0,x)} dF(\xi) \int_{0+}^{x-} \{(\xi - \sigma - i\eta)^{-\rho} - (\xi - \sigma + i\eta)^{-\rho}\} d\sigma \\
=& \frac{1}{2\pi i}\int_{0+}^{x-} \{F(\xi) - F(0)\}\{(\xi - x - i\eta)^{-\rho} - (\xi - x + i\eta)^{-\rho}\} d\xi \\
\to& \frac{\sin\rho\pi}{\pi}\int_{0+}^{x-} \{F(\xi) - F(0)\}(x - \xi)^{-\rho} d\xi \\
=& \frac{1}{\Gamma(\rho)}\mathcal{I}^{1-\rho}[F](x).
\end{aligned}$$

The convergence is justified by

$$\int_{0+}^{x-} (x-\xi)^{-\rho} d\xi < \infty. \tag{4.1}$$

It easily follows that S_2 converges to zero from (4.1) and

$$\lim_{\eta \to 0+} \{(x-\sigma-i\eta)^{-\rho} - (x-\sigma+i\eta)^{-\rho}\} = 0$$

if $x - \sigma > 0$.

Finally, note that

$$S_3 = \frac{1}{2\pi i} \int_{0+}^{x-} d\sigma \int_{[2x,\infty)} dF(\xi) \int_{-\eta}^{\eta} i\rho(\xi-\sigma-i\zeta)^{-\rho-1} d\zeta.$$

Thus,

$$\begin{aligned} |S_3| \leq & \frac{\eta}{\pi} \int_{[2x,\infty)} dF(\xi) \int_{0+}^{x-} 2\eta\rho(\xi-\sigma)^{-\rho-1} d\sigma \\ = & \frac{\eta}{\pi\rho} \int_{[2x,\infty)} \{(\xi-x)^{-\rho} - \xi^{-\rho}\} dF(\xi) \\ \to & 0 \end{aligned}$$

as $\eta \to 0+$. □

In the case of $F(x) = G_{\alpha,p}(x), 0 < \alpha < 1, 0 < p < 1$, we have

$$\mathcal{S}_\alpha[dF](x) = \frac{1}{p(1+x)^\alpha + (1-p)x^\alpha}.$$

Then we can compute $\phi(\sigma)$ as

$$\phi(\sigma) = \frac{\sin \alpha\pi}{\pi} \frac{(1-p)\sigma^\alpha}{p^2(1-\sigma)^{2\alpha} + (1-p)^2\sigma^{2\alpha} + 2p(1-p)\sigma^\alpha(1-\sigma)^\alpha \cos \alpha\pi}$$

and hence obtain (3.2).

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References

[1] Barlow, M. T., Pitman, J. W. and Yor, M., Une extension multidimentionnelle de la loi de l'arc sinus, *Séminaire de Prob.XXIII*, 294-314, Lecture Notes in Math., 1372, Springer, Berlin-Heidelberg-New York, 1989.

[2] Feller, W., *An Introduction to Probability Theory and its Applications*, Vol. II, Wiley, 1971.

[3] Itô, K. and McKean, H. P., Jr., *Diffusion Processes and their Sample Paths*, Springer, Berlin-Heidelberg-New York, 1965.

[4] Kasahara, Y., Spectral theory of generalized second order differential operators and its applications to Markov processes, *Japan J. Math.*, **1** (1975), 67-84.

[5] Kasahara, Y. and Yano, Y., On a generalized arc-sine law for one-dimensional diffusion processes, *Osaka J. Math.*, **42** (2005), 1-10.

[6] Kotani, S. and Watanabe, S., Krein's spectral theory of strings and generalized diffusion processes, *Functional Analysis in Markov processes*, 235-259, Lecture Notes in Math., 923, Springer, Berlin-Heidelberg-New York, 1982.

[7] Lamperti, J., An occupation time theorem for a class of stochastic processes, *Trans. Amer. Math. Soc.*, **88** (1958), 380-387.

[8] McKean, H. P., Jr., Elementary solutions for certain parabolic partial differential equations, *Trans. Amer. Math. Soc.*, **82** (1956), 519-548.

[9] Sumner, D. B., An inversion formula for the generalized Stieltjes transform, *Bull. Amer. Math. Soc.*, **55** (1949), 174-183.

[10] Watanabe, S., Generalized arc-sine laws for one-dimensional diffusion processes and random walks, *Stochastic analysis* (Ithaca, NY, 1993), 157-172, Proc. Sympos. Pure Math., 57, Amer. Math. Soc., Providence, RI, 1995.

[11] Widder, D. V., The Stieltjes transform, *Trans. Amer. Math. Soc.*, **43** (1938), 7-60.

[12] ———, *The Laplace transform*, Princeton Univ. Press, 1941.