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Motion of Nonadmissible Convex Polygons by Crystalline Curvature

By

Shigetoshi YAZAKI*

Abstract

Behavior of convex solution polygons to a general crystalline motion is investigated. A polygon is called admissible if the set of its normal angles equals that of the Wulff shape. We prove that if the initial polygon is not an admissible polygon, then all edges disappear simultaneously, or edge disappearing occurs at most finitely many instants and eventually a convex solution polygon becomes an admissible convex polygon. In the latter case, the normal angle of disappearing edge does not belong to the set of the normal angles of the Wulff shape. We also show five typical examples of this motion.

§1. Introduction and a Main Result

We consider an evolution equation of a closed convex polygon $\mathcal{P}(t)$ in the plane $\mathbb{R}^2$:

$$v_j = g\left(\theta_j, \frac{l_f(\theta_j)}{d_j}\right) \tag{1.1}$$

at time t with the normal angle of the j-th edge being $\theta_j \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ (numbered $j = 0, 1, \ldots$ counterclockwise). Here $v_j = v_j(t)$ denotes the normal

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*Faculty of Engineering, University of Miyazaki, 1-1 Gakuen Kibanadai Nishi, Miyazaki 889-2192, Japan.
e-mail: yazaki@cc.miyazaki-u.ac.jp

velocity of the j-th edge of $\mathcal{P}(t)$ in the direction of the inward unit normal $\boldsymbol{n}_j = -{}^t(\cos\theta_j, \sin\theta_j)$ and $d_j = d_j(t)$ is the length of the j-th edge. The meaning of l_f and g are as follows: We assume that an interfacial energy density σ is defined on $\mathcal{P}(t)$ and that σ is a convex function on $\mathbb{R}^2$ and satisfies $\sigma(r\cos\nu, r\sin\nu) = rf(\nu)$ ($r \geq 0$, $\nu \in S^1$) by a positive function $f \in C(S^1)$. In the present paper, we consider only those σ where the Wulff shape of σ, defined as $\mathcal{W}_f = \bigcap_{\nu\in S^1}\{(x,y) \in \mathbb{R}^2 \,|\, x\cos\nu + y\sin\nu \leq f(\nu)\}$, is a convex polygon. In this case, σ is called *crystalline energy* and the Wulff shape becomes

$$\mathcal{W}_f = \bigcap_{0\leq j<n} \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x\cos\nu_j + y\sin\nu_j \leq f(\nu_j) \right\},$$

where ν_j is the normal angle of the j-th edge of n-sided polygon $\mathcal{W}_f$. Let the set of the normal angles of $\mathcal{W}_f$ be

$$\Theta_f = \{\nu_0 < \nu_1 < \cdots < \nu_{n-1} < \nu_0 + 2\pi\}.$$

Since $\mathcal{W}_f$ is convex, $\nu_j - \nu_{j-1} < \pi$ holds for all j. In (1.1), $l_f(\theta_j)$ is the (positive) length of the j-th edge of $\mathcal{W}_f$ if $\theta_j \in \Theta_f$ and $l_f(\theta_j) = 0$ if $\theta_j \notin \Theta_f$. We assume that

$$\text{(A0)} \quad \begin{cases} \text{the function } g(\theta_j, \lambda) \text{ is a given positive function for } \lambda > 0, \\ g(\theta_j, \lambda) \text{ is monotone nondecreasing in } \lambda, \text{ and} \\ \lim_{\lambda\to\infty} g(\theta_j, \lambda) = \infty \text{ and } g(\theta_j, 0) \equiv 0 \text{ hold for all } \theta_j. \end{cases}$$

A typical example is $g(\theta_j, \lambda) = a(\theta_j)\lambda^\alpha$ with a positive function $a(\cdot)$ and a parameter $\alpha > 0$. Moreover, we assume that

(A1) the map $\lambda \mapsto g(\theta_j, \lambda)$ ($\theta_j \in \Theta_0$) is locally Lipschitz continuous on $\mathbb{R}_+$.

We will use (A1) in order to prove the time local existence of a solution polygon (see Lemma 3.1). Under these assumptions, if the j-th normal angle θ_j of $\mathcal{P}(t)$ belongs to Θ_f, then $v_j > 0$, and if $\theta_j \notin \Theta_f$, then $v_j = 0$. The second variable $l_f(\theta_j)/d_j$ (in g) is called *crystalline curvature*.

A polygon is called an *admissible* polygon if its set of normal angles equals Θ_f. In this paper, if the normal angle of an edge belongs to Θ_f, then we call the edge *admissible* edge, and if not, we call the edge *nonadmissible*. See Figure 1.

In the physical context, the region enclosed by the polygon $\mathcal{P}$ represents the crystal. Motion of admissible polygons or crystal by the evolution equation (1.1) is called *crystalline motion* or motion by crystalline curvature. See the

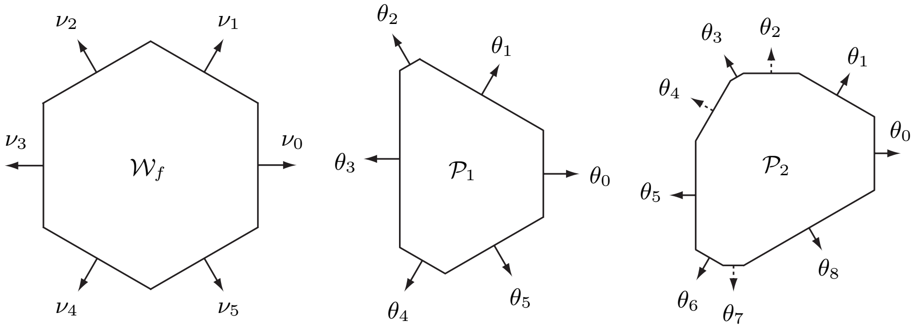

Figure 1. The Wulff shape $\mathcal{W}_f$ (left); admissible polygon $\mathcal{P}_1$ (middle); nonadmissible polygon $\mathcal{P}_2$ (right). The set of the normal angles of $\mathcal{P}_i$ ($i = 1, 2$), say Θ_i, satisfy $\Theta_1 = \Theta_f$ and $\Theta_2 \supsetneq \Theta_f$, respectively. In the right figure, for $j = 2, 4, 7$, the j-th edge is nonadmissible since $\theta_j \notin \Theta_f$.

pioneer works by Angenent and Gurtin [2], Taylor [14, 15], and Taylor, Cahn and Handwerker [16], Gurtin [5] for the background story of this motion.

Let Θ_0 be a set of normal angles of the initial convex polygon $\mathcal{P}(0) = \mathcal{P}_0$. Our aim in the present paper is to show the behavior of a solution polygon in the case where

(A2) $\Theta_0 \supsetneq \Theta_f$,

i.e., $\mathcal{P}_0$ is nonadmissible. The main result is as follows.

Theorem 1.1. *Assume* (A0), (A1) *and* (A2). *Then there exists a constant $t_1 > 0$ and a unique nonadmissible convex solution polygon $\mathcal{P}(t)$ of* (1.1) *on the interval $t \in [0, t_1)$ with the initial nonadmissible convex polygon $\mathcal{P}(0) = \mathcal{P}_0$. Let Θ_t be a set of normal angles of a solution polygon $\mathcal{P}(t)$. Then one and only one of the following three cases holds as t tends to t_1*:

(1) $\mathcal{P}(t)$ *converges to an admissible convex polygon $\mathcal{P}(t_1)$*:

$$\Theta_t \supsetneq \Theta_{t_1} = \Theta_f \quad (t \in [0, t_1));$$

(2) $\mathcal{P}(t)$ *converges to a nonadmissible convex polygon $\mathcal{P}(t_1)$*:

$$\Theta_t \supsetneq \Theta_{t_1} \supsetneq \Theta_f \quad (t \in [0, t_1));$$

(3) $\mathcal{P}(t)$ *shrinks to a single point.*

This results say that no edges of a solution polygon $\mathcal{P}(t)$ disappear at $t \in [0, t_1)$ starting with nonadmissible convex polygon $\mathcal{P}_0$. If some edges disappear at time t_1, then they are nonadmissible; or else if some admissible edges disappear, then all edges disappear simultaneously. See Example 5 for case (3).

In the case (1), after the time t_1, a solution polygon $\mathcal{P}(t)$ with initial admissible polygon $\mathcal{P}(t_1)$ evolves, while its admissibility is preserved, and eventually it shrinks to a single point (single point extinction phenomenon, *PE* in short) or collapses to a lines segment with positive length (degenerate pinching phenomenon, *DP* in short) in a finite time, say $T > t_1$, depending on the growth condition of $g(\nu_j, \lambda)$ with respect to λ. No edges of $\mathcal{P}(t)$ disappear for $t \in [t_1, T)$. This result was proved by M.-H. Giga and Y. Giga [3]. Theorem 1.1 asserts that degenerate pinching does not occur without becoming an admissible polygon. This is a contrast to the case where the initial polygon is admissible. See Example 1 and Example 2 for cases (1) to PE and (1) to DP, respectively. Andrews [1] showed a condition for an initial admissible polygon to tend to a degenerate pinching. Moreover, Ishiwata and the author [12, 13] showed that, in the case where $g(\nu_j, \lambda) = a(\nu_j)\lambda^\alpha$ with a positive function $a(\cdot)$ and $\alpha \in (0, 1)$, the blow-up order of $\max_j v_j$ is $(T - t)^{-\alpha}$ in degenerate pinching phenomenon under a monotonicity assumption on g. See also conjectures in [6].

In the case (2), there exists $t_2 > t_1$ such that a solution polygon $\mathcal{P}(t)$ with initial nonadmissible polygon $\mathcal{P}(t_1)$ evolves until time t_2 and the similar three cases (as in Theorem 1.1) occur as t tends to t_2. After the time t_2, even if case (1) is kept being selected, since number of edges is finite, edge disappearing occurs at most finitely many instants $0 < t_1 < \cdots < t_m$ and eventually $\Theta_0 \supsetneq \Theta_{t_1} \supsetneq \cdots \supsetneq \Theta_{t_m} = \Theta_f$ holds. See Example 3 in case $m = 2$ and PE, and Example 4 in case $m = 2$ and DP.

In the next Section 2, we will present five examples of this motion. The main theorem will be proved in the last section 3.

Recent progress of related research. Hontani, Giga, Giga and Deguchi [9] constructed a selfsimilar expanding solution to a crystalline flow starting from an arbitrary (nonconvex) polygonal curve (see also [4]). They called a polygonal curve an *essentially admissible crystal* if its set of normal angles satisfies (A2). If the initial curve is not necessarily an essentially admissible crystal (and is not admissible), then there exists a corner of the curve which omit normal angles in Θ_f. A reasonable way to solve a crystalline flow from such initial curve is that one inserts zero-length edges into the curve at that corner. It is proved that these edges agree with the initial data which is the limit of a unique selfsimilar expanding solution as time tends to $+0$. The similar strategy can be found in [15, §2.2].

In the case where the initial polygonal curve is nonconvex, even if it is admissible, the asymptotic behavior is not simple. For example, one can construct nonconvex self-similar solutions [11, 8] (which means that PE occurs without becoming convex), and explicit solutions which yield "whisker"-type and split-type DP singularities [7, 8]. In [3], they showed that if PE and DP (including self-intersection) do not occur, then the admissibility is preserved. They also presented sufficient conditions of non-DP depending on the growth rate of g or on the symmetry of $\mathcal{W}_f$.

§2. Examples

We present five examples: Example 1, 2, Example 3, 4 and Example 5 are typical examples of Theorem 1.1 (1), (2) and (3), respectively. In each figures, time evolution of a solution polygon moves inward starting from the outermost polygon to the inside. Throughout this paper we use the notation $\dot{u}(t)$ for $du(t)/dt$.

A simple calculation shows that $d_j(t)$'s satisfy a system of ordinary differential equations:

(2.1)
$$\dot{d}_j(t) = (\cot(\theta_{j+1}-\theta_j)+\cot(\theta_j-\theta_{j-1}))\, v_j - \frac{v_{j+1}}{\sin(\theta_{j+1}-\theta_j)} - \frac{v_{j-1}}{\sin(\theta_j-\theta_{j-1})}.$$

Here $\theta_j \in \Theta_t$. See, e.g., Angenent and Gurtin [2, Fig. 10C] and Gurtin [5, (12.29)].

Example 1 (PE in two stages: $\Theta_0 \supsetneqq \Theta_{t_1} = \Theta_f$). Let $g(\theta_j,\lambda)=\lambda$, and let the Wulff shape be a square with $\Theta_f = \{\nu_j = \pi j/2\ (j=0,1,2,3)\}$ and $l_f(\nu_j) = 1\ (\forall j)$. See Figure 2 (left).

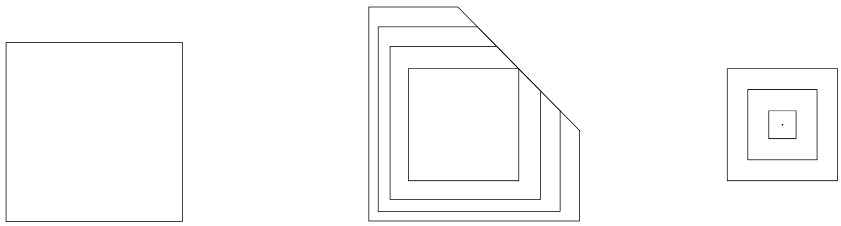

Figure 2. The Wulff square $\mathcal{W}_f$ (left); time evolution from $\mathcal{P}_0$ to $\mathcal{P}(t_1)$ (middle); time evolution from $\mathcal{P}(t_1)$ to a single point (right).

Initial data and the first stage. Let $\mathcal{P}_0$ be a symmetric pentagon with $\Theta_0 = \{\theta_0 = 0 < \pi/4 < \pi/2 < \pi < 3\pi/2\}$ and $d_0(0) = d_2(0)$, $d_3(0) = d_4(0) =$

$d_0(0)+d_1(0)/\sqrt{2}$. See Figure 2 (the outermost pentagon in the middle). From the symmetry and $v_1=0$, evolution equations are $\dot{d}_0=v_0-v_3$, $\dot{d}_1=-2\sqrt{2}v_0$ and $\dot{d}_3=-v_0-v_3$. Here $v_i=1/d_i$ $(i=0,3)$. Put $C(t)=d_3(t)^2+2d_0(t)d_3(t)-d_0(t)^2$. Then $\dot{C}(t)=-8$ holds and we have solutions

$$d_1(t)=\sqrt{2}(d_3(t)-d_0(t)),\quad d_3(t)=-d_0(t)+\sqrt{2d_0(t)^2+C(0)-8t}.$$

Hence there exists a $t_1>0$ satisfying $C(0)=8t_1+2d_0(t_1)^2$, and it holds that $d_1(t_1)=0$, $d_0(t_1)=d_3(t_1)>0$ and that $\Theta_t\equiv\Theta_0$ for $0\le t<t_1$. See Figure 2 (middle).

The final stage starts from an admissible square $\mathcal{P}(t_1)$: $\Theta_{t_1}=\Theta_f$ and $d_0(t_1)=d_i(t_1)$ $(i=1,2,3)$ (renumbered). See Figure 2 (the outermost square in the right). From the symmetry, an evolution equation is $\dot{d}_0=-2v_0$ and $v_0=1/d_0$. Then we have the exact solution $d_0(t)=2\sqrt{T-t}$ $(t_1\le t<T)$ where $T=t_1+d_0(t_1)^2/4$. A solution polygon shrinks to a single point as $t\to T$ and $\Theta_t\equiv\Theta_f$ holds for $t_1\le t<T$.

Example 2 (DP in two stages: $\Theta_0\supsetneq\Theta_{t_1}=\Theta_f$)**.** Let $g(\theta_j,\lambda)=\lambda^\alpha$ with $\alpha\in(0,1)$, and let the Wulff shape be a square with $\Theta_f=\{\nu_j=\pi/4+\pi j/2\ (j=0,1,2,3)\}$ and $l_f(\nu_j)=1$ $(\forall j)$. See Figure 3 (left).

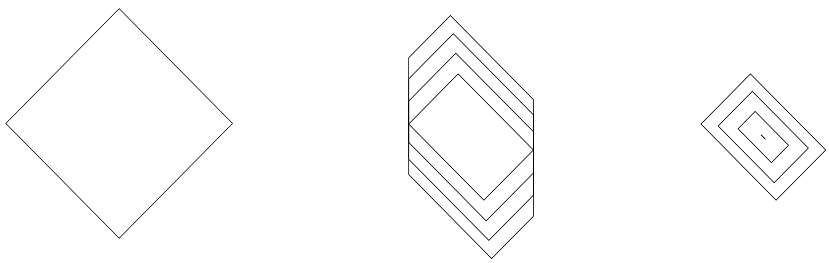

Figure 3. The Wulff square $\mathcal{W}_f$ (left); time evolution from $\mathcal{P}_0$ to $\mathcal{P}(t_1)$ (middle); time evolution from $\mathcal{P}(t_1)$ to a line segment with positive length (right).

Initial data and the first stage. Let $\mathcal{P}_0$ be a symmetric hexagon with $\Theta_0=\{\theta_0=0,\ \theta_1=\pi/4,\ \theta_2=3\pi/4,\ \theta_j=\theta_{j-3}+\pi\ (j=3,4,5)\}$ and $d_j(0)=d_{j-3}(0)$ $(j=3,4,5)$. Assume that $d_2(0)<d_1(0)$. See Figure 3 (the outermost hexagon in the middle). From the symmetry and $v_0=0$, evolution equations are $\dot{d}_0=-\sqrt{2}(v_1+v_2)$, $\dot{d}_1=v_1-v_2$ and $\dot{d}_2=-v_1+v_2$. Here $v_i=d_i^{-\alpha}$ $(i=1,2)$. The last two evolution equations yield $d_1(t)+d_2(t)=d_1(0)+d_2(0)=C_0$. Then $d_i(t)\le C_0$ $(i=1,2)$. Hence $\dot{d}_0\le-2\sqrt{2}C_0^{-\alpha}$ and we have $d_0(t)\le d_0(0)-2\sqrt{2}C_0^{-\alpha}t$. From this inequality (or by Lemma 3.2 in general), there exists a $t_1\in(0,C_0^\alpha d_0(0)/2\sqrt{2}]$ such that $d_i(t)>0$ $(\forall i)$ holds for $0\le t<t_1$

and $\min_{i=0,1,2} d_i(t_1) = 0$ holds. From the assumption and the uniqueness of solutions, we have $d_2(t) < d_1(t)$ for any t. Then $\dot{d}_2 \geq -v_1 = -d_1^{-\alpha} > -d_2^{-\alpha}$ and therefore $d_2(t_1)^{1+\alpha} > d_2(0)^{1+\alpha} - (1+\alpha)t_1 \geq d_2(0)^{1+\alpha} - (1+\alpha)C_0^\alpha d_0(0)/2\sqrt{2} > 0$ if $d_0(0) < 2\sqrt{2}d_2(0)^{1+\alpha}/(1+\alpha)C_0^\alpha$. Hence $d_0(t_1) = 0 < d_2(t_1) < d_1(t_1)$ holds. See Figure 3 (middle). Put $\mu = d_2(t_1)/d_1(t_1) < 1$.

The final stage starts from an admissible rectangle $\mathcal{P}(t_1)$: $\Theta_{t_1} = \Theta_f$ and $d_1(t_1) = \mu d_0(t_1)$, $d_i(t_1) = d_{i-2}(t_1)$ $(i = 2, 3)$ (renumbered). See Figure 3 (the outermost rectangle in the right). From the symmetry, evolution equations are $\dot{d}_0 = -2v_1$, $\dot{d}_1 = -2v_0$ and $v_i = d_i^{-\alpha}$ $(i = 0, 1)$. Since $d_0(t) > d_1(t)$ holds, there exists a $T > t_1$ satisfying $d_0(T) \geq d_1(T) = 0$, while we have $d_0(t)^{1-\alpha} = d_1(t)^{1-\alpha} + C_1$ with $C_1 = d_0(t_1)^{1-\alpha}(1-\mu^{1-\alpha}) > 0$. Hence degenerate pinching occurs: $d_0(T) = C_1^{1/(1-\alpha)} > 0 = d_1(T)$ holds at the final time $T = t_1 + \dfrac{1}{2}\displaystyle\int_0^{d_1(t_1)} (\xi^{1-\alpha} + C_1)^{\alpha/(1-\alpha)}\, d\xi$, and $\Theta_t \equiv \Theta_f$ holds for $t_1 \leq t < T$. See Figure 3 (right).

Example 3 (PE in three stages: $\Theta_0 \supsetneq \Theta_{t_1} \supsetneq \Theta_{t_2} = \Theta_f$)**.** Let $g(\theta_j, \lambda) = \lambda^\alpha$ with $\alpha > 0$, and let the Wulff shape be the same as in Example 1. See Figure 4 (far left).

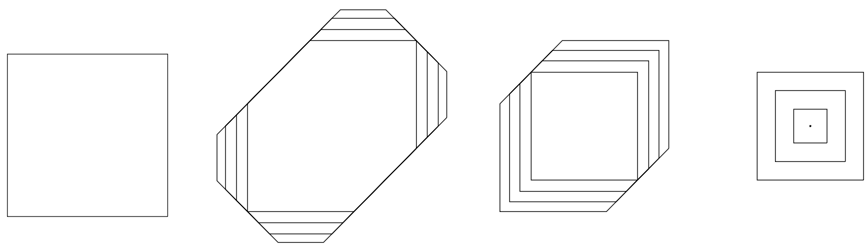

Figure 4. The Wulff square $\mathcal{W}_f$ (far left); time evolution from $\mathcal{P}_0$ to $\mathcal{P}(t_1)$ (left); time evolution from $\mathcal{P}(t_1)$ to $\mathcal{P}(t_2)$ (right); time evolution from $\mathcal{P}(t_2)$ to a single point (far right).

Initial data and the first stage. Let $\mathcal{P}_0$ be a symmetric octagon with $\Theta_0 = \{\theta_j = \pi j/4\ (j = 0, 1, \ldots, 7)\}$ and $d_0(0) = d_i(0)$ $(i = 2, 4, 6)$, $d_1(0) = d_5(0)$, $d_3(0) = d_7(0)$. See Figure 4 (the outermost octagon in the left). Assume $d_3(0) > d_1(0)$. From the symmetry and $v_1 = v_3 = 0$, evolution equations are $\dot{d}_0 = 2v_0$, $\dot{d}_1 = \dot{d}_3 = -2\sqrt{2}v_0$ and $v_0 = d_0^{-\alpha}$. Then we have explicit solutions

$$d_0(t) = \left(d_0(0)^{\alpha+1} + 2(\alpha+1)t\right)^{1/(\alpha+1)}, \quad d_i(t) = d_i(0) + \sqrt{2}(d_0(0) - d_0(t))$$

for $i = 1, 3$ and $0 \leq t < t_1 = ((d_0(0) + d_1(0)/\sqrt{2})^{\alpha+1} - d_0(0)^{\alpha+1})/2(\alpha+1)$. Therefore it holds that $d_i(t) > 0$ $(\forall i)$ for $0 \leq t < t_1$, $d_0(t_1) = d_0(0) + d_1(0)/\sqrt{2} >$

0, $d_1(t_1) = 0$, $d_3(t_1) = d_3(0) - d_1(0) > 0$ and that $\Theta_t \equiv \Theta_0$ for $0 \le t < t_1$.

The second stage. The initial polygon $\mathcal{P}(t_1)$ is a symmetric hexagon with $\Theta_{t_1} = \{\theta_0 = 0 < \pi/2 < 3\pi/4 < \pi < 3\pi/2 < 7\pi/4\}$ and $d_0(t_1) = d_i(t_1)$ ($i = 1, 3, 4$), $d_2(t_1) = d_5(t_1)$ (renumbered). See Figure 4 (the outermost hexagon in the right). From the symmetry and $v_2 = 0$, evolution equations are $\dot{d}_0 = 0$, $\dot{d}_2 = -2\sqrt{2}v_0$ and $v_0 = d_0^{-\alpha}$. Then we have explicit solutions

$$d_0(t) \equiv d_0(t_1), \quad d_2(t) = \frac{2\sqrt{2}}{d_0(t_1)^\alpha}(t_2 - t) \quad (t_1 \le t < t_2).$$

Here $t_2 = t_1 + d_0(t_1)^\alpha d_2(t_1)/2\sqrt{2}$. Hence it holds that $d_2(t_2) = 0$ and that $\Theta_t \equiv \Theta_{t_1}$ for $t_1 \le t < t_2$.

The final stage starts from an admissible square $\mathcal{P}(t_2)$: $\Theta_{t_2} = \Theta_f$ and $d_0(t_2) = d_i(t_2)$ ($i = 1, 2, 3$). See Figure 4 (the outermost square in the far right). From the symmetry, an evolution equation is $\dot{d}_0 = -2v_0$ and $v_0 = d_0^{-\alpha}$ (renumbered). Then we have an explicit solution

$$d_0(t) = (2(\alpha + 1)(T - t))^{1/(\alpha+1)} \quad (t_2 \le t < T).$$

Here $T = t_2 + d_0(t_2)^{\alpha+1}/2(\alpha + 1)$. A solution polygon shrinks to a single point as $t \to T$ and $\Theta_t \equiv \Theta_f$ holds for $t_2 \le t < T$.

Example 4 (DP in three stages: $\Theta_0 \supsetneq \Theta_{t_1} \supsetneq \Theta_{t_2} = \Theta_f$). Let $g(\theta_j, \lambda) = \lambda^\alpha$ with $\alpha \in (0, 1)$, and let the Wulff shape be the same as in Example 1. See Figure 5 (far left).

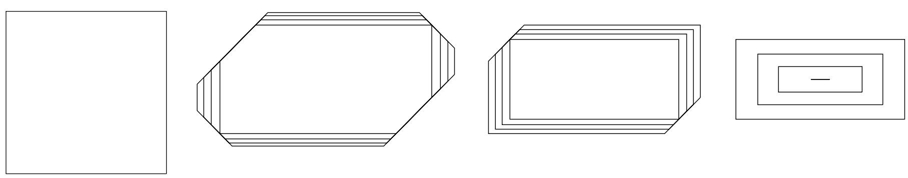

Figure 5. The Wulff square $\mathcal{W}_f$ (far left); time evolution from $\mathcal{P}_0$ to $\mathcal{P}(t_1)$ (left); time evolution from $\mathcal{P}(t_1)$ to $\mathcal{P}(t_2)$ (right); time evolution from $\mathcal{P}(t_2)$ to a line segment with positive length (far right).

Initial data and the first stage. Let $\mathcal{P}_0$ be a symmetric octagon with $\Theta_0 = \{\theta_j = \pi j/4$ ($j = 0, 1, \ldots, 7$)$\}$ and $d_i(0) = d_{i+4}(0)$ ($i = 0, 1, 2, 3$). See Figure 5 (left). Assume $d_2(0) > d_0(0)$, $d_3(0) > d_1(0)$. From the symmetry and $v_1 = v_3 = 0$, evolution equations are $\dot{d}_0 = 2v_0$, $\dot{d}_1 = \dot{d}_3 = -\sqrt{2}(v_0 + v_2)$,

$\dot{d}_2 = 2v_2$ and $v_i = d_i^{-\alpha}$ $(i = 0, 2)$. Then solutions are written $d_i(t) = \left(d_i(0)^{\alpha+1} + 2(\alpha+1)t\right)^{1/(\alpha+1)}$ and $d_j(t) = d_j(0) + (d_0(0) + d_2(0) - d_0(t) - d_2(t))/\sqrt{2}$ for $i = 0, 2$ and $j = 1, 3$. Then there exists a $t_1 > 0$ such that $d_2(t_1) > d_0(t_1) \geq d_0(0) > 0$ and $d_3(t_1) > d_1(t_1) = 0$ hold. Put $\mu = d_2(t_1)/d_0(t_1) > 1$.

The second stage. The initial polygon $\mathcal{P}(t_1)$ is a symmetric hexagon with $\Theta_{t_1} = \{\theta_0 = 0 < \pi/2 < 3\pi/4 < \pi < 3\pi/2 < 7\pi/4\}$ and $d_i(t_1) = d_{i+3}(t_1)$ $(i = 0, 1, 2)$, $d_1(t_1) = \mu d_0(t_1)$ (renumbered). See Figure 5 (the outermost hexagon in the right). From the symmetry and $v_2 = 0$, evolution equations are $\dot{d}_0 = -\dot{d}_1 = v_0 - v_1$, $\dot{d}_2 = -\sqrt{2}(v_0 + v_1)$ and $v_i = d_i^{-\alpha}$ $(i = 0, 1)$. Then we have $d_0(t) + d_1(t) = d_0(0) + d_1(0)$. Hence there exists a $t_2 > t_1$ such that $d_2(t_2) = 0 < d_0(t_2) < d_1(t_2)$ and $\Theta_t \equiv \Theta_{t_1}$ for $t_1 \leq t < t_2$ hold. Put $\eta = d_1(t_2)/d_0(t_2) > 1$.

The final stage starts from an admissible rectangle $\mathcal{P}(t_2)$: $\Theta_{t_2} = \Theta_f$ and $d_1(t_2) = \eta d_0(t_2)$, $d_i(t_2) = d_{i-2}(t_2)$ $(i = 2, 3)$. See Figure 5 (the outermost rectangle in the far right). From the symmetry, evolution equations are $\dot{d}_0 = -2v_1$, $\dot{d}_1 = -2v_0$ and $v_i = d_i^{-\alpha}$ $(i = 0, 1)$ (renumbered). Then we have $d_1(t)^{1-\alpha} = d_0(t)^{1-\alpha} + C_0$. Here $C_0 = (\eta^{1-\alpha} - 1)d_0(t_2) > 0$ since $\alpha \in (0, 1)$. Hence there exists a $T > t_2$ such that a solution polygon collapses to a line segment with the length $d_1(T) = C_0^{1/(1-\alpha)} > 0 = d_0(T)$ and $\Theta_t \equiv \Theta_f$ holds for $t_2 \leq t < T$.

Example 5 (direct PE in case: $0 \in \Theta_f, \pi \notin \Theta_f \subsetneq \Theta_0 \ni 0, \pi$). Let the Wulff shape be a symmetric pentagon (circumscribed about the unit circle) with $\Theta_f = \{\nu_0 = 0, \nu_j = \pi j/2 - \pi/4\ (j = 1, 2, 3, 4)\}$ and $l_f(\nu_0) = 2(\sqrt{2} - 1)$, $l_f(\nu_1) = l_f(\nu_4) = \sqrt{2}$, $l_f(\nu_2) = l_f(\nu_3) = 2$. See Figure 6 (left). Let $a(\cdot)$ be a positive function satisfying $a(\nu_0) = 4(2 + \sqrt{2})$, $a(\nu_1) = a(\nu_4) = 2(1 + 2\sqrt{2})$, $a(\nu_2) = a(\nu_3) = \sqrt{2}$, and let $g(\theta_j, \lambda) = a(\theta_j)\lambda$.

Figure 6. The Wulff pentagon $\mathcal{W}_f$ (left); time evolution from $\mathcal{P}_0$ to a single point (right).

PE occurs directly. Let $\mathcal{P}_0$ be a symmetric hexagon with $\Theta_0 = \{\theta_0 =$

$0,\ \theta_1 = \pi/4,\ \theta_2 = 3\pi/4,\ \theta_j = \theta_{j-3} + \pi\ (j = 3,4,5)\}$ and $d_0(0) = d_i(0)$ $(i = 1,2,\dots,5)$. See Figure 6 (the outermost hexagon in the right). From the symmetry and $v_3 = 0$, evolution equations are $\dot{d}_0 = 2(v_0 - \sqrt{2}v_1)$, $\dot{d}_1 = -\sqrt{2}v_0 + v_1 - v_2$, $\dot{d}_2 = -v_1 + v_2$, $\dot{d}_3 = -2\sqrt{2}v_2$. Here $v_0 = 8\sqrt{2}/d_0$, $v_1 = 2(4+\sqrt{2})/d_1$, $v_2 = 2\sqrt{2}/d_2$. Then we have a solution $d_0(t) = d_i(t)$ $(i = 1,2,\dots,5)$ satisfying the evolution equation $\dot{d}_0 = -8/d_0$. Hence $d_0(t) = 4\sqrt{t_1 - t}$ with $t_1 = d_0(0)^2/16$. This is a self-similar solution: A solution polygon shrinks to a single point homothetically. See Figure 6 (right).

§3. Proof of Theorem 1.1

Combining (1.1) and (2.1), we obtain the local existence theorem from a general theory.

Lemma 3.1. *Assume* (A1) *and* (A2)*. Then there is a constant $t_* > 0$ and a unique convex solution polygon $\mathcal{P}(t)$ of* (1.1) *with a prescribed initial convex polygon $\mathcal{P}(0) = \mathcal{P}_0$ and the set of normal angles $\Theta_t \equiv \Theta_0$ for $t \in [0, t_*)$.*

We will see that some edges disappear in a finite time. In what follows, we assume (A0) additionally. Let $\mathcal{L}(t)$ be a total length of $\mathcal{P}(t)$:

$$\mathcal{L}(t) = \sum_{\theta_j \in \Theta_0} d_j(t) \quad (t \in [0, t_*)).$$

From (2.1), we have

$$\dot{\mathcal{L}}(t) = -\sum_{\theta_j \in \Theta_0} \gamma_j v_j = -\sum_{\theta_j \in \Theta_f} \gamma_j v_j \quad (t \in [0, t_*)),$$

since $v_j = 0$ for $\theta_j \in \Theta_0 \backslash \Theta_f$. Here

$$\gamma_j = \frac{1 - \cos(\theta_{j+1} - \theta_j)}{\sin(\theta_{j+1} - \theta_j)} + \frac{1 - \cos(\theta_j - \theta_{j-1})}{\sin(\theta_j - \theta_{j-1})} = \tan\frac{\theta_{j+1} - \theta_j}{2} + \tan\frac{\theta_j - \theta_{j-1}}{2}.$$

Note that $0 < \theta_j - \theta_{j-1} < \pi$ holds by convexity of $\mathcal{P}(t)$ and then $\gamma_j > 0$ for all j. Therefore $\dot{\mathcal{L}} \le 0$ holds and we have $\mathcal{L}(t) \le \mathcal{L}(0)$. Obviously, $d_j(t) \le \mathcal{L}(t)$ holds for all j. Since $g(\theta_j, \lambda)$ is monotone nondecreasing in λ, if $\theta_j \in \Theta_f$, then g is bounded from below by a positive constant, say C_0:

$$g\left(\theta_j, \frac{l_f(\theta_j)}{d_j}\right) \ge \min_{\theta_k \in \Theta_f} g\left(\theta_k, \frac{l_f(\theta_k)}{d_k}\right) \ge \min_{\theta_k \in \Theta_f} g\left(\theta_k, \frac{l_f(\theta_k)}{\mathcal{L}(0)}\right) = C_0 > 0 \qquad (\theta_j \in \Theta_f).$$

Therefore there exist a $t_1 \le t_*$ and a positive constant C_1 satisfying

$$\dot{\mathcal{L}}(t) \le -nC_0 \min_{\theta_j \in \Theta_f} \gamma_j = -C_1 < 0 \quad (t \in [0, t_1)),$$

and then it holds that

$$\min_{\theta_j \in \Theta_0} d_j(t) \le \mathcal{L}(t) \le \mathcal{L}(0) - C_1 t \quad (t \in [0, t_1)).$$

Hence we have the following lemma.

Lemma 3.2. *Assume* (A0), (A1) *and* (A2). *Then there exist a* $\theta_k \in \Theta_0$ *and a* $t_1 > 0$ *such that* $\lim_{t\to t_1} d_k(t) = 0$ *and* $d_j(t) > 0$ *hold for all* $\theta_j \in \Theta_0$ *and* $t \in [0, t_1)$. *The limit* $\lim_{t\to t_1} d_k(t) = 0$ *follows from a weaker condition* $\liminf_{t\to t_1} d_k(t) = 0$.

Theorem 1.1 follows from the following lemma.

Lemma 3.3. *Assume* (A0), (A1) *and* (A2). *Let* t_1 *be the same as in Lemma* 3.2. *Put*

$$\mathcal{J} = \left\{ \theta_j \in \Theta_0 \;\middle|\; \lim_{t\to t_1} d_j(t) = 0 \right\}.$$

If $\mathcal{J} \ne \Theta_0$, *then* $\mathcal{J} \subseteq \Theta_0 \backslash \Theta_f$ *holds.*

Proof. One can represent $\mathcal{J}$ as a disjoint sum of $\mathcal{J}_k$; namely $\mathcal{J} = \bigoplus_k \mathcal{J}_k$, where $\mathcal{J}_k$'s are maximal subsets having m_k consecutive elements θ_j of the form

$$\mathcal{J}_k = \{\theta_j \in \mathcal{J} \,|\, j = j_k, j_{k+1}, \ldots, j_k + m_k - 1\},$$

with the boundary of $\mathcal{J}_k$:

$$\partial\mathcal{J}_k = \{\theta_j \,|\, j = j_k - 1, j_k + m_k\}.$$

By the definition, $m_k \ge 1$ holds for each k. If $\mathcal{J} \ne \Theta_0$, then $\partial\mathcal{J}_k \subseteq \Theta_0 \backslash \mathcal{J}$, i.e., $\inf_{0<t<t_1} d_j(t) > 0$ holds for $\theta_j \in \bigoplus_k \partial\mathcal{J}_k$.

Let $L_j(t)$ be the straight line extending the j-th edge of $\mathcal{P}(t)$ for $\theta_j \in \Theta_0$, and let $\boldsymbol{B}_j(t)$ be the intersection point of $L_j(t)$ and $L_{j-1}(t)$, i.e., $\boldsymbol{B}_j(t)$ is the j-th vertex of $\mathcal{P}(t)$. We denote $p = j_k - 1$ and $q = j_k + m_k$ for simplicity. By the definition of $\mathcal{J}_k$, vertices $\boldsymbol{B}_{p+1}(t), \ldots, \boldsymbol{B}_q(t)$ converge to a point, say $\boldsymbol{B}_*$, as $t \to t_1$:

$$\boldsymbol{B}_* \in \bigcap_{0\le t<t_1} \bigcap_{p\le j\le q} \left\{ \boldsymbol{x} \in \mathbb{R}^2 \;\middle|\; \langle \boldsymbol{x} - \boldsymbol{B}_j(t), \boldsymbol{n}_j \rangle \ge 0 \right\}.$$

Here $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product. Note that the intersection is taken over $p \le j \le q$ since the sign of v_j is nonnegative for all $p \le j \le q$. See, e.g., Ishii and Soner [10, Fig. 3]. We denote $|\mathcal{J}_k| = |\theta_p - \theta_q|$.

Claim. $|\mathcal{J}_k| \le \pi$ holds.

Suppose $|\mathcal{J}_k| > \pi$. Without loss of generality, we may assume that $\pi < \theta_q - \theta_p < 2\pi$. Then we have

$$\langle \boldsymbol{B}_{p+1} - \boldsymbol{B}_q, \boldsymbol{n}_q \rangle = \langle \boldsymbol{B}_p - \boldsymbol{B}_q, \boldsymbol{n}_q \rangle + d_p \langle \boldsymbol{t}_p, \boldsymbol{n}_q \rangle \ge \inf_{0<t<t_1} d_p(t) \sin(\theta_p - \theta_q) > 0,$$

where $\boldsymbol{t}_j = {}^t(-\sin\theta_j, \cos\theta_j) = (\boldsymbol{B}_{j+1} - \boldsymbol{B}_j)/d_j$ is the unit tangent vector on the j-th edge. Therefore $\inf_{0<t<t_1} \langle \boldsymbol{B}_{p+1}(t) - \boldsymbol{B}_q(t), \boldsymbol{n}_q \rangle > 0$ holds, which contradicts $\lim_{t\to t_1} \boldsymbol{B}_j(t) = \boldsymbol{B}_*$ for $j = p+1, q$. Hence assertion holds.

If $\mathcal{J} \subseteq \Theta_0 \backslash \Theta_f$ does not hold, then we may choose a k such that $\mathcal{J}_k \cap \Theta_f \ne \emptyset$. Then there exists at least one normal angle, say $\theta_r \in \mathcal{J}_k \cap \Theta_f$, such that $p < r < q$ holds, and $\inf_{0<t<t_1} v_r(t) > 0$ and $\lim_{t\to t_1} v_r(t) = \lim_{t\to t_1} g(\theta_r, l_f(\theta_r)/d_r(t)) = \infty$ hold.

Case $|\mathcal{J}_k| < \pi$. Let $\boldsymbol{y}(t)$ be the intersection point of $L_p(t)$ and $L_q(t)$:

$$\boldsymbol{y}(t) = \boldsymbol{B}_{p+1}(t) + \frac{\langle \boldsymbol{B}_q(t) - \boldsymbol{B}_{p+1}(t), \boldsymbol{t}_p - \mu \boldsymbol{t}_q \rangle}{1 - \mu^2} \boldsymbol{t}_p, \quad \mu = \langle \boldsymbol{t}_p, \boldsymbol{t}_q \rangle = \cos(\theta_p - \theta_q).$$

Note that $|\mu| < 1$ holds since $0 < |\theta_p - \theta_q| < \pi$, and that $\boldsymbol{y}(t)$ converges to $\boldsymbol{B}_*$ as $t \to t_1$. An evolution equation of the j-th vertex is

$$\dot{\boldsymbol{B}}_j = v_{j-1} \boldsymbol{n}_{j-1} + \frac{v_{j-1}\cos(\theta_j - \theta_{j-1}) - v_j}{\sin(\theta_j - \theta_{j-1})} \boldsymbol{t}_{j-1} \tag{3.1}$$

$$= v_j \boldsymbol{n}_j + \frac{v_{j-1} - \cos(\theta_j - \theta_{j-1}) v_j}{\sin(\theta_j - \theta_{j-1})} \boldsymbol{t}_j. \tag{3.2}$$

By using $\dot{\boldsymbol{B}}_{p+1}$ with (3.1) and $\dot{\boldsymbol{B}}_q$ with (3.2), we have

$$\dot{\boldsymbol{y}} = v_p \boldsymbol{n}_p + \frac{\langle v_q \boldsymbol{n}_q - v_p \boldsymbol{n}_p, \boldsymbol{t}_p - \mu \boldsymbol{t}_q \rangle}{1 - \mu^2} \boldsymbol{t}_p.$$

If $\theta_p, \theta_q \in \Theta_0 \backslash \Theta_f$, then $v_p = v_q = 0$ and $\dot{\boldsymbol{y}} = \boldsymbol{0}$ hold, which contradicts to convergence of $\boldsymbol{y}$ to $\boldsymbol{B}_*$. So either $\theta_p \in \Theta_f$ or $\theta_q \in \Theta_f$ hold. Since $\theta_p, \theta_q \notin \mathcal{J}$, $\sup_{0<t<t_1} v_j(t)$ is bounded from above ($j = p, q$), and therefore there exists a positive constant, say C_*, such that $\sup_{0<t<t_1} |\dot{\boldsymbol{y}}(t)| \le C_*$ holds. We define

$$a(t) = \langle \boldsymbol{B}_* - \boldsymbol{y}(t), \boldsymbol{n}_r \rangle, \quad b(t) = \mathrm{dist}(\boldsymbol{B}_*, L_r(t)) = \langle \boldsymbol{B}_* - \boldsymbol{B}_r(t), \boldsymbol{n}_r \rangle.$$

Then $a(t) \geq b(t)$ holds for $t \in [0, t_1)$ and $\lim_{t \to t_1} a(t) = \lim_{t \to t_1} b(t) = 0$ holds.

Therefore by $\dot{a}(t) = -\langle \dot{\boldsymbol{y}}(t), \boldsymbol{n}_r \rangle$, $|\dot{a}(t)| \leq C_*$ and $\dot{b} = -\langle \dot{\boldsymbol{B}}_r(t), \boldsymbol{n}_r \rangle = -v_r$, there exists $\eta \in (t, t_1)$ such that

$$0 < \int_t^{t_1} v_r(\tau)\, d\tau = -\int_t^{t_1} \dot{b}(\tau)\, d\tau = b(t) \leq a(t) = -\dot{a}(\eta)(t_1 - t) \leq C_*(t_1 - t).$$

This contradicts the fact $v_r \to \infty$ as $t \to t_1$.

Hence $\mathcal{J}_k \cap \Theta_f = \emptyset$ for all k, i.e., $\mathcal{J} \subseteq \Theta_0 \backslash \Theta_f$ holds, in other words, only some nonadmissible edges disappear (any admissible edges do not disappear) at time t_1. See Example 1, 2, 3 and 4.

Case $|\mathcal{J}_k| = \pi$. By a geometric inspection, there exist exactly two sets $\mathcal{J}_1, \mathcal{J}_2$ such that $\mathcal{J} = \bigoplus_{k=1}^{2} \mathcal{J}_k$ and $\Theta_0 \backslash \mathcal{J} = \{\theta_p, \theta_q\}$ hold. If $\{\theta_p, \theta_q\} \cap \Theta_f = \emptyset$, then $v_p = v_q = 0$, which is impossible. Therefore $\{\theta_p, \theta_q\} \cap \Theta_f \neq \emptyset$ holds. Since $\theta_p, \theta_q \notin \mathcal{J}$, $\sup_{0<t<t_1} v_j(t)$ is bounded from above $(j = p, q)$.

Assume that $\theta_{p+1}, \theta_{q-1} \in \mathcal{J}_1$ and $\theta_{q+1}, \theta_{p-1} \in \mathcal{J}_2$.

Claim 1. $\{\theta_{p+1}, \theta_{q-1}\} \cap \Theta_f \neq \emptyset$ and $\{\theta_{q+1}, \theta_{p-1}\} \cap \Theta_f \neq \emptyset$ hold.

We may assume without loss of generality that $\theta_p = \nu_0$. Then $\theta_p < \nu_1 < \theta_q = \theta_p + \pi$ holds. Suppose $\{\theta_{p+1}, \theta_{q-1}\} \cap \Theta_f = \emptyset$. If $\mathcal{J}_1 = \{\theta_{p+1} = \theta_{q-1}\}$, then $\theta_{p+1} = \theta_{q-1} = \nu_1 \in \Theta_f$, which is a contradiction. If $\mathcal{J}_1 = \{\theta_{p+1} < \theta_{p+2} = \theta_{q-1}\}$, then either $\theta_{p+1} = \nu_1$ or $\theta_{q-1} = \nu_1$ holds. This is also a contradiction. If $\mathcal{J}_1 = \{\theta_{p+1} < \theta_{p+2} < \cdots < \theta_{q-1}\}$, then there exists $\theta_r = \nu_1$ such that $\theta_{p+1} < \theta_r < \theta_{q-1}$, and then $v_{p+1} = v_{q-1} = 0$ and $\inf_{0<t<t_1} v_r > 0$ hold. Therefore we have

$$\dot{\boldsymbol{B}}_{p+1} = \frac{v_p}{\sin(\theta_{p+1} - \theta_p)} \boldsymbol{t}_{p+1}, \quad \dot{\boldsymbol{B}}_q = -\frac{v_q}{\sin(\theta_q - \theta_{q-1})} \boldsymbol{t}_{q-1}$$

from (3.2) and (3.1), respectively. Hence $\boldsymbol{B}_{p+1}$ and $\boldsymbol{B}_q$ converge, as $t \to t_1$, to the intersection point of L_{p+1} and L_{q-1}, say $\boldsymbol{y}$:

$$\boldsymbol{y} = \boldsymbol{B}_{p+1} + \frac{\langle \boldsymbol{B}_q - \boldsymbol{B}_{p+1}, \boldsymbol{t}_{p+1} - \mu \boldsymbol{t}_{q-1} \rangle}{1 - \mu^2} \boldsymbol{t}_{p+1},$$

$$\mu = \langle \boldsymbol{t}_{p+1}, \boldsymbol{t}_{q-1} \rangle = \cos(\theta_{p+1} - \theta_{q-1}).$$

Note that $|\mu| < 1$ holds since $\theta_p < \theta_{p+1} < \theta_{q-1} < \theta_q = \theta_p + \pi$, and that $\dot{\boldsymbol{y}} = \boldsymbol{0}$ holds. The r-th vertex $\boldsymbol{B}_r$ $(\theta_{p+1} < \theta_r < \theta_{q-1})$ is given by

$$\boldsymbol{B}_r = \boldsymbol{B}_{p+1} + \sum_{m=p+1}^{r-1} d_m \boldsymbol{t}_m = \boldsymbol{B}_q - \sum_{m=r}^{q-1} d_m \boldsymbol{t}_m.$$

Then we have

$$\langle \boldsymbol{B}_r - \boldsymbol{B}_{p+1}, \boldsymbol{n}_{p+1} \rangle = \sum_{m=p+1}^{r-1} d_m \langle \boldsymbol{t}_m, \boldsymbol{n}_{p+1} \rangle = \sum_{m=p+1}^{r-1} d_m \sin(\theta_m - \theta_{p+1}) > 0,$$

and

$$\langle \boldsymbol{B}_r - \boldsymbol{B}_q, \boldsymbol{n}_{q-1} \rangle = -\sum_{m=r}^{q-1} d_m \langle \boldsymbol{t}_m, \boldsymbol{n}_{q-1} \rangle = -\sum_{m=r}^{q-1} d_m \sin(\theta_m - \theta_{q-1}) > 0,$$

from $\theta_{p+1} < \theta_{r-1} < \theta_r < \theta_{q-1}$ (we have $\boldsymbol{B}_r = \boldsymbol{B}_{p+2}$ if $\theta_{p+1} = \theta_{r-1}$). Therefore $\boldsymbol{B}_r$ is in the sector $\boldsymbol{B}_{p+1}\boldsymbol{y}\boldsymbol{B}_q$ or on its boundary except for L_{q-1}. Then $\boldsymbol{B}_r \neq \boldsymbol{y}$ holds, since the number of elements of $\mathcal{J}_1$ is greater than or equal to three. From $\mathrm{dist}(\boldsymbol{y}, L_r) = -\langle \boldsymbol{y} - \boldsymbol{B}_r, \boldsymbol{n}_r \rangle > 0$,

$$\frac{d}{dt} \mathrm{dist}(\boldsymbol{y}, L_r) = -\langle \dot{\boldsymbol{y}} - \dot{\boldsymbol{B}}_r, \boldsymbol{n}_r \rangle \geq \inf_{0<t<t_1} v_r > 0$$

holds. Hence $\inf\limits_{0<t<t_1} \mathrm{dist}(\boldsymbol{y}, L_r) > 0$ holds, i.e., $\boldsymbol{B}_r$ does not converge to $\boldsymbol{y}$. This is a contradiction. Then our assertion holds and also $\{\theta_{q+1}, \theta_{p-1}\} \cap \Theta_f \neq \emptyset$ holds.

Claim 2. $\mathcal{J} \subsetneq \Theta_f$ holds.

Suppose $\mathcal{J} \not\subseteq \Theta_f$. Then there exists $\theta_r \in \mathcal{J} \cap \Theta_0 \backslash \Theta_f$. Without loss of generality, assume that $\theta_r \in \mathcal{J}_1$ and that $\theta_{p+1} \in \Theta_f$ (by Claim 1). Then $p+1 < r < q$ holds. Let $\boldsymbol{y}$ be the intersection point of L_p and L_r. Since $|\mathcal{J}_1| = \pi$ and $\theta_r \in \Theta_0 \backslash \Theta_f$, vertices $\boldsymbol{B}_{p+1}, \ldots, \boldsymbol{B}_q$ and $\boldsymbol{y}$ converge to a point $\boldsymbol{B}_*$ which is on the r-th edge. Put $a(t) = \langle \boldsymbol{B}_* - \boldsymbol{y}, \boldsymbol{n}_{p+1} \rangle$ and $b(t) = \langle \boldsymbol{B}_* - \boldsymbol{B}_{p+1}, \boldsymbol{n}_{p+1} \rangle$. One can repeat the same argument as in Case $|\mathcal{J}_k| < \pi$ (since $v_{p+1} \to \infty$ as $t \to t_1$), which leads us to a contradiction. Then $\mathcal{J} \subseteq \Theta_f$ holds and also $\mathcal{J} \neq \Theta_f$ holds since $\{\theta_p, \theta_q\} \cap \Theta_f \neq \emptyset$.

From Claim 2, if $\{\theta_p, \theta_q\} \subsetneq \Theta_f$, then $\Theta_0 = \Theta_f$ holds, which contradicts the assumption $\Theta_0 \supsetneq \Theta_f$. Therefore, either $\Theta_0 = \{\theta_p\} \oplus \Theta_f$ or $\Theta_0 = \{\theta_q\} \oplus \Theta_f$ holds. Assume that $\Theta_0 = \{\theta_q\} \oplus \Theta_f$, i.e., nonadmissible edge is only the q-th edge.

By the closedness of $\mathcal{W}_f$ and $0 < \theta_i - \theta_{i-1} < \pi$ $(i = q, q+1)$, we have $0 < \theta_{q+1} - \theta_{q-1} < \pi$. Then either $0 < \theta_q - \theta_{q-1} < \pi/2$ or $0 < \theta_{q+1} - \theta_q < \pi/2$ holds. Assume that $0 < \theta_q - \theta_{q-1} < \pi/2$. For $i = p, q$, let $\boldsymbol{y}_i$ be the intersection point between L_i and the straight line in the direction of $\boldsymbol{n}_{q-1}$ which passes $\boldsymbol{B}_{q-1}$. Vertices $\boldsymbol{B}_{p+1}, \ldots, \boldsymbol{B}_q$ and $\boldsymbol{y}_q$ converge to a point, say $\boldsymbol{B}_*$, as $t \to t_1$: $\boldsymbol{B}_*$ is on the q-th edge and $\boldsymbol{B}_* = \boldsymbol{B}_q + \alpha \boldsymbol{t}_q$ holds with $\alpha > 0$ (since $\inf_{0<t<t_1} v_{q-1}(t) > 0$).

Let $w(t)$ be the width between L_p and L_q. Put $a(t) = |\boldsymbol{y}_p - \boldsymbol{y}_q| = w(t)/\cos(\theta_q - \theta_{q-1})$ and $b(t) = \langle \boldsymbol{B}_* - \boldsymbol{B}_q, \boldsymbol{n}_{q-1}\rangle = \alpha\sin(\theta_q - \theta_{q-1})$. Note that $\dot{w} = -v_p$, $\dot{b} = -v_{q-1}$ and $\lim_{t\to t_1} v_{q-1} = \infty$. Therefore one can repeat the same argument as in Case $|\mathcal{J}_k| < \pi$, which leads us to a contradiction. Consequently, the case $|\mathcal{J}_k| = \pi$ has been excluded, i.e., degenerate pinching does not occur. □

From Lemma 3.3 it follows that $\mathcal{J} = \Theta_0\backslash\Theta_f$, $\mathcal{J} \subsetneq \Theta_0\backslash\Theta_f$ or $\mathcal{J} = \Theta_0$ holds exclusively. These correspond to Theorem 1.1 (1), (2) and (3), respectively. Hence the proof of Theorem 1.1 is completed.

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