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The $\overline{\partial}$-theory for Inverse Problems Associated with Schrödinger Operators on Hyperbolic Spaces

By

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Abstract

An analogue of the Faddeev scattering amplitude is introduced for Schrödinger operators on hyperbolic spaces. It satisfies a $\overline{\partial}$-equation and enables us to derive an integral representation of the potential.

§1. Introduction

In the present paper, we are concerned with the inverse problem associated with Schrödinger operators on hyperbolic spaces. The most fundamental object in scattering theory is the S-matrix. For the Schrödinger operator in $\mathbf{R}^n$, it is a unitary operator on $L^2(S^{n-1})$ having the following expression

$$S(E)f(\theta) = f(\theta) + C_E \int_{S^{n-1}} \widetilde{A}(E;\theta,\omega) f(\omega) d\omega, \quad f \in L^2(S^{n-1}), \tag{1.1}$$

where C_E is a constant depending only on the energy $E > 0$ and the scattering amplitude $\widetilde{A}(E;\theta,\omega)$ is observed from the asymptotic behavior of the solution to the Schrödinger equation

$$(-\Delta + V(x))\varphi = E\varphi \tag{1.2}$$

in the following manner :

$$\varphi(x;E,\omega) \sim e^{i\sqrt{E}\omega\cdot x} + \widetilde{C}_E \frac{e^{i\sqrt{E}r}}{r^{(n-1)/2}} \widetilde{A}(E;\theta,\omega) \tag{1.3}$$

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as $r = |x| \to \infty, \theta = x/r$. This φ is obtained by solving the Lippman-Schwinger equation :

$$\varphi(x) = e^{i\sqrt{E}\omega\cdot x} - \int_{\mathbf{R}^n} G_0(x-y,E)V(y)\varphi(y)dy, \tag{1.4}$$

where $G_0(x,E)$ is the Green function for $-\Delta - E$ defined by

$$G_0(x,E) = (2\pi)^{-n}\int_{\mathbf{R}^n} \frac{e^{ix\cdot\xi}}{\xi^2 - E - i0}d\xi. \tag{1.5}$$

Here and in the sequel for $\zeta = (\zeta_1,\cdots,\zeta_n) \in \mathbf{C}^n$, we denote $\zeta^2 = \sum_{i=1}^n \zeta_i^2$.

The inverse problem for the Schrödinger operator aims at constructing $V(x)$ from the S-matrix. When $n = 1$, the well-known theory of Gel'fand-Levitan-Marchenko provides us with the necessary and sufficient condition for a function $S(E)$ to be the S-matrix of a Schrödinger operator and an algorithm for the reconstruction of $V(x)$.

The multi-dimensional inverse problem has not been solved yet completely as in the 1-dimensional case. The main difficulty arises from the overdeterminacy ; the scattering amplitude $\tilde{A}(E;\theta,\omega)$ is a function of $2n-1$ parameters while the potential $V(x)$ depends on n variables. Therefore for a function $f(E,\theta,\omega)$ on $(0,\infty)\times S^{n-1}\times S^{n-1}$ to be the scattering amplitude associated with a Schrödinger operator, f must satisfy a sort of compatibility condition, which is still unknown. However, there is a series of deep results related to inverse problems in multi-dimensions, the main idea of which consists in using exponentially growing solutions for the Schrödinger equation (1.2). In inverse boundary value problems in a bounded domain, it is called the complex geometrical optics solution (see [SyUh]). In the inverse scattering problem, it is commonly called the $\overline{\partial}$-theory ([Na1], [Na2], [KhNo]), although the pioneering work of Faddeev [Fa] does not use this term.

In the $\overline{\partial}$-approach of inverse scattering, instead of $\widetilde{A}(E;\theta,\omega)$, one uses Faddeev's scattering amplitude :

$$A(\xi,\zeta) = \int_{\mathbf{R}^n} e^{-ix\cdot(\xi+\zeta)}V(x)\psi(x,\zeta)dx, \quad \xi\in\mathbf{R}^n, \quad \zeta\in\mathbf{C}^n \tag{1.6}$$

where $\zeta^2 = E$, and $\psi(x,\zeta)$ is a solution to the equation

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \int_{\mathbf{R}^n} G(x-y,\zeta)V(y)\psi(y,\zeta)dy, \tag{1.7}$$

$G(x,\zeta)$ being Faddeev's Green function defined by

$$G(x,\zeta) = (2\pi)^{-n}\int_{\mathbf{R}^n} \frac{e^{ix\cdot(\xi+\zeta)}}{\xi^2 + 2\zeta\cdot\xi}d\xi. \tag{1.8}$$

This function $A(\xi,\zeta)$ has the following features :

(i) It is natural to regard $A(\xi,\zeta)$ as a function on the fiber bundle $\mathcal{M} = \cup_\xi\{\xi\}\times\mathcal{V}_\xi$, where ξ varies over the base space $\mathbf{R}^n$ and the fiber $\mathcal{V}_\xi$ is defined by

$$\mathcal{V}_\xi = \{\zeta\in\mathbf{C}^n;\zeta^2 = E,\xi^2+2\zeta\cdot\xi = 0,\operatorname{Im}\zeta\neq 0\}. \tag{1.9}$$

As a 1-form on $\mathcal{M}$, it satisfies a $\overline{\partial}$-equation

$$\overline{\partial_\zeta}A(\xi,\zeta) = -(2\pi)^{1-n}\sum_{j=1}^n\left(\int_{\mathbf{R}^n}A(\xi-\eta,\zeta+\eta)A(\eta,\zeta)\eta_j\delta(\eta^2+2\zeta\cdot\eta)d\eta\right)d\overline{\zeta_j}. \tag{1.10}$$

(ii) When $n\geq 3$, the Fourier transform of the potential V is recovered from $A(\xi,\zeta)$ in the following way :

$$\hat{V}(\xi) = (2\pi)^{-n/2}\lim_{|\zeta|\to\infty,\,\zeta\in\mathcal{V}_\xi}A(\xi,\zeta). \tag{1.11}$$

Consequently, by virtue of a generalization of Bochner-Martinelli's formula on $\mathcal{V}_\xi$, we have an integral representation of $V(x)$ in terms of $A(\xi,\zeta)$.

(iii) The $\overline{\partial}$-equation characterizes the Faddeev scattering amplitude. Namely, the equation (1.10) is a necessary and sufficient condition for a function $A(\xi,\zeta)$ on the fiber bundle $\mathcal{M}$ to be the scattering amplitude associated with a Schrödinger operator on $\mathbf{R}^n$.

These ideas have been found and confirmed in various levels. For the details see [NaAb], [BeCo], [Na1], [No], [Gr] and especially the introduction of [KhNo]. See also [Ha], [We].

The purpose of the present paper is to generalize (a part of) these results for the Schrödinger operator on the hyperbolic space $\mathbf{H}^n$. In our previous works [Is1], [Is2], we have seen a close connection between the inverse problem on the Euclidean space and that on the hyperbolic space. Namely, the inverse boundary value problem in $\mathbf{R}^n$ is equivalent to the one in $\mathbf{H}^n$, or the hyperbolic quotient manifolds under the action of discrete group of translations, which then turns out to be equivalent to the inverse scattering problem. When $n\geq 3$, this latter can be solved by passing to the Faddeev scattering amplitude for the Floquet operators. However, the inversion procedures are not quite constructive and essentially the uniqueness has been proven.

In this paper we show that the Green function for the gauge-transformed Laplacian on $\mathbf{H}^n$ satisfies a $\overline{\partial}$-equation (Theorem 2.7). We then introduce an analogue of Faddeev scattering amplitude and derive a $\overline{\partial}$-formula for it

(Theorem 3.4). When $n = 3$, this leads to an integral representation of the potential in terms of the Faddeev scattering amplitude (Theorems 3.5 and 3.7). The counter part of $A(\xi, \zeta)$ introduced in this paper is a triple $\{B_{II}, B_{IJ}, B_{JI}\}$ ((3.26)–(3.28)) living on a simple line bundle and one can make use of the standard generalized Cauchy formula on $\mathbf{C}$ to derive the integral representation of the potential. We allow the potential V to be complex-valued.

Two interesting problems remain open. One is the relation between the physical scattering amplitude and the Faddeev scattering amplitude. When the potential is compactly supported, these two scattering amplitudes determine each other through the Dirichlet-Neumann map for the boundary value problem on a bounded domain which contains the support of the potential. However, a direct link between them in the case of potentials of long-tail is still unknown. The other problem is the characterization of Faddeev scattering amplitude in terms of the $\overline{\partial}$-equation. We shall return to these problems elsewhere.

We mainly work in $\mathbf{H}^3$, although many preliminary results are proven in general dimensions. The reason of the restriction to $n = 3$ is that the decay estimate for the Green operator (Theorems 2.8) is proved only when $n \leq 3$.

In §2, we prepare basic estimates for the Green operator of the gauge-transformed Laplacian on $\mathbf{H}^n$ and derive the $\overline{\partial}$-equation. In §3 we introduce the Faddeev scattering amplitude and derive its $\overline{\partial}$-equation and integral representation formulas of the potential. In §4, we show that the Faddeev scattering amplitude and the Dirichlet-Neumann map of the boundary value problem on a bounded domain determine each other.

§2. Green Operators

§2.1. Modified Bessel functions

Let $J_\nu(y)$ be the Bessel function of order ν. For $y > 0$ modified Bessel functions are defined by

$$I_\nu(y) = e^{-\nu\pi i/2} J_\nu(iy), \quad \nu \in \mathbf{C}, \tag{2.1}$$

$$K_\nu(y) = \frac{\pi}{2} \frac{I_{-\nu}(y) - I_\nu(y)}{\sin(\nu\pi)}, \quad \nu \notin \mathbf{Z}, \tag{2.2}$$

$$K_n(z) = K_{-n}(z) = \lim_{\nu \to n} K_\nu(z), \quad n \in \mathbf{Z}. \tag{2.3}$$

They are linearly independent solutions of the equation

$$y^2 u'' + y u' - (y^2 + \nu^2) u = 0. \tag{2.4}$$

They are analytic in the complex plane with cut along the negative real axis and

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!\,\Gamma(\nu+k+1)}. \tag{2.5}$$

If $-\pi < \arg z < \pi$, we have

$$I_\nu(e^{m\pi i}z) = e^{\nu m\pi i} I_\nu(z),$$
$$K_\nu(e^{m\pi i}z) = e^{-\nu m\pi i} K_\nu(z) - \pi i \frac{\sin(\nu m\pi)}{\sin(\nu\pi)} I_\nu(z).$$

(See [Wa], p. 80). In particular, for $r > 0$,

$$I_\nu(ir) = e^{\nu\pi i} I_\nu(-ir) = e^{\nu\pi i/2} J_\nu(r), \tag{2.6}$$

$$K_\nu(ir) = e^{-\nu\pi i} K_\nu(-ir) - \pi i I_\nu(-ir). \tag{2.7}$$

The following asymptotic expansions are well-known (see [Wa], p. 202) :

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} + \frac{e^{-z+(\nu+1/2)\pi i}}{\sqrt{2\pi z}}, \quad |z| \to \infty, \quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}, \tag{2.8}$$

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} + \frac{e^{-z-(\nu+1/2)\pi i}}{\sqrt{2\pi z}}, \quad |z| \to \infty, \quad -\frac{3\pi}{2} < \arg z < \frac{\pi}{2}, \tag{2.9}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \quad |z| \to \infty, \quad -\pi < \arg z < \pi. \tag{2.10}$$

The formula (2.5) implies

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu, \quad z \to 0, \tag{2.11}$$

and for $\nu \notin \mathbf{Z}$,

$$K_\nu(z) \sim \frac{\pi}{2\sin(\nu\pi)} \left(\frac{1}{\Gamma(1-\nu)} \left(\frac{z}{2}\right)^{-\nu} - \frac{1}{\Gamma(1+\nu)} \left(\frac{z}{2}\right)^{\nu} \right), \quad z \to 0. \tag{2.12}$$

When $n = 0, 1, 2, \cdots$, $K_n(z)$ has the following expression (see [Le], p. 110) :

$$\begin{aligned} K_n(z) &= (-1)^{n-1} I_n(z) \log \frac{z}{2} \\ &+ \frac{1}{2} \sum_{k=0}^{n-1} \frac{(-1)^k (n-k-1)!}{k!} \left(\frac{z}{2}\right)^{2k-n} \\ &- \frac{(-1)^{n-1}}{2} \sum_{k=0}^{\infty} \frac{\psi(k+1) + \psi(k+n+1)}{k!(k+n)!} \left(\frac{z}{2}\right)^{2k+n}, \end{aligned} \tag{2.13}$$

$$\psi(1) = -\gamma, \quad \psi(k+1) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k}, \quad k = 1, 2, \cdots, \tag{2.14}$$

γ being Euler's constant. This implies that when $z \to 0$

$$K_n(z) \sim \begin{cases} -\log z, & n = 0, \\ 2^{n-1}(n-1)!z^{-n}, & n = 1, 2, \cdots. \end{cases} \tag{2.15}$$

As was discussed in [Is1] and [Is2], one can solve the inverse boundary value problem in $\mathbf{R}^n$ by imbedding it into $\mathbf{H}^n$, where we encounter the case $\nu = 1/2$, and the above functions are written in terms of elementary functions :

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z, \tag{2.16}$$

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}, \tag{2.17}$$

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z. \tag{2.18}$$

§2.2. 1-dimensional operator

Let

$$E = \frac{(n-1)^2}{4} - \nu^2, \tag{2.19}$$

and for a complex parameter $\zeta \neq 0$ satisfying $\mathrm{Re}\,\zeta \geq 0$, consider the differential operator

$$L_0(\zeta) = y^2(-\partial_y^2 + \zeta^2) + (n-2)y\partial_y - E \tag{2.20}$$

on $(0, \infty)$, where $n \geq 2$ is an integer and $\partial_y = \partial/\partial y$. By (2.4)

$$y^{(n-1)/2} I_\nu(\zeta y), \quad y^{(n-1)/2} K_\nu(\zeta y)$$

are linearly independent solutions of $L_0(\zeta)u = 0$.

The above constant E corresponds to the energy parameter for the Laplace-Beltrami operator H_0 on $\mathbf{H}^n$, whose spectrum is the interval $[(n-1)^2/4, \infty)$. Accordingly, we consider two cases :

$$\nu \in i\mathbf{R} \setminus \{0\}, \quad \text{or} \quad \nu \in (0, \infty) \setminus \mathbf{Z}. \tag{2.21}$$

In the former case E belongs to the spectrum of H_0, and in the latter case to the resolvent set. In this paper we do not deal with the case $\nu \in \mathbf{Z}$ in order to avoid the logarithmic singularity of $K_n(z)$. We define the Green kernel of the 1-dimensional operator (2.20) by

$$G_0(y,y';\zeta) = \begin{cases} (yy')^{(n-1)/2}K_\nu(\zeta y)I_\nu(\zeta y'), & y > y' > 0, \\ (yy')^{(n-1)/2}I_\nu(\zeta y)K_\nu(\zeta y'), & y' > y > 0, \end{cases} \tag{2.22}$$

and introduce the Green operator

$$G_0(\zeta)f(y) = \int_0^\infty G_0(y,y';\zeta)f(y')\frac{dy'}{(y')^n}. \tag{2.23}$$

Since

$$\begin{aligned} &\Big(y^{(n-1)/2}I_\nu(\zeta y)\Big)\Big(y^{(n-1)/2}K_\nu(\zeta y)\Big)' \\ &- \Big(y^{(n-1)/2}I_\nu(\zeta y)\Big)'\Big(y^{(n-1)/2}K_\nu(\zeta y)\Big) = -y^{n-2}, \end{aligned}$$

we have for $f \in C_0^\infty((0,\infty))$

$$L_0(\zeta)G_0(\zeta)f = f. \tag{2.24}$$

Throughout the paper *we assume that* ν *satisfies* (2.21), although we do not mention it specifically.

Lemma 2.1. *The Green function* $G_0(y,y';\zeta)$ *is analytic in* ζ *when* $\operatorname{Re}\zeta > 0$. *There exists a constant* $C = C_\nu > 0$ *such that the inequalities*

$$|G_0(y,y';\zeta)| \le C(yy')^{(n-1)/2}, \tag{2.25}$$

$$|G_0(y,y';\zeta)| \le \frac{C}{|\zeta|}(yy')^{(n-2)/2}, \tag{2.26}$$

$$\Big|\frac{\partial}{\partial\zeta}G_0(y,y';\zeta)\Big| \le \frac{C}{|\zeta|}(yy')^{(n-2)/2}(y+y') \tag{2.27}$$

hold for $y,\ y' > 0$ *and* ζ *such that* $\operatorname{Re}\zeta \ge 0$.

Proof. The analyticity is obvious. By virtue of (2.8)–(2.15), we have

$$|I_\nu(z)| \le C\left(\frac{|z|}{1+|z|}\right)^{\operatorname{Re}\nu}(1+|z|)^{-1/2}e^{\operatorname{Re}z}, \tag{2.28}$$

$$|K_\nu(z)| \le C\left(\frac{|z|}{1+|z|}\right)^{-\operatorname{Re}\nu}(1+|z|)^{-1/2}e^{-\operatorname{Re}z}. \tag{2.29}$$

Since $t/(1+t)$ is monotone increasing for $t \geq 0$, by (2.28), (2.29) and $\mathrm{Re}\,\nu \geq 0$ we have if $y > y' > 0$,

$$|K_\nu(\zeta y)I_\nu(\zeta y')| \leq C\frac{e^{-\mathrm{Re}\,\zeta(y-y')}}{(1+|\zeta y|)^{1/2}(1+|\zeta y'|)^{1/2}}.$$

It then follows that

$$|G_0(y,y';\zeta)| \leq C(yy')^{(n-1)/2}\frac{e^{-\mathrm{Re}\,\zeta|y-y'|}}{(1+|\zeta y|)^{1/2}(1+|\zeta y'|)^{1/2}}, \tag{2.30}$$

which proves (2.25) and (2.26). Using

$$2I'_\nu(z) = I_{\nu-1}(z) + I_{\nu+1}(z), \tag{2.31}$$

$$-2K'_\nu(z) = K_{\nu-1}(z) + K_{\nu+1}(z), \tag{2.32}$$

(see [Le], p. 110) and (2.28), (2.29) as well as (2.15), we have

$$|zI'_\nu(z)| \leq C\left(\frac{|z|}{1+|z|}\right)^{\mathrm{Re}\,\nu}(1+|z|)^{1/2}e^{\mathrm{Re}\,z}, \tag{2.33}$$

$$|zK'_\nu(z)| \leq C\left(\frac{|z|}{1+|z|}\right)^{-\mathrm{Re}\,\nu}(1+|z|)^{1/2}e^{-\mathrm{Re}\,z}, \tag{2.34}$$

which imply

$$\Big|\frac{\partial}{\partial\zeta}I_\nu(\zeta y)\Big| \leq \frac{C}{|\zeta|}\left(\frac{|\zeta y|}{1+|\zeta y|}\right)^{\mathrm{Re}\,\nu}(1+|\zeta y|)^{1/2}e^{\mathrm{Re}\,\zeta y}, \tag{2.35}$$

$$\Big|\frac{\partial}{\partial\zeta}K_\nu(\zeta y)\Big| \leq \frac{C}{|\zeta|}\left(\frac{|\zeta y|}{1+|\zeta y|}\right)^{-\mathrm{Re}\,\nu}(1+|\zeta y|)^{1/2}e^{-\mathrm{Re}\,\zeta y}. \tag{2.36}$$

This together with (2.28), (2.29) and an elementary inequality

$$\left(\frac{1+|\zeta y'|}{1+|\zeta y|}\right)^{1/2} \leq \left(1+\frac{y'}{y}\right)^{1/2} \leq \frac{y+y'}{(yy')^{1/2}}$$

proves (2.27). □

§2.3. Green operator on $\mathbf{H}^n$

Let us construct a Green operator of

$$H_0(\theta) = y^2(-\partial_y^2 + (-i\partial_x+\theta)^2) + (n-2)y\partial_y. \tag{2.37}$$

For $\theta, \theta' \in \mathbf{C}^{n-1}$, we put

$$\theta \cdot \theta' = \sum_{i=1}^{n-1} \theta_i {\theta_i}', \quad \theta^2 = \theta \cdot \theta,$$

and define for $\xi \in \mathbf{R}^{n-1}$

$$\zeta(\xi, \theta) = \sqrt{(\xi + \theta)^2}, \tag{2.38}$$

where we take the branch of $\sqrt{\cdot}$ such that $\mathrm{Re}\sqrt{\cdot} \geq 0$, i.e. $\sqrt{z} = \sqrt{r}e^{i\varphi/2}$ for $z = re^{i\varphi}, -\pi < \varphi < \pi$. We define

$$\mathbf{G}_0(\theta)f(x,y) = (2\pi)^{-(n-1)/2} \int_{\mathbf{R}^{n-1}} e^{ix\cdot\xi} \left(G_0(\zeta(\xi,\theta))\hat{f}(\xi,\cdot) \right)(y) d\xi, \tag{2.39}$$

$$\hat{f}(\xi, y) = (2\pi)^{-(n-1)/2} \int_{\mathbf{R}^{n-1}} e^{-ix\cdot\xi} f(x,y) dx. \tag{2.40}$$

By (2.24) and (2.25), we have

$$(H_0(\theta) - E)\mathbf{G}_0(\theta) f = f, \quad \forall f \in C_0^\infty(\mathbf{R}^n_+). \tag{2.41}$$

Let us remark that when $\theta \in \mathbf{R}^{n-1}$ and $\nu = i\sigma$ with $\sigma > 0$ (or $\sigma < 0$), $\mathbf{G}_0(\theta)$ is the incoming (or outgoing) Green operator of $H_0(\theta) - E$:

$$\mathbf{G}_0(\theta) = (H_0(\theta) - (E \mp i0))^{-1},$$

where the right-hand side exists on a certain Banach space (see [Is1]), which we now explain.

For $s \in \mathbf{R}$, we introduce the following function space :

$$f \in L^{2,s} \Longleftrightarrow \int_{\mathbf{R}^n_+} (1 + |\log y|)^{2s} |f(x,y)|^2 \frac{dxdy}{y^n} < \infty \tag{2.42}$$

equipped with the obvious norm. In the following, for two Banach spaces X and Y, $\mathbf{B}(X;Y)$ denotes the totality of bounded operators from X to Y.

Lemma 2.2. *Let $s > 1/2$. Then there exists a constant $C = C_{s,\nu} > 0$ such that*

$$\|\mathbf{G}_0(\theta)\|_{\mathbf{B}(L^{2,s},L^{2,-s})} \leq C, \quad \forall \theta \in \mathbf{C}^{n-1}.$$

Proof. Let $u = \mathbf{G}_0(\theta)f$. Since

$$\hat{u}(\xi, y) = \int_0^\infty G_0(y, y'; \zeta(\xi,\theta))\, \hat{f}(\xi, y') \frac{dy'}{(y')^n},$$

we have by using (2.25),

$$|\hat{u}(\xi, y)|^2 \leq C y^{n-1} \int_0^\infty (1 + |\log y'|)^{2s} |\hat{f}(\xi, y')|^2 \frac{dy'}{(y')^n}.$$

Integration with respect to ξ and y proves the lemma. □

This Green operator $\mathbf{G}_0(\theta)$ has an integral kernel

$$\mathbf{G}_0(x, y, x', y'; \theta) = (2\pi)^{-(n-1)} \int_{\mathbf{R}^{n-1}} e^{i(x-x')\cdot\xi} G_0(y, y'; \zeta(\xi, \theta)) d\xi. \tag{2.43}$$

It has the following estimates.

Lemma 2.3. (1) *For any $\theta \in \mathbf{C}^{n-1}$, there exists a constant $C = C_\theta > 0$ independent of x, x', y, y' such that*

$$|\mathbf{G}_0(x, y, x', y'; \theta)| \leq C \frac{(yy')^{(n-1)/2}}{|y - y'|^{n-1}} e^{C|y-y'|}.$$

(2) *Fix $y, y' > 0$ and $\theta \in \mathbf{C}^{n-1}$ arbitrarily. Then there exist $h_1(x), h_2(x)$ such that*

$$\mathbf{G}_0(x, x', y, y'; \theta) = h_1(x - x') + h_2(x - x'),$$

where $h_1(x) \in H^m(\mathbf{R}^{n-1})$, $\forall m > 0$, $h_2(x) \in C^\infty(\mathbf{R}^{n-1} \setminus \{0\})$ and satisfies

$$|\partial_x^\alpha h_2(x)| \leq C_\alpha |x|^{-N}, \quad \forall \alpha, \ N > n - 2.$$

Proof. There exists a constant $C > 0$ such that $\mathrm{Re}\, \zeta(\xi, \theta) \geq |\xi|/2 - C$. This together with the estimate $|G_0(y, y'; \zeta)| \leq C(yy')^{(n-1)/2} e^{-\mathrm{Re}\, \zeta|y-y'|}$ (see (2.30)) proves the assertion (1).

Take $\chi_0, \chi_\infty \in C^\infty(\mathbf{R}^{n-1})$ such that $\chi_0(\xi) + \chi_\infty(\xi) = 1$ on $\mathbf{R}^{n-1}$ and $\chi_0(\xi) = 1$ for $|\xi| < C$, $\chi_0(\xi) = 0$ for $|\xi| > 2C$, where C is chosen large enough so that $\zeta(\xi, \theta)$ is smooth on $|\xi| > C$. Split $G_0(y, y'; \zeta(\xi, \theta))$ into two parts :

$$g_0(\xi) = \chi_0(\xi) G_0(y, y'; \zeta(\xi, \theta)), \quad g_\infty(\xi) = \chi_\infty(\xi) G_0(y, y'; \zeta(\xi, \theta)).$$

We then have

$$\mathbf{G}_0(x, x', y, y'; \theta) = \hat{g}_0(x' - x) + \hat{g}_\infty(x' - x),$$

and $\hat{g}_0 \in H^m(\mathbf{R}^{n-1})$, $\forall m > 0$, by (2.25). Let $\hat{\xi} = \xi/|\xi|$. Using

$$\zeta(\xi, \theta) = |\xi| + \hat{\xi} \cdot \theta + O(1/|\xi|), \quad \text{as} \quad |\xi| \to \infty,$$

we have the asymptotic expansions

$$I_\nu(\zeta(\xi,\theta)y) = \frac{e^{(|\xi|+\hat{\xi}\cdot\theta)y}}{\sqrt{2\pi|\xi|y}}\Big(1 + \frac{a_1(\hat{\xi})}{|\xi|} + \frac{a_2(\hat{\xi})}{|\xi|^2} + \cdots\Big),$$

$$K_\nu(\zeta(\xi,\theta)y) = \sqrt{\frac{\pi}{2|\xi|y}}e^{-(|\xi|+\hat{\xi}\cdot\theta)y}\Big(1 + \frac{b_1(\hat{\xi})}{|\xi|} + \frac{b_2(\hat{\xi})}{|\xi|^2} + \cdots\Big),$$

$a_m(\hat{\xi}), b_m(\hat{\xi})$ being smooth functions. Hence we have the asymptotic expansion

$$g_\infty(\xi) \sim \chi_\infty(\xi)e^{-|\xi||y-y'|}\sum_{m=1}^{\infty}\frac{c_m(\hat{\xi})}{|\xi|^m},$$

where $c_m(\hat{\xi})$ is a smooth function. By induction one can show

$$|\partial_\xi^\alpha e^{-|\xi||y-y'|}| \leq C_\alpha|\xi|^{-|\alpha|}.$$

Hence we have

$$\Big|\partial_\xi^\alpha\Big(\chi_\infty(\xi)e^{-|\xi||y-y'|}\frac{c_m(\hat{\xi})}{|\xi|^m}\Big)\Big| \leq C_\alpha(1+|\xi|)^{-m-|\alpha|}.$$

We now use $e^{ix\cdot\xi} = -|x|^{-2}\Delta_\xi e^{ix\cdot\xi}$ and integrate by parts to see that

$$\Big|\int e^{ix\cdot\xi}\chi_\infty(\xi)e^{-|\xi||y-y'|}\frac{c_m(\hat{\xi})}{|\xi|^m}d\xi\Big| \leq C_N|x|^{-N},$$

which proves the assertion (2). □

§2.4. $\overline{\partial}$-equation

For $\theta = \theta_R + i\theta_I \in \mathbf{C}^{n-1}$, let $\overline{\partial_\theta}$ be defined as follows :

$$\overline{\partial_\theta} = \left(\frac{\partial}{\partial\overline{\theta}_1},\cdots,\frac{\partial}{\partial\overline{\theta}_{n-1}}\right), \quad \frac{\partial}{\partial\overline{\theta}_j} = \frac{1}{2}\left(\frac{\partial}{\partial\theta_{Rj}} + i\frac{\partial}{\partial\theta_{Ij}}\right). \tag{2.44}$$

We are going to compute $\overline{\partial_\theta}\mathbf{G}_0(\theta)$. Note that if $f(z)$ is analytic, $f(\zeta(\xi,\theta))$ has singularities on the set $\{\theta\in\mathbf{C}^{n-1}\,;\,(\xi+\theta)^2\leq 0\}$.

Lemma 2.4. *Let $f(z)$ be an analytic function on $\{z\in\mathbf{C}\,;\,\mathrm{Re}\,z > 0\}$ satisfying*

$$\sup_{|z|<r}|f(z)| < \infty, \quad \forall r > 0.$$

For $\theta = \theta_R + i\theta_I \in \mathbf{C}^{n-1}$ *such that* $\theta_I \neq 0$ *we put*

$$r_\theta(\xi) = \sqrt{|\theta_I|^2 - |\xi + \theta_R|^2}, \tag{2.45}$$

$$M_\theta = \Big\{\xi \in \mathbf{R}^{n-1}\,;\, \theta_I \cdot (\xi + \theta_R) = 0,\, |\xi + \theta_R| < |\theta_I|\Big\}, \tag{2.46}$$

and define a compactly supported distribution $T_\theta(\xi)$ *by*

$$\langle T_\theta(\xi), \varphi(\xi)\rangle = \int_{M_\theta} \varphi(\xi) \frac{i(\xi + \overline{\theta})}{2|\theta_I|} dM_\theta(\xi), \quad \forall \varphi \in C_0^\infty(\mathbf{R}^{n-1}), \tag{2.47}$$

$dM_\theta(\xi)$ *being the measure on* M_θ *induced from the Lebesgue measure* $d\xi$ *on* $\mathbf{R}^{n-1}$. *Then regarding* $f(\zeta(\xi,\theta))$ *as a distribution with respect to* $\xi \in \mathbf{R}^{n-1}$ *depending on a parameter* $\theta \in \mathbf{C}^{n-1}$, *we have for* $\theta_I \neq 0$

$$\overline{\partial_\theta}\, f(\zeta(\xi,\theta)) = [f(ir_\theta(\xi)) - f(-ir_\theta(\xi))]\, T_\theta(\xi). \tag{2.48}$$

Proof. Take $\chi(t) \in C^\infty(\mathbf{R})$ such that $\chi(t) = 1$ ($|t| > 2$), $\chi(t) = 0$ ($|t| < 1$) and let $\chi_\epsilon(t) = \chi(t/\epsilon)$. Since $\zeta(\xi,\theta)$ is analytic with respect to θ if $\theta_I \cdot (\xi + \theta_R) \neq 0$, we have

$$\overline{\partial_\theta}\, \chi_\epsilon(\theta_I \cdot (\xi + \theta_R)) f(\zeta(\xi,\theta)) = \frac{i}{2\epsilon} \chi'\Big(\frac{\theta_I \cdot (\xi + \theta_R)}{\epsilon}\Big)(\xi + \overline{\theta}) f(\zeta(\xi,\theta)).$$

We put

$$\tau = |\theta_I|, \quad \alpha = \theta_I/|\theta_I|, \quad p^\perp = p - (p\cdot\alpha)\alpha, \quad p \in \mathbf{R}^{n-1}.$$

Let $k_1 = \alpha \cdot (\xi + \theta_R), k_2 = (\xi + \theta_R)^\perp$. Again letting $k_1 = \epsilon\eta_1$ with $\epsilon > 0$, we have

$$\zeta(\xi,\theta)^2 = \epsilon^2\eta_1^2 + k_2^2 - \tau^2 + 2i\epsilon\tau\eta_1.$$

Therefore when $\epsilon \to 0$

$$\zeta(\xi,\theta) \to \begin{cases} \sqrt{k_2^2 - \tau^2} & \text{if} \quad |k_2| \geq \tau, \\ \operatorname{sgn}(\eta_1)\, i\sqrt{\tau^2 - k_2^2} & \text{if} \quad |k_2| < \tau, \end{cases}$$

where $\operatorname{sgn}(\eta_1) = 1$ if $\eta_1 > 0$, $\operatorname{sgn}(\eta_1) = -1$ if $\eta_1 < 0$. Let

$$g(k) = f(\zeta(\xi,\theta)), \quad \psi(k) = \varphi(\xi).$$

Then we have

$$\begin{aligned} &\overline{\partial_\theta} \int \chi_\epsilon(\theta_I \cdot (\xi + \theta_R)) f(\zeta(\xi,\theta))\varphi(\xi) d\xi \\ &\quad = \frac{i}{2} \int \chi'(\tau\eta_1)\big(\epsilon\eta_1\alpha + k_2 - i\theta_I\big) g(\epsilon\eta_1, k_2)\psi(\epsilon\eta_1, k_2) d\eta_1 dk_2. \end{aligned}$$

Let us recall here that the boundary value on the imaginary axis $f(is) = \lim_{\epsilon\to 0} f(is+\epsilon)$ exists almost every where (see e.g. [Hof], p. 38). Since $\int_{-\infty}^{\infty}\chi'(\tau\eta_1)d\eta_1 = 0$, we have

$$\int_{|k_2|>\tau}\chi'(\tau\eta_1)\big(\epsilon\eta_1\alpha + k_2 - i\theta_I\big)g(\epsilon\eta_1,k_2)\psi(\epsilon\eta_1,k_2)d\eta_1 dk_2$$
$$\to \int_{|k_2|>\tau}\chi'(\tau\eta_1)\big(k_2 - i\theta_I\big)f\Big(\sqrt{k_2^2-\tau^2}\Big)\psi(0,k_2)d\eta_1 dk_2 = 0.$$

On the other hand, since $\int_0^\infty \chi'(\tau\eta_1)d\eta_1 = 1/\tau$, $\int_{-\infty}^0\chi'(\tau\eta_1)d\eta_1 = -1/\tau$, the integral over the region $\{|k_2|<\tau\}$ converges to

$$\int_{\eta_1>0,|k_2|<\tau}\chi'(\tau\eta_1)\frac{i(k_2-i\theta_I)}{2}f\Big(i\sqrt{\tau^2-k_2^2}\Big)\psi(0,k_2)d\eta_1 dk_2$$
$$+\int_{\eta_1<0,|k_2|<\tau}\chi'(\tau\eta_1)\frac{i(k_2-i\theta_I)}{2}f\Big(-i\sqrt{\tau^2-k_2^2}\Big)\psi(0,k_2)d\eta_1 dk_2$$
$$=\frac{1}{\tau}\int_{|k_2|<\tau}\Big[f\Big(i\sqrt{\tau^2-k_2^2}\Big)-f\Big(-i\sqrt{\tau^2-k_2^2}\Big)\Big]\frac{i(k_2-i\theta_I)}{2}\psi(0,k_2)dk_2.$$

From this the lemma follows immediately. □

We put

$$D_\theta(\xi) = [G_0(ir_\theta(\xi)) - G_0(-ir_\theta(\xi)]\,\frac{i(\xi+\overline{\theta})}{2|\theta_I|}. \tag{2.49}$$

By virtue of (2.22), (2.39) and (2.48), we have formally

$$\overline{\partial_\theta}\,\mathbf{G}_0(\theta)f(x,y) = (2\pi)^{-(n-1)/2}\int_{M_\theta}e^{ix\cdot\xi}\Big(D_\theta(\xi)\hat{f}(\xi,\cdot)\Big)(y)dM_\theta(\xi). \tag{2.50}$$

Let us give a precise meaning to this operator. For $t, s\in\mathbf{R}$, we introduce the function spaces :

(2.51)
$$\mathcal{H}^{(\pm)}_{t,s}\ni f \Longleftrightarrow \int_{\mathbf{R}^n_+}\left[(1+|x|)^t(1+|\log y|)^s\frac{(1+y)}{y^{1/2}}\right]^{\pm 2}|f(x,y)|^2\frac{dxdy}{y^n}<\infty,$$

$$\mathcal{H}^{(\pm)}_{s} = \mathcal{H}^{(\pm)}_{s,s} \tag{2.52}$$

equipped with the obvious norm. We define the operator $\chi_\epsilon(\theta_I\cdot(D+\theta_R))$ by

$$\big(\chi_\epsilon(\theta_I\cdot(D+\theta_R))\varphi\big)(x) = (2\pi)^{-(n-1)/2}\int_{\mathbf{R}^{n-1}}e^{ix\cdot\xi}\chi_\epsilon(\theta_I\cdot(\xi+\theta_R))\hat{\varphi}(\xi)d\xi, \tag{2.53}$$

where χ_ϵ is the function given in the proof of Lemma 2.4, and put

$$\mathbf{G}_{0,\epsilon}(\theta) = \chi_\epsilon(\theta_I \cdot (D + \theta_R))\mathbf{G}_0(\theta). \tag{2.54}$$

The following theorem gives the procedure for defining $\overline{\partial_\theta}\, \mathbf{G}_0(\theta)$.

Theorem 2.5. *Let $s > 1/2$ and suppose $f \in \mathcal{H}^{(+)}_s$. We put $u_\epsilon = \mathbf{G}_{0,\epsilon}(\theta) f$, $u = \mathbf{G}_0(\theta) f$ and*

$$v(x, y; \theta) = (2\pi)^{-(n-1)/2} \int_{M_\theta} e^{ix\cdot\xi} \left(D_\theta(\xi)\hat{f}(\xi, \cdot)\right)(y)dM_\theta(\xi).$$

Then
(1) $u_\epsilon \to u$ *in* $L^{2,-s}$.
(2) u_ϵ *is strongly differentiable in* $\mathcal{H}^{(-)}_{0,s}$ *with respect to* θ *if* $\theta_I \neq 0$.
(3) $\overline{\partial_\theta}\, u_\epsilon \to v$ *in* $\mathcal{H}^{(-)}_s$.

Proof. The assertion (1) is obvious. Let us prove (2). If $\epsilon > 0$ is fixed and θ varies over a bounded set, we have on supp $\chi_\epsilon(\theta_I \cdot (\xi + \theta_R))$

$$1/|\zeta(\xi, \theta)| \leq C_\epsilon(1 + |\xi|)^{-1}.$$

Let $d_\theta = \partial/\partial\theta$ or $\partial/\partial\overline{\theta}$. Then by virtue of (2.26), (2.27) and the above estimate, we have

$$|d_\theta\, \hat{u}_\epsilon(\xi, y; \theta)| \leq C_\epsilon \int_0^\infty (yy')^{(n-2)/2}(1 + y)(1 + y')|\hat{f}(\xi, y')| \frac{dy'}{(y')^n},$$

which implies for $s > 1/2$

$$\frac{y}{(1+y)^2} \frac{|d_\theta\, \hat{u}_\epsilon(\xi, y; \theta)|^2}{y^n} \leq \frac{C_\epsilon}{y} \int_0^\infty (1 + |\log y'|)^{2s} \frac{(1 + y')^2}{y'} |\hat{f}(\xi, y')|^2 \frac{dy'}{(y')^n}.$$

Therefore by Lebesgue's theorem, $u_\epsilon(x, y; \theta)$ is a $\mathcal{H}^{(-)}_{0,s}$-valued C^1-function of θ if $\theta_I \neq 0$, and we have

$$\overline{\partial_\theta}\, u_\epsilon(x, y; \theta) = (2\pi)^{-(n-1)/2} \int_{\mathbf{R}^n_+} e^{ix\cdot\xi} \frac{i}{2\epsilon} \chi'\Big(\frac{\theta_I \cdot (\xi + \theta_R)}{\epsilon}\Big)(\xi + \overline{\theta}) \times G_0(y, y'; \zeta(\xi, \theta))\hat{f}(\xi, y') \frac{d\xi dy'}{(y')^n}. \tag{2.55}$$

Let us show the assertion (3). Let θ vary over the ball $\{|\theta| < C_0\}$ and pick $\psi_0, \psi_\infty \in C^\infty(\mathbf{R}^n)$ such that $\psi_0(\xi) = 1$ if $|\xi| < 5C_0$, $\psi_0(\xi) = 0$ if $|\xi| > 6C_0$, and $\psi_\infty(\xi) = 1 - \psi_0(\xi)$. We put

$$\hat{v}^{(0)}_\epsilon(\xi, y; \theta) = \psi_0(\xi)\overline{\partial_\theta}\, \hat{u}_\epsilon(\xi, y; \theta), \tag{2.56}$$

$$\hat{v}_\epsilon^{(\infty)}(\xi, y; \theta) = \psi_\infty(\xi)\overline{\partial_\theta}\, \hat{u}_\epsilon(\xi, y; \theta). \tag{2.57}$$

We first show that $v_\epsilon^{(\infty)} \to 0$ in $\mathcal{H}_s^{(-)}$ as $\epsilon \to 0$. Assume for the notational simplicity that $\theta_I/|\theta_I| = \alpha = (1, 0, \cdots, 0)$, and let $k_1 = \alpha \cdot (\xi + \theta_R), k_2 = (\xi + \theta_R)^\perp$. We put

$$w_\epsilon^{(\infty)}(x_1, k_2, y; \theta) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ix_1 k_1} \hat{v}_\epsilon^{(\infty)}(k_1, k_2, y; \theta) dk_1. \tag{2.58}$$

By the change of variable $k_1 = \epsilon\eta_1$, we then have letting $\tau = |\theta_I|$

(2.59)
$$w_\epsilon^{(\infty)}(x_1, k_2, y; \theta) = (2\pi)^{-1/2} \frac{i}{2} \int_{\mathbf{R}_+^2} e^{i\epsilon\eta_1 x_1} \chi'(\tau\eta_1)(\epsilon\eta_1\alpha + k_2 - i\theta_I)$$
$$\times\, \psi_\infty(\epsilon\eta_1, k_2) G_0(y, y'; \zeta(\epsilon\eta_1, k_2, \theta)) \hat{f}(\epsilon\eta_1, k_2, y') \frac{d\eta_1 dy'}{(y')^n}.$$

On $\operatorname{supp} \psi_\infty(\epsilon\eta_1, k_2)$, we have

$$1/|\zeta(\epsilon\eta_1, k_2, \theta))| \leq C(1 + |k_2|)^{-1},$$

where C is independent of $\epsilon > 0$. We also note that letting

$$\tilde{f}(x_1, k_2, y) = (2\pi)^{-(n-2)/2} \int_{\mathbf{R}^{n-2}} e^{-ix_2 \cdot k_2} f(x_1, x_2, y) dx_2,$$

we have for $s > 1/2$

$$\sup_{k_1} |\hat{f}(k_1, k_2, y)| \leq C \left(\int_{-\infty}^{\infty} \langle x_1 \rangle^{2s} |\tilde{f}(x_1, k_2, y)|^2 dx_1 \right)^{1/2}. \tag{2.60}$$

Here and in the following we write $\langle x \rangle = (1 + |x|^2)^{1/2}$. Therefore by (2.26) the integrand of the right-hand side of (2.59) is dominated from above by

$$C|\chi'(\tau\eta_1)|(yy')^{(n-2)/2} \frac{|\hat{f}(\epsilon\eta_1, k_2, y')|}{(y')^n} \leq Cy^{(n-2)/2} |\chi'(\tau\eta_1)| \frac{(1 + |\log y'|)^{-s}}{\sqrt{y'}}$$
$$\times \left(\int_{-\infty}^{\infty} \langle x_1' \rangle^{2s} (1 + |\log y'|)^{2s} \frac{|\tilde{f}(x_1', k_2, y')|^2}{(y')^{n+1}} dx_1' \right)^{1/2}.$$

We then see by Schwarz' inequality that the right-hand side is integrable with respect to η_1 and y' for a.e. k_2. Therefore by Lebesgue's convergence theorem and the argument in the proof of Lemma 2.4, $w_\epsilon^{(\infty)} \to 0$ pointwise as $\epsilon \to 0$. We also have

$$
\begin{aligned}
&|w_\epsilon^{(\infty)}(x_1, k_2, y; \theta)| \\
&\quad \le C y^{(n-2)/2} \left(\int_{\mathbf{R}_+^2} (1 + |\log y'|)^{2s} \frac{\langle x_1' \rangle^{2s}}{y'} |\tilde{f}(x_1', k_2, y')|^2 \frac{dx_1' dy'}{(y')^n} \right)^{1/2}.
\end{aligned}
$$

Again by virtue of Lebesgue's theorem we have

$$
\int_{\mathbf{R}_+^n} \langle x_1 \rangle^{-2s} (1 + |\log y|)^{-2s} \frac{|w_\epsilon^{(\infty)}(x_1, k_2, y; \theta)|^2}{y^{n-1}} dx_1 dk_2 dy \to 0,
$$

which proves $v_\epsilon^{(\infty)} \to 0$ in $\mathcal{H}_s^{(-)}$.

We finally consider $v_\epsilon^{(0)}(\xi, y; \theta)$. Let

$$
\begin{aligned}
&w_\epsilon^{(0)}(x_1, k_2, y; \theta) \\
&\quad = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ix_1 k_1} \hat{v}_\epsilon^{(0)}(k_1, k_2, y; \theta) dk_1 \\
&\quad = (2\pi)^{-1/2} \frac{i}{2} \int_{\mathbf{R}_+^2} e^{i\epsilon\eta_1 x_1} \chi'(\tau\eta_1)(\epsilon\eta_1\alpha + k_2 - i\theta_I) \psi_0(\epsilon\eta_1, k_2) \\
&\qquad \times \ G_0(y, y'; \zeta(\epsilon\eta_1, k_2, \theta)) \hat{f}(\epsilon\eta_1, k_2, y') \frac{d\eta_1 dy'}{(y')^n}.
\end{aligned}
$$

Then by the same argument as in the proof of Lemma 2.4 we have if $|k_2| > \tau$,

$$
w_\epsilon^{(0)}(x_1, k_2, y; \theta) \to 0
$$

pointwise as $\epsilon \to 0$, and if $|k_2| < \tau$,

$$
\begin{aligned}
w_\epsilon^{(0)}(x_1, k_2, y; \theta) \to \frac{(2\pi)^{-1/2}}{\tau} \int_0^\infty &\Big[G_0\Big(y, y'; i\sqrt{\tau^2 - k_2^2}\Big) \\
&- G_0\Big(y, y'; -i\sqrt{\tau^2 - k_2^2}\Big) \Big] \frac{i(k_2 - i\theta_I)}{2} \hat{f}(0, k_2, y') \frac{dy'}{(y')^n}
\end{aligned}
$$

pointwise as $\epsilon \to 0$. Moreover by (2.25) and (2.60), we have

$$
\begin{aligned}
&|w_\epsilon^{(0)}(x_1, k_2, y; \theta)| \\
&\quad \le C y^{(n-1)/2} \times \left(\int_{\mathbf{R}_+^2} (1 + |\log y'|)^{2s} \langle x_1' \rangle^{2s} |\tilde{f}(x_1', k_2, y')|^2 \frac{dx_1' dy'}{(y')^n} \right)^{1/2}.
\end{aligned}
$$

By Lebesgue's convergence theorem, $v_\epsilon^{(0)} \to v$ in $\mathcal{H}_s^{(-)}$. □

Let us rewrite $\overline{\partial_\theta}\, \mathbf{G}_0(\theta)$ more explicitly.

Lemma 2.6. *For $r > 0$*

$$\Big[G_0(ir) - G_0(-ir)\Big] f(y) = -\pi i \int_0^\infty (yy')^{(n-1)/2} J_\nu(ry) J_\nu(ry') f(y') \frac{dy'}{(y')^n}.$$

Proof. In view of (2.6) and (2.7), we have

$$\begin{aligned} &K_\nu(iry) I_\nu(iry') - K_\nu(-iry) I_\nu(-iry') \\ &\quad = -e^{\nu\pi i} \pi i I_\nu(-iry) I_\nu(-iry') \\ &\quad = -\pi i J_\nu(ry) J_\nu(ry'). \end{aligned}$$

The lemma then follows from (2.22). □

The above lemma and (2.49) together with Theorem 2.5 then imply the following

Theorem 2.7. *Let $f \in \mathcal{H}_s^{(+)}$ with $s > 1/2$ and suppose $\theta_I \neq 0$. Then :*
(1) *For $\epsilon > 0$, $\mathbf{G}_{0,\epsilon}(\theta) f$ is an $\mathcal{H}_{0,s}^{(-)}$-valued C^1-function of θ.*
(2) *$\mathbf{G}_{0,\epsilon}(\theta) f \to \mathbf{G}_0(\theta) f$ in $L^{2,-s}$ as $\epsilon \to 0$.*
(3) *When $\epsilon \to 0$, $\overline{\partial_\theta}\, \mathbf{G}_{0,\epsilon}(\theta) f$ converges in $\mathcal{H}_s^{(-)}$. Denoting this limit by $\overline{\partial_\theta}\, \mathbf{G}_0(\theta) f$, we have the following formula :*

$$\begin{aligned} \overline{\partial_\theta}\, \mathbf{G}_0(\theta) f = &-\frac{\pi i}{(2\pi)^{(n-1)/2}} \iint_{M_\theta \times (0,\infty)} e^{ix\cdot k}\, (yy')^{(n-1)/2} \\ &\cdot J_\nu(r_\theta(k) y) J_\nu(r_\theta(k) y') \hat{f}(k, y') \frac{i\left(k + \overline{\theta}\right)}{2|\theta_I|} \frac{dM_\theta(k) dy'}{(y')^n}. \end{aligned}$$

Let us remark that in Theorem 2.1 of [IsUh], the analyticity with respect to $z \in \mathbf{C}$ of the Green operator $\mathbf{G}_0(z\alpha)$, $\alpha \in S^{n-1}$ is stated. It is not correct and the above Theorem 2.7 gives the correct assertion.

§2.5. Perturbed Green operator

From now on we restrict the space dimension to 2 or 3. For $n = 3$ and $s, t \geq 0$, we introduce the following function spaces :

$$(2.61) \quad \mathcal{W}_{t,s}^{(\pm)} \ni u \Longleftrightarrow \int_{\mathbf{R}_+^3} \left[(1 + |x|)^{2t} \frac{(1 + |\log y|)^{2s}}{y} \right]^{\pm 1} |u(x, y)|^2 \frac{dxdy}{y^3} < \infty$$

equipped with the obvious norm. For $n = 2$ and $s \geq 0$, we define

$$(2.62) \qquad \mathcal{W}_s^{(\pm)} \ni u \Longleftrightarrow \int_{\mathbf{R}_+^2} \left[\frac{(1 + |\log y|)^{2s}}{y} \right]^{\pm 1} |u(x, y)|^2 \frac{dxdy}{y^2} < \infty.$$

Theorem 2.8. (1) *Let $n = 3$ and $s > 1$. Then there exists a constant $C_s > 0$ such that for $0 \le t \le s$*

$$\|\mathbf{G}_0(\theta)f\|_{\mathcal{W}^{(-)}_{t,s}} \le C_s \left(\frac{\log|\theta_I|}{|\theta_I|}\right)^{1/2} \|f\|_{\mathcal{W}^{(+)}_{s-t,s}} \quad \text{if} \quad |\theta_I| > 2. \tag{2.63}$$

(2) *Let $n = 2$ and $s > 1/2$. Then we have*

$$\|\mathbf{G}_0(\theta)f\|_{\mathcal{W}^{(-)}_{s}} \le \frac{C_s}{|\theta_I|}\|f\|_{\mathcal{W}^{(+)}_{s}} \quad \text{if} \quad |\theta_I| > 2. \tag{2.64}$$

Proof. First let us note that in order to prove (2.63), we have only to show

$$\|\mathbf{G}_0(\theta)f\|_{\mathcal{W}^{(-)}_{0,s}} \le C_s \left(\frac{\log|\theta_I|}{|\theta_I|}\right)^{1/2} \|f\|_{\mathcal{W}^{(+)}_{s,s}} \quad \text{if} \quad |\theta_I| > 2. \tag{2.65}$$

Once this is proved, by taking the adjoint we also have

$$\|\mathbf{G}_0(\theta)f\|_{\mathcal{W}^{(-)}_{s,s}} \le C_s \left(\frac{\log|\theta_I|}{|\theta_I|}\right)^{1/2} \|f\|_{\mathcal{W}^{(+)}_{0,s}} \quad \text{if} \quad |\theta_I| > 2. \tag{2.66}$$

Interpolating (2.65) and (2.66), we get (2.63).

Let $k = \theta_R$, $\tau = |\theta_I|$, $\alpha = \theta_I/\tau$ and rewrite θ as $\theta = k + i\tau\alpha$. Then we have

$$e^{ix\cdot k}\mathbf{G}_0(\theta)e^{-ix\cdot k} = \mathbf{G}_0(i\tau\alpha).$$

This reduces the theorem to the case that $\theta_R = 0$. The rest of the proof is essentially the same as that of [IsUh], Theorem 2.3, which we reproduce here for the reader's convenience. Without loss of generality, we assume that $\alpha = (0,1)$ if $n = 3$, or $\alpha = 1$ if $n = 2$. Let

$$\Omega_1 = \left\{|\xi|^2 < \frac{\tau^2}{2}\right\} \cup \left\{|\xi|^2 > \frac{3\tau^2}{2}\right\},$$
$$\Omega_2 = \left\{\frac{\tau^2}{2} < |\xi|^2 < \frac{3\tau^2}{2}\right\} \cap \left\{|\xi_2| > \frac{\tau}{10}\right\},$$
$$\Omega_3 = \left\{\frac{\tau^2}{2} < |\xi|^2 < \frac{3\tau^2}{2}\right\} \cap \left\{|\xi_2| < \frac{\tau}{10}\right\}.$$

Note that if $n = 2$, $\xi_2 = \xi$ and Ω_3 is empty. We put

$$u_j(x,y) = (2\pi)^{-1}\int_{\Omega_j} e^{ix\cdot\xi}\big(G_0(i\tau\alpha)\hat{f}(\xi,\cdot)\big)(y)d\xi.$$

On $\Omega_1 \cup \Omega_2$, $|\zeta| \geq C\tau$ for $C > 0$. Hence using (2.26), we have for $j = 1, 2$,

$$|\hat{u}_j(\xi, y)| \leq \frac{C}{\tau} y^{(n-2)/2} \int_0^\infty \frac{|\hat{f}(\xi, y')|}{(y')^{n/2+1}} dy'.$$

We then have for $s > 1/2$

$$\int_{\mathbf{R}^{n-1}} |u_j(x,y)|^2 dx \leq \frac{C}{\tau^2} y^{n-2} \int_{\mathbf{R}^n_+} \frac{(1 + |\log y'|)^{2s}}{y'} |f(x', y')|^2 \frac{dx'dy'}{(y')^n}.$$

Therefore u_1 and u_2 satisfy
(2.67)

$$\int_{\mathbf{R}^n_+} \frac{y}{(1 + |\log y|)^{2s}} |u_j(x,y)|^2 \frac{dxdy}{y^n} \leq \frac{C}{\tau^2} \int_{\mathbf{R}^n_+} \frac{(1 + |\log y|)^{2s}}{y} |f(x,y)|^2 \frac{dxdy}{y^n}.$$

When $n = 2$, the proof is finished.

To estimate u_3, we let

$$\Omega'_3 = \left\{ \frac{1}{2} < |\eta|^2 < \frac{3}{2} \right\} \cap \left\{ |\eta_2| < \frac{1}{10} \right\}.$$

By the change of variable $\xi = \tau\eta$, $\zeta(\xi, \theta) = \tau(\eta^2 - 1 + i\eta_2)^{1/2}$. Then arguing as above, we have

$$\int_{\mathbf{R}^2} |u_3(x,y)|^2 dx \leq Cy \int_{\Omega'_3 \times (0,\infty)} \frac{(1 + |\log y'|)^{2s}}{y'} \frac{|\hat{f}(\tau\eta, y')|^2}{|\eta^2 - 1| + |\eta_2|} \frac{d\eta dy'}{(y')^3}.$$

We further split Ω'_3 into two parts :

$$\Omega'_3 = (\Omega'_3 \cap \{\eta_1 > 0\}) \cup (\Omega'_3 \cap \{\eta_1 < 0\}) =: \Omega'_{3,+} \cup \Omega'_{3,-},$$

and estimate the integral over $\Omega'_{3,+} \times (0, \infty)$.

Let $\eta_0 = (1, 0)$ and put

$$h_\tau(x,y) = e^{-i\tau\eta_0 \cdot x} f(x,y).$$

Then by the change of variables $\eta = \eta_0 + k$

$$\int_{\Omega'_{3,+} \cap \{|\eta - \eta_0| < \epsilon\}} \frac{|\hat{f}(\tau\eta, y)|^2}{|\eta^2 - 1| + |\eta_2|} d\eta = \int_{\Omega''_{3,+} \cap \{|k| < \epsilon\}} \frac{|\hat{h}_\tau(\tau k, y)|^2}{|k^2 + 2k_1| + |k_2|} dk,$$

where $\Omega''_{3,+} = \{1/2 < |k + \eta_0|^2 < 3/2\} \cap \{|k_2| < 1/10, k_1 > -1\}$. By choosing ϵ small enough,

$$|k^2 + 2k_1| + |k_2| \geq (|k_1| + |k_2|)/2$$

for $|k| < \epsilon$. We have, therefore,

$$\int_{\Omega''_{3,+}\cap\{|k|<\epsilon\}} \frac{|\hat{h}_\tau(\tau k, y)|^2}{|k^2+2k_1|+|k_2|}dk \leq C\int_0^\epsilon \Big(\int_{S^1}|\hat{h}_\tau(\tau r\omega, y)|^2 d\omega\Big)dr. \tag{2.68}$$

By virtue of [IsUh] Lemma 2.2 (2), we have for $s > 1$

$$\int_{S^1}|\hat{h}_\tau(\tau r\omega, y)|^2 d\omega \leq C(1+\tau r)^{-1}\|\langle\cdot\rangle^s h_\tau(\cdot, y)\|^2_{L^2(\mathbf{R}^2)}.$$

Therefore the right-hand side of (2.68) is dominated from above by

$$\frac{\log\tau}{\tau}\int_{\mathbf{R}^2}\langle x\rangle^{2s}|h_\tau(x,y)|^2 dx = \frac{\log\tau}{\tau}\int_{\mathbf{R}^2}\langle x\rangle^{2s}|f(x,y)|^2 dx$$

for $s > 1$.

On the other hand there is a constant $C > 0$ such that $|k^2+2k_1|+|k_2| \geq C$ on $\Omega''_{3,+}\cap\{|k| > \epsilon\}$. Therefore we have for some $\epsilon_0 > 0$

$$\begin{aligned}\int_{\Omega''_{3,+}\cap\{|k|>\epsilon\}} \frac{|\hat{h}_\tau(\tau k, y)|^2}{|k^2+2k_1|+|k_2|}dk &\leq C\int_\epsilon^{\epsilon_0}\Big(\int_{S^1}|\hat{h}_\tau(\tau r\omega, y)|^2 d\omega\Big)dr \\ &\leq \frac{C}{\tau}\int_{\mathbf{R}^2}\langle x\rangle^{2s}|f(x,y)|^2 dx.\end{aligned}$$

The integral over $\Omega'_{3,-}\times\mathbf{R}$ is estimated in a similar manner. We have thus shown that u_3 satisfies

$$\begin{aligned}&\int_{\mathbf{R}^3_+}\frac{y}{(1+|\log y|)^{2s}}|u_3(x,y)|^2\frac{dxdy}{y^3} \\ &\quad\leq C\frac{\log\tau}{\tau}\int_{\mathbf{R}^3_+}\frac{(1+|\log y|)^{2s}}{y}\langle x\rangle^{2s}|f(x,y)|^2\frac{dxdy}{y^3}.\end{aligned} \tag{2.69}$$

The inequalities (2.67) and (2.69) prove the theorem. □

We now define the perturbed Green operator. We assume that V satisfies

$$|V(x,y)| \leq C(1+|x|)^{-2s}(1+|\log y|)^{-2s}(1+y)^{-2}y \tag{2.70}$$

for some $s > 1$. Let us remark here that throughout the paper we allow V to be complex-valued, although different decay assumptions ((2.70), (3.11), (4.3)) are imposed in each section. Since

$$\mathcal{H}_s^{(+)} \subset \mathcal{W}_{t,s}^{(+)}, \quad \mathcal{W}_{s-t,s}^{(-)} \subset \mathcal{H}_s^{(-)}, \quad 0 \leq t \leq s$$

with continuous inclusions, and $V \in \mathbf{B}(\mathcal{H}_s^{(-)};\mathcal{H}_s^{(+)})$, the following theorem is easily proved by Theorem 2.8.

Theorem 2.9. *Let $s > 1$ be the constant in (2.70). Let $\mathbf{G}_V(\theta)$ be defined by*

$$\mathbf{G}_V(\theta) = (1 + \mathbf{G}_0(\theta)V)^{-1}\mathbf{G}_0(\theta)$$

for sufficiently large $|\theta_I|$. Then there exists a constant $C_s > 0$ such that for $n = 3$

$$\|\mathbf{G}_V(\theta)\|_{\mathbf{B}(\mathcal{W}^{(+)}_{s-t,s};\mathcal{W}^{(-)}_{t,s})} \leq C_s \left(\frac{\log|\theta_I|}{|\theta_I|}\right)^{1/2}, \quad |\theta_I| > C_s,$$

for $0 \leq t \leq s$, and for $n = 2$

$$\|\mathbf{G}_V(\theta)\|_{\mathbf{B}(\mathcal{W}^{(+)}_s;\mathcal{W}^{(-)}_s)} \leq \frac{C_s}{|\theta_I|}, \quad |\theta_I| > C_s.$$

Let us notice that by virtue of Theorem 2.8, we have

$$\|\mathbf{G}_0(\theta)f\|_{W^{(-)}_{0,s}} \leq C_s \left(\frac{\log|\theta_I|}{|\theta_I|}\right)^{1/2} \|f\|_{\mathcal{H}^{(+)}_s}, \quad |\theta_I| > 2 \tag{2.71}$$

for $n = 3$, $s > 1$, and also

$$\|\mathbf{G}_0(\theta)f\|_{W^{(-)}_s} \leq \frac{C_s}{|\theta_I|}\|f\|_{\mathcal{H}^{(+)}_s}, \quad |\theta_I| > 2 \tag{2.72}$$

for $n = 2$ and $s > 1/2$. Since the weights in $\mathcal{W}^{(-)}_{0,s}$ and $\mathcal{W}^{(-)}_s$ do not depend on x, the inequalities (2.71) and (2.72) also hold for $\mathbf{G}_{0,\epsilon}(\theta)$ independently of $\epsilon > 0$. This in particlular implies that for $s > 1$

$$\|\mathbf{G}_{0,\epsilon}(\theta)\|_{\mathbf{B}(\mathcal{H}^{(+)}_{\mathbf{s}};\mathcal{H}^{(-)}_{\mathbf{s}})} \leq C_s \begin{cases} \left(\dfrac{\log|\theta_I|}{|\theta_I|}\right)^{1/2}, & n = 3, \\ \dfrac{1}{|\theta_I|}, & n = 2, \end{cases} \tag{2.73}$$

where the constant C_s is independent of $\epsilon > 0$. Taking account of this inequality, we put

$$\mathbf{G}_{V,\epsilon}(\theta) = (1 + \mathbf{G}_{0,\epsilon}(\theta)V)^{-1}\mathbf{G}_{0,\epsilon}(\theta). \tag{2.74}$$

Then $\mathbf{G}_{\mathbf{V},\epsilon}(\theta)$ is well-defined as an operator $\in \mathbf{B}(\mathcal{H}^{(+)}_s;\mathcal{H}^{(-)}_s)$ with $s > 1$ for $|\theta_I| > C$, C being independent of $\epsilon > 0$. By Theorem 2.7, for $f \in \mathcal{H}^{(+)}_s$ and $g \in \mathcal{H}^{(-)}_s$,

$$\mathbf{G}_{0,\epsilon}(\theta)f \to \mathbf{G}_0(\theta)f, \quad \overline{\partial_\theta}\,\mathbf{G}_{0,\epsilon}(\theta)Vg \to \overline{\partial_\theta}\,\mathbf{G}_0(\theta)Vg \quad \text{in} \quad \mathcal{H}^{(-)}_s,$$

since $V \in \mathbf{B}(\mathcal{H}^{(-)}_{\mathbf{s}};\mathcal{H}^{(+)}_{\mathbf{s}})$. Then $\overline{\partial_\theta}\,\mathbf{G}_{\mathbf{V}}(\theta)$ is defined as the limit of $\overline{\partial_\theta}\,\mathbf{G}_{\mathbf{V},\epsilon}(\theta)$ when $\epsilon \to 0$.

Lemma 2.10. *The following equalities hold* :

$$\begin{aligned}\overline{\partial_\theta}\,\mathbf{G}_V(\theta) &= (1+\mathbf{G}_0(\theta)V)^{-1}\left(\overline{\partial_\theta}\,\mathbf{G}_0(\theta)\right)(1-V\mathbf{G}_V(\theta)) \\ &= (1-\mathbf{G}_V(\theta)V)\left(\overline{\partial_\theta}\,\mathbf{G}_0(\theta)\right)(1-V\mathbf{G}_V(\theta)).\end{aligned}$$

Proof. The first equality is derived from the definition of $\mathbf{G}_{V,\epsilon}(\theta)$. To prove the second, we have only to note

$$(1+\mathbf{G}_0(\theta)V)^{-1}(1+\mathbf{G}_0(\theta)V-\mathbf{G}_0(\theta)V) = 1-\mathbf{G}_V(\theta)V.$$

□

§3. $\overline{\partial}$-theory for Scattering Amplitudes

§3.1. Scattering matrix in quantum mechanics

The wave function associated with the Schrödinger operator in quantum mechanics on $\mathbf{R}^n$ is a *bounded* solution to the equation $(-\Delta + V(x))\phi = E\phi$. It is also the case for the hyperbolic space $\mathbf{H}^n$. Suppose $\nu = i\sigma,\ \sigma \in \mathbf{R}\setminus\{0\}$. Then the wave function for the equation

$$H\Phi := \left[-y^2(\partial_y^2+\Delta)+(n-2)y\partial_y+V(x,y)\right]\Phi = E\Phi \tag{3.1}$$

is defined as follows. Let for $\eta \in \mathbf{R}^{n-1}$

$$\begin{aligned}\Phi_0(x,y,\eta) &= e^{ix\cdot\eta}y^{(n-1)/2}K_\nu(|\eta|y), \\ \Phi(x,y,\eta) &= \Phi_0(x,y,\eta)-v, \\ v(x,y,\eta) &= \mathbf{G}_V(0)[V(x,y)\Phi_0(x,y,\eta)], \\ E &= \frac{(n-1)^2}{4}-\nu^2.\end{aligned}$$

Then Φ solves (3.1), behaves like $e^{ix\cdot\eta}(c_1y^{(n-1)/2+i\sigma}+c_2y^{(n-1)/2-i\sigma})$ as $y\to 0$, and gives an eigenfunction expansion associated with H (see e.g. [Hi] or [Is1]). By observing the behavior of the Fourier transform of v with respect to x, we get

$$\hat{v}(\xi,y,\eta) \sim (2\pi)^{-(n-1)/2}\left(\frac{|\xi|}{2}\right)^{i\sigma}\frac{y^{i\sigma+(n-1)/2}}{\Gamma(i\sigma+1)}\tilde{A}(\xi,\eta), \quad y\to 0. \tag{3.2}$$

This $\tilde{A}(\xi,\eta)$ is (after a suitable unitary transformation) the scattering amplitude in the quantum mechanical scattering problem (see [Is1], Theorem 3.7).

§3.2. Exponentially growing solutions

In the $\overline{\partial}$-approach, contrary to the above quantum mechanical problem, we seek exponentially growing solutions to the equation (3.1). We assume ν to satisfy (2.21). We put for $\eta \in \mathbf{R}^{n-1}$ and $\theta \in \mathbf{C}^{n-1}$,

$$\psi_0(x, y; \eta, \theta) = e^{ix\cdot\theta}\Psi_0(x, y; \eta, \theta), \tag{3.3}$$

$$\Psi_0(x, y; \eta, \theta) = e^{ix\cdot\eta}y^{(n-1)/2}I_\nu(\zeta(\eta, \theta)y). \tag{3.4}$$

It satisfies the Schrödinger equation

$$H_0\psi_0 := \left[-y^2(\partial_y^2 + \Delta_x) + (n-2)y\partial_y\right]\psi_0 = E\psi_0, \tag{3.5}$$

and behaves like $e^{ix\cdot(\theta+\eta)}y^{(n-1)/2+\nu}$ as $y \to 0$. Hence if $\theta = 0$ and $y \to 0$, ψ_0 is bounded. However it grows up exponentially as $y \to \infty$.

We seek a solution of the perturbed Schrödinger equation

$$(H_0 + V(x, y))\psi = E\psi, \tag{3.6}$$

which behaves like ψ_0 at infinity. It is defined as

$$\psi(x, y; \eta, \theta) = \psi_0(x, y; \eta, \theta) - e^{ix\cdot\theta}u, \tag{3.7}$$

$$u = \mathbf{G}_V(\theta)\left[V(x, y)\Psi_0(x, y; \eta, \theta)\right]. \tag{3.8}$$

Note that for $n \geq 4$, $\mathbf{G}_V(\theta)$ exists for small V. Since $\mathbf{G}_V(\theta) = \mathbf{G}_0(\theta) - \mathbf{G}_0(\theta)V\mathbf{G}_V(\theta)$, by passing to the Fourier transformation with respect to x, we have (at least formally)

$$\hat{u}(\xi, y; \eta, \theta) \sim (2\pi)^{-(n-1)/2}y^{(n-1)/2}K_\nu(\zeta(\xi, \theta)y)A(\xi, \eta; \theta), \quad y \to \infty, \tag{3.9}$$

$$\begin{aligned} A(\xi, \eta; \theta) = &\int_{\mathbf{R}^n_+} e^{-ix\cdot\xi}y^{(n-1)/2}I_\nu(\zeta(\xi, \theta)y)V(x, y)\Psi_0(x, y; \eta, \theta)\frac{dxdy}{y^n}, \\ &- \int_{\mathbf{R}^n_+} e^{-ix\cdot\xi}y^{(n-1)/2}I_\nu(\zeta(\xi, \theta)y)V(x, y)u(x, y; \eta, \theta)\frac{dxdy}{y^n}. \end{aligned} \tag{3.10}$$

Our scattering amplitude will be defined to be this $A(\xi, \eta; \theta)$. One can also introduce similar scattering amplitudes using $K_\nu(z)$ in place of $I_\nu(z)$ in (3.4). The following arguments also work well for this choice. However, Theorem 3.7 does not seem to hold.

§3.3. Scattering amplitudes and the $\overline{\partial}$-equation

In the remaining part of this section, we exclusively deal with the case $n=3$. The potential $V(x,y)$ is assumed to satisfy the following condition.

There exist $\alpha > 2$ and $\beta > 3/2$ such that for any $N > 0$

$$|V(x,y)| \leq C_N(1+|x|)^{-\alpha}y^{\beta}e^{-Ny} \tag{3.11}$$

holds on $\mathbf{R}^3_+$ for a constant $C_N > 0$.

We put

$$\Psi_I^{(0)}(x,y;\xi,\theta) = \zeta(\xi,\theta)^{-\nu}e^{ix\cdot\xi}yI_\nu(\zeta(\xi,\theta)y), \tag{3.12}$$

$$\Psi_I(x,y;\xi,\theta) = \Psi_I^{(0)}(x,y;\xi;\theta) - \left(\mathbf{G}_V(\theta)(V\Psi_I^{(0)}(\xi,\theta))\right)(x,y), \tag{3.13}$$

$$\Psi_J^{(0)}(x,y;\xi,\theta) = r_\theta(\xi)^{-\nu}e^{ix\cdot\xi}yJ_\nu(r_\theta(\xi)y), \tag{3.14}$$

$$\Psi_J(x,y;\xi,\theta) = \Psi_J^{(0)}(x,y;\xi;\theta) - \left(\mathbf{G}_V(\theta)(V\Psi_J^{(0)}(\xi,\theta))\right)(x,y), \tag{3.15}$$

where $\Psi_I^{(0)}(\xi,\theta) = \Psi_I^{(0)}(x,y;\xi,\theta)$, $\Psi_J^{(0)}(\xi,\theta) = \Psi_J^{(0)}(x,y;\xi,\theta)$.

Lemma 3.1. (1) *$\Psi_I^{(0)}(x,y;\xi,\theta)$ is smooth with respect to x,y,ξ,θ. In particular*

$$\overline{\partial_\theta}\,\Psi_I^{(0)}(x,y;\xi,\theta) = 0.$$

(2) *$\Psi_I^{(0)}(x,y;k,\theta) = \Psi_J^{(0)}(x,y;k,\theta)$ for $k \in M_\theta$.*

Proof. By (2.5), we have

$$\zeta(\xi,\theta)^{-\nu}I_\nu(\zeta(\xi,\theta)y) = \left(\frac{y}{2}\right)^\nu \sum_{k=0}^{\infty}\frac{((\xi+\theta)^2y^2/4)^k}{k!\Gamma(\nu+k+1)},$$

$$r_\theta(k)^{-\nu}J_\nu(r_\theta(k)y) = \left(\frac{y}{2}\right)^\nu \sum_{k=0}^{\infty}\frac{(-r_\theta(k)^2y^2/4)^k}{k!\Gamma(\nu+k+1)}.$$

If $k \in M_\theta$, we have $\zeta(k,\theta)^2 = (k+\theta)^2 = -r_\theta(k)^2$. These properties imply the lemma. □

Note that without the regularizing factor $\zeta(\xi,\theta)^{-\nu}$ we have by Lemma 2.4

$$\overline{\partial_\theta}\,I_\nu(\zeta(\xi,\theta)y) = 2i\sin\left(\frac{\nu\pi}{2}\right)J_\nu(r_\theta(\xi)y)T_\theta(\xi). \tag{3.16}$$

Definition 3.2. We define the scattering amplitude by

$$A(\xi,\eta;\theta) = \int_{\mathbf{R}^3_+} \Psi_I^{(0)}(x,y;-\xi,-\theta)V(x,y)\Psi_I(x,y;\eta,\theta)\frac{dxdy}{y^3}. \tag{3.17}$$

By the assumption (3.11), $A(\xi,\eta;\theta)$ is well-defined for large θ_I. By Lemma 3.1, if $\xi \in M_\theta$ ($\eta \in M_\theta$), $\Psi_I^{(0)}$ (Ψ_I) is replaced by $\Psi_J^{(0)}$ (Ψ_J). The following theorem shows that one can uniquely reconstruct $V(x,y)$ from this scattering amplitude. In fact, extend $\widetilde{V}(x,y) := V(x,y)/y^2$ to $\mathbf{R}^3$ as an even function of y. Then by the following theorem, one can uniquely reconstruct the Fourier transform of $\widetilde{V}$ (through analytic continuation) from the knowledge of $A(\xi,\eta;\theta)$.

Theorem 3.3. *Let $\alpha = \theta_I/|\theta_I|$. Suppose $\alpha\cdot(\xi+\theta_R) > 0$, $\alpha\cdot(\eta+\theta_R) > 0$. Then*

$$\lim_{|\theta_I|\to\infty} |\theta_I|^{1+2\nu} A(\xi,\eta;\theta) = \frac{1}{\pi}\int_{\mathbf{R}^3_+} e^{-ix\cdot(\xi-\eta)}\cosh(ay)V(x,y)\frac{dxdy}{y^2},$$

where $a = \alpha\cdot(\xi-\eta)$.

Proof. If $\alpha\cdot(\xi+\theta_R) > 0$, we have as $\tau \to \infty$,

$$\zeta(\xi,\theta) = \tau i + \alpha\cdot(\xi+\theta_R) + O(1/\tau).$$

This implies by (2.8)

$$\sqrt{2\pi\tau y}I_\nu(\zeta(\xi,\theta)y) = e^{-\pi i/4}e^{i\tau y}e^{\alpha\cdot(\xi+\theta_R)y} + e^{\pi i/4}e^{i\nu\pi}e^{-i\tau y}e^{-\alpha\cdot(\xi+\theta_R)y} + O\left(\frac{1}{\tau}\right).$$

Hence if both ξ and η satisfy the above condition

$$2\pi\tau y I_\nu(\zeta(\xi,\theta)y)I_\nu(\zeta(\eta,\theta)y) = 2e^{i\nu\pi}\cosh(ay) + e^{-\pi i/2}e^{2i\tau y}e^{by} + e^{\pi i/2}e^{2i\nu\pi}e^{-2i\tau y}e^{-by} + O\left(\frac{1}{\tau}\right),$$

where $a = \alpha\cdot(\xi-\eta)$, $b = \alpha\cdot(\xi+\eta+2\theta_R)$. Thus by using Theorem 2.9 and Riemann-Lebesgue Lemma, we have

$$\begin{aligned}2\pi\tau^{1+2\nu}A(\xi,\eta;\theta) &\sim e^{-\nu\pi i}\int_{\mathbf{R}^3} e^{-ix\cdot(\xi-\eta)}2\pi\tau y I_\nu(\zeta(\xi,\theta)y)I_\nu(\zeta(\eta,\theta))V(x,y)\frac{dxdy}{y^2} \\ &\sim 2\int_{\mathbf{R}^3_+} e^{-ix\cdot(\xi-\eta)}\cosh(ay)V(x,y)\frac{dxdy}{y^2}.\end{aligned}$$

□

Our next aim is to compute $\overline{\partial_\theta}\,A(\xi,\eta;\theta)$. Here to compute $\overline{\partial_\theta}\,\Psi_I(x,y;\xi,\theta)$, we replace $\mathbf{G}_V(\theta)$ by $\mathbf{G}_{V,\epsilon}(\theta)$ from (2.74) and take the limit $\epsilon \to 0$.

Theorem 3.4. *For all* $\xi,\eta \in \mathbf{R}^2$, *we have*

(3.18)

$$\overline{\partial_\theta}\,\Psi_I(x,y;\xi,\theta) = -\frac{1}{8\pi}\int_{M_\theta}\Psi_I(x,y;k,\theta)A(k,\xi;\theta)\frac{r_\theta(k)^{2\nu}(k+\overline{\theta})}{|\theta_I|}dM_\theta(k).$$

$$\overline{\partial_\theta}\,A(\xi,\eta;\theta) = -\frac{1}{8\pi}\int_{M_\theta}A(\xi,k;\theta)A(k,\eta;\theta)\frac{r_\theta(k)^{2\nu}(k+\overline{\theta})}{|\theta_I|}dM_\theta(k). \tag{3.19}$$

Proof. By Lemma 3.1 (2) and Lemma 2.10, we have

$$\begin{aligned}\overline{\partial_\theta}\,\Psi_I(\xi,\theta) &= -(1-\mathbf{G}_V(\theta)V)(\overline{\partial_\theta}\,\mathbf{G}_0(\theta))(1-V\mathbf{G}_V(\theta))V\Psi_I^{(0)}(\xi,\theta)\\ &= -(1-\mathbf{G}_V(\theta)V)(\overline{\partial_\theta}\,\mathbf{G}_0(\theta))V\Psi_I(\xi,\theta),\end{aligned}$$

where we have used (3.13). By Theorem 2.7, this is equal to

$$\begin{aligned}&-\frac{1}{8\pi}(1-\mathbf{G}_V(\theta)V)\int_{M_\theta}e^{ix\cdot k}yJ_\nu(r_\theta(k)y)\frac{(k+\overline{\theta})}{|\theta_I|}\\ &\qquad\times\left(\int e^{-ix'\cdot k}y'J_\nu(r_\theta(k)y')V(x',y')\Psi_I(x',y';\xi,\theta)\frac{dx'dy'}{(y')^3}\right)dM_\theta(k)\\ &= -\frac{1}{8\pi}\int_{M_\theta}\Psi_J(k;\theta)A(k,\xi;\theta)\frac{r_\theta(k)^{2\nu}(k+\overline{\theta})}{|\theta_I|}dM_\theta(k),\end{aligned}$$

since $(1-\mathbf{G}_V(\theta)V)\left(e^{ix\cdot k}yJ_\nu(r_\theta(k)y)\right) = r_\theta(k)^\nu\Psi_J(x,y;k,\theta)$ by (3.14) and (3.15), where we have used $r_\theta(k) = r_{-\theta}(-k)$. This and Lemma 3.1 (1) prove (3.18). Using (3.17) and Lemma 3.1 (2), we get (3.19). □

§3.4. Integral representation of the potential

The above $\overline{\partial}$-equation enables us to derive integral representations of the potential $V(x,y)$ in terms of $A(\xi,\eta;\theta)$.

Let $\alpha,\alpha^\perp \in S^1$ be such that $\alpha\cdot\alpha^\perp = 0$. For a sufficiently large constant $T_0 > 0$, let Ω be the set of $\theta = \theta_R + i\theta_I \in \mathbf{C}^2$ satisfying the following condition :

$$|\theta_R - T_0\alpha| < 1, \quad \alpha\cdot\theta_I > T_0, \quad |\alpha^\perp\cdot\theta_I| < 1. \tag{3.20}$$

Let us note that for $\theta \in \Omega$

$$\frac{\theta_I}{|\theta_I|} \to \alpha \quad \text{as} \quad |\theta_I| \to \infty, \quad \theta_R\cdot\theta_I > 0 \quad \forall\theta\in\Omega. \tag{3.21}$$

Theorem 3.5. *Let ξ, η be such that $\theta_I \cdot (\xi + \theta_R) > 0,\ \theta_I \cdot (\eta + \theta_R) > 0,\ \forall \theta \in \Omega$. We put $\theta^{4+2\nu} = (\theta^2)^{2+\nu}$, $K(\theta) = \theta_1 d\theta_2 - \theta_2 d\theta_1,\ L(\theta) = d\theta_1 \wedge d\theta_2$, and $a = \alpha \cdot (\xi - \eta)$. Then we have for $\theta_0 \in \Omega$,*

$$
\begin{aligned}
&e^{\nu\pi i} \int_{\mathbf{R}^3_+} e^{-ix\cdot(\xi-\eta)} \cosh(ay) V(x,y) \frac{dxdy}{y^2} \\
&\quad = 2(\theta_0)^{4+2\nu} A(\xi, \eta; \theta_0) \\
&\quad - \frac{1}{4} \int_{\partial\Omega} A(\xi, \eta; \theta) \frac{\theta^{4+2\nu} K(\overline{\theta} - \overline{\theta_0})}{|\theta - \theta_0|^4} \wedge L(\theta) \\
&\quad - \frac{1}{32\pi} \int_{\Omega} \left(\int_{M_\theta} A(\xi, k; \theta) A(k, \eta; \theta) \frac{r_\theta(k)^{2\nu} \big(k + \overline{\theta}\big)}{|\theta_I|} dM_\theta(k) \right) N(\theta), \\
&\quad N(\theta) = d\overline{\theta} \wedge \frac{\theta^{4+2\nu} K(\overline{\theta} - \overline{\theta_0})}{|\theta - \theta_0|^4} \wedge L(\theta),
\end{aligned}
$$

where the integral is performed in the sense of improper integral.

Proof. Recall the Bochner-Martinelli formula

$$
f(z) = \frac{1}{8} \int_{\partial D} f(\zeta) \frac{K(\overline{\zeta} - \overline{z})}{|\zeta - z|^4} \wedge L(\zeta) - \frac{1}{8} \int_{D} \big(\overline{\partial} f(\zeta)\big) \wedge \frac{K(\overline{\zeta} - \overline{z})}{|\zeta - z|^4} \wedge L(\zeta),
$$

which holds on a bounded domain $D \subset \mathbf{C}^2$ and $z \in D$ (see [Kr], p. 22). Replace f by $\theta^{4+2\nu} A(\xi, \eta; \theta)$, D by $\Omega \cap \{|\theta_I| < T\}$. Note that $(\theta)^{4+2\nu}$ is analytic on Ω. Let $T \to \infty$ and use Theorem 3.3. On $S_T = \Omega \cap \{|\theta_I| = T\}$, we have $d\theta_{1I} \wedge d\theta_{2I} = 0$. Hence

$$
\begin{aligned}
K(\overline{\theta}) \wedge L(\theta) = {} & 2(\theta_{2I} d\theta_{1I} - \theta_{1I} d\theta_{2I}) \wedge d\theta_{1R} \wedge d\theta_{2R} \\
& + 2i(\theta_{2R} d\theta_{1I} - \theta_{1R} d\theta_{2I}) \wedge d\theta_{1R} \wedge d\theta_{2R}.
\end{aligned}
$$

Since θ_R varies over a bounded set, we have

$$
\int_{S_T} K(\overline{\theta}) \wedge L(\theta) \sim 2 \int_{S_T} (\theta_{2I} d\theta_{1I} - \theta_{1I} d\theta_{2I}) \wedge d\theta_{1R} \wedge d\theta_{2R} \sim 4\pi T,
$$

which proves the theorem. □

§3.5. Restriction to lower dimensional submanifolds

Let us recall that in the Euclidean case, the Faddeev scattering amplitude $A(\xi, \zeta)$ is first defined on a 7-dim. manifold $\mathbf{R}^3 \times \{\zeta \in \mathbf{C}^3\,;\, \zeta^2 = E\}$, and then restricted to the 5-dim. manifold $\cup_\xi \{\xi\} \times \mathcal{V}_\xi$. In the hyperbolic space case,

$A(\xi, \eta; \theta)$ is a function on a 8-dim. manifold $\mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{C}^2$. However, noting the formula

$$e^{-ix\cdot k}\mathbf{G}_0(\theta)e^{ix\cdot k} = \mathbf{G}_0(\theta + k), \quad \forall k \in \mathbf{R}^2, \tag{3.22}$$

and the resulting equation

$$A(\xi - k, \eta - k; \theta + k) = A(\xi, \eta; \theta), \quad \forall k \in \mathbf{R}^2, \tag{3.23}$$

one can see that $A(\xi, \eta; \theta)$ actually depends on 6 parameters. Let us restrict $A(\xi, \eta; \theta)$ to a 5-dim. manifold.

In the Euclidean case, the fibre $\mathcal{V}_\xi$ defined by (1.9) has a natural complex structure. The condition $\xi^2 + 2\zeta \cdot \xi = 0$ stems from the singularities of the integrand of the Green function (1.8). In the hyperbolic space case, the corresponding singularities appear from $\sqrt{(\xi+\theta)^2}$, which gives rise to the condition $\mathrm{Im}\,(\xi+\theta)^2 = 2\theta_I \cdot (\xi + \theta_R) = 0$. Since the set of all θ satisfying this condition is of 3-dimension, we should look for a 2-dim. submanifold for θ. We try a simple choice of $\mathbf{C}\hat{\xi}_\perp$ to be defined below. Note that this set is not included in the above set of singularities.

For $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2 \setminus \{0\}$, we put

$$\hat{\xi}_\perp = \Big(-\frac{\xi_2}{|\xi|}, \frac{\xi_1}{|\xi|}\Big) \tag{3.24}$$

and for $z \in \mathbf{C}$, we define

$$\theta(\xi, z) = z\hat{\xi}_\perp. \tag{3.25}$$

For $\xi \in \mathbf{R}^2 \setminus \{0\},\ z \in \mathbf{C}$ such that $\mathrm{Re}\, z \neq 0$ and $|\mathrm{Im}\, z|$ is sufficiently large, and $k \in M_{\theta(\xi,z)}$, we put

$$B_{II}(\xi, z) = z^{2+2\nu} A\Big(\frac{\xi}{2}, -\frac{\xi}{2}; \theta(\xi, z)\Big), \tag{3.26}$$

$$B_{IJ}(\xi, k, z) = z^{2+2\nu} A\Big(\frac{\xi}{2}, k; \theta(\xi, z)\Big), \tag{3.27}$$

$$B_{JI}(k, \xi, z) = z^{2+2\nu} A\Big(k, -\frac{\xi}{2}; \theta(\xi, z)\Big). \tag{3.28}$$

Since $\mathrm{Re}\, z \neq 0$, $\pm\xi/2 \notin M_{\theta(\xi,z)}$. Note that $B_{II}(\xi, z)$ is a function on (an open set of) the product space $\mathbf{R}^2 \times \mathbf{C}$ and $B_{IJ}(\xi, k, z)$, $B_{JI}(k, \xi, z)$ are functions on (an open set of) the line bundle with base space $\mathbf{R}^2 \times \mathbf{C}$ and fibre $M_{\theta(\xi,z)}$. Or it may be better to regard $\mathbf{R}^2$ as base space and $\mathbf{C}\hat{\xi}_\perp \times M_{\theta(\xi,z)}$ as fibre.

Lemma 3.6. *The following equation holds :*

$$\overline{\partial_z}\, B_{II}(\xi, z) = \frac{i\,\epsilon(z)}{8\pi z^{2+2\nu}} \int_{M_\theta} B_{IJ}(\xi, k, z) B_{JI}(k, \xi, z) r_\theta(k)^{2\nu} dM_\theta(k),$$

where $\theta = \theta(\xi, z)$ *and* $\epsilon(z) = 1$ *if* $\operatorname{Im} z > 0$, $\epsilon(z) = -1$ *if* $\operatorname{Im} z < 0$.

Proof. Note that

$$\overline{\partial_z}\, A(\xi, \eta; \theta(\xi, z)) = (\overline{\partial_\theta}\, A(\xi, \eta; \theta(\xi, z)) \cdot \overline{\partial_z\, \theta(\xi, z)}$$

and $\overline{\partial_z\, \theta(\xi, z)} = \overline{\theta(\xi, z)}/\overline{z}$. Then we have by Theorem 3.4

$$\begin{aligned}\overline{\partial_z}\, &B_{II}(\xi, z)\\ &= \frac{-1}{8\pi \overline{z} z^{2+2\nu} |\theta_I|} \int_{M_\theta} B_{IJ}(\xi, k; z) B_{JI}(k, \xi, z) r_\theta(k)^{2\nu} (k + \overline{\theta}) \cdot \overline{\theta} dM_\theta(k),\end{aligned}$$

where $\theta = \theta(\xi, z)$. By a simple computation, we have $(k + \overline{\theta(\xi, z)}) \cdot \overline{\theta(\xi, z)} = -i\overline{z} \operatorname{Im} z$, which proves the lemma. □

Take $T_0 > 0$ large enough and put

$$D = \{z = t + i\tau\,; 1 < t < 2,\ T_0 < \tau < \infty\}. \tag{3.29}$$

Theorem 3.7. *For* $w \notin \overline{D}$, *we have in the sense of improper integral*

$$\begin{aligned} e^{\nu\pi i} \int_{\mathbf{R}^3_+} e^{-ix\cdot\xi} V(x, y) \frac{dxdy}{y^2} &= -\pi i \int_{\partial D} \frac{B_{II}(\xi, z)}{z - w} dz \\ &\quad + \frac{1}{8} \int_D F(\xi, z) \frac{dz \wedge d\overline{z}}{z^{2+2\nu}(z - w)}, \\ F(\xi, z) &= \int_{M_\theta} B_{IJ}(\xi, k, z) B_{JI}(k, \xi, z) r_\theta(k)^{2\nu} dM_\theta(k), \end{aligned}$$

where $\theta = \theta(\xi, z)$.

Proof. Let

$$\begin{aligned} D_T &= \{t + i\tau\,; 1 < t < 2,\ T_0 < \tau < T\}, \\ C'_T &= \{t + iT\,; 1 < t < 2\}, \\ C_T &= \partial D_T \setminus C'_T. \end{aligned}$$

Then if $w \notin \overline{D}$, by Green's formula

$$\int_{\partial D_T} \frac{B_{II}(\xi, z)}{z - w} dz + \int_{D_T} \frac{\overline{\partial_z}\, B_{II}(\xi, z)}{z - w} dz \wedge d\overline{z} = 0.$$

For $z \in D$, we have

$$\zeta\left(\pm\xi/2, \theta(\xi, z)\right) = \sqrt{z^2 + \xi^2/4} = z + O(1/z) \quad \text{as} \quad z \to \infty.$$

Therefore by (2.8)

$$2\pi z y I_\nu\Big(\zeta\Big(\frac{\xi}{2}, \theta(\xi, z)\Big)y\Big) I_\nu\Big(\zeta\Big(-\frac{\xi}{2}, \theta(\xi, z)\Big)y\Big) \sim e^{2zy} + 2ie^{i\nu\pi} - e^{-2zy}e^{2i\nu\pi}.$$

By the Riemann-Lebesgue Lemma, we then have as $T \to \infty$

$$\int_{C_T'} \frac{B_{II}(\xi, z)}{z - w} dz \to \frac{i}{\pi} e^{i\nu\pi} \int_{\mathbf{R}^3_+} e^{-ix\cdot\xi} V(x, y) \frac{dxdy}{y^2},$$

which completes the proof. □

§3.6. Radon transform

Let Π be a 2-dimensional plane orthogonal to $\{y = 0\}$, and $d\Pi_E$ be the measure induced on Π from the Euclidean metric $(dx)^2 + (dy)^2$. By Theorem 3.7 one can reconstruct

$$\int_\Pi V(x, y) \frac{d\Pi_E}{y^2} \tag{3.30}$$

from $B_{II}(\xi; z), B_{IJ}(\xi, k; z), B_{JI}(k, \xi; z)$. Let S be any hemisphere in $\mathbf{R}^3_+$ with center at $\{y = 0\}$ and take an isometry ϕ on $\mathbf{H}^3$ mapping S to Π. Then from the Faddeev scattering amplitude of $H_\phi = \phi \circ H \circ \phi^{-1}$, one can recover (3.30). Therefore one can recover $\int_S V(x, y)dS$, dS being the measure on S induced from the hyperbolic metric. If one knows the scattering amplitude $A^{(\phi)}(\xi, \eta; \theta)$ of H_ϕ for all ϕ, one can then reconstruct $V(x, y)$ by virtue of the inverse Radon transform on $\mathbf{H}^3$ (see e.g. [He] or [BeTa]). For this to be possible, one must be able to compute $A^{(\phi)}(\xi, \eta; \theta)$ for all ϕ from a given Faddeev scattering amplitude. This does not seem to be an obvious problem in general. If V is compactly supported, however, this is possible via the Dirichlet-Neumann map, which we explain in the next section.

§4. Inverse Boundary Value Problem

In this section, we shall work in $\mathbf{H}^n$ with $n \geq 2$. Let Ω be a relatively compact open set in $\mathbf{H}^n$, and consider the boundary value problem

$$\begin{cases} \left(-y^2(\partial_y^2 + \Delta_x) + (n-2)y\partial_y + q(x, y)\right) u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega. \end{cases} \tag{4.1}$$

We assume that $q(x,y)$ is bounded on Ω and that 0 is not a Dirichlet eigenvalue of $H_0 + q$ on Ω. The Dirichlet-Neumann map is defined by

$$\Lambda(q)f = y^2(n_y\partial_y + n_x \cdot \nabla_x)u\big|_{\partial\Omega}, \tag{4.2}$$

where u is a solution to (4.1) and $n = (n_x, n_y)$ is the outer unit normal to $\partial\Omega$ with respect to the Euclidean metric $(dx)^2 + (dy)^2$. Let a constant $E \neq (n-1)^2/4$ be fixed, and put

$$V(x,y) = \begin{cases} q(x,y) + E & \text{on} \quad \Omega, \\ 0 & \text{on} \quad \Omega^C. \end{cases} \tag{4.3}$$

We show that from the DN map $\Lambda(q)$ one can construct the scattering amplitude $A(\xi,\eta;\theta)$ for $H_0 + V$. The idea is essentially the same as [Na1]. Since there are many technical differences, however, we reproduce the proof.

For a surface S in $\mathbf{H}^n$, let dS be the measure on S induced from the hyperbolic metric $y^{-2}\left((dx)^2 + (dy)^2\right)$. This is equal to dS_E/y^n, where dS_E is the measure induced on S from the Euclidean metric $(dx)^2 + (dy)^2$.

§4.1. Modified radiation condition

Let $\mathbf{G}_0(x,y,x',y';\theta)$ be the Green function defined by (2.43).

Lemma 4.1. *Put $z = (x,y), z' = (x',y'), z_0 = (x_0,y_0) \in \mathbf{R}^n_+$, and*

$$u(z) = \int_{\partial\Omega} \mathbf{G}_0(x,y,x',y';\theta) f(x',y') dS_{x'y'}, \tag{4.4}$$

$$v(z_0,z) = \mathbf{G}_0(x_0,y_0,x,y;\theta), \tag{4.5}$$

where $f \in L^2(\partial\Omega)$. Then for any $z_0 \in \mathbf{R}^n_+$, there exists a constant $R_0 > 0$ such that

$$\int_{y=R} \left\{ \frac{\partial u}{\partial y}(z) v(z_0,z) - u(z)\frac{\partial v}{\partial y}(z_0,z) \right\} dx = 0$$

if $R > R_0$ or $R < 1/R_0$.

Proof. We consider the case that R is large enough. By taking suitable local coordinates on $\partial\Omega$, $\hat{u}(\xi,y)$ is written as

$$\hat{u}(\xi,y) = y^{(n-1)/2} K_\nu(\zeta(\xi,\theta)y) g(\xi,\theta),$$

where $g(\xi,\theta)$ is a sum of the following terms

$$\int_U e^{-ix'(\omega)\cdot\xi} y'(\omega)^{(n-1)/2} I_\nu(\zeta(\xi,\theta)y'(\omega)) f(x'(\omega),y'(\omega)) J(\omega) d\omega,$$

U being an open set in $\mathbf{R}^{n-1}$ and $x'(\omega), y'(\omega), J(\omega) \in C_0^\infty(U)$. Since $\zeta(-\xi, -\theta)$ $= \zeta(\xi, \theta)$, we have

$$\begin{aligned}(2\pi)^{(n-1)/2}\hat{v}(z_0, \xi, y) &= e^{-ix_0\cdot\xi} G_0(y_0, y; \zeta(\xi, -\theta)) \\ &= e^{-ix_0\cdot\xi}(yy_0)^{(n-1)/2} K_\nu(\zeta(\xi, -\theta)y) I_\nu(\zeta(\xi, -\theta)y_0).\end{aligned}$$

Using $\int f(x)g(x)dx = \int \hat{f}(\xi)\hat{g}(-\xi)d\xi$, we get the lemma. When R is small, we have only to exchange K_ν and I_ν. □

Here let us note that

$$G_0(y, y'; \zeta) = G_0(y', y\,; \zeta),$$

which follows from the definition (2.22). Hence by (2.43)

$$\mathbf{G}_0(x, y, x', y'; \theta) = \mathbf{G}_0(x', y', x, y\,; -\theta). \tag{4.6}$$

This and (2.37) imply that for any $z_0 = (x_0, y_0) \in \mathbf{R}^n_+$

$$(H_0(-\theta) - E)\, \mathbf{G}_0(x_0, y_0, x, y\,; \theta) = \delta(z - z_0), \tag{4.7}$$

where $\delta(z - z_0)$ is the delta-function with respect to the measure $dxdy/y^n$.

We use the following notation :

$$(u, v)_\Omega = \int_\Omega u\overline{v}\, \frac{dxdy}{y^n}, \tag{4.8}$$

$$(f, g)_{\partial\Omega} = \int_{\partial\Omega} f\overline{g}\, \frac{dS_E}{y^n}, \tag{4.9}$$

$$B(\theta) = y^2 \left(n_y\partial_y + n_x \cdot (\partial_x + i\theta)\right). \tag{4.10}$$

Green's formula implies the following lemma.

Lemma 4.2. *For any $u, v \in H^2(\Omega)$, we have*

$$(H_0(\theta)u, v)_\Omega - (u, H_0(\overline{\theta})v)_\Omega = -(B(\theta)u, v)_{\partial\Omega} + (u, B(\overline{\theta})v)_{\partial\Omega}. \tag{4.11}$$

Or, equivalently,

$$\begin{aligned}&\int_\Omega (H_0(\theta)u)\, v\frac{dxdy}{y^n} - \int_\Omega u\, (H_0(-\theta)v)\, \frac{dxdy}{y^n} \\ &\quad = -\int_{\partial\Omega} (B(\theta)u)\, v\frac{dS_E}{y^n} + \int_{\partial\Omega} u\, (B(-\theta)v)\, \frac{dS_E}{y^n}.\end{aligned} \tag{4.12}$$

Now we consider the gauge transform of $\Lambda(q)$:

$$\Lambda(\theta; q) = e^{-ix\cdot\theta}\Lambda(q)e^{ix\cdot\theta}. \tag{4.13}$$

Note that $\Lambda(\theta; q)$ is the Dirichlet-Neumann map for the following *interior* boundary value problem :

$$\Lambda(\theta; q)f = B(\theta)u, \tag{4.14}$$

where u is a solution to the interior Dirichlet problem

$$\begin{cases} (H_0(\theta) + q)u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \partial\Omega. \end{cases} \tag{4.15}$$

The following lemma is an easy consequence of Lemma 4.2.

Lemma 4.3. *For $f, g \in H^{1/2}(\partial\Omega)$,*

$$(\Lambda(\theta; q)f, g)_{\partial\Omega} = (f, \Lambda(\overline{\theta}; \overline{q})g)_{\partial\Omega}. \tag{4.16}$$

Or, equivalently,

$$\int_{\partial\Omega} (\Lambda(\theta; q)f)\, g\frac{dS_E}{y^n} = \int_{\partial\Omega} f\, (\Lambda(-\theta; q)g)\, \frac{dS_E}{y^n}. \tag{4.17}$$

We match this interior problem with the following *exterior* problem. Let

$$\psi_0(x, y; \eta, \theta) = \zeta(\eta, \theta)^{-\nu} e^{ix\cdot\eta} y^{(n-1)/2} I_\nu(\zeta(\eta, \theta)y), \tag{4.18}$$

$$\Omega_{ex} = \mathbf{R}^n_+ \setminus \overline{\Omega}, \tag{4.19}$$

$$\Omega^y_{ex} = \{x \in \mathbf{R}^{n-1}\,;\, (x, y) \in \Omega_{ex}\}. \tag{4.20}$$

The problem we address is the following :

$$(H_0(\theta) - E)\,\psi = 0 \quad \text{in} \quad \Omega_{ex}, \tag{4.21}$$

$$B(\theta)\psi = \Lambda(\theta; q)h \quad \text{on} \quad \partial\Omega_{ex}, \quad \text{where} \quad h = \psi\big|_{\partial\Omega}, \tag{4.22}$$

and $u = \psi - \psi_0(x, y; \eta, \theta)$ satisfies

$$\partial_y^m u(\cdot, y) \in H^{1-m}(\Omega^y_{ex}), \quad m = 0, 1, \quad \forall y > 0, \tag{4.23}$$

$$\int_{y=R} \left\{\frac{\partial u}{\partial y}(z)v(z_0, z) - u(z)\frac{\partial v}{\partial y}(z_0, z)\right\}\frac{dx}{y^{n-2}} \to 0 \tag{4.24}$$

as $R \to \infty$ and $R \to 0$ for any $z_0 \in \mathbf{R}^n_+$, where $v(z_0, z)$ is defined by (4.5). Let us remark that n in $B(\theta)$ of (4.22) is the inner unit normal to Ω_{ex}.

§4.2. Single and double layer potentials

For $z = (x, y) \in \Omega_{ex}$, we introduce the single and double layer potentials by

$$\mathcal{S}_\theta f(z) = \int_{\partial\Omega} \mathbf{G}_0(x, y, x', y'; \theta) f(x', y') \frac{dS_E}{(y')^n}, \tag{4.25}$$

$$\mathcal{D}_\theta f(z) = \int_{\partial\Omega} \frac{\partial \mathbf{G}_0}{\partial n(z')}(x, y, x', y'; \theta) f(x', y') \frac{dS_E}{(y')^n}, \tag{4.26}$$

where $n(z')$ is the unit normal at $z' = (x', y') \in \partial\Omega$ pointing toward the direction exterior to Ω. We also put for $z \in \partial\Omega$

$$\mathcal{B}_\theta f(z) = \text{p.v.} \int_{\partial\Omega} \frac{\partial \mathbf{G}_0}{\partial n(z')}(x, y, x', y'; \theta) f(x', y') \frac{dS_E}{(y')^n}. \tag{4.27}$$

$$\mathcal{B}_\theta^\dagger f(z) = \text{p.v.} \int_{\partial\Omega} \frac{\partial \mathbf{G}_0}{\partial n(z)}(x, y, x', y'; \theta) f(x', y') \frac{dS_E}{(y')^n}. \tag{4.28}$$

Let $\mathbf{G}_0^E$ be the Green operator of $-\Delta$ in $\mathbf{R}^n$. Then we have

$$\mathbf{G}_0(\theta) = \mathbf{G}_0^E \frac{1}{y^2} + \mathbf{G}_0^E \Big(2i\theta \cdot \partial_x - \theta^2 - \frac{n-2}{y}\partial_y + \frac{E}{y^2} \Big) \mathbf{G}_0(\theta). \tag{4.29}$$

The second term of the right-hand side is continuous from $L^2(\partial\Omega)$ to $H^{5/2}_{loc}(\mathbf{R}^n_+)$. Therefore the jumps of $\mathcal{D}_\theta f$, $\partial\mathcal{S}_\theta/\partial n$ across $\partial\Omega$ come from the first term of the right-hand side. This implies the following two lemmas (see [Na1], Lemmas 2.4 and 2.5).

Lemma 4.4. *Let $u = \mathcal{S}_\theta f$, $f \in H^{1/2}(\partial\Omega)$. Then u satisfies $(H_0(\theta) - E)u = 0$ on Ω_{ex}. Moreover the non-tangential limits $(\partial u/\partial n)_+$ (and $(\partial u/\partial n)_-$) of $\partial u/\partial n$ on the boundary from outside (respectively inside) Ω are given by the formula*

$$\left(\frac{\partial u}{\partial n}\right)_\pm = \mp \frac{1}{2y^2} f(z) + \mathcal{B}_\theta^\dagger f(z) \quad \text{on} \quad \partial\Omega. \tag{4.30}$$

In particular

$$\left(\frac{\partial u}{\partial n}\right)_+ - \left(\frac{\partial u}{\partial n}\right)_- = -\frac{1}{y^2} f(z) \quad \text{on} \quad \partial\Omega. \tag{4.31}$$

Lemma 4.5. *Let $w = \mathcal{D}_\theta f$, $f \in H^{3/2}(\partial\Omega)$. Then w satisfies $(H_0(\theta) - E)w = 0$ on Ω_{ex}. Moreover the non-tangential limits w_+ (and w_-) of w on the boundary from outside (respectively inside) Ω are given by the formula*

$$w_\pm = \pm \frac{1}{2y^2} f(z) + \mathcal{B}_\theta f(z) \quad \text{on} \quad \partial\Omega. \tag{4.32}$$

In particular

$$w_+ - w_- = \frac{1}{y^2} f(z) \quad \text{on} \quad \partial\Omega. \tag{4.33}$$

Lemma 4.6. (1) *Let* ψ *be a solution to the exterior problem* (4.21)–(4.24)*. Then its trace on the boundary* $f = \psi|_{\partial\Omega}$ *satisfies*

$$f = \psi_0(x, y; \eta, \theta) - \left[\mathcal{S}_\theta\Big(\Lambda(\theta; q) + i\theta \cdot n_x y^2\Big) - \mathcal{B}_\theta y^2 - \frac{1}{2}\right] f. \tag{4.34}$$

(2) *Suppose that* E *is not a Dirichlet eigenvalue of* H_0 *on* Ω*, and that* f *is a solution to* (4.34)*. Then the function* ψ *defined on* Ω_{ex} *by*

$$\psi = \psi_0(x, y; \eta, \theta) - \left[\mathcal{S}_\theta\Big(\Lambda(\theta; q) + i\theta \cdot n_x y^2\Big) - \mathcal{D}_\theta y^2\right] f \tag{4.35}$$

solves the exterior problem (4.21)–(4.24)*, and* $\psi|_{\partial\Omega} = f$*.*

Proof. We first prove the assertion (1). Put $\Omega_{ex}^{R,r} = \Omega_{ex} \cap \{|x| < r, \frac{1}{R} < y < R\}$, and take $z_0 \in \Omega_{ex}^{R,r}$. Apply Lemma 4.2 with Ω replaced by $\Omega_{ex}^{R,r}$, u by $\psi - \psi_0$ and v by $v(z_0, z)$ from (4.5). Then taking notice of the direction of n and (4.7), we have

$$u(z_0) = \int_{\partial\Omega_{ex}^{R,r}} u\,(B(-\theta)v)\,\frac{dS_E}{y^n} - \int_{\partial\Omega_{ex}^{R,r}} (B(\theta)u)\,v\frac{dS_E}{y^n}.$$

Let us note that $\partial\Omega_{ex}^{R,r} = \partial\Omega_{ex} \cup \{|x| < r,\ y = R \text{ or } y = 1/R\} \cup \{|x| = r,\ 1/R < y < R\}$ and that $B(\theta) = y^2\hat{x} \cdot (\partial_x + i\theta), \hat{x} = x/|x|$, on $\{|x| = r,\ 1/R < y < R\}$. By virtue of Lemma 2.3 and (4.23), $u, \partial_x u, v, \partial_x v \in L^2(\{|x| > r, 1/R < y < R\})$ for large $r > 0$. Therefore the integral over the sphere $|x| = r_j$ tends to 0 as $r_j \to \infty$ along a suitable sequence $\{r_j\}$. The integrals over the plane $y = R$, $y = 1/R$ tend to 0 as $R \to \infty$ by the condition (4.24). We have, therefore,

$$\begin{aligned}
u(z_0) &= \int_{\partial\Omega_{ex}} u\,(B(-\theta)v)\,\frac{dS_E}{y^n} - \int_{\partial\Omega_{ex}} (B(\theta)u)\,v\frac{dS_E}{y^n} \\
&= \int_{\partial\Omega_{ex}} \psi\,(B(-\theta)v)\,\frac{dS_E}{y^n} - \int_{\partial\Omega_{ex}} (B(\theta)\psi)\,v\frac{dS_E}{y^n} \\
&\quad - \int_{\partial\Omega_{ex}} \psi_0\,(B(-\theta)v)\,\frac{dS_E}{y^n} + \int_{\partial\Omega_{ex}} (B(\theta)\psi_0)\,v\frac{dS_E}{y^n}.
\end{aligned}$$

Since v satisfies $(H_0(-\theta) - E)v = 0$ on Ω, we have $B(-\theta)v = \Lambda(-\theta; 0)v$. By the same reasoning, $B(\theta)\psi_0 = \Lambda(\theta; 0)\psi_0$. Therefore, in view of of (4.17), we have

$$-\int_{\partial\Omega_{ex}} \psi_0\,(B(-\theta)v)\,\frac{dS_E}{y^n} + \int_{\partial\Omega_{ex}} (B(\theta)\psi_0)\,v\frac{dS_E}{y^n} = 0.$$

This implies

$$u(z_0) = \int_{\partial\Omega} \psi\,(B(-\theta)v)\,\frac{dS_E}{y^n} - \int_{\partial\Omega} (B(\theta)\psi)\,v\frac{dS_E}{y^n} \tag{4.36}$$
$$= \mathcal{D}_\theta y^2\psi - \mathcal{S}_\theta\left(i\theta\cdot n_x y^2\psi + \Lambda(\theta;q)\psi\right),$$

where we have used $B(-\theta) = y^2\partial/\partial n - i\theta\cdot n_x y^2$, and the boundary condition (4.22). Letting $z_0 \to z \in \partial\Omega$ in (4.36) and using (4.32), we get the assertion (1).

Let us prove (2). The equation (4.21) is easily checked. Lemmas 2.3 and 4.1 imply (4.23) and (4.24), respectively. By Lemma 4.5 and (4.34), $\psi\Big|_{\partial\Omega} = f$. By the same computation as above to derive (4.36), one can show that

$$\psi = \psi_0 + \mathcal{D}_\theta y^2\psi - \mathcal{S}_\theta\left(i\theta\cdot n_x y^2 + B(\theta)\right)\psi. \tag{4.37}$$

Since $\psi = f$ on $\partial\Omega$, comparing (4.35) and (4.37), we have

$$\mathcal{S}_\theta\left(B(\theta)\psi - \Lambda(\theta;q)f\right) = 0 \quad \text{on} \quad \Omega_{ex}. \tag{4.38}$$

Now if we put $w = \mathcal{S}_\theta\left(B(\theta)\psi - \Lambda(\theta;q)f\right)$, w satisfies $(H_0(\theta) - E)w = 0$ in Ω and $w = 0$ on $\partial\Omega$. Since $H_0(\theta)$ is a gauge transform of H_0, this means that $w = 0$ in Ω by our assumption that E is not the Dirichlet eigenvalue of H_0 on Ω. Therefore the equation (4.38) holds on $\mathbf{R}^n_+$. In view of (4.31), we obtain $B(\theta)\psi - \Lambda(\theta;q)f = 0$, which proves (4.22). □

The solvability of the exterior problem is closely related to the whole space problem. Let us consider the equation

$$\psi = \psi_0(x, y; \eta, \theta) - \mathbf{G}_0(\theta)V\psi \quad \text{on} \quad \mathbf{R}^n_+, \tag{4.39}$$

and for some $s > 1/2$

$$\psi - \psi_0(x, y; \eta, \theta) \in L^{2,-s}(\mathbf{R}^n_+). \tag{4.40}$$

Lemma 4.7. *We assume that E is not a Dirichlet eigenvalue of H_0+V on Ω.*
(1) *Suppose ψ is a solution to* (4.39), (4.40)*. Then ψ also solves the exterior problem* (4.21)–(4.24)*.*
(2) *Conversely, if ψ solves* (4.21)–(4.24)*, there exists a unique solution $\tilde{\psi}$ to* (4.39), (4.40) *such that $\tilde{\psi} = \psi$ on Ω_{ex}.*

Proof. Let us prove (1). The equations (4.21) and (4.22) are easy to check. Put $u = \psi - \psi_0$. Then since

$$\hat{u}(\xi, y) = -\int_0^\infty G_0(y, y'; \zeta(\xi,\theta))\widehat{V\psi}(\xi, y')\frac{dy'}{(y')^n},$$

$$\hat{v}(z_0, -\xi, y) = G_0(y_0, y; \zeta(\xi, \theta))e^{ix_0\cdot\xi},$$

the properties (4.23) and (4.24) can be proved by a direct computation.

Let us prove (2). If there are two such solutions $\tilde{\psi}_1, \tilde{\psi}_2$, we have that $(H_0(\theta)+V-E)\tilde{\psi}_1 = (H_0(\theta)+V-E)\tilde{\psi}_2 = 0$ on $\mathbf{R}^n_+$ and $\tilde{\psi}_1 = \tilde{\psi}_2$ on $\partial\Omega$. Since E is not a Dirichlet eigenvalue of $H_0(\theta) + V$, $\tilde{\psi}_1 = \tilde{\psi}_2$ in Ω. By the unique continuation theorem, they coincide on $\mathbf{R}^n_+$. Lets us prove the existence. For a solution ψ to the exterior problem (4.21)–(4.24), we define $\tilde{\psi} = \psi$ on Ω_{ex} and $\tilde{\psi} = \varphi$ on Ω, where φ is the solution of the interior Dirichlet problem $(H_0(\theta) + V - E)\varphi = 0$ in Ω satisfying $\varphi = \psi$ on $\partial\Omega$. Then $\tilde{\psi}$ is continuous across $\partial\Omega$, and $B(\theta)\tilde{\psi}$ computed as the limit from Ω_{ex} and that from Ω coincide. Then $\partial\tilde{\psi}/\partial n$ is also continuous across $\partial\Omega$. This shows that $\tilde{\psi} \in H^2_{loc}(\mathbf{R}^n_+)$ and $(H_0(\theta)+V-E)\tilde{\psi} = 0$ on $\mathbf{R}^n_+$. Let $u = \tilde{\psi}-\psi_0$ and $U_{R,r} = \{|x| < r, \frac{1}{R} < y < R\}$. Then by using (4.12), one has with $v = v(z_0, z)$ from (4.5)

$$\begin{aligned} u(z_0) = &-\int_{U_{R,r}} V\tilde{\psi}v\frac{dxdy}{y^n} \\ &+ \int_{\partial U_{R,r}} (B(\theta)u)\, v\frac{dS_E}{y^n} - \int_{\partial U_{R,r}} u\,(B(-\theta)v)\,\frac{dS_E}{y^n} \end{aligned} \tag{4.41}$$

We let $R, r \to \infty$. Then by virtue of the conditions (4.23) and (4.24), the second and the third terms of the right-hand side vanish. Hence we have $u = -\mathbf{G}_0(\theta)V\tilde{\psi}$, which proves (4.39). The condition (4.40) follows from Lemma 2.2. ☐

§4.3. D-N map and the scattering amplitude

Lemma 4.8. *The operator $\mathcal{S}_\theta\left(\Lambda(\theta; q) + i\theta\cdot n_x y^2\right) - \mathcal{B}_\theta y^2 - \frac{1}{2}$ is compact on $H^{3/2}(\partial\Omega)$.*

Proof. Let $u = Tf$ be the solution to the Dirichlet problem (4.15), and v be as in (4.5). Then by virtue of (4.12), we have for $z_0 \in \Omega$,

$$u(z_0) = -\mathbf{G}_0(\theta)VTf + \mathcal{S}_\theta\left(\Lambda(\theta; q) + i\theta\cdot n_x y^2\right) f - \mathcal{D}_\theta y^2 f.$$

Letting z_0 approach $\partial\Omega$, we have by Lemma 4.5

$$\mathcal{S}_\theta\left(\Lambda(\theta; q) + i\theta\cdot n_x y^2\right) - \mathcal{B}_\theta y^2 - \frac{1}{2} = \mathbf{G}_0(\theta)VT.$$

The right-hand side is compact on $H^{3/2}(\partial\Omega)$, since T is compact from $H^{3/2}(\partial\Omega)$ to $L^2(\Omega)$ and $\mathbf{G}_0(\theta)V$ is bounded from $L^2(\Omega)$ to $H^{3/2}(\partial\Omega)$. ☐

Taking into account that $\mathbf{G}_0(\theta)V$ is compact on $L^{2,-s}(\mathbf{R}^n_+)$ for $s > 1/2$, we introduce the following assumption :

(A) $\quad -1 \notin \sigma_p(\mathbf{G}_0(\theta)V)$.

Under this assumption $\mathbf{G}_V(\theta) = (1+\mathbf{G}_0(\theta)V)^{-1}\mathbf{G}_0(\theta)$ is well-defined. We define $\Psi_I^{(0)}$ by (4.18) and put Ψ_I and $A(\xi,\eta;\theta)$ in the same way as in (3.13) and (3.17).

Notice that when $n \le 3$, this assumption is satisfied for large $|\theta_I|$ by virtue of Theorem 2.8.

Lemma 4.9. *Under the assumption* (A) *and the condition that* E *is not a Dirichlet eigenvalue of* H_0 *on* Ω, $\mathcal{S}_\theta\left(\Lambda(\theta;q) + i\theta\cdot n_x y^2\right) - \mathcal{B}_\theta y^2 + \frac{1}{2}$ *has a trivial null space on* $H^{3/2}(\partial\Omega)$.

Proof. Suppose f is in this null space. Let $\psi = -\Big[\mathcal{S}_\theta\left(\Lambda(\theta;q) + i\theta\cdot n_x y^2\right) -\mathcal{D}_\theta y^2\Big] f$. Then by the same arguments as in the proof of Lemma 4.6 (2), ψ solves the exterior problem (4.21)–(4.24), where in this case ψ_0 is taken to be 0. Then by the same argument as in the proof of Lemma 4.7 (2), there exists a solution to the problem

$$\tilde{\psi} = -\mathbf{G}_0(\theta)V\tilde{\psi} \quad \text{on} \quad \mathbf{R}^n_+, \quad \tilde{\psi} \in L^{2,-s}(\mathbf{R}^n_+),$$

and $\tilde{\psi} = \psi$ on Ω_{ex}. The assumption (A) implies $\tilde{\psi} = 0$. Since $\psi = f$ on $\partial\Omega$, we then have $f = 0$. □

We are now in a position to construct $A(\xi,\eta;\theta)$ from the DN map Λ.

Theorem 4.10. *Let* H_0^D *be* H_0 *on* Ω *with Dirichlet boundary condition. Assume* (A) *and that* $E \notin \sigma_p(H_0^D)$, $0 \notin \sigma_p(H_0^D + q)$. *Define* V *by* (4.3). *Let* $\psi_0(\eta,\theta)$ *be defined by* (4.18). *Then there exists a unique* $f \in H^{3/2}(\partial\Omega)$ *such that*

$$f = \psi_0 - \Big[\mathcal{S}_\theta\left(\Lambda(\theta;q) + i\theta\cdot n_x y^2\right) - \mathcal{B}_\theta y^2 - \frac{1}{2}\Big] f,$$

and a unique solution $\psi(\eta,\theta)$ *of the exterior problem* (4.21)−(4.24) *satisfying* $\psi = f$ *on* $\partial\Omega$. *Moreover the scattering amplitude* $A(\xi,\eta;\theta)$ *is represented as*

$$A(\xi,\eta;\theta) = \int_{\partial\Omega}\Big[\psi_0(-\xi,-\theta)\Lambda(\theta;q)\psi(\eta,\theta) - \left(B(-\theta)\psi_0(-\xi,-\theta)\right)\psi(\eta,\theta)\Big]\frac{dS_E}{y^n}.$$

Proof. By Lemmas 4.8 and 4.9, there exists a unique f as above. By Lemma 4.6 (2), there exists a unique ψ as above. Lemma 4.7 (2) implies the existence of $\tilde{\psi}$. By (3.13), we have $\tilde{\psi} = \Psi_I(x, y; \eta, \theta)$. Therefore by (3.17),

$$A(\xi, \eta; \theta) = \int_{\Omega} \psi_0(-\xi, -\theta) V \tilde{\psi}(\eta, \theta) \frac{dxdy}{y^n}.$$

Using (4.12), we complete the proof of the theorem. □

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