A Torres Condition for Twisted Alexander Polynomials

Dedicated to Professor Tomoyuki Wada on his 60th birthday

By

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Abstract

As a generalization of a fundamental result about the Alexander polynomial of links, we give a description of a Torres condition for the twisted Alexander polynomial of links associated to a unimodular representation.

§1.   Introduction

The theory of twisted Alexander polynomial was introduced by Lin [13] and Wada [18]. Lin defined it for knots in the 3-sphere using regular Seifert surfaces. On the other hand, Wada defined the twisted Alexander polynomial for finitely presentable groups, which include the link groups. In particular, as an application, Wada told the Kinoshita-Terasaka knot from the Conway knot by means of his invariant. Shortly afterward, several significant results on the original Alexander polynomial were generalized to the twisted case. For example, equivalence of the twisted Alexander polynomial and the Reidemeister torsion, and its symmetry [9], [7], sliceness obstruction for knots and a relation to the Casson-Gordon invariant [7], [8], monicness of the twisted Alexander polynomial for fibered knots [1], [2] and so on. Recently the twisted Alexander polynomials are extensively investigated. See for instance [3], [4], [5], [6], [10], [11], [12], [14], [15] and [16].
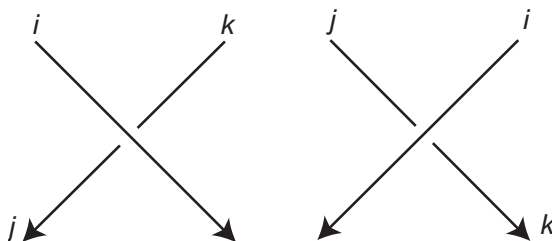
Figure 1.

However, almost all results mentioned above are basically about knots in the 3-sphere and it seems that there are few generalized results on links. The purpose of the present paper is to give a generalization of the following well-known formula for the Alexander polynomial of links.

Theorem 1.1 (Torres [17]). *The Alexander polynomial $\Delta_L(t_1, \ldots, t_\mu)$ of a μ-component link $L = L_1 \cup \cdots \cup L_\mu$ satisfies*

$$\Delta_L(t_1, \ldots, t_{\mu-1}, 1) = \begin{cases} \dfrac{t_1^{l_1} - 1}{t_1 - 1} \Delta_{L'}(t_1) & \text{if } \mu = 2 \\ (t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}} - 1)\Delta_{L'}(t_1, \ldots, t_{\mu-1}) & \text{if } \mu > 2, \end{cases}$$

where $L' = L_1 \cup \cdots \cup L_{\mu-1}$ is the link obtained from L by removing L_μ and l_i denotes the linking number of the components L_i and L_μ.

More precisely, we give a description of a Torres condition for the twisted Alexander polynomial of links associated to a unimodular representation. In the next section, we briefly recall the definition of the twisted Alexander polynomial for a link group. The precise statement and the proof of the main theorem of this paper are given in Section 3.

§2.   Twisted Alexander Polynomial for Links

Let $L = L_1 \cup \cdots \cup L_\mu$ be a μ-component link in the 3-sphere. We denote the fundamental group of its exterior E by $G(L)$. Namely, we put $G(L) = \pi_1(E)$ and call it the link group. We choose and fix a Wirtinger presentation of $G(L)$. That is, given a regular projection of the link L, we assign to each overpass a generator x_i as in Figure 1, a relator $x_i x_k x_i^{-1} x_j^{-1}$ or $x_i^{-1} x_j x_i x_k^{-1}$. Thus we obtain a presentation of $G(L)$ with u generators and u relators,

$$\langle x_1, \ldots, x_u \mid r_1, \ldots, r_u \rangle.$$

After some reordering of the indicies, the relators $r_1, \ldots, r_u$ satisfy

$$\prod_{i=1}^{u} r_i^{\pm 1} = 1.$$

This means that any one of the relators is a consequence of the other $u-1$ relators. We remove one of the relators and call the resulting presentation

$$G(L) = \langle x_1, \ldots, x_u \mid r_1, \ldots, r_{u-1} \rangle$$

a Wirtinger presentation of $G(L)$.

The abelianization homomorphism

$$\alpha : G(L) \to H_1(E; \mathbb{Z}) \cong \mathbb{Z}^{\oplus \mu} = \langle t_1 \rangle \oplus \cdots \oplus \langle t_\mu \rangle$$

is given by assigning to each generator x_i the meridian element $t_k \in H_1(E; \mathbb{Z})$ of the corresponding component L_k of L. In this paper, we consider a linear representation $\rho : G(L) \to SL(n; F)$, where F denotes a field.

These maps naturally induce two ring homomorphisms $\tilde{\rho} : \mathbb{Z}[G(L)] \to M(n; F)$ and $\tilde{\alpha} : \mathbb{Z}[G(L)] \to \mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$, where $\mathbb{Z}[G(L)]$ is the group ring of $G(L)$ over $\mathbb{Z}$ and $M(n; F)$ is the matrix algebra of degree n over F. Taking the tensor of $\tilde{\rho}$ and $\tilde{\alpha}$, we obtain a ring homomorphism

$$\tilde{\rho} \otimes \tilde{\alpha} : \mathbb{Z}[G(L)] \to M\left(n; F[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]\right).$$

Let F_u denote the free group on generators $x_1, \ldots, x_u$ and

$$\Phi : \mathbb{Z}[F_u] \to M\left(n; F[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]\right)$$

the composite of the surjection $\mathbb{Z}[F_u] \to \mathbb{Z}[G(L)]$ induced by the presentation and the map $\tilde{\rho} \otimes \tilde{\alpha}$.

Let us consider the $(u-1) \times u$ matrix $M = M(t_1, \ldots, t_\mu)$ whose (i,j)th component is the $n \times n$ matrix

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M\left(n; F[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]\right),$$

where $\partial/\partial x$ denotes the free differential calculus. This matrix M is called the Alexander matrix of $G(L)$ associated to the representation ρ.

For $1 \leq j \leq u$, let us denote by $M_j = M_j(t_1, \ldots, t_\mu)$ the $(u-1) \times (u-1)$ matrix obtained from M by removing the column corresponding to a generator x_j. We also regard M_j as an $n(u-1) \times n(u-1)$ matrix with coefficients in $F[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$.

Then Wada's twisted Alexander polynomial of a link L for a representation $\rho : G(L) \to SL(n; F)$ is defined to be a rational function

$$\Delta_{L,\rho}(t_1, \ldots, t_\mu) = \frac{|M_j|}{|\Phi(x_j - 1)|},$$

where $|M_j|$ denotes the determinant of the matrix M_j, and it is well-defined up to a factor $\pm t_1^{nk_1} \cdots t_\mu^{nk_\mu}$ $(k_i \in \mathbb{Z})$ if n is odd and up to only $t_1^{nk_1} \cdots t_\mu^{nk_\mu}$ if n is even (see [18] Section 5 for details).

Remark 2.1.   In general, the twisted Alexander polynomial for a finitely presentable group is a rational function, but it is actually a polynomial for a link group (see [18] Proposition 9 and [10] Theorem 3.1).

§3.   A Torres Condition

In this section, we state and prove a generalized Torres condition for the twisted Alexander polynomial of links. An advantage of our description here is that we need not separate the case for $\mu = 2$ from the one for $\mu > 2$. We first prove the theorem in the case of an $SL(2; F)$-representation. After reading the proof for it, one can easily show the similar result for general cases.

Theorem 3.1.   *Let $L = L_1 \cup \cdots \cup L_\mu$ be a μ-component link and $L' = L_1 \cup \cdots \cup L_{\mu-1}$. For a given representation $\rho' : G(L') \to SL(2; F)$, it holds that*

$$\Delta_{L,\rho}(t_1, \ldots, t_{\mu-1}, 1) = \{(t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}})^2 + \varepsilon_{\rho'} t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}} + 1\} \Delta_{L',\rho'}(t_1, \ldots, t_{\mu-1}),$$

where $\rho : G(L) \to SL(2; F)$ is the composite of the natural surjection $G(L) \to G(L')$ and ρ', l_i denotes the linking number of L_i and L_μ, and $\varepsilon_{\rho'}$ is an element of F.

Proof.   For the link group $G(L)$, we choose a Wirtinger presentation:

$$G(L) = \langle x_{ij} \mid r_{kl} \rangle,$$

where $x_{i1}, x_{i2}, \ldots, x_{ij_i}$ $(1 \leq i \leq \mu)$ are generators corresponding to the component L_i and the relator

$$r_{kl} = x_{k'l'} x_{kl} x_{k'l'}^{-1} x_{k,l+1}^{-1} \quad \text{or} \quad x_{k'l'}^{-1} x_{kl} x_{k'l'} x_{k,l+1}^{-1}$$

corresponds to a crossing of $L_{k'}$ over L_k. In the above presentation, we arrange the generators and relators in lexicographic order, which is determined by the

order of components $L_1, \ldots, L_\mu$ and the orientation of each component L_i. We should note that the link group $G(L)$ has the deficiency one (namely, the number of relators is less than that of generators).

Let us consider the Alexander matrix of $G(L)$ associated to the representation $\rho : G(L) \to SL(2; F)$:

$$M(t_1, \ldots, t_\mu) = \left(\Phi \left(\frac{\partial r_{kl}}{\partial x_{ij}} \right) \right)$$

$$= \begin{pmatrix} \Phi \left(\dfrac{\partial r_{kl}}{\partial x_{ij}} \right)_{k, i \neq \mu} & \Phi \left(\dfrac{\partial r_{kl}}{\partial x_{\mu j}} \right)_{k \neq \mu, 1 \leq j \leq j_\mu} \\ \Phi \left(\dfrac{\partial r_{\mu l}}{\partial x_{ij}} \right)_{i \neq \mu, 1 \leq l \leq j_\mu} & \Phi \left(\dfrac{\partial r_{\mu l}}{\partial x_{\mu j}} \right)_{1 \leq j, l \leq j_\mu} \end{pmatrix}.$$

Then we know that if we remove the column corresponding to a generator x_{ij} $(i \neq \mu)$,

$$|M_{ij}(t_1, \ldots, t_\mu)| = |\Phi(x_{ij} - 1)|\Delta_{L,\rho}(t_1, \ldots, t_\mu)$$

holds. Thus setting $t_\mu = 1$ in $M(t_1, \ldots, t_\mu)$, it follows that

$$|M_{ij}(t_1, \ldots, t_{\mu-1}, 1)| = |\Phi(x_{ij} - 1)|\Delta_{L,\rho}(t_1, \ldots, t_{\mu-1}, 1)$$

if $i \neq \mu$.

Now the generators $\{x_{\mu j}\}$ $(1 \leq j \leq j_\mu)$ appear in the following two kinds of relators:

$$\text{(i) } r_{\mu j} = x_{vw}^{\pm 1} x_{\mu j} x_{vw}^{\mp 1} x_{\mu, j+1}^{-1} \quad \text{and} \quad \text{(ii) } r_{pq} = x_{\mu l}^{\pm 1} x_{pq} x_{\mu l}^{\mp 1} x_{p, q+1}^{-1},$$

where the relator (i) corresponds to crossings of L_v over L_μ and (ii) corresponds to that of L_μ over L_p. Let us see which are the contributions of these relators to the matrix $M(t_1, \ldots, t_{\mu-1}, 1)$.

Claim 1.   *The contributions of $r_{\mu j}$ are as follows:*

(i) $\Phi \left(\dfrac{\partial r_{\mu j}}{\partial x_{vw}} \right)_{t_\mu = 1} = O,$

(ii) $\Phi \left(\dfrac{\partial r_{\mu j}}{\partial x_{\mu j}} \right)_{t_\mu = 1} = \begin{cases} t_v^{\pm 1} \rho(x_{vw})^{\pm 1} & \text{if } \mu \neq v \\ I & \text{if } \mu = v, \end{cases}$

(iii) $\Phi \left(\dfrac{\partial r_{\mu j}}{\partial x_{\mu, j+1}} \right)_{t_\mu = 1} = -I,$

where O and I denote the zero and the identity matrix respectively.

Proof. (i) An easy calculation shows that

$$\frac{\partial r_{\mu j}}{\partial x_{vw}} = 1 - x_{vw}x_{\mu j}x_{vw}^{-1} \quad \text{or} \quad -x_{vw}^{-1} + x_{vw}^{-1}x_{\mu j}.$$

Putting $t_\mu = 1$, we obtain

$$\Phi\left(\frac{\partial r_{\mu j}}{\partial x_{vw}}\right)_{t_\mu=1} = I - t_v t_v^{-1}\rho(x_{vw})\rho(x_{\mu j})\rho(x_{vw})^{-1} = O$$

or

$$\Phi\left(\frac{\partial r_{\mu j}}{\partial x_{vw}}\right)_{t_\mu=1} = -t_v^{-1}\rho(x_{vw})^{-1} + t_v^{-1}\rho(x_{vw})^{-1}\rho(x_{\mu j}) = O,$$

because $\rho(x_{\mu j}) = I$ for $1 \le j \le j_\mu$. (ii) and (iii) follow from the similar calculation. This completes the proof of Claim 1. ◻

Claim 2.   *The contributions of* r_{pq} *are as follows*:

(i) $\Phi\left(\dfrac{\partial r_{pq}}{\partial x_{\mu l}}\right)_{t_\mu=1} = \pm(I - t_p\rho(x_{pq}))$,

(ii) $\Phi\left(\dfrac{\partial r_{pq}}{\partial x_{pq}}\right)_{t_\mu=1} = I$,

(iii) $\Phi\left(\dfrac{\partial r_{pq}}{\partial x_{p,q+1}}\right)_{t_\mu=1} = -\rho(x_{pq})\rho(x_{p,q+1})^{-1}$ *if* $p \ne \mu$

and the case $p = \mu$ *has already been considered.*

Proof. We only show (iii). Since

$$\frac{\partial r_{pq}}{\partial x_{p,q+1}} = -x_{\mu l}^{\pm 1}x_{pq}x_{\mu l}^{\mp 1}x_{p,q+1}^{-1},$$

putting $t_\mu = 1$ and using $\rho(x_{\mu l}) = I$, we have

$$\Phi\left(\frac{\partial r_{pq}}{\partial x_{p,q+1}}\right)_{t_\mu=1} = -t_p t_p^{-1}\rho(x_{\mu l})^{\pm 1}\rho(x_{pq})\rho(x_{\mu l})^{\mp 1}\rho(x_{p,q+1})^{-1}$$

$$= -\rho(x_{pq})\rho(x_{p,q+1})^{-1},$$

if $p \ne \mu$. The proof of Claim 2 is completed. ◻

From the above two claims, we see that the matrix $M(t_1,\ldots,t_{\mu-1},1)$ has the following form:

$$M(t_1,\ldots,t_{\mu-1},1) = \begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where

$$A = \left(\Phi\left(\frac{\partial r_{kl}}{\partial x_{ij}}\right)_{t_\mu=1}\right) \; (k, i \neq \mu), \; B = \left(\Phi\left(\frac{\partial r_{kl}}{\partial x_{\mu j}}\right)_{t_\mu=1}\right) \; (k \neq \mu, \; 1 \leq j \leq j_\mu),$$

and

$$C = \left(\Phi\left(\frac{\partial r_{\mu l}}{\partial x_{\mu j}}\right)_{t_\mu=1}\right) \; (1 \leq j, l \leq j_\mu)$$

$$= \begin{pmatrix} t_{v_1}^{\delta_{v_1}} \rho(x_{v_1 w_1})^{\delta_{v_1}} & -I & & & & \\ & t_{v_2}^{\delta_{v_2}} \rho(x_{v_2 w_2})^{\delta_{v_2}} & -I & & & \\ & & t_{v_3}^{\delta_{v_3}} \rho(x_{v_3 w_3})^{\delta_{v_3}} & & & \\ & & & \ddots & -I & \\ -I & & & & t_{v_{j_\mu}}^{\delta_{v_{j_\mu}}} \rho(x_{v_{j_\mu} w_{j_\mu}})^{\delta_{v_{j_\mu}}} \end{pmatrix}.$$

Here $\delta_{v_i} = 1$ or -1 according to crossings of L_{v_i} over L_μ.

Claim 3. *The determinant of the submatrix C is given by*

$$|C| = (t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}})^2 + \varepsilon_{\rho'} t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}} + 1,$$

where $\varepsilon_{\rho'}$ is an element of F.

Proof. By definition of the determinant of a matrix, we have

$$|C| = \prod_{i=1}^{j_\mu} |\rho(x_{v_i w_i})^{\delta_{v_i}}| t_{v_i}^{2\delta_{v_i}} + (-1)^{j_\mu} \sum_{\sigma \in S} (\text{sgn } \sigma) \gamma_1^\sigma \cdots \gamma_{j_\mu}^\sigma t_{v_1}^{\delta_{v_1}} \cdots t_{v_{j_\mu}}^{\delta_{v_{j_\mu}}}$$
$$+ (\text{sgn } \sigma_0)(-1)^{2j_\mu},$$

where $\gamma_i^\sigma \in F$ denotes a component of the 2×2-matrix $\rho(x_{v_i w_i})^{\delta_{v_i}}$ determined by a permutation σ, S is a subset of the symmetric group $\mathfrak{S}_{2j_\mu}$ consisting of permutations which choose just one component from each submatrix $\rho(x_{v_i w_i})^{\delta_{v_i}}$ and

$$\sigma_0 = (135 \ldots 2j_\mu - 1)(246 \ldots 2j_\mu) \in \mathfrak{S}_{2j_\mu}.$$

For example, when $j_\mu = 2$, the permutation $\sigma = (1342) \in S \subset \mathfrak{S}_4$ assigns the coefficient

$$\gamma_1^\sigma \gamma_2^\sigma = c_1 b_2,$$

where c_1, b_2 are components of the images

$$\rho(x_{v_i w_i})^{\delta_{v_i}} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad (i = 1, 2).$$

On the other hand, in the matrix C, there is an appearance of $t_i^{\delta_i}$ for each crossing of L_i over L_μ ($1 \le i \le \mu$) and $\delta_i = 1$ or -1 according as L_i crosses over L_μ from left to right or from right to left. Thus $t_{v_1}^{\delta_{v_1}} \cdots t_{v_{j_\mu}}^{\delta_{v_{j_\mu}}} = t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}}$ holds. Since $\mathrm{sgn}\, \sigma_0 = 1$, if we put

$$\varepsilon_{\rho'} = (-1)^{j_\mu} \sum_{\sigma \in S} (\mathrm{sgn}\, \sigma) \gamma_1^\sigma \cdots \gamma_{j_\mu}^\sigma \in F,$$

we obtain

$$|C| = (t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}})^2 + \varepsilon_{\rho'} t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}} + 1.$$

This completes the proof of Claim 3. $\square$

Next the matrix A is equivalent to the Alexander matrix $M'(t_1,\ldots,t_{\mu-1})$ of $G(L')$ associated to the representation $\rho' : G(L') \to SL(2;F)$. Hence if we remove a column corresponding to a generator x_{ij} ($i \neq \mu$), then we have

$$
\begin{aligned}
|M_{ij}(t_1,\ldots,t_{\mu-1},1)| &= |A_{ij}||C| \\
&= \{(t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}})^2 + \varepsilon_{\rho'} t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}} + 1\}|M'_{ij}(t_1,\ldots,t_{\mu-1})|,
\end{aligned}
$$

where A_{ij} is the matrix obtained from A by removing the column corresponding to x_{ij}. Therefore, by definition of the twisted Alexander polynomial, we see that

$$\Delta_{L,\rho}(t_1,\ldots,t_{\mu-1},1) = \{(t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}})^2 + \varepsilon_{\rho'} t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}} + 1\}\Delta_{L',\rho'}(t_1,\ldots,t_{\mu-1}).$$

This completes the proof of Theorem 3.1. $\square$

Remark 3.2.   The fact that $\Delta_{L,\rho}(t_1,\ldots,t_{\mu-1},1)$ is divisible by $\Delta_{L',\rho'}(t_1,\ldots,t_{\mu-1})$ also follows from a recent result of Kitano, Suzuki and Wada in [12]. However, we can have no detailed information on the quotient from their result.

A linear representation $\rho : G(L) \to GL(n;F)$ is called reducible if it has a nontrivial invariant subspace in F^n. In this case, we can obtain a piece of information about the coefficient $\varepsilon_{\rho'}$.

Corollary 3.3.   *Under the setting as in Theorem* 3.1, *if* $\rho' : G(L') \to SL(2;F)$ *is a reducible representation, then we have*

$$\varepsilon_{\rho'} = - \left(\prod_{i=1}^{\mu-1} \lambda_i^{l_i} + \prod_{i=1}^{\mu-1} \lambda_i^{-l_i} \right),$$

where λ_i *is an eigenvalue of the image of a generator* x_{ij} ($i \neq \mu$) *of* $G(L')$.

Proof. First we can assume that the images of generators in a Wirtinger presentation of $G(L)$ have the following forms:

$$\rho(x_{ij}) = \begin{pmatrix} a_{ij} & b_{ij} \\ 0 & a_{ij}^{-1} \end{pmatrix} \ (i \neq \mu) \quad \text{and} \quad \rho(x_{\mu j}) = I,$$

where $a_{ij} \in F^{\times}$ and $b_{ij} \in F$. Because the representation ρ' has a 1-dimensional invariant subspace in F^2.

Since $x_{ij}x_{ik}^{-1}$ $(i \neq \mu,\ j \neq k)$ is an element of the commutator subgroup of $G(L)$, $[G(L), G(L)]$, we see that $a_{ij} = a_{ik}$ holds for these generators. We then put $\lambda_i = a_{ij}$ for simplicity. Each lower left component of $\rho(x_{ij})$ is zero, so that the nontrivial terms appeared in the coefficient of $t_{v_1}^{\delta_{v_1}} \cdots t_{v_{j_\mu}}^{\delta_{v_{j_\mu}}}$ are just

$$(-1)^{j_\mu}(\mathrm{sgn}\ \sigma_1)\lambda_{v_1}^{\delta_{v_1}} \cdots \lambda_{v_{j_\mu}}^{\delta_{v_{j_\mu}}} + (-1)^{j_\mu}(\mathrm{sgn}\ \sigma_2)\lambda_{v_1}^{-\delta_{v_1}} \cdots \lambda_{v_{j_\mu}}^{-\delta_{v_{j_\mu}}},$$

where $\sigma_1 = (246\ldots 2j_\mu)$ and $\sigma_2 = (135\ldots 2j_\mu - 1)$ are elements of the symmetric group $\mathfrak{S}_{2j_\mu}$. Then it is easy to check that $(-1)^{j_\mu}\mathrm{sgn}\ \sigma_1 = (-1)^{j_\mu}\mathrm{sgn}\ \sigma_2 = -1$ holds. Therefore we can have the desired formula. This completes the proof. □

Example 3.4. Let $\rho' : G(L') \to SL(2; F)$ be a reducible representation of a knot $L' = L_1$. Then the twisted Alexander polynomial of L' associated to ρ' is given by

$$\Delta_{L', \rho'}(t_1) = \frac{\Delta_{L'}(\lambda t_1)\Delta_{L'}(\lambda^{-1}t_1)}{(t_1 - \lambda)(t_1 - \lambda^{-1})},$$

where $\Delta_{L'}(t_1)$ is the original Alexander polynomial of L' and λ is an eigenvalue of the image of a generator of $G(L')$ (see the proof of [10] Theorem 3.1 for instance). Hence we have

$$\begin{aligned} \Delta_{L, \rho}(1, 1) &= \{2 - (\lambda^{l_1} + \lambda^{-l_1})\}\Delta_{L', \rho'}(1) \\ &= \frac{(1 - \lambda^{l_1})(1 - \lambda^{-l_1})}{(1 - \lambda)(1 - \lambda^{-1})}\Delta_{L'}(\lambda)\Delta_{L'}(\lambda^{-1}) \\ &= (1 + \lambda + \cdots + \lambda^{l_1 - 1})(1 + \lambda^{-1} + \cdots + \lambda^{-(l_1 - 1)})\Delta_{L'}(\lambda)\Delta_{L'}(\lambda^{-1}). \end{aligned}$$

In particular, if ρ' has the eigenvalue $\lambda = 1$, then we obtain $\Delta_{L, \rho}(1, 1) = l_1{}^2$ (because $\Delta_{L'}(1) = \pm 1$).

Example 3.5. Let $\rho' : G(L') \to SL(2; F)$ be the trivial representation. In this case $\varepsilon_{\rho'} = -2$ holds, so that we have

$$\Delta_{L, \rho}(t_1, \ldots, t_{\mu-1}, 1) = (t_1^{l_1} \cdots t_{\mu-1}^{l_{\mu-1}} - 1)^2 \Delta_{L', \rho'}(t_1, \ldots, t_{\mu-1}).$$

This formula corresponds to the square of Torres' original formula in Theorem 1.1. In particular, $\Delta_{L,\rho}(1,\ldots,1) = 0$ holds for $\mu > 2$.

If we slightly modify the proof of Theorem 3.1, we obtain the following general formula for a unimodular representation $\rho' : G(L') \to SL(n;F)$. We omit here the repetitious proof.

Theorem 3.6. *Let $L = L_1 \cup \cdots \cup L_\mu$ be a μ-component link and $L' = L_1 \cup \cdots \cup L_{\mu-1}$. For a given representation $\rho' : G(L') \to SL(n;F)$, it holds that*

$$\Delta_{L,\rho}(t_1,\ldots,t_{\mu-1},1) = \left\{ (t_1^{l_1}\cdots t_{\mu-1}^{l_{\mu-1}})^n + \sum_{k=1}^{n-1} \varepsilon_{k,\rho'}(t_1^{l_1}\cdots t_{\mu-1}^{l_{\mu-1}})^{n-k} + (-1)^n \right\}$$
$$\times \Delta_{L',\rho'}(t_1,\ldots,t_{\mu-1}),$$

where $\rho : G(L) \to SL(n;F)$ is the composite of the natural surjection $G(L) \to G(L')$ and ρ', l_i denotes the linking number of L_i and L_μ, and $\varepsilon_{k,\rho'}$ $(1 \le k \le n-1)$ are elements of F.

Remark 3.7. If ρ is a representation to the general linear group $GL(n;F)$, then the coefficient of the leading term $(t_1^{l_1}\cdots t_{\mu-1}^{l_{\mu-1}})^n$ becomes a unit element $\varepsilon_{0,\rho'} \in F^\times$.

Finally, we extend Corollary 3.3 when all the images of the representation $\rho' : G(L') \to SL(n;F)$ are upper triangle matrices.

Corollary 3.8. *Under the setting as in Theorem 3.6, if $\mathrm{Im}(\rho')$ are upper triangle matrices, then the coefficient $\varepsilon_{k,\rho'}$ is given by*

$$\varepsilon_{k,\rho'} = (-1)^k \sum_{1 \le j_1 < \cdots < j_k \le n} \prod_{i=1}^{\mu-1} (\lambda_{i1}\cdots\hat{\lambda}_{ij_1}\cdots\hat{\lambda}_{ij_k}\cdots\lambda_{in})^{l_i},$$

where λ_{im} $(1 \le m \le n)$ are the eigenvalues of the image of a generator x_{ij} $(i \ne \mu)$ of $G(L')$ and $\hat{\lambda}_{im}$ implies that λ_{im} is removed from the product.

Acknowledgements

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