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On the Number of Enriques Quotients of a K3 Surface

By

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§0. Introduction

A $K3$ *surface* X is a compact complex surface with $K_X \sim 0$ and $H^1(X, \mathcal{O}_X) = 0$. An *Enriques surface* is a compact complex surface with $H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0$ and $2K_Y \sim 0$. The universal covering of an Enriques surface is a $K3$ surface. Conversely every quotient of a $K3$ surface by a free involution is an Enriques surface. Here a *free involution* is an automorphism of order 2 without any fixed points.

The moduli space of Enriques surfaces is constructed using the periods of their covering $K3$ surfaces. Precisely speaking, an Enriques surface determines a lattice-polarized $K3$ surface and vice versa, so that the moduli space of Enriques surfaces can be described by the moduli space of lattice-polarized $K3$ surfaces. We note that even if we do not fix any polarization on Enriques surfaces, their covering $K3$ surfaces automatically have a lattice-polarization. Then, what happens if we drop the lattice-polarization of the covering $K3$ surface?

We will call two Enriques quotients of a $K3$ surface *distinct* if they are not isomorphic to each other as varieties. In his paper [3], Kondo discovered a $K3$ surface with two distinct Enriques quotients. He computed the automorphism groups of the two quotients. Since then, as far as the author knows, no other examples have been found.

In this paper we investigate this phenomenon. We show that $K3$ surfaces with more than one distinct Enriques quotients have 9-dimensional components (neither irreducible nor closed) in the period domain. Moreover we compute the exact number of distinct Enriques quotients at a very general point of

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each component (Proposition 3.6). This generalizes Kondo's example in an arithmetic way and results in the following unboundedness theorem.

Theorem 0.1. *For any nonnegative integer l, there exists a $K3$ surface X with exactly 2^{l+10} distinct Enriques quotients. In particular, there does not exist a universal bound for the number of distinct Enriques quotients of a $K3$ surface.*

We also generalize Kondo's example in a geometric way. Its construction is due to Mukai [4]. We introduce his construction and show that a generic Kummer surface X of product type (see Section 4) has exactly 15 distinct Enriques quotients, which can be constructed from classical Lieberman's involution and Kondo-Mukai's involution.

Theorem 0.2. *X has exactly 15 distinct Enriques quotients which are naturally in one-to-one correspondence with nonzero elements of the discriminant group of $NS(X)$.*

From the theoretical point of view, we first show the following finiteness theorem on the automorphism group of a $K3$ surface X using a theorem of Borel.

Theorem 0.3. *In $\mathrm{Aut}(X)$ there are only finitely many conjugacy classes of finite subgroups.*

This theorem concerns us because it bounds the number of distinct Enriques quotients for any $K3$ surfaces.

Corollary 0.4. *Every $K3$ surface X has only finitely many distinct Enriques quotients.*

The usage of the theorem of Borel is suggested by an anonimous referee. We remark that Corollary 0.4 follows also from our counting method described in Section 2. There, to count the exact number of distinct Enriques quotients, we consider more directly the embeddings of the Enriques lattice $U(2) \oplus E_8(2)$ into Néron-Severi lattices.

Notations and Convention. Our main tool is the theory of lattices and their discriminant forms. Here we collect some basic definitions about them. See [7] for the detailed exposition.

A *lattice* L is a free $\mathbb{Z}$-module of finite rank equipped with a $\mathbb{Z}$-valued symmetric bilinear form. L is said to be *even* if for all $l \in L, l^2 \in 2\mathbb{Z}$. In this

paper we treat only even lattices, so that we sometimes omit mentioning the evenness. For a lattice L, there is a natural homomorphism $c : L \to L^* = \mathrm{Hom}(L, \mathbb{Z})$ defined by $l \mapsto (l, \cdot)$. L is said to be *nondegenerate* if c is injective, and *unimodular* if c is bijective. For $m \in \mathbb{Q}$, $L(m)$ denotes the same underlying group equipped with the form multiplied by m, assuming that it is $\mathbb{Z}$-valued. U, E_8 and D_4 denote the lattices given by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, the Dynkin diagrams of type E_8 and D_4 respectively. We understand the latter two to be negative definite.

A *finite quadratic form* is a triple (A, q, b) where A is a finite abelian group, q is a map $A \to \mathbb{Q}/2\mathbb{Z}$ and b is a bilinear map $A \times A \to \mathbb{Q}/\mathbb{Z}$ which is symmetric and satisfies

$$q(x + y) = q(x) + q(y) + 2b(x, y), \ x, y \in A.$$

In the following we abbreviate $b(x, y)$ (resp. $q(x)$) to xy (resp. x^2) and sometimes (A, q, b) to (A, q). We call x^2 the norm of x. As in the lattice case, we have a natural homomorphism $c : A \to A^* = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$ defined by using b. (A, q) is said to be *nondegenerate* if c is bijective.

For an even nondegenerate lattice L, we can canonically associate a finite quadratic form (A_L, q_L), called the discriminant quadratic form of L, by putting $A_L := L^*/L$ and q_L is the one naturally induced from the linear extension of the form on L to $L^* \subset L \otimes \mathbb{Q}$. The discriminant group of $U(2)$ (resp. D_4) is denoted by $u(2)$ (resp. $v(2)$).

For a lattice L, $O(L)$ (resp. $O(q_L)$) denotes the integral orthogonal group of L (resp. of (A_L, q_L)). We note that there is a natural homomorphism $\sigma_L : O(L) \to O(q_L)$. $L_{\mathbb{R}}$ (resp. $L_{\mathbb{C}}$) is the scalar extension of L to $\mathbb{R}$ (resp. $\mathbb{C}$).

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§1. Finiteness of Conjugacy Classes of Finite Subgroups

First we collect some basic definitions about $K3$ surfaces. Let X be a $K3$ surface. It is known that all $K3$ surfaces are diffeomorphic. *A $K3$ lattice* is a lattice isomorphic to $H^2(X, \mathbb{Z}) = U^{\oplus 3} \oplus E_8^{\oplus 2}$. ω_X is the period of X, namely $\mathbb{C}\omega_X = H^{2,0}(X)$. $NS(X) = \omega_X^{\perp} \cap H^2(X, \mathbb{Z})$ is the Néron-Severi lattice of X. $T_X = (NS(X))^{\perp}$ is the transcendental lattice of X.

We recall the structure of the integral automorphism group $O(NS)$ of $NS(X)$.

Definition 1.1.

1. The *positive cone* $\mathcal{C}_X$ is the connected component of $\{x \in NS(X)\otimes\mathbb{R} |\ x^2 > 0\ \}$ which contains an ample divisor.
2. The *ample cone* $\mathcal{A}_X$ is the subcone of $\mathcal{C}_X$ generated as a semigroup by ample divisors multiplied by positive real numbers.

Definition 1.2.

1. The *Weyl group* W_X of X is the subgroup of $O(NS)$ generated by automorphisms of the form $s_l : x \mapsto x + (xl)l$ for all elements $l \in NS(X)$ with $l^2 = -2$.
2. $O^{\uparrow}(NS) := \{\varphi \in O(NS) | \varphi(\mathcal{C}_X) = \mathcal{C}_X\}$.
3. $O^{+}(NS) := \{\varphi \in O(NS) | \varphi(\mathcal{A}_X) = \mathcal{A}_X\}$.
4. $O_0(NS) := \ker(\sigma_{NS} : O(NS) \to O(q_{NS}))$.

Further we use the abbreviations

$$O_0^{\uparrow}(NS) = O_0(NS) \cap O^{\uparrow}(NS) \quad \text{and} \quad O_0^{+}(NS) = O_0(NS) \cap O^{+}(NS).$$

$O^{\uparrow}(NS)$ is of index 2 in $O(NS)$. The inclusions $O_0(NS) \subset O(NS)$ and $O_0^{+}(NS) \subset O^{+}(NS)$ are of finite index since $O(q_{NS})$ is a finite group.

The following relation between these subgroups are important.

Proposition 1.3. *We have* (1) $W_X \subset O_0^{\uparrow}(NS)$, (2) $O^{\uparrow}(NS) = W_X \rtimes O^{+}(NS)$ *and* (3) $O_0^{\uparrow}(NS) = W_X \rtimes O_0^{+}(NS)$.

Proof. Since the generator $s_l \in W_X$ acts trivially on the hyperplane H_l orthogonal to l which intersects with the positive cone, W_X preserves the positive cone. If $x \in NS^*$, then $xl \in \mathbb{Z}$ and $s_l(x) = x$ modulo $\mathbb{Z}l$. This proves (1). We have the semidirect product decomposition as in (2) because W_X and $O^{\uparrow}(NS)$ are discrete subgroups of the isometry group of the Lobačevskiĭ space modeled in $\mathcal{C}_X$ and W_X is a reflection group with ample cone as its fundamental domain. See [12]. The proof of (3) is the same. □

We proceed to the proof of Theorem 0.3. For brevity, we say that a group G has property (FP) if G has only finitely many conjugacy classes of finite subgroups. For example, let G be an algebraic group defined over $\mathbb{Q}$. Then $G_{\mathbb{Z}}$ has property (FP) by [11, Theorem 4.3] which we call the theorem of Borel.

Lemma 1.4.
(1) *Let $\alpha : G \to K$ be a homomorphism of groups. If $\operatorname{im}\alpha$ has property (FP) and $\ker\alpha$ is finite, then G has property (FP).*
(2) *Let $G = W \rtimes K$ be a semidirect decomposition of a group G. If two finite subgroups $F_1, F_2 \subset K$ are G-conjugate, then they are also K-conjugate. In particular if G has property (FP), then so does K.*
(3) *Let $H \subset G$ be a subgroup of finite index. If G has property (FP), then so does H.*

Proof. (1) Let $P_1, \cdots, P_n$ be the complete representatives of conjugacy classes of finite subgroups of $\operatorname{im}\alpha$. Then any conjugacy class of finite subgroups of G has a representative included in at least one of $\alpha^{-1}(P_j)$, $(j = 1, \cdots, n)$.
(2) Assume $F_2 = gF_1g^{-1}$. g can be written as $g = wk$, $w \in W$, $k \in K$. If $f_i \in F_i$ satisfy $f_2 = gf_1g^{-1}$, then we have $(kf_1k^{-1})^{-1}f_2 = (kf_1^{-1}k^{-1}wkf_1k^{-1})w^{-1} \in W \cap K = \{1\}$. Hence $kf_1k^{-1} = f_2$. Thus F_1 and F_2 are conjugate by $k \in K$.
(3) Again let $P_1, \cdots, P_n$ be the complete representatives of conjugacy classes of finite subgroups of G. We put $G/H = \{a_1H, \cdots, a_rH\}$. Then the conjugacy classes of finite subgroups of H are represented by $\{a_i^{-1}P_ja_i | i = 1, \cdots, r \text{ and } j = 1, \cdots, n\}$. □

Now we show Theorem 0.3. In our words,

Theorem 1.5. *Let X be a $K3$ surface. Then $\operatorname{Aut}(X)$ has property (FP).*

Proof. First we assume that X is projective. Consider the representation $r : \operatorname{Aut}(X) \to O(NS)$. Since every automorphism in $\ker r$ fixes an ample divisor, $\ker r$ is finite. Thus it is enough to show that $\operatorname{im} r$ has the property (FP) by Lemma 1.4 (1).

By the theorem of Borel above, $O(NS)$ has property (FP). Then, by Lemma 1.4 and Proposition 1.3, the property (FP) goes down to $O^{\uparrow}(NS)$ and $O^{+}(NS)$. Now by the global Torelli theorem [10], $\operatorname{im} r$ contains $O_0^{+}(NS)$, since $\varphi \in O_0^{+}(NS)$ preserves the ample cone $\mathcal{A}_X$ and can be extended to an isometry of $H^2(X, \mathbb{Z})$ which acts trivially on T_X. Thus we obtain

$$O_0^{+}(NS) \subset \operatorname{im} r \subset O^{+}(NS)$$

and these inclusions are of finite index. We see that $\operatorname{im} r$ has the property (FP) by Lemma 1.4 (3).

Next we assume that X is not projective. Nikulin [8] shows that any automorphism of X of finite order acts on T_X trivially. Therefore it is enough to show that $G = \ker(\mathrm{Aut}(X) \to O(T_X))$ has property (FP). We consider the representation $r : G \to O(NS)$.

If $\mathrm{alg.dim}(X) = 0$, then $NS(X)$ is negative definite. Hence $NS(X) \oplus T_X$ is of finite index in $H^2(X, \mathbb{Z})$, r is injective and G is finite since $O(NS)$ is a finite group.

If $\mathrm{alg.dim}(X) = 1$, then $NS(X)$ has one-dimensional kernel $\mathbb{Z}e$ and $Q := NS(X)/\mathbb{Z}e$ is negative definite. Every element of G fixes e since $e^2 = 0$ and exactly one of e and $-e$ is represented by an effective cycle. Thus r induces $s : G \to O(Q)$. Let $g \in G$ be an element of finite order. Since the fixed part $H^2(X, \mathbb{Z})^g$ is nondegenerate by the lemma below, it follows that if $s(g) = \mathrm{id}_Q$ then $g = \mathrm{id}_X$. On the other hand, $O(Q)$ is a finite group. Thus $\mathrm{Aut}(X)$ has only finitely many elements of finite order and $\mathrm{Aut}(X)$ has property (FP). □

Lemma 1.6. *Let L be a nondegenerate lattice and let g be an isometry of L of finite order n. Let $M = L^g = \{x \in L | gx = x\}$ be the fixed lattice. Then M is nondegenerate.*

Proof. Let $0 \neq x \in M$. Since L is nondegenerate, there exists $y \in L$ with $xy \neq 0$. Put $z = y + g(y) + \cdots + g^{n-1}(y)$. Obviously $z \in M$ and we have

$$xz = xy + g(x)g(y) + \cdots + g^{n-1}(x)g^{n-1}(y) = nxy \neq 0.$$

Therefore M is nondegenerate. □

As a corollary, Corollary 0.4 follows. In fact, if two free involutions i_1 and i_2 are conjugate by an automorphism g, then g induces an isomorphism between X/i_1 and X/i_2.

§2. Number of Distinct Enriques Quotients

The isomorphism classes of Enriques quotients are exactly the conjugacy classes of free involutions by the next proposition.

Proposition 2.1. *Let X be a $K3$ surface and let i_1 and i_2 be free involutions on X. Then, X/i_1 and X/i_2 are isomorphic if and only if there exists an automorphism g of X such that $gi_1g^{-1} = i_2$.*

Proof. The "if" part is trivial; see the sentence after Lemma 1.6. Conversely, let h be an isomorphism from $Y_1 := X/i_1$ to $Y_2 := X/i_2$. It induces an isomorphism of the canonical line bundles $h^*\mathcal{K}_{Y_2} \to \mathcal{K}_{Y_1}$. Since X and Y_1, Y_2 are related as $X = \mathrm{Spec}(\mathcal{O}_{Y_j} \oplus \mathcal{K}_{Y_j})$ $j = 1, 2$, h induces the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X/i_1 & \xrightarrow[h]{} & X/i_2 \end{array}$$

where π denotes the covering map. It is clear that g is the desired automorphism of X. □

We put

$$\mathfrak{M} := \left\{ M \subset NS \;\middle|\; \begin{array}{l} \text{a primitive sublattice which satisfies} \\ \text{(A)}: \ M \cong U(2) \oplus E_8(2) \\ \text{(B)}: \ \text{No vector of square } -2 \text{ in } NS(X) \text{ is orthogonal to } M. \end{array} \right\}.$$

Recall that $U \oplus E_8$ is the Enriques lattice (modulo torsion) and $U(2) \oplus E_8(2)$ is the pullback in the covering $K3$ lattice. For each $M \in \mathfrak{M}$, we define an isometry $i_M : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ by $i_M(m) = m$ when $m \in M$ and $i_M(n) = -n$ when n is orthogonal to M. This is well-defined because $M \cong U(2) \oplus E_8(2)$ is 2-elementary.

Proposition 2.2. *On a $K3$ surface X, there is a one-to-one correspondence between free involutions on X and primitive sublattices M of $NS(X)$ which satisfy* (A) *and* (B) *above and the following*

(C) *M contains an ample divisor.*

In other words, i_M defined above is a free involution if and only if M contains an ample divisor. Also any free involution can be written in the form i_M.

Proof. We associate a free involution with its invariant sublattice in $H^2(X, \mathbb{Z})$. The statement follows from [6, Corollary 2.5], [9, Theorem 4.2.2, p. 1426] and the strong Torelli theorem for $K3$ surfaces [10]. In [9], the assumption is slightly different from ours, but the same proof goes. □

To count the number of distinct Enriques quotients, we consider the natural action of $O(NS)$ on $\mathfrak{M}$,

$$O(NS) \ni \varphi : M \mapsto \varphi(M) \in \mathfrak{M}.$$

Corresponding lattice automorphisms satisfy $i_{\varphi(M)} = \varphi i_M \varphi^{-1}$.

In the following, $\mathrm{Aut}(T_X, \omega_X)$ is the subgroup of $O(T_X)$ consisting of the integral orthogonal transformations whose scalar extention to $\mathbb{C}$ preserves the period $\mathbb{C}\omega_X \subset T_X \otimes \mathbb{C}$.

Theorem 2.3. *Let $M_1, \cdots, M_k \in \mathfrak{M}$ be a (finite) complete set of representatives for the action of $O(NS)$ on $\mathfrak{M}$. For each $j = 1, \cdots, k$, let*

$$K^{(j)} = \{\varphi \in O(NS) | \varphi(M_j) = M_j\}$$

be the stabilizer subgroup of M_j and $\sigma(K^{(j)})$ its canonical image in $O(q_{NS})$. We put

$$B_0 = \sum_{j=1}^{k} \#(O(q_{NS})/\sigma(K^{(j)})).$$

(1) *The number of distinct Enriques quotients of X does not exceed B_0.*

(2) *If $\sigma : O(NS) \to O(q_{NS})$ is surjective and if $\mathrm{Aut}(T_X, \omega_X) = \{\pm \mathrm{id}\}$, then X has exactly B_0 distinct Enriques quotients.*

Proof. First we remark that by Proposition 1.15.1 in [7], the set of representatives is always finite. In view of Proposition 2.1, we can count the number of distinct Enriques quotients separately for each orbit $O(NS) \cdot M_j$. Hence, for simplicity, we fix an orbit and omit the index j so that we use the symbols $M := M_j$, $\mathcal{O} := O(NS) \cdot M$ and $K = K^{(j)}$.

Step 1. $\mathcal{O}$ contains an element which corresponds to a free involution.

Proof. The following is a standard argument used in [10]. Our proof is taken from [6]. By the condition (A) in Proposition 2.2, $M \cap \mathcal{C}_X \neq \emptyset$. Consider in $\mathcal{C}_X$ countably many hyperplanes $H_d = \{x \in NS_{\mathbb{R}} | xd = 0\}$, where d runs over (-2) vectors in NS. The union $\cup H_d$ is a locally finite closed subset in $\mathcal{C}_X$ and does not contain M by the condition (B). The complement $\mathcal{C}_X - \cup H_d$ is a collection of (at most) countably many connected open sets, namely chambers, which corresponds to the elements of W_X in one-to-one way. The ample cone $\mathcal{A}_X$ equals one of the chambers. Thus if we choose $v \in M \cap \mathcal{C}_X - \cup H_d$, there exists $\varphi \in W_X \subset O(NS)$ such that $\varphi(v)$ is an ample divisor. □

Thus we can assume that i_M is already a free involution of X. Next we set

$$\mathfrak{N} := \{M' \in \mathcal{O} | i_{M'} \text{ is a free involution}\}.$$

Step 2. $\mathfrak{N} = O^+(NS) \cdot M$.

Proof. $\supset$ follows from Proposition 2.2. Let $\varphi \in O(NS)$ and suppose $i_{\varphi(M)}$ is a free involution. We can assume $\varphi \in O^{\uparrow}(NS)$, since otherwise $-\varphi \in O^{\uparrow}(NS)$ and $(-\varphi)(M) = \varphi(M)$. By Proposition 1.3, we can write $\varphi = w\psi$ with $w \in W_X$ and $\psi \in O^{+}(NS)$. Using Lemma 1.4 (2), $i_{\varphi(M)} = w i_{\psi(M)} w^{-1}$ implies $i_{\varphi(M)} = i_{\psi(M)}$. Therefore $\varphi(M) = \psi(M)$. □

Step 3. Let $\psi_1, \psi_2 \in O^{+}(NS)$. If $\sigma(\psi_j)$ have the same class in $O(q_{NS})/\sigma(K)$, then $i_{\psi_1(M)}$ and $i_{\psi_2(M)}$ are conjugate in $\mathrm{Aut}(X)$.

Proof. By the assumption $\sigma(\psi_1^{-1}\psi_2) \in \sigma(K)$, so there exists $\varphi \in K$ such that $\sigma(\varphi) = \sigma(\psi_1^{-1}\psi_2)$. It follows that $\sigma(\psi_1\varphi\psi_2^{-1}) = \mathrm{id}$, so that $\psi_1\varphi\psi_2^{-1}|_{NS}$ together with id_{T_X} gives an automorphism a of X, by the Torelli theorem. It follows that

$$a i_{\psi_2(M)} a^{-1} = i_{a\psi_2(M)} = i_{\psi_1\varphi(M)} = i_{\psi_1(M)}.$$ □

By now, we have proved that $\mathcal{O}$ contains at most $\#O(q_{NS})/\sigma(K)$ distinct Enriques quotients. Assertion (1) follows.

Step 4. If $\mathrm{Aut}(T_X, \omega_X) = \{\pm \mathrm{id}\}$, then the converse of Step 3 holds.

Proof. Assume there exists $\varphi \in \mathrm{Aut}(X)$ such that $\varphi i_{\psi_1(M)} \varphi^{-1} = i_{\psi_2(M)}$, which is equivalent to $\varphi\psi_1(M) = \psi_2(M)$ and to $\psi_2^{-1}\varphi\psi_1 \in K$. By the assumption, $\sigma(\varphi) = \pm \mathrm{id}$ which is contained in the center of $O(q_{NS})$. It follows that $\pm\sigma(\psi_2^{-1}\psi_1) \in \sigma(K)$. We remark that $-\mathrm{id} \in \sigma(K)$ since $\sigma(i_M) = -\mathrm{id}$. Therefore we get $\sigma(\psi_2^{-1}\psi_1) \in \sigma(K)$. □

Step 5. If σ is surjective, then the restriction $\sigma|_{O^{+}(NS)}$ is also surjective.

Proof. Put $N = M^{\perp}$ in NS. Since M is 2-elementary, $t = (-\mathrm{id}_M, \mathrm{id}_N)$ extends to an isometry of NS. t doesn't preserve the positive cone. Therefore $O(NS)$ is generated by t and $O^{\uparrow}(NS)$. This implies the surjectivity of $O^{\uparrow}(NS) \to O(q_{NS})$. By Proposition 1.3, the assertion follows. □

Now the proof is complete. □

Lastly we mention a useful theorem of Nikulin in [7] which saves us from checking one of the conditions in Theorem 2.3 (2).

Theorem 2.4 (Nikulin)**.** *Let T be an even indefinite nondegenerate lattice satisfying the following two conditions*:

(1) $\mathrm{rank}(T) \geq l(A_{T_p}) + 2$ *for all prime numbers p except for* 2.

(2) *if* $\mathrm{rank}(T) = l(A_{T_2})$, *then q_{T_2} contains a component $u(2)$ or $v(2)$.*

Then the genus of T contains only one class, and the homomorphism $O(T) \to O(q_T)$ is surjective. Here A_{T_p} denotes the p-component of the finite abelian group A_T and l denotes the number of minimal generators. □

§3. Enriques Quotients of $K3$ Surfaces in the Heegner Divisors

In this section we prove Theorem 0.1. We deal with certain divisors of the period domain $\mathfrak{D}$ of $U(2) \oplus E_8(2)$-polarized marked $K3$ surfaces. Fix the unique primitive embedding of $U(2) \oplus E_8(2)$ in the $K3$ lattice Λ. Then $\mathfrak{D}$ is by definition

$$\mathfrak{D} := \{[\omega] \in \mathbb{P}((U(2) \oplus E_8(2))^{\perp}_{\mathbb{C}}) | \omega^2 = 0,\ \omega\overline{\omega} > 0\}.$$

Here $\mathbb{P}(V)$ means the associated projective space of a complex vector space V, which consists of all lines through the origin. It follows from the surjectivity of the period map that every point of $\mathfrak{D}$ corresponds to a $K3$ surface X with a marking $H^2(X, \mathbb{Z}) \cong \Lambda$.

Let $S \subset \Lambda$ be a primitive sublattice of rank 11 containing the lattice $U(2) \oplus E_8(2)$ fixed above. Then the subset

$$\mathfrak{D}(S) := \{[\omega] \in \mathbb{P}(S^{\perp}_{\mathbb{C}}) | \omega^2 = 0,\ \omega\overline{\omega} > 0\}$$

is called *the Heegner divisor of type* S in $\mathfrak{D}$. Let X be a marked $K3$ surface whose period is in $\mathfrak{D}(S)$. Since $NS(X)$ is written as $\Lambda \cap \omega_X^{\perp}$, $NS(X)$ contains the primitive sublattice S.

Proposition 3.1. *If X corresponds to a very general point of $\mathfrak{D}(S)$, namely to a point in the complement of a union of countably many closed analytic subset of $\mathfrak{D}(S)$, then we have $NS(X) = S$ and $\mathrm{Aut}(T_X, \omega_X) = \{\pm \mathrm{id}\}$.*

Proof. This is a well-known fact. For the latter, the same proof as in [1, Lemma 2.9] works. □

We consider the case when

$$S = U(2) \oplus E_8(2) \oplus \langle -2N \rangle,$$

where $\langle -2N \rangle$ is the rank 1 lattice whose generator g has $g^2 = -2N$. It is easy to see that the $K3$ lattice Λ contains S as a primitive sublattice. We fix it once and for all. The discriminant form of S is isomorphic to $q = u(2)^{\oplus 5} \oplus c(-2N)$, where $c(-2N)$ is the discriminant form of $\langle -2N \rangle$.

Let the integer N be $N = 4p_1 \cdots p_l$, where $p_1, \cdots, p_l$ are distinct odd prime numbers. In the next we compute the order of $O(q)$.

Lemma 3.2.

$$\#\{x \in c(-2N) | \operatorname{ord}(x) = 2N, x^2 \equiv -1/2N \pmod{2\mathbb{Z}}\} = 2^{l+1},$$
$$\#\{x \in c(-2N) | \operatorname{ord}(x) = 2N, x^2 \equiv 1 - 1/2N \pmod{2\mathbb{Z}}\} = 2^{l+1}.$$

Proof. The left-hand-side of the first equality is

$$\begin{aligned}
&\#\{k \in \mathbb{Z} | (k, 2N) = 1, 1 \leq k \leq 2N-1 \text{ and } -k^2/2N = -1/2N \in \mathbb{Q}/2\mathbb{Z}\} \\
&\quad = (1/2)\#\{k \in \mathbb{Z} | (k, 4N) = 1, 1 \leq k \leq 4N-1 \text{ and } k^2 - 1 \equiv 0 (\mathrm{mod}\, 4N)\} \\
&\quad = (1/2)\#\{x \in (\mathbb{Z}/4N\mathbb{Z})^\times | \mathrm{ord}(x) = 1 \text{ or } 2\}.
\end{aligned}$$

Then we can use the structure theorem of the unit group $(\mathbb{Z}/4N\mathbb{Z})^\times$. Similarly the left-hand-side of the latter is

$$\begin{aligned}
&\#\{k \in \mathbb{Z} | (k, 2N) = 1, 1 \leq k \leq 2N-1 \text{ and } -k^2/2N = 1 - 1/2N \in \mathbb{Q}/2\mathbb{Z}\} \\
&\quad = (1/2)\#\{k \in \mathbb{Z} | (k, 4N) = 1, 1 \leq k \leq 4N-1 \text{ and } k^2 \equiv 1 - 2N (\mathrm{mod}\, 4N)\} \\
&\quad = (1/2)\#\{k \in \mathbb{Z} | (k, 4N) = 1, 1 \leq k \leq 4N-1, k^2 \\
&\quad \equiv 1 (\mathrm{mod}\, 2N) \text{ and } k^2 \not\equiv 1 (\mathrm{mod}\, 4N)\}.
\end{aligned}$$

Using the commutative diagram

$$\begin{array}{ccc}
(\mathbb{Z}/4N\mathbb{Z})^\times & \xrightarrow{\sim} & (\mathbb{Z}/2^4\mathbb{Z})^\times \oplus (\mathbb{Z}/p_1 \cdots p_l\mathbb{Z})^\times \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow \qquad {\scriptstyle\sim}\downarrow \\
(\mathbb{Z}/2N\mathbb{Z})^\times & \xrightarrow{\sim} & (\mathbb{Z}/2^3\mathbb{Z})^\times \oplus (\mathbb{Z}/p_1 \cdots p_l\mathbb{Z})^\times
\end{array}$$

where α, β are both 2:1 maps, we can count the number of elements which have order 2 in the bottom row but do not in the top row. □

Proposition 3.3. *$O(q)$ acts transitively on the set of elements $x \in q$ with $x^2 \equiv -1/2N \pmod{2\mathbb{Z}}$. There are 2^{11+l} such elements.*

Proof. Such element x generates a subgroup $\langle x \rangle$ isomorphic to $c(-2N)$. Since it is nondegenerate, $\langle x \rangle$ is a direct summand in q. This implies the transitivity. If we put the number of elements in $u(2)^{\oplus 5}$ with norm 0 to be A and norm 1 $2^{10} - A$, we can compute the length of the orbit as $2^{l+1} \cdot A + 2^{l+1} \cdot (2^{10} - A) = 2^{11+l}$. □

Under these computations we can prove

Theorem 3.4. *For any nonnegative integer l, there exists a $K3$ surface X with exactly 2^{l+10} distinct Enriques quotients.*

Proof. By Proposition 3.1, there exists a $K3$ surface X such that $NS(X) \cong S$ and $\mathrm{Aut}(T_X, \omega_X) = \{\pm \mathrm{id}\}$. We show that the primitive embedding of

$U(2) \oplus E_8(2)$ into $NS(X)$ is unique under the action of $O(NS)$. In fact, since $NS(1/2)$ is again an even lattice, we have a natural identification

$$\mathrm{Hom}(U(2) \oplus E_8(2), NS) = \mathrm{Hom}(U \oplus E_8, NS(1/2)).$$

We see that any primitive embedding as above is a direct summand. This clearly implies the uniqueness.

Obviously $NS(X)$ has a primitive sublattice M isomorphic to $U(2) \oplus E_8(2)$ and $M^\perp = \langle -2N \rangle$. Let K be the stabilizer group of M and $\sigma(K)$ its canonical image in $O(q_{NS})$. Since $NS \cong M \oplus M^\perp$ we see that $K = O(M) \times O(M^\perp) = O(M) \times \{\pm \mathrm{id}_{M^\perp}\}$. On the other hand by Theorem 2.4 $\sigma_M : O(M) \to O(q_M)$ is surjective. This shows $\sigma(K) = O(u(2)^{\oplus 5}) \times \{\pm \mathrm{id}\} \subset O(q_M \oplus q_{M^\perp}) = O(q_{NS})$. Thus $\#(O(q_{NS})/\sigma(K)) = \#O(q_{NS})/2\#O(u(2)^{\oplus 5}) = 2^{10+l}$ by Proposition 3.3. This together with Theorems 2.3 and 2.4 completes the proof. □

In fact we can classify all the possible Néron-Severi lattices of a $K3$ surface with Picard number 11 having an Enriques quotient. In each case, we can compute the number of Enriques quotients as follows by an explicit calculation. Details are omitted. The result is as follows.

Proposition 3.5. *Let X be a $K3$ surface with Picard number 11 having an Enriques quotient. Then the Néron-Severi lattice of X is one of the followings.*

$$\begin{aligned} &\text{Type I}: \ U(2) \oplus E_8(2) \oplus \langle -2N \rangle \qquad (N \geq 2) \\ &\text{Type II}: \ U \oplus E_8(2) \oplus \langle -4M \rangle \qquad (M \geq 1). \end{aligned}$$

If we put $2N = 2^e p_1^{e_1} \cdots p_l^{e_l}$ in type I, or $4M = 2^e p_1^{e_1} \cdots p_l^{e_l}$ in type II, the bound B_0 in Theorem 2.3 is as follows.

Proposition 3.6.

$$B_0 = \begin{cases} 2^{l-1} & \text{in Type I and } e = 1 \\ (2^5+1) \cdot 2^{l+4} & \text{in Type I and } e = 2 \\ 2^{l+10} & \text{in Type I and } e \geq 3 \\ 1 & \text{in Type II and } e = 2, \quad l = 0 \\ 2^{l-1} & \text{in Type II and } e = 2, \quad l > 0 \\ 2^{2l+5} & \text{in Type II and } e \geq 3 \end{cases}$$

The lattice S we used fits in the third case.

§4. Enriques Quotients of Generic Kummer Surfaces of Product Type

Kondo found the first example of a $K3$ surface which has two distinct Enriques quotients in [3, Remark 3.5.3], where he computed the automorphism groups of the two quotients. Recently Mukai generalized Kondo's example which we now describe.

Kummer surfaces of product type. Let C_1 and C_2 be elliptic curves and construct the Kummer surface as $X = Km(C_1 \times C_2)$. We put the 2-torsion points of C_1 (resp. C_2) as $\{b_1 = 0, b_2, b_3, b_4\}$ (resp. $\{c_1 = 0, c_2, c_3, c_4\}$) and denote by δ the natural rational map of $C_1 \times C_2$ to X. Let E_k (resp. F_k) be the image of $C_1 \times \{c_k\}$ (resp. $\{b_k\} \times C_2$) by δ. Then X has the configuration of 24 smooth rational curves as in Figure 1, where G_{ij} is the exceptional curve corresponding to $(b_i, c_j) \in C_1 \times C_2$.

Sometimes it is called the double Kummer configuration. In the following we introduce two kinds of free involutions on X with parameters.

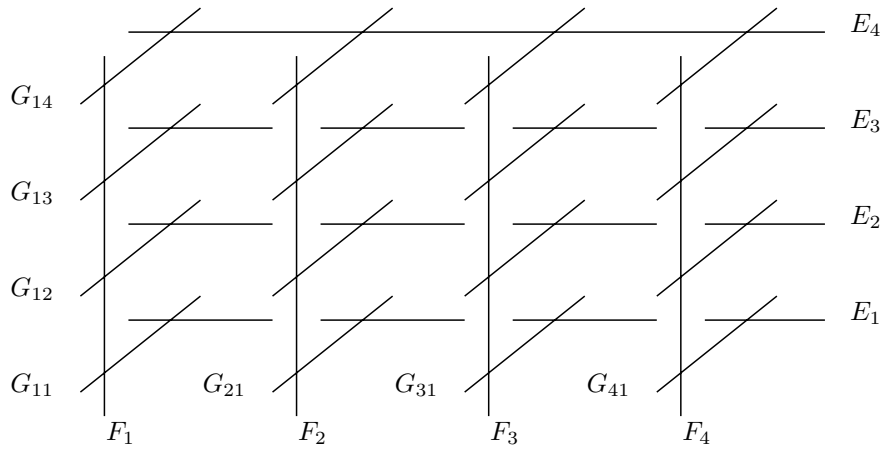

Figure 1. The double Kummer configuration

Involutions of Lieberman type. Let $a = (b_i, c_j) \in C_1 \times C_2$ be a 2-torsion point not lying on $C_1 \times \{0\}$ or $\{0\} \times C_2$. Let τ (resp. ρ) be the involution of X induced by the automorphism $(-\mathrm{id}_{C_1}, \mathrm{id}_{C_2})$ (resp. the translation by a) of $C_1 \times C_2$. Then $\sigma = \tau\rho$ is a free involution. We remark that a has the parameter i, j with $2 \leq i, j \leq 4$.

Involutions of Kondo-Mukai type. Let τ be as above. X/τ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with sixteen points blown up, which correspond to G_{ij}'s in Figure 1. We regard $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth quadric in $\mathbb{P}^3$ so that there are morphisms

$$X \longrightarrow X/\tau \xrightarrow[\varepsilon]{} \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3.$$

Choose two permutations $I = \{i_1, i_2, i_3, i_4\}$ and $J = \{j_1, j_2, j_3, j_4\}$ of $\{1, 2, 3, 4\}$ and we put $g_k = \varepsilon(G_{i_k j_k}) \in \mathbb{P}^1 \times \mathbb{P}^1$. We project $\mathbb{P}^1 \times \mathbb{P}^1$ onto a hyperplane $\cong \mathbb{P}^2$ of $\mathbb{P}^3$ from g_4. This birational map ε' contracts two curves whose images we denote by P and $Q \in \mathbb{P}^2$. Let $\overline{g}_k = \varepsilon'(g_k),\ k = 1, 2, 3$. Then we obtain two involutions on X: One is the covering transformation ν of the degree 2 map $X \to \mathbb{P}^2$. The other involution μ is induced by the unique Cremona transformation of $\mathbb{P}^2$ centered at $\overline{g}_1, \overline{g}_2, \overline{g}_3$ interchanging P and Q. Then $\sigma = \nu\mu$ is a free involution. It can be shown that σ depends only on the set $\{G_{i_k j_k}\}_{1 \le k \le 4}$. See [4] for such canonical description of this involution. Therefore the parameter is the set $\{G_{i_k j_k}\}_{1 \le k \le 4}$. There are twenty-four choices of parameters in appearance, but we will reveal that there are essentially six.

If C_1 and C_2 are chosen to be general enough, we see that

$$NS(X) \cong U \oplus E_8 \oplus D_4^{\oplus 2},\ T_X \cong U(2)^{\oplus 2},\ A_{NS(X)} = u(2)^{\oplus 2}$$
$$\text{and} \quad \mathrm{Aut}(T_X, \omega_X) = \{\pm \mathrm{id}\}.$$

We will call such X *a generic Kummer surface of product type.* In the following main theorem of this section we classify the all free involutions on X.

Theorem 4.1. *Let X be a generic Kummer surface of product type. Then X has exactly fifteen distinct Enriques quotients which are naturally in one-to-one correspondence with nonzero elements of $A_{NS(X)}$. Moreover all of them can be geometrically constructed from the preceding examples by choosing appropriate parameters.*

We remark that the Lieberman involutions correspond to nine elements of norm 0 of $A_{NS(X)}$ and Kondo-Mukai involutions to six elements of norm 1. In the rest of this section we prove Theorem 4.1. First we determine the isomorphism classes of primitive embeddings of $M := U(2) \oplus E_8(2) \subset S = U \oplus E_8 \oplus D_4^{\oplus 2}$. This step is purely lattice-theoretic. We use the following theorem of Nikulin [7, Propositions 1.5.1 and 1.15.1].

Theorem 4.2. *Let $M \subset S$ be a primitive embedding of even nondegenerate lattices. Put $N := M^{\perp}$. Then the following isomorphisms exist.* (*Note*

that (1) *expresses* A_S *in terms of* A_M *and* A_N *while* (2) *expresses* A_N *in terms of* A_M *and* A_S.)
(1) *There are subgroups* $\Gamma_M \subset A_M, \Gamma_N \subset A_N$ *and a sign-reversing isometry* $\gamma : \Gamma_M \to \Gamma_N$ *such that if* Γ *is the pushout of* γ, *namely* $\Gamma = \{(x, \gamma(x)) \in A_M \oplus A_N | x \in \Gamma_M\}$, *then*

$$q_S \cong (q_M \oplus q_N|_{\Gamma^\perp}/\Gamma).$$

(2) *There are subgroups* $\Gamma_M \subset A_M, \Gamma_S \subset A_S$ *and a sign-reversing isometry* $\gamma : \Gamma_M \to \Gamma_S$ *such that if* Γ *is the pushout of* γ, *then*

$$q_N \cong (-q_M \oplus q_S|_{\Gamma^\perp}/\Gamma).$$

Proposition 4.3. *Let*

$$S = U \oplus E_8 \oplus D_4^{\oplus 2} \text{ and } M = U(2) \oplus E_8(2).$$

Assume M *is a primitive sublattice of* S *which is orthogonal to no* (-2) *vectors of* S. *Then*
(1) $N := M^\perp$ *is isomorphic to* $E_8(2)$.
(2) *There are exactly two such primitive sublattices up to the action of* $O(S)$.

Proof. By Theorem 4.2 (2), A_N is a 2-elementary abelian group and q_N takes only integral values on A_N. On the other hand, N is a negative definite lattice of rank 8. This implies

$$A_N \cong \begin{cases} u(2)^{\oplus a}, & 0 \le a \le 4, \text{ or} \\ v(2) \oplus u(2)^{\oplus a}, & 0 \le a \le 3. \end{cases}$$

by the decomposition theorem of 2-elementary finite quadratic forms (see [7, Proposition 1.8.1]).

Checking the signature of (A_N, q_N), we see that the latter does not occur. In all other cases, we find that N has a unimodular overlattice of rank 8, i.e., E_8. The index $[E_8 : N]$ is given by 2^a. □

Claim. Let $N \subset E_8$ be an overlattice and assume N contains no (-2) vectors. Then $[E_8 : N] \ge 9$.

Proof. We take a basis of E_8 as in Figure 2. Consider the elements $f_0 := 0,\ f_j := e_1 + \cdots + e_j,\ 1 \le j \le 7$ and $f_8 := 2e_1 + 3e_2 + 4e_3 + 5e_4 + 6e_5 + 4e_6 + 2e_7 + 3e_8$. It is easy to see that any difference $f_j - f_i,\ 0 \le i < j \le 8$ has

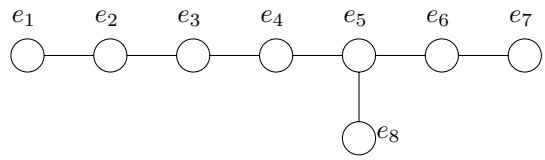

Figure 2. Dynkin diagram of type E_8

norm -2. This means that every f_i is a distinct element of the reminder class group E_8/N. □

Thus we obtain $A_N = u(2)^{\oplus 4}$. Therefore N has the maximal number of minimal generators of the discriminant group. This implies that for any $n \in N$, $n/2 \in N^*$. In particular $N(1/2)$ is a unimodular lattice. Thus we see that $N \cong E_8(2)$ and (1) is proved.

For (2), we use the notation introduced in Theorem 4.2 (1). Since $q_M \oplus q_N$ is nondegenerate $\#\Gamma = \#\Gamma_N = (\#A_M \cdot \#A_N/\#A_S)^{1/2} = 2^7$. Therefore $\#\Gamma_N^{\perp} = \#A_N/\#\Gamma = 2$ and we can put $\Gamma_N^{\perp} = \{0, z_N\}$. There are two cases where $q_N(z_N) = 0,\ 1$. Thus there are at least two distinct primitive embedding of $M \subset S$. On the other hand, the canonical homomorphism $\sigma_M : O(M) \to O(q_M)$ and $\sigma_N : O(N) \to O(q_N)$ are both surjective by Proposition 2.4 and [1]. Thus the primitive embedding of $M \subset S$ is classified by the invariant $q_N(z_N)$ as we see from the following argument.

Claim. Let M_1 and M_2 be two primitive sublattices of S satisfying the assumption. For each $k = 1, 2$, we use the same notation as above, indexed by k. If $q_{N_1}(z_{N_1}) = q_{N_2}(z_{N_2})$, then there exists $\varphi \in O(S)$ which transforms M_1 onto M_2.

Proof. By Witt's theorem on the finite quadratic forms, there exist ψ_M and ψ_N fitting in the square inside.

$$\begin{array}{ccccc} A_{M_1} & \supset & \Gamma_{M_1} & \xrightarrow{\gamma_1} & \Gamma_{N_1} \subset A_{N_1} \\ {\scriptstyle\varphi_M}\downarrow & {\scriptstyle\psi_M}\downarrow & & & \downarrow{\scriptstyle\psi_N} \quad \downarrow{\scriptstyle\varphi_N} \\ A_{M_2} & \supset & \Gamma_{M_2} & \xrightarrow{\gamma_2} & \Gamma_{N_2} \subset A_{N_2} \end{array}$$

Again by Witt's theorem we can extend ψ_M (resp. ψ_N) to φ_M (resp. φ_N) in the diagram. By the surjectivity of σ_M and σ_N mentioned above, these isomorphisms lift to an isomorphism between $M_1 \oplus N_1$ and $M_2 \oplus N_2$ which preserves the overlattice S. This was the assertion. □

Remark 4.4. The same but geometric situation of the proposition is considered in [5]. There (1) is proved by a geometric method.

Now we regard S as the Néron-Severi lattice NS of a generic Kummer surface of product type. To see the natural correspondence in Theorem 4.1, we have to associate a free involution with an element of $A_{NS(X)} = A_{NS}$.

Definition 4.5. Let $M \cong U(2) \oplus E_8(2) \subset NS$ be a primitive sublattice. We use the same notation as in Proposition 4.3. Then we define the *patching element* $v_M \in A_{NS}$ associated with the primitive sublattice M to be the image of the element $(0, z_N) \in A_M \oplus A_N$ in A_{NS} under the isomorphism of Theorem 4.2 (1).

We note that $v_M \neq 0$. Also we note the equality $q_N(z_N) = q_{NS}(v_M)$. Using v_M, we can describe the group $\sigma_{NS}(K)$ in the Theorem 2.3. Recall that $K \subset O(NS)$ is the stabilizer subgroup of M.

Lemma 4.6. *$\sigma_{NS}(K)$ is equal to the stabilizer subgroup $G_{v_M} \subset O(q_{NS})$ of v_M.*

Proof. According to the isomorphism in Theorem 4.2 (1), we consider the group

$$H := \{(\alpha_M, \alpha_N) \in O(q_M) \times O(q_N) | \alpha_M(\Gamma_M) = \Gamma_M, \alpha_N(\Gamma_N) = \Gamma_N, \alpha_M \gamma = \gamma \alpha_N\}$$

and decompose σ_{NS} as $K \xrightarrow{p} H \xrightarrow{q} O(q_{NS})$. Since an automorphism in K preserves Γ_M and Γ_N, it is clear that $\sigma_{NS}(K) \subset G_{v_M}$.

We prove the converse, $\sigma_{NS}(K) \supset G_{v_M}$. First we note that p is surjective since σ_M and σ_N are both surjective as seen in the proof of Proposition 4.3. Thus it is enough to see $\operatorname{im} q \supset G_{v_M}$. By Theorem 4.2, we have an isomorphism

$$q_M \cong (q_N \oplus q_{NS}|\Gamma'^{\perp})/\Gamma' \tag{1}$$

where Γ' is an isotropic subgroup of $q_N \oplus q_{NS}$ which is a pushout of an isomorphism $\gamma' : \Gamma'_N \to \Gamma'_{NS}$ between subgroups of A_N and A_{NS} respectively. Using the same notation of Proposition 4.3, it is easy to see that $\Gamma' = \{0, (z_N, v_M)\}$.

Suppose we are given an element $\beta \in G_{v_M}$. Then the automorphism

$$(\mathrm{id}_N, \beta) \in O(q_N) \times O(q_{NS})$$

clearly preserves Γ' and induces an element $\alpha_M \in O(q_M)$ under the isomorphism (1). Since the element $(0, v_M)$ of the right-hand-side of (1) corresponds

to z_M, α_M preserves z_M. By construction $\alpha_M\gamma = \gamma\,\mathrm{id}_N$ holds and therefore $(\alpha_M, \mathrm{id}_N) \in H$. □

Proposition 4.7. *Let M_1 and M_2 be the fixed lattices of two free involutions i_{M_1}, i_{M_2} on X. Then they give the isomorphic quotients if and only if their patching elements coincide.*

Proof. Suppose $X/i_{M_1} \cong X/i_{M_2}$. There exists $\varphi \in \mathrm{Aut}(X)$ such that $\varphi(M_1) = M_2$. φ preserves the overlattice NS so that we have $\varphi(z_{M_1}) = z_{M_2}$ and $\sigma_{NS}(\varphi)(v_{M_1}) = v_{M_2}$. On the other hand, φ acts on A_{NS} trivially by the assumption of the theorem. Thus we see $v_{M_1} = v_{M_2}$.

Conversely assume the patching elements $v_M \in A_{NS}$ concide. By Proposition 4.3, the primitive embeddings M_1 and M_2 are isomorphic and there exists $\varphi \in O(NS)$ such that $\varphi(M_1) = M_2$, namely $\varphi i_{M_1}\varphi^{-1} = i_{M_2}$. By assumption, $\sigma_{NS}(\varphi)(v_M) = v_M$. We can assume $\varphi \in O^{\uparrow}(NS)$ by replacing φ by $-\varphi$ if necessary. According to Proposition 1.3 (2), $\varphi = w\psi$ where $w \in W_X, \psi \in O^+(NS)$. Then Lemma 1.4 (2) implies $\psi(M_1) = (M_2)$. On the other hand, since w acts on A_{NS} trivially, $\sigma_{NS}(\psi) = \sigma_{NS}(\varphi)$ and this element fixes v_M. We apply Theorem 2.3, Step 3 to $M = M_1, \psi_1 = \psi, \psi_2 = \mathrm{id}_{NS}$. Both ψ_1 and id stabilize v_M, therefore their images by σ_{NS} have the same class in $O(q_{NS})/\sigma_{NS}(K)$ by the previous lemma. Thus the conclusion holds. □

Next we compute the patching elements of involutions of Lieberman and Kondo-Mukai. They involve parameters as mentioned in the beginning of this section and we have to consider the dependence of patching elements on the parameters. This is directly done. We take the following basis of A_{NS}.

$e_1 = (G_{11} + G_{13} + G_{31} + G_{33})/2, \quad f_1 = (G_{22} + G_{23} + G_{32} + G_{33})/2,$

$e_2 = (G_{21} + G_{23} + G_{31} + G_{33})/2, \quad f_2 = (G_{12} + G_{13} + G_{32} + G_{33})/2.$

Then the result is as in Figures 3 and 4.

From these figures we can see which patching element we obtain when we choose a parameter of a geometrically constructed free involution. In Figure 4, we normalized the cases to only $i_4 = j_4 = 4$.

Since the vectors in Figures 3 and 4 run all over $A_{NS} - \{0\}$, we obtain

Proposition 4.8. *The two kinds of free involutions gives the all distinct Enriques quotients of X.*

The proof of Theorem 4.1 is completed.

Remark 4.9. (1) The involution τ acts trivially on $NS(X)$. So it induces a numerically trivial involution on the fifteen Enriques quotients. The Kondo-Mukai case of this is the last and missing result of [5], first found in [3].

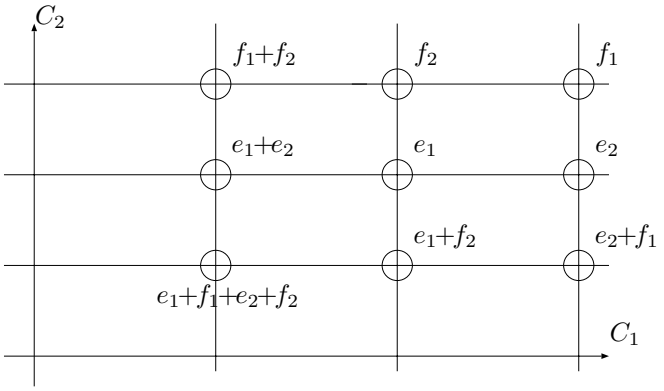

Figure 3. Lieberman involutions

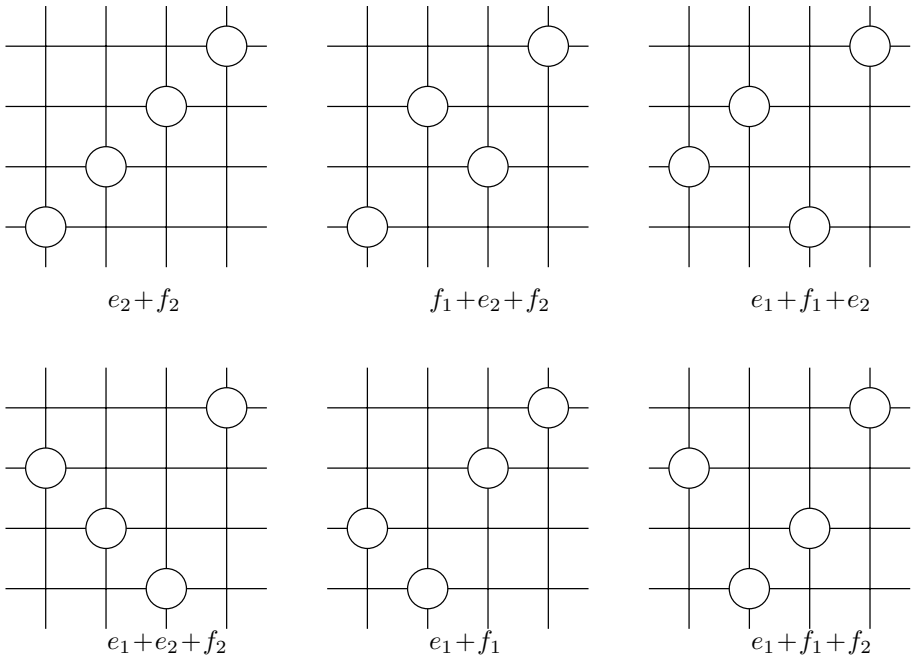

Figure 4. Kondo-Mukai involutions

(2) The number of Enriques quotients can be computed in other 2-elementary cases, using the argument of this section.

When $NS(X) \cong U(2) \oplus E_8^{\oplus 2}$, the Barth-Peters case, the number B_0 is equal to 1.

When $NS(X) \cong U(2) \oplus E_8(2)$, then X has only one Enriques quotient. Finally using the result of [5], we see that in other 2-elementary cases X has no Enriques quotients.

(3) The generators of the whole automorphism group $\mathrm{Aut}(X)$ are found in [2].

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