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The Irregularity of the Direct Image of Some $\mathcal{D}$-modules

By

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Abstract

Let f and g be two regular functions on U smooth affine variety. Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_U$-module. We are interested in the irregularity of the complex $f_+(\mathcal{M}e^g)$. More precisely, we relate the irregularity number at c of the systems $\mathcal{H}^k f_+(\mathcal{M}e^g)$ with the characteristic cycles of the systems $\mathcal{H}^k(f,g)_+(\mathcal{M})$.

§1. Introduction

• Let U be a smooth affine variety over $\mathbb{C}$ and $g : U \to \mathbb{C}$ be a regular function on U. We denote by $\mathcal{O}_U$ the sheaf of regular functions on U and by $\mathcal{D}_U$ the sheaf of algebraic differential operators on U.

Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_U$-module. We denote by $\mathcal{M}e^g$ the $\mathcal{D}_U$-module obtained from $\mathcal{M}$ by twisting by e^g. If $\bigtriangledown$ is the connection defined by the $\mathcal{D}_U$-module structure of $\mathcal{M}$, $\bigtriangledown + dg$ is the one associated with $\mathcal{M}e^g$. Although $\mathcal{M}$ is regular, $\mathcal{M}e^g$ is not regular in general. Here, regular means that there exists a smooth compactification X of U and an extension of $\mathcal{M}e^g$ as $\mathcal{D}_X$-module which is regular holonomic on X. In [10], C. Sabbah describes a comparison theorem for these $\mathcal{D}$-modules twisted by an exponential. This theorem gives a relation between the irregularity complex of $\mathcal{M}e^g$ (see [6]) and some topological data given by g and $\mathcal{M}$.

In this paper, we consider two regular functions $f, g : U \to \mathbb{C}$. We are interested in the irregularity of the cohomology modules of the direct image by f of a $\mathcal{D}_U$-module, $\mathcal{M}e^g$, where $\mathcal{M}$ is regular and holonomic.

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• In Section 2, we recall the definitions of a $\mathcal{D}$-module twisted by an exponential of a meromorphic function. We will need the definition in the case of meromorphic function during the proof of the main theorem.

Then, we will consider the case where $\mathcal{M}$ is the sheaf of regular function $\mathcal{O}_U$. According to [4], the system $\mathcal{H}^k(f_+(\mathcal{O}_U e^g))$ extends vector bundle with flat holomorphic connection such that the generic fiber of the sheaf of their horizontal sections is canonically isomorphic to the cohomology group $H^{k+n-1}_{\phi_t}(f^{-1}(t)^{an}, \mathbb{C})$, where ϕ_t is the family of closed subsets of $f^{-1}(t)$ on which e^{-g} is rapidly decreasing. Using this result, we will motivate the study of the irregularity of the systems $\mathcal{H}^k f_+(\mathcal{O}_U e^g)$ by observations on some integrals.

• The main theorem of this paper gives us a formula for the irregularity number of the systems $\mathcal{H}^k(f_+(\mathcal{M}e^g))$ at finite distance and at infinity.

In the case where f and g are two polynomials in two variables which are algebraically independents and $\mathcal{M} = \mathcal{O}_{\mathbb{C}^2}$, the complex $f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$ is concentrated in degree 0 except at a finite number of points (see [9]). Then, the irregularity number at a point $c \in \mathbb{C} \cup \{\infty\}$ of the system $\mathcal{H}^0 f_+(\mathcal{O}_{\mathbb{C}^2} e^g)$ can be expressed in terms of some geometric data associated with f and g (see [9]).

In this paper, we calculate the irregularity number at $c \in \mathbb{C} \cup \{\infty\}$ of the systems $\mathcal{H}^k(f_+(\mathcal{M}e^g))$ with the help of the characteristic cycle of the systems $\mathcal{H}^k(f,g)_+(\mathcal{M})$, in the general case where f and g are any regular functions.

In the following, we identify $\mathbb{C} \cup \{\infty\}$ with $\mathbb{P}^1$. Let i be the inclusion of $\mathbb{C}^2$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $c \in \mathbb{P}^1$ and $V = V_1 \times V_2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ a neighbourhood of (c, ∞).

Let $Cch(c,k)$ be the characteristic cycle of $\mathcal{H}^k i_+(f,g)_+(\mathcal{M})$ in the neighbourhood V:
$Cch(c,k) = mT^*_V V + m' T^*_{(c,\infty)} V + m'' T^*_{\{c\}\times V_2} V + m''' T^*_{V_1 \times \{\infty\}} V + \sum m_l T^*_{Z_l} V$,
where Z_l are some germs of irreducible curves in a neighbourhood of (c,∞) distinct from $V_1 \times \{\infty\}$ and $\{c\} \times V_2$.

Theorem 1.1. *The irregularity number of $\mathcal{H}^k f_+(\mathcal{M}e^g)$ at c is equal to $\sum_l m_l I_{(c,\infty)}(Z_l, \mathbb{P}^1 \times \{\infty\})$, where $I_{(c,\infty)}(Z_l, \mathbb{P}^1 \times \{\infty\})$ is the intersection multiplicity of Z_l and $\mathbb{P}^1 \times \{\infty\}$ at (c,∞).*

• The theorem of commutation between the irregularity functor and the direct image functor ([6]) allows us to rephrasing Theorem 1.1 in terms of an irregularity complex of a regular holonomic $\mathcal{D}$-module twisted by an exponential (cf. Lemma 3.2).

Then, using the comparison theorem of [10], we are led to calculate the Euler characteristic of a germ of a complex of nearby cycles.

§2. The Complex $f_+(\mathcal{M}e^g)$

§2.1. Regular holonomic $\mathcal{D}$-modules twisted by an exponential

Let X be a smooth algebraic variety over $\mathbb{C}$.

We identify $\mathbb{P}^1$ with $\mathbb{C}\cup\{\infty\}$. Let $h : X \to \mathbb{P}^1$ be a meromorphic function.

Definition 2.1. We define the $\mathcal{D}_X$-module $\mathcal{O}_X[*h^{-1}(\infty)]e^h$ as a $\mathcal{D}_X$-module which is isomorphic to $\mathcal{O}_X[*h^{-1}(\infty)]$ as $\mathcal{O}_X$-module. The original connection $\bigtriangledown$ on $\mathcal{O}_X[*h^{-1}(\infty)]$ is replaced with the connection $\bigtriangledown + dh$ on $\mathcal{O}_X[*h^{-1}(\infty)]e^h$.

Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module.

Definition 2.2. We define the $\mathcal{D}_X$-module $\mathcal{M}[*h^{-1}(\infty)]e^h$ as the $\mathcal{D}_X$-module $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{O}_X[*h^{-1}(\infty)]e^h$.

Remark. $\mathcal{O}_X[*h^{-1}(\infty)]e^h$ is the direct image by an open immersion of a vector bundle with integrable connection. Then, it is a holonomic $\mathcal{D}_X$-module as algebraic direct image of a holonomic $\mathcal{D}$-module.

$\mathcal{M}[*h^{-1}(\infty)]e^h$ is a holonomic left $\mathcal{D}_X$-module as tensor product of two holonomic left $\mathcal{D}_X$-modules (cf. Theorem 4.6 of [2]).

We have analogous definitions in the analytic case. We just have to transpose in the analytic setting.

§2.2. On the solutions of the systems $\mathcal{H}^k(f_+(\mathcal{O}_Ue^g))$

The generic fiber of the sheaf of horizonal sections of $\mathcal{H}^k(f_+(\mathcal{O}_Ue^g))$ can be describe as follows:

Theorem 2.1 ([4]). *There exists a finite subset $\Sigma\subset\mathbb{C}$ such that*

- $\mathcal{H}^k(f_+(\mathcal{O}_Ue^g))_{|\mathbb{C}\setminus\Sigma}$ *is a vector bundle with flat holomorphic connection.*
- *For all $t\in\mathbb{C}\setminus\Sigma$, $i_t^+\mathcal{H}^{k-n+1}(f_+(\mathcal{O}_Ue^g))\simeq H^k_{\phi_t}((f^{-1}(t))^{an},\mathbb{C})$, where i_t is the inclusion of $\{t\}$ in $\mathbb{C}$ and ϕ_t is the family of closed subsets of $f^{-1}(t)$ on which e^{-g} is rapidly decreasing.*

More precisely, the family ϕ_t is defined as follow. Let $\pi : \widetilde{\mathbb{P}^1}\to\mathbb{P}^1$ be the oriented real blow-up of $\mathbb{P}^1$ at infinity. $\widetilde{\mathbb{P}^1}$ is diffeomorphic to $\mathbb{C}\cup S^1$, where S^1 is the circle of directions at infinity. A is in ϕ_t if A is a closed subset of $f^{-1}(t)$ and the closure of $g(A)$ in $\mathbb{C}\cup S^1$ intersects S^1 in $]-\frac{\pi}{2},\frac{\pi}{2}[$.

Let us motivate the study of this complex by observations on some integrals. Concerning Gauss-Manin systems, we can express their solutions as period integrals of the type $\int_{\gamma(t)} w_{|f^{-1}(t)}$, where $\gamma(t)$ is an horizontal family of cycles in the fibres $f^{-1}(t)$ and w is a relative algebraic differential form. As the Gauss-Manin connection is regular, these integrals have moderate growth in the neighbourhood of their singularities. In our case, some solutions can also be expressed as integrals.

Let Ψ_t be the family of closed subsets A of $f^{-1}(t)$ such that for all R big enough, $A \setminus g^{-1}(\{t \in \mathbb{C} \mid Re(-t) > R\})$ is compact. We consider the complex of semi-algebraic chains with support in Ψ_t (see [8]). We denote by $H_{k,\Psi_t}(f^{-1}(t)^{an}, \mathbb{C})$ the k-th homology group associated with this complex. We can now integrate forms in $H^k_{\Phi_t}(f^{-1}(t)^{an}, \mathbb{C})$ on cycles in $H_{k,\Psi_t}(f^{-1}(t)^{an}, \mathbb{C})$.

According to Theorem 1.4 of [1], since f is a submersion outside Σ, we have an isomorphism

$$\mathcal{H}^{k-n+1}(f_+(\mathcal{O}_U e^g))_{|\mathbb{C}\setminus\Sigma} \simeq R^k f_*(DR_{\mathbb{C}^n/\mathbb{C}}(\mathcal{O}_U)e^g)_{|\mathbb{C}\setminus\Sigma}.$$

Thus, we can extend the integration defined before to a form $we^g_{|f^{-1}(t)}$, where w is a relative algebraic differential form. Indeed, by the definition of Ψ_t, e^g is rapidly decreasing on the cycles and semi-algebraic chains with support in Ψ_t behave well at infinity.

In this way, to $\gamma(t)$, horizontal family of cycles in $H_{k,\Psi_t}(f^{-1}(t)^{an}, \mathbb{C})$, we can associate a solution of the $\mathcal{D}_{\mathbb{C}\setminus\Sigma}$-module $\mathcal{H}^{k-n+1}(f_+(\mathcal{O}_U e^g))_{|\mathbb{C}\setminus\Sigma}$. It is a morphism α of $\mathcal{D}_{\mathbb{C}\setminus\Sigma}$-modules defined by $\alpha([we^g]) = \int_{\gamma(t)} we^g_{|f^{-1}(t)}$.

The study of the irregularity of the systems $\mathcal{H}^{k-n+1}(f_+(\mathcal{O}_U e^g))$ gives us informations about the growth of these integrals in the neighbourhood of their singularities.

§3. On the Irregularity of the Complex $f_+(\mathcal{M}e^g)$

In the following, we will identify $\mathbb{C} \cup \{\infty\}$ with $\mathbb{P}^1$ and we consider the canonical immersion $j : \mathbb{C} \to \mathbb{P}^1$. Let us fix $k \in \mathbb{Z}$ and $c \in \mathbb{P}^1$.

We are interested in the number $IR_{c,k}$, it being the irregularity number at $c \in \mathbb{P}^1$ of the system $\mathcal{H}^k j_+ f_+(\mathcal{M}e^g)$.

The first step of the proof of Theorem 1.1 consists in rephrasing it using an irregularity complex of a $\mathcal{D}_{\mathbb{C}^2}$-module twisted by an exponential.

For the definition of irregularity complex along an hypersurface, we refer the reader to [6] and [7]. We adopt the following notations. If $\mathfrak{M}$ is a complex of $\mathcal{D}_X$-modules and Z is an hypersurface of X, we denote by $IR_Z(\mathfrak{M})$ the irregularity complex of $\mathfrak{M}$ along Z. For simplicity of notations, we write $IR^k_Z(\mathfrak{M})$ instead of $\mathcal{H}^k(IR_Z(\mathfrak{M}))$.

According to [5], the irregularity number $IR_{c,k}$ is equal to the dimension of the $\mathbb{C}$-vector space $IR^0(\mathcal{H}^k j_+ f_+(\mathcal{M}e^g))_c$.

Denote by $\mathcal{N}^\bullet$ the complex of $\mathcal{D}_{\mathbb{P}^1\times\mathbb{P}^1}$-modules $i_+(f,g)_+(\mathcal{M})$. In the way of rephrasing Theorem 1.1, we need the following lemma:

Lemma 3.1. *Let $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the second projection and $D = (\mathbb{P}^1 \times \mathbb{P}^1) \setminus i(\mathbb{C}^2)$.*

$$IR_c(\mathcal{H}^k j_+ f_+(\mathcal{M}e^g))_c = IR_{\{c\}\times\mathbb{P}^1}(\mathcal{H}^k(\mathcal{N}^\bullet)[*D]e^{\pi_2})_{(c,\infty)}[+1].$$

Proof. • Reduction to the case of the two projections.

Let $p_1 : \mathbb{C}^2 \to \mathbb{C}$ and $p_2 : \mathbb{C}^2 \to \mathbb{C}$ be the two canonical projections.

As $f = p_1 \circ (f,g)$, we have $f_+(\mathcal{M}e^g) = p_{1+}(f,g)_+(\mathcal{M}e^g)$. Moreover, $(f,g)_+(\mathcal{M}e^g) = (f,g)_+(\mathcal{M})e^{p_2}$.

Finally, we obtain that $f_+(\mathcal{M}e^g) \simeq p_{1+}((f,g)_+(\mathcal{M})e^{p_2})$.

• In this paragraph, we denote by ${}^p\mathcal{H}$ the perverse cohomology. According to Corollary 2-1-8 of [6], we have:

$$IR_c(\mathcal{H}^k j_+ f_+(\mathcal{M}e^g)) = {}^p\mathcal{H}^k IR_c(j_+ f_+(\mathcal{M}e^g)).$$

Consider the following diagrams:

$$\begin{array}{ccc} \mathbb{C}^2 & \xhookrightarrow{i} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle \pi_1} \\ \mathbb{C} & \xhookrightarrow{j} & \mathbb{P}^1 \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{C}^2 & \xhookrightarrow{i} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \Big\downarrow{\scriptstyle p_2} & & \Big\downarrow{\scriptstyle \pi_2} \\ \mathbb{C} & \xhookrightarrow{j} & \mathbb{P}^1 \end{array} \quad .$$

Then,

$$\begin{aligned} IR_c(j_+ f_+(\mathcal{M}e^g)) &= IR_c(j_+ p_{1+}((f,g)_+(\mathcal{M})e^{p_2})) \\ &= IR_c(\pi_{1+} i_+((f,g)_+(\mathcal{M})e^{p_2})) \\ &= IR_c(\pi_{1+} i_+(\mathcal{N}^\bullet[*D]e^{\pi_2})). \end{aligned}$$

Then, $IR_c(\mathcal{H}^k j_+ f_+(\mathcal{M}e^g)) = {}^p\mathcal{H}^k IR_c(\pi_{1+} i_+(\mathcal{N}^\bullet[*D]e^{\pi_2}))$.

• According to Proposition 3-6-4 of [7], the irregularity functor commutes with the direct image functor. Thus:

$$\begin{aligned} IR_c(\pi_{1+} i_+(\mathcal{N}^\bullet[*D]e^{\pi_2}))_c &= \mathbb{R}\pi_{1*} IR_{\{c\}\times\mathbb{P}^1}(\mathcal{N}^\bullet[*D]e^{\pi_2})_c[+1] \\ &= \mathbb{R}\Gamma(\{c\}\times\mathbb{P}^1, IR_{\{c\}\times\mathbb{P}^1}(\mathcal{N}^\bullet[*D]e^{\pi_2}))[+1]. \end{aligned}$$

• Then, we remark that π_2 is holomorphic out of (c,∞) and $\mathcal{N}^\bullet$ is regular holonomic (direct image complex of an algebraic regular holonomic $\mathcal{D}$-module). Then, $IR_{\{c\}\times\mathbb{P}^1}(\mathcal{N}^\bullet[*D]e^{\pi_2})$ has its support in (c,∞) and we have an isomorphism of complexes of vector spaces

$$IR_c(\pi_{1+}i_+(\mathcal{N}^\bullet[*D]e^{\pi_2}))_c = IR_{\{c\}\times\mathbb{P}^1}(\mathcal{N}^\bullet[*D]e^{\pi_2})_{(c,\infty)}[+1].$$

• We conclude that

$$\begin{aligned} IR_c(\mathcal{H}^k j_+ f_+(\mathcal{M}e^g))_c &=^p \mathcal{H}^k IR_c(\pi_{1+}i_+(\mathcal{N}^\bullet[*D]e^{\pi_2}))_c \\ &=^p \mathcal{H}^k IR_{\{c\}\times\mathbb{P}^1}(\mathcal{N}^\bullet[*D]e^{\pi_2})_{(c,\infty)}[+1] \\ &= IR_{\{c\}\times\mathbb{P}^1}(\mathcal{H}^k(\mathcal{N}^\bullet[*D]e^{\pi_2}))_{(c,\infty)}[+1] \\ &= IR_{\{c\}\times\mathbb{P}^1}(\mathcal{H}^k(\mathcal{N}^\bullet)[*D]e^{\pi_2})_{(c,\infty)}[+1]. \end{aligned}$$

□

Now, we can rephrase Theorem 1.1.

Let us choose some local coordinates (x,z) of $\mathbb{P}^1\times\mathbb{P}^1$ in a neighbourhood of (c,∞) such that:

- (c,∞) has for coordinates $(0,0)$,
- $\{c\}\times\mathbb{P}^1$ has equation $x=0$ in a neighbourhood of (c,∞),
- $\mathbb{P}^1\times\{\infty\}$ has equation $z=0$ in a neighbourhood of (c,∞).

In these coordinates, π_2 is equal to $\frac{1}{z}$ in a neighbourhood of (c,∞). Then, according to Lemma 3.1, we are led to prove the following lemma:

Lemma 3.2. *Let $\mathfrak{M}$ be a holonomic regular $\mathcal{D}_{\mathbb{C}^2}$-module. We denote the characteristic cycle of $\mathfrak{M}$ in a neighbourhood of $(0,0)$ by:*

$$Cch(\mathfrak{M}) = mT^*_{\mathbb{C}^2}\mathbb{C}^2 + m^{'}T^*_{(c,\infty)}\mathbb{C}^2 + m^{''}T^*_{x=0}\mathbb{C}^2 + m^{'''}T^*_{z=0}\mathbb{C}^2 + \sum m_l T^*_{Z_l}\mathbb{C}^2,$$

where Z_l are some germs of irreducible curves in a neighbourhood of $(0,0)$ distinct from $x=0$ and $z=0$.

Then,

$$\chi(IR_{x=0}(\mathfrak{M}[\frac{1}{z}]e^{\frac{1}{z}})_{(0,0)} = -\sum_l m_l I_{(c,\infty)}(Z_l,\{z=0\}).$$

§4. Proof of Lemma 3.2

We break up the proof of Lemma 3.2 in three steps:

Lemma 4.1. $\chi(IR_{x=0}(\mathfrak{M}[\frac{1}{z}]e^{\frac{1}{z}})_{(0,0)}) = \chi(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})_{(0,0)})$.

We denote by $\Psi_z(\mathfrak{M}[\frac{1}{x}])$ the complex of nearby cycles of $\mathfrak{M}[\frac{1}{x}]$ relative to z. It is a complex of constructible sheaves on $\widetilde{\mathbb{C}\times}\{0\}$ defined as follows.

Let η small enough. We denote by $\widetilde{D^*(0,\eta)}$ the universal covering of $D^*(0,\eta)$. Let $(E,\pi,\tilde{z})$ be the fiber product over $D^*(0,\eta)$ of $\mathbb{C}\times D^*(0,\eta)$ and $\widetilde{D^*(0,\eta)}$. Then, we have the following diagram:

$$\begin{tikzcd}
\mathbb{C}\times\{0\} \arrow[r, hook, "\alpha"] & \mathbb{C}^2 & \mathbb{C}\times D^*(0,\eta) \arrow[l, hook', "\tilde{i}"'] \arrow[d, "z"'] & E \arrow[l, "\pi"'] \arrow[d] \\
 & & D^*(0,\eta) & \widetilde{D^*(0,\eta)} \arrow[l] .
\end{tikzcd}$$

$$\Psi_z(\mathfrak{M}[\frac{1}{x}]) = \alpha^{-1}R(\tilde{i}\circ\pi)_*(\tilde{i}\circ\pi)^{-1}(DR(\mathfrak{M}[\frac{1}{x}])).$$

Lemma 4.2. $\chi(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})_{(0,0)}) = \chi(\Psi_z(\mathfrak{M}[\frac{1}{x}])_{(0,0)})$.

Lemma 4.3. $\chi(\Psi_z(\mathfrak{M}[\frac{1}{x}])_{(0,0)}) = -\sum_l m_l I_{(c,\infty)}(Z_l, \mathbb{P}^1\times\{\infty\})$.

Proof of Lemma 4.1. Let us first show that

$$IR_{x=0}(\mathfrak{M}[\frac{1}{z}]e^{\frac{1}{z}}) = R\Gamma_{x=0}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})).$$

Let η be the inclusion of $\mathbb{C}\times\mathbb{C}^*$ in $\mathbb{C}^2$. By definition,

$$\begin{aligned}
IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}}) &= cone\left(DR(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}}) \to R\eta_*\eta^{-1}(DR(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}}))\right)\\
&= cone\left(DR(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}}) \to R\eta_*(DR(\mathfrak{M}[\frac{1}{x}])_{|\mathbb{C}\times\mathbb{C}^*})\right).
\end{aligned}$$

Now, consider the following diagram:

$$\begin{tikzcd}
\mathbb{C}\times\mathbb{C}^* \arrow[r, hook, "\eta"] & \mathbb{C}^2 \\
\mathbb{C}^*\times\mathbb{C}^* \arrow[u, hook, "J'"] \arrow[r, hook, "\eta'"] & \mathbb{C}^*\times\mathbb{C} \arrow[u, hook, "J"'] .
\end{tikzcd}$$

As $\mathfrak{M}$ is regular, we have:

$$\begin{aligned} R\eta_*(DR(\mathfrak{M}[\frac{1}{x}])_{|\mathbb{C}\times\mathbb{C}^*}) &= R\eta_* RJ'_*(DR(\mathfrak{M})_{|\mathbb{C}^*\times\mathbb{C}^*}) \\ &= RJ_* R\eta'_*(DR(\mathfrak{M})_{|\mathbb{C}^*\times\mathbb{C}^*}). \end{aligned}$$

But $R\Gamma_{x=0}RJ_* = 0$. Then:

$$\begin{aligned} R\Gamma_{x=0}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})) &= R\Gamma_{x=0}(DR(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})) \\ &= IR_{x=0}(\mathfrak{M}(\frac{1}{z}]e^{\frac{1}{z}}), \end{aligned}$$

by definition of irregularity complex.

Then, we are led to prove that the complexes $R\Gamma_{x=0}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}}))$ and $IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})$ have the same characteristic function at $(0,0)$.

Using the following distinguished triangle,

$$\begin{tikzcd} & RJ_*J^{-1}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})) \arrow[dl, "{[+1]}"'] & \\ R\Gamma_{x=0}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}})) \arrow[rr] & & IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}}), \arrow[ul] \end{tikzcd}$$

it is sufficient to show that the characteristic function on $\{x=0\}$ of the complex $RJ_*J^{-1}(IR_{z=0}(\mathfrak{M}[\frac{1}{xz}]e^{\frac{1}{z}}))$ is zero.

Now, if $\mathcal{F}$ is a constructible sheaf on X and $P \in \{x=0\}$,

$$\chi((RJ_*J^{-1}\mathcal{F})_P) = \chi((\mathbb{D}(J_!J^{-1}\mathbb{D}\mathcal{F}))_P) = \chi((J_!J^{-1}\mathbb{D}\mathcal{F})_P) = 0,$$

where $\mathbb{D}$ is the Verdier duality (see [11]). □

Proof of Lemma 4.2. This is a particular case of a result of C. Sabbah (cf. Corollary 5-2 of [10]). □

Proof of Lemma 4.3. Denote by $\mathcal{C}^\bullet$ the complex $\Psi_z(\mathfrak{M}[\frac{1}{x}])$. By definition, $\mathcal{C}^\bullet_{(0,0)} = R(\tilde{i}\circ\pi)_*(\tilde{i}\circ\pi)^{-1}(DR(\mathfrak{M}[\frac{1}{x}]))_{(0,0)}$.

• Then, for all $k \in \mathbb{Z}$,

$$\mathcal{H}^k\mathcal{C}^\bullet_{(0,0)} = \underset{(0,0)\in U open}{indlim} \mathbb{R}^k\Gamma(U, R(\tilde{i}\circ\pi)_*(\tilde{i}\circ\pi)^{-1}(DR(\mathfrak{M}[\frac{1}{x}]))).$$

As $\{D(0,\eta_1)\times D(0,\eta_2)\}_{\eta_1,\eta_2}$ is a fundamental system of neighbourhoods of $(0,0)$, we have

$$\mathcal{H}^k\mathcal{C}^\bullet_{(0,0)} = \underset{\eta_1,\eta_2}{indlim} \mathbb{R}^k\Gamma(D(0,\eta_1)\times D(0,\eta_2), R(\tilde{i}\circ\pi)_*(\tilde{i}\circ\pi)^{-1}(DR(\mathfrak{M}[\frac{1}{x}]))).$$

• Let Σ be a Whitney stratification associated with the constructible sheaf $DR(\mathfrak{M}[\frac{1}{x}])$. Then, for η_1 and η_2 small enough, there exists a homotopy equivalence $p : (\tilde{i} \circ \pi)^{-1}(D(0, \eta_1) \times D(0, \eta_2)) \to D(0, \eta_1) \times \{\tilde{\eta}\}$ compatible with Σ. Thus, $\mathcal{H}^k\mathcal{C}^\bullet_{(0,0)} = \underset{\eta_1,\eta_2}{indlim} \mathbb{R}\Gamma(D(0, \eta_1) \times \{\tilde{\eta}\}, DR(\mathfrak{M}[\frac{1}{x}]))$.

• Now, as $\mathfrak{M}$ is regular, $DR(\mathfrak{M}[\frac{1}{x}]) = RJ_*J^{-1}(DR(\mathfrak{M}))$. Then,

$$\begin{aligned}\mathcal{H}^k\mathcal{C}^\bullet_{(0,0)} &= \underset{\eta_1,\eta_2}{indlim} \mathbb{R}\Gamma(D(0, \eta_1) \times \{\tilde{\eta}\}, RJ_*J^{-1}(DR(\mathfrak{M})))\\ &= \underset{\eta_1,\eta_2}{indlim} \mathbb{R}\Gamma(D^*(0, \eta_1) \times \{\tilde{\eta}\}, J^{-1}(DR(\mathfrak{M}))).\end{aligned}$$

• Let fix η_1 and $\tilde{\eta}$ small enough such that the singular support of $\mathfrak{M}$ in $D^*(0, \eta_1) \times \{\tilde{\eta}\}$ is a finite number of points. Denote by $P_1, \ldots, P_s$ these points. They are the intersection points of $D^*(0, \eta_1) \times \{\tilde{\eta}\}$ and $\cup Z_l$. As $DR(\mathfrak{M})_{|D^*(0,\eta_1)\times\{\tilde{\eta}\}}$ is a complex of constructible sheaves with respect to the stratification $\{D^*(0, \eta_1)\times\{\tilde{\eta}\}\setminus\{P_1, \ldots, P_l\}, P_1, \ldots, P_l\}$, the Euler characteristic of $\mathbb{R}\Gamma(D^*(0, \eta_1) \times \{\tilde{\eta}\}, J'^{-1}(DR(\mathfrak{M}))$ is equal to:

$$\chi(\mathbb{R}\Gamma(D^*(0, \eta_1) \times \{\tilde{\eta}\} \setminus \{P_1, \ldots, P_l\}, DR(\mathfrak{M}))) + \sum_{i=1}^{l} \chi(DR(\mathfrak{M})_{P_i}).$$

Then, according to the index theorem of Kashiwara (cf. [3]),

$$\begin{aligned}&\chi(\mathbb{R}\Gamma(D^*(0, \eta_1) \times \{\tilde{\eta}\}, J'^{-1}(DR(\mathfrak{M})))\\ &\quad= rk(\mathfrak{M}) \sum_l I_{(0,0)}(Z_l, \{z = 0\}) + \sum_l (rk(\mathfrak{M}) - m_l) I_{(0,0)}(Z_l, \{z = 0\})\\ &\quad= -\sum_l m_l I_{(0,0)}(Z_l, \{z = 0\}).\end{aligned}$$

□

Remark. If f and g are two polynomials in two variables, we can compare Theorem 1.1 and Theorem 1 of [9]. Let us recall this theorem:

Let $\mathbb{X}$ be a smooth projective compactification of $\mathbb{C}^2$ such that there exists $F, G : \mathbb{X} \to \mathbb{P}^1$, two meromorphic maps, which extend f and g. Let us denotes by D the divisor $\mathbb{X} \setminus \mathbb{C}^2$. Let Γ be the critical locus of (F, G).

Let $c \in \mathbb{P}^1$. We denote by Δ_1 the cycle in $\mathbb{P}^1 \times \mathbb{P}^1$ which is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $(F, G)(\Gamma) \cap (\mathbb{C}^2 \setminus \{c\} \times \mathbb{C})$, where the image is counted with multiplicity and by Δ_2 the cycle in $\mathbb{P}^1 \times \mathbb{P}^1$ which is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of $(F, G)(D) \cap (\mathbb{C}^2 \setminus \{c\} \times \mathbb{C})$, where the image is counted with multiplicity.

Theorem 4.1. *If f and g are algebraically independent, the irregularity number of $\mathcal{H}^0(f_+(\mathcal{O}_{\mathbb{C}^2}e^g))$ is equal to*

$$I_{(c,\infty)}(\Delta_1, \mathbb{P}^1 \times \{\infty\}) + I_{(c,\infty)}(\Delta_2, \mathbb{P}^1 \times \{\infty\}).$$

Then, we can prove that the germs Z_l of irreducible curves in Theorem 1.1 are the germs at (c, ∞) of the irreducible branches of $\Delta_1 \cup \Delta_2$. The multiplicity m_l of $i_+(f, g)_+(\mathcal{O}_{\mathbb{C}^2})$ on $T^*_{Z_l}V$ are the multiplicity of Z_k in $\Delta_1 \cup \Delta_2$.

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