Further Examples of Explicit Krein Representations of Certain Subordinators

By

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Abstract

In a previous paper [1], we have shown that the gamma subordinators may be represented as inverse local times of certain diffusions. In the present paper, we give such representations for other subordinators whose Lévy densities are of the form $\frac{c}{(\sinh(y))^\gamma}$, $0 < \gamma < 2$, and the more general family obtained from those by exponential tilting. These densities are closely linked with those of the inverse local times of the squared radial Ornstein-Uhlenbeck processes.

§1. Aim of the Paper and Summary of [1]

§1.1

In this paper, we continue the program started in [1], that is to represent as many subordinators $(\mathcal{S}_\ell, \ell \geq 0)$, i.e. increasing Lévy processes started at 0, as possible as inverse local times $(\tau_\ell, \ell \geq 0)$ of some particular $\mathbb{R}_+$-valued diffusion (X_t), such that 0 is regular for itself, relatively to X. More precisely, assume that

$$E[\exp(-\lambda \mathcal{S}_\ell)] = \exp(-\ell \Psi(\lambda)), \quad \lambda, l \geq 0,$$

where $\Psi(\lambda) = \int_0^\infty \nu(dy)(1 - e^{-\lambda y})$. If the Lévy measure $\nu(dy)$ of $(\mathcal{S}_\ell, \ell \geq 0)$ is of the form

$$(1.1) \qquad \nu(dy) = h(y)dy, \quad \text{with} \quad h(y) = \int_0^\infty d\sigma(x)\, e^{-xy},$$

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for some positive σ-finite measure $\sigma(dx)$ on $\mathbb{R}_+$, then, it is known, as a consequence of Krein's theory (cf. Knight [5], Kotani-Watanabe [6]), that there exists a unique diffusion (X_t) taking values in $\mathbb{R}_+$ whose inverse local time $(\tau_\ell, \ell \geq 0)$ at 0 is distributed as $(\mathcal{S}_\ell, \ell \geq 0)$. Finding X when ν is given is called (here) *Krein representation problem*.

In our paper [1], we could fill in the following:

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Table 1

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$h(y)$	Generator of (X_t)	Distribution
(1) $\quad \frac{\mathcal{C}}{y^{\alpha+1}}$ $\quad\quad (0 < \alpha < 1)$	$\mathcal{L}_{-\alpha} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{\delta-1}{2x}\frac{d}{dx}\,; \quad \delta = 2(1-\alpha)$	P^δ
(2) $\quad \frac{\mathcal{C}}{y^{\alpha+1}}\, e^{-\mu y}$ $\quad (0 < \alpha < 1; \mu > 0)$	$\mathcal{L}_{-\alpha}^{\mu\downarrow} = \mathcal{L}_{-\alpha} + \sqrt{2\mu}\, \frac{\widehat{K}'_\alpha(\sqrt{2\mu}x)}{\widehat{K}_\alpha(\sqrt{2\mu}x)}\frac{d}{dx}$ where $\widehat{K}_\alpha(y) = y^\alpha K_\alpha(y),\, y > 0$	$P^{\delta;\mu\downarrow}$
(3) $\quad \frac{\mathcal{C}}{y}\, e^{-\mu y}$ $\quad\quad (\mu > 0)$	$\mathcal{L}_0^{\mu\downarrow} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{1}{2x} + \sqrt{2\mu}\frac{K'_0}{K_0}\left(\sqrt{2\mu}x\right)\right)\frac{d}{dx}$	$P^{2;\mu\downarrow}$

The measures σ_i $(i = 1, 2, 3)$ representing h in (1), (2), (3) via formula (1.1) are

$$\sigma_1(dx) = \frac{\mathcal{C}}{\Gamma(\alpha+1)}x^\alpha dx;\ \sigma_2(dx) = \frac{\mathcal{C}}{\Gamma(\alpha+1)}(x-\mu)_+^\alpha dx;\ \sigma_3(dx) = \mathcal{C}1_{(x\geq\mu)}dx.$$

Before we give some details about this Table, we explain our method of finding diffusions which solve some Krein representation problems.

§1.2   Krein problem and the construction of associated diffusions

Let $(X_t)_{t\geq 0}$ be an $\mathbb{R}_+$-valued diffusion, with family of laws $(P_x)_{x\in\mathbb{R}_+}$. We assume that 0 is regular, hence X admits a local time at 0, which we denote by $(L_t, t \geq 0)$. We use the notation

$$\tau_\ell = \inf\{t : L_t > \ell\}, \qquad \ell \geq 0,$$

for the corresponding inverse local time at 0, $\nu(dy)$ for its Lévy measure and $(\Psi(\theta), \theta \geq 0)$ for its Lévy exponent. We look for a new Markovian family $(\tilde{P}_x)_{x \in \mathbb{R}_+}$ such that

$$(1.2) \qquad \tilde{P}_0|_{\mathcal{F}_{\tau_l}} = \exp(\Psi(\theta)l - \theta\tau_l) \cdot P_0|_{\mathcal{F}_{\tau_l}}.$$

Since the process $(\exp(\Psi(\theta)L_t - \theta t), t \geq 0)$, which occurs in (1.2), is not a $(\mathcal{F}_t)$ martingale, it seems natural to look for a function $(\varphi(x); x \geq 0)$ such that

$$(1.3) \qquad \tilde{P}_x|_{\mathcal{F}_t} = \frac{\varphi(X_t)}{\varphi(x)} \exp(\Psi(\theta)L_t - \theta t) \cdot P_x|_{\mathcal{F}_t}.$$

In particular, if $T_0(X)$ denotes the first hitting time of 0 by X, then,

$$\frac{\varphi(X_{t \wedge T_0})}{\varphi(x)} \exp(-\theta(t \wedge T_0))$$

is a $(P_x, \mathcal{F}_t)$ martingale, and assuming φ to be decreasing, Doob's optional stopping theorem implies

$$\varphi(x) = \varphi_{\theta\downarrow}(x) := E_x[\exp(-\theta T_0(X))]$$

(we have normalized φ so that $\varphi(0) = 1$). Thus, we are led naturally to consider the diffusion, denoted by $(X_t^{\theta\downarrow}, t \geq 0)$, with laws $(P_x^{\theta\downarrow}, x \geq 0)$, such that

$$(1.4) \qquad P_x^{\theta\downarrow}|_{\mathcal{F}_t} = \frac{\varphi_{\theta\downarrow}(X_t)}{\varphi_{\theta\downarrow}(x)} \exp(\Psi(\theta)L_t - \theta t) \cdot P_x|_{\mathcal{F}_t}.$$

As follows clearly from the starting point (1.2) of our method, the inverse local time $(\tau_\ell, \ell \geq 0)$ of $(X_t^{\theta\downarrow}, t \geq 0)$ satisfies

$$E_0^{\theta\downarrow}(\exp(-\lambda\tau_\ell)) = \exp(-\ell(\Psi(\lambda + \theta) - \Psi(\theta))),$$

i.e. this inverse local time is the θ-Esscher transform of $(\tau_\ell, \ell \geq 0)$ under P_0, or equivalently its Lévy measure is $e^{-\theta y}\nu(dy)$.

It is also noteworthy that, under some adequate restriction of their domains, the infinitesimal generators $\mathcal{L}^{\theta\downarrow}$ and $\mathcal{L}$ are related by

$$\mathcal{L}^{\theta\downarrow} = \mathcal{L} + \frac{d}{dx}\left(\log(\varphi_{\theta\downarrow}(x))\right) \cdot \frac{d}{dx},$$

assuming for simplicity that

$$\mathcal{L} = \frac{1}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx}.$$

In the next section, we shall show, essentially with the help of the recipe (1.4), how to construct a diffusion, indexed by the three parameters (α, μ, k), which solves the *Krein representation problem for $\nu_{\mu, \alpha, k}$*.

§1.5

Among the new diffusions we are finding as solutions of Krein's problem, some are related by time change to the diffusions we found in [1] . We first discovered this relationship by applying the analytical identity

$$(1.7) \qquad W_{0, \beta}(z) = \sqrt{\frac{z}{\pi}} \, K_\beta \, \left(\frac{z}{2} \right)$$

between $W_{0, \beta}$, the Whittaker function with parameters $(0, \beta)$, and K_β, the Bessel-Macdonald function with parameter β (see the Appendix where we also refer to Lebedev [7] for definitions and properties of the special functions involved). Thus, a part of our present discussion may be considered as giving a probabilistic interpretation to (1.7).

We also develop a similar discussion for the analytical identity

$$M_{0, \beta}(z) = 4^\beta \Gamma(\beta + 1) \sqrt{z} \, I_\beta \left(\frac{z}{2} \right)$$

between the Whittaker function $M_{0,\beta}$ and the modified Bessel function I_β.

§2. Solving Krein's Problem for $\nu_{\mu, \alpha, k}$

§2.1

We take up the notation in Subsection 1.3; in fact, it is more convenient to consider the family of radial Ornstein-Uhlenbeck processes (and not their squares), which we shall denote as $(R^{\delta, \mu}(t), t \geq 0)$, and their laws $(P_r^{\delta, \mu}, r \geq 0)$. It will be helpful, for the sequel, to have the following formula at hand for the infinitesimal generator $\mathcal{L}_{-\alpha, \mu}$ of $R^{\delta, \mu}$:

$$(2.1) \qquad \mathcal{L}_{-\alpha, \mu} = \frac{1}{2} \, \frac{d^2}{dx^2} \, + \left(\frac{\delta - 1}{2x} - \mu x \right) \, \frac{d}{dx}.$$

It is well-known (see, e.g., Pitman-Yor [9], p. 454, formula (6.b)) that there is the relationship

$$(2.2) \qquad R^{\delta, \mu}(t) = e^{-\mu t} R^\delta \left(\frac{e^{2\mu t} - 1}{2\mu} \right), \qquad t \geq 0,$$

where, on the RHS, $(R^\delta(u), u \geq 0)$ denotes a δ-dimensional Bessel process. Thus, we obtain

$$(2.3) \qquad \left(\frac{e^{2\mu T_0} - 1}{2\mu} \, ; \, P_x^{\delta,\,\mu} \right) \underset{(a)}{\overset{(law)}{=}} (T_0 \, ; \, P_x^\delta) \underset{(b)}{\overset{(law)}{=}} \frac{x^2}{2\gamma_\alpha}$$

where, on the RHS, γ_α denotes a gamma variable with parameter $\alpha = 1 - \delta/2$. (a) follows from (2.2), while (b) is well-known and goes back to Getoor [3]. See, e.g., Yor [13], for some variants...

We thus deduce the following formula from (2.3) with the help of elementary computations: for $-2\alpha\mu < \theta$,

$$
\begin{aligned}
E_x^{\delta,\,\mu}&(\exp(-\theta T_0)) \\
&= \frac{1}{\Gamma(\alpha)(\mu x^2)^{\theta/2\mu}} \int_0^\infty \frac{t^{\alpha-1+\theta/2\mu} e^{-t}}{(1 + \frac{t}{\mu x^2})^{\theta/2\mu}} \, dt \\
(2.4) \qquad &= \frac{\Gamma(\alpha + \theta/2\mu)}{\Gamma(\alpha)} (\mu x^2)^{(\alpha-1)/2} e^{\mu x^2/2} W_{(1-\alpha-\theta/\mu)/2,\,\alpha/2}(\mu x^2),
\end{aligned}
$$

where $W_{a,\,b}$ denotes the Whittaker function with parameters (a, b). See [7, p. 279] for the integral representation of $W_{a,b}$.

We now write

$$\nu_{\mu,\,\alpha,\,k}(dy) = \mathcal{C} \left(\frac{\mu}{\sinh(\mu y)} \right)^{\alpha+1} \exp(\mu k y) dy$$

$$\equiv \mathcal{C} \left(\frac{\mu}{\sinh(\mu y)} \right)^{\alpha+1} \exp\left(\mu \frac{\delta}{2} y \right) \exp(-\theta y) dy \,,$$

where $\theta = \mu \left(\frac{\delta}{2} - k \right), \; k < 1 + \alpha$. According to the preceding computation (2.4), we now find that the diffusion with infinitesimal generator

$$(2.5)$$
$$\mathcal{L}_{-\alpha,\,\mu}^{\theta\downarrow} \equiv \mathcal{L}_{-\alpha,\,\mu} + \frac{d}{dx} \log \left\{ (\mu x^2)^{(\alpha-1)/2} e^{\mu x^2/2} W_{(1-\alpha-\theta/\mu)/2,\,\alpha/2}(\mu x^2) \right\} \cdot \frac{d}{dx}$$

solves Krein's representation problem for $\nu_{\mu,\,\alpha,\,k}$. We note in fact that $(1 - \alpha - \theta/\mu)/2 = k/2$, hence $W_{(1-\alpha-\theta/\mu)/2,\alpha/2}(\xi) \equiv W_{k/2,\,\alpha/2}(\xi)$.

The case where $k = 0$ is particularly interesting, since on one hand

$$\nu_{\mu,\alpha,0}(dy) = \mathcal{C} \left(\frac{\mu}{\sinh(\mu y)} \right)^{\alpha+1} dy$$

and on the other hand (see Appendix)

$$(2.6) \qquad\qquad W_{0,\alpha/2}(\xi) = \sqrt{\frac{\xi}{\pi}} \, K_{\alpha/2} \left(\frac{\xi}{2} \right),$$

so that the diffusion which solves Krein's representation problem for $\nu_{\mu,\alpha,0}(dy)$ is the solution to

$$(2.7) \qquad dX_t = dB_t + \left[\frac{\delta-1}{2X_t} + \mu X_t \left(\frac{\widehat{K}'_{\alpha/2}}{\widehat{K}_{\alpha/2}} \right) \left(\mu \frac{X_t^2}{2} \right) \right] dt.$$

Here, we need to give some details about this computation:

a) We deduce from formula (2.5), in the particular case $k = 0$, i.e. $\theta = \delta\mu/2$ that

$$\mathcal{L}_{-\alpha,\mu}^{\theta\downarrow} = \mathcal{L}_{-\alpha,\mu} + \left(\frac{\alpha}{x} + \mu x + \mu x \frac{K'_{\alpha/2}}{K_{\alpha/2}} \left(\mu \frac{x^2}{2} \right) \right) \cdot \frac{d}{dx}$$

with the help of formula (2.6).

b) Now, trivially

$$\mathcal{L}_{-\alpha,\mu} + \mu x \frac{d}{dx} = \mathcal{L}_{-\alpha}$$

and

$$\frac{\alpha}{x} + \mu x \frac{K'_{\alpha/2}}{K_{\alpha/2}} \left(\mu \frac{x^2}{2} \right) = \mu x \frac{\widehat{K}'_{\alpha/2}}{\widehat{K}_{\alpha/2}} \left(\mu \frac{x^2}{2} \right),$$

which translates into the stochastic differential equation form (2.7).

To summarize the above and as an introduction to the next discussion in Subsection 2.2, we write down

Table 2

$h(y)$	Generator of (X_t)	Distribution
$\frac{C}{y^{\alpha+1}}\, e^{-\mu y}$ $(0 < \alpha < 1; \mu > 0)$	$\mathcal{L}^{\mu\downarrow}_{-\alpha} = \mathcal{L}_{-\alpha} + \sqrt{2\mu}\, \frac{\widehat{K}'_{\alpha}}{\widehat{K}_{\alpha}}(\sqrt{2\mu}x)\frac{d}{dx}$	$P^{\delta;\mu\downarrow}$ $(\delta = 2(1-\alpha))$
$C\left(\frac{\mu}{\sinh(\mu y)}\right)^{\alpha+1} e^{\mu\frac{\delta}{2}y}$ $(0 < \delta = 2(1-\alpha) < 2; \mu > 0)$	$\mathcal{L}_{-\alpha,\,\mu} \equiv \mathcal{L}_{-\alpha} - \mu x \frac{d}{dx}$	$P^{\delta,\mu}$
$C\left(\frac{\mu}{\sinh(\mu y)}\right)^{\alpha+1}$ $(0 < \alpha < 1; \mu > 0)$	$\mathcal{L}^{\theta\downarrow}_{-\alpha,\,\mu} \equiv \mathcal{L}_{-\alpha} + \mu x \frac{\widehat{K}'_{\alpha/2}}{\widehat{K}_{\alpha/2}}\left(\mu\frac{x^2}{2}\right)\frac{d}{dx}$ $\left(\theta = \frac{\delta\mu}{2}\right)$	$P^{\delta,\mu;\frac{\delta\mu}{2}\downarrow}$

The first row is simply taken from Table 1 (second row there). As said above, the second row here follows from Pitman-Yor [11]. In the third row, we have written $\mathcal{L}^{\theta\downarrow}_{-\alpha,\,\mu}$ for the infinitesimal generator of the process which is defined as the radial Ornstein-Uhlenbeck process with dimension $\delta = 2(1-\alpha)$ and drift parameter $(-\mu)$, pushed downwards with parameter $\theta = \delta\mu/2$. That this infinitesimal generator may be expressed in terms of $\widehat{K}_{\alpha/2}$ has already been discussed after (2.7). Another argument is given in the next Subsection 2.2.

§2.2

We shall now prove a remarkable relationship between the two families of diffusions whose infinitesimal generators are found in Table 2. This relationship explains precisely why (Row 1) may be deduced from (Row 2), and vice-versa.

Proposition 2.1. *Let $0 < \alpha < 1$ and $\delta = 2(1-\alpha)$. The following relationship holds with $\theta = \delta\mu/2$,*

$$(2.8) \qquad X^2_{-\alpha,\,\mu;\,\theta\downarrow}(t) = X_{-\frac{\alpha}{2};\,\frac{\mu^2}{8}\downarrow}\left(4\int_0^t X^2_{-\alpha,\,\mu;\,\theta\downarrow}(u)\,du\right),$$

meaning that, starting from $X \equiv X_{-\alpha,\,\mu;\,\theta\downarrow}$ *on the LHS, there exists* $\left(X_{-\frac{\alpha}{2},\,\frac{\mu^2}{8}\downarrow}(u),\, u \geq 0 \right)$ *such that the relationship* (2.8) *holds.*

Comment about our notation. In formula (2.8), and possibly several times below, we have written $X_{i;\,\theta\downarrow}$, etc... instead of $X_i^{\theta\downarrow}$, for some index i. It seemed more appropriate here, because of the power 2 which occurs on both sides of (2.8). There should be no confusion for the reader between the different diffusions $X_{i,\,\theta}$ and $X_{i;\,\theta\downarrow}$.

Proof. We start from the stochastic differential equation satisfied by $(X_{-\alpha,\,\mu;\,\theta\downarrow}(t), t \geq 0)$ as described implicitly in Row 2 of Table 2. Taking squares, we obtain

$$X_t^2 = x^2 + 2 \int_0^t X_s dB_s + \delta t + 2 \int_0^t (\mu X_s^2) \frac{\widehat{K}'_{\alpha/2}}{\widehat{K}_{\alpha/2}} \left(\mu \frac{X_s^2}{2} \right)\, ds.$$

We now define $(Y_u \equiv Y(u),\, u \geq 0)$ via

$$X_t^2 = Y\left(4 \int_0^t X_s^2 ds \right), \quad t \geq 0,$$

and find that Y satisfies

$$Y_u = x^2 + \beta_u + \left(\frac{\delta}{4} \right) \int_0^u \frac{ds}{Y_s} + \frac{1}{2} \int_0^u \mu\, \frac{\widehat{K}'_{\alpha/2}}{\widehat{K}_{\alpha/2}} \left(\frac{\mu}{2} Y_s \right)\, ds$$

for some Brownian motion (β_u). Since $\delta/4 = (\widehat{\delta} - 1)/2$, with $\widehat{\delta} = 2 - \alpha = 2(1 - \alpha/2)$, it now appears that $(Y_u,\, u \geq 0)$ is precisely the diffusion with infinitesimal generator $\mathcal{L}^{\nu\downarrow}_{-\alpha/2}$, with $\sqrt{2\nu} = \mu/2$, i.e. $\nu = \mu^2/8$. $\square$

Let us note that the proof we have just given for the Proposition relies upon the identification of the infinitesimal generator of the diffusion $X_{-\alpha,\,\mu;\,(\delta\mu/2)\downarrow}$ as given in Table 2. This identification was obtained from an analytical identity between $W_{0,\,\boldsymbol{\cdot}}$ and $K_{\boldsymbol{\cdot}}$. (see formula (2.6)).

We now explain and prove the Proposition without relying on such identities, but rather on absolute continuity relationships between the different laws involved. We now find it a little more convenient to refer to the laws $\{Q_z^{\delta,\,\mu}\}$ and the main absolute continuity result we need is

(2.9) $$Q_z^{\delta,\,\mu}|_{\mathcal{F}_t} = \exp\left(-\frac{\mu}{2}(Z_t - \delta t - z) - \frac{\mu^2}{2} \int_0^t Z_s ds \right) . \, Q_z^{\delta}|_{\mathcal{F}_t}.$$

Here, $(Z_t, t \geq 0)$ denotes the coordinate process on the canonical space $C(\mathbb{R}_+, \mathbb{R}_+)$. We now combine this relation (2.9) with that of the "push downwards"

with parameter θ, so that, with notations which we shall explain after writing the formula

$$(2.10) \quad Q_z^{\delta,\,\mu;\,\theta\downarrow}|_{\mathcal{F}_t} = \frac{\varphi_{\theta\downarrow}(Z_t)}{\varphi_{\theta\downarrow}(z)} \exp(\Psi(\theta)L_t - \theta t) \,\cdots$$
$$\exp\left(-\frac{\mu}{2}(Z_t - \delta t - z) - \frac{\mu^2}{2}\int_0^t ds Z_s\right) \cdot Q_z^{\delta}|_{\mathcal{F}_t},$$

where we note that, for $\theta = \delta\mu/2$, this relation simplifies as
$$(2.11)$$
$$Q_z^{\delta,\,\mu;\,\frac{\delta\mu}{2}\downarrow}|_{\mathcal{F}_t} = \frac{\exp\left(-\frac{\mu}{2}Z_t\right)\varphi_{\theta\downarrow}(Z_t)}{\exp\left(-\frac{\mu}{2}z\right)\varphi_{\theta\downarrow}(z)} \exp\left(\Psi(\theta)L_t - \frac{\mu^2}{2}\int_0^t ds Z_s\right) \cdot Q_z^{\delta}|_{\mathcal{F}_t}.$$

The due explanation of formula (2.10) is that we have combined the "push-downwards" formula (1.4), relative to $\{Q_z^{\delta,\,\mu}\}$, i.e. the function $\varphi_{\theta\downarrow}$ and $\Psi(\theta)$ are relative to that diffusion, with the preceding formula (2.9). From now on, we keep $\theta = \delta\mu/2$.

We now consider what becomes of formula (2.11), once we time change both hand sides with the inverse of $\left(4\int_0^t Z_u du, \, t \geq 0\right)$, so that, by a slight abuse of notation, the process of reference is now $(\widehat{Z}(h), \, h \geq 0)$, with $\widehat{Z}$ defined by

$$(2.12) \qquad Z_t = \widehat{Z}\left(4\int_0^t Z_u du\right).$$

Thus, we obtain

$$(2.13) \quad \widehat{Q}_z^{\delta,\,\mu;\,\frac{\delta\mu}{2}\downarrow}|_{\widehat{\mathcal{F}}_u} = \frac{\exp\left(-\frac{\mu}{2}\widehat{Z}_u\right)\varphi_{\theta\downarrow}(\widehat{Z}_u)}{\left(e^{-\frac{\mu}{2}z}\varphi_{\theta\downarrow}(z)\right)} \exp\left(\Psi(\theta)\widehat{L}_u - \frac{\mu^2}{8}u\right) \cdot \widehat{Q}_z^{\delta}|_{\widehat{\mathcal{F}}_u}.$$

From the well-known property of time change for Bessel processes (see [12], Chapter XI, Proposition 1.11), $\widehat{Q}_z^{\delta}$ is the distribution of a Bessel process of index $\alpha/2$, i.e. of dimension $\hat{\delta} = 2 - \alpha$, that is $\widehat{Q}_z^{\delta} = P_z^{\hat{\delta}}$. Again, with obvious notation, the RHS of (2.13) may be written

$$\frac{\widehat{\varphi}_{\frac{\mu^2}{8}\downarrow}(\widehat{Z}_u)}{\widehat{\varphi}_{\frac{\mu^2}{8}\downarrow}(z)} \exp\left(\widehat{\Psi}\left(\frac{\mu^2}{8}\right)\widehat{L}_u - \frac{\mu^2 u}{8}\right) \cdot \widehat{Q}_z^{\delta}|_{\widehat{\mathcal{F}}_u},$$

and we discover that
$$(2.14) \qquad \begin{cases} \widehat{\varphi}_{\frac{\mu^2}{8}\downarrow}(z) = e^{-\frac{\mu}{2}z}\varphi_{\theta\downarrow}(z) \\ \widehat{\Psi}\left(\frac{\mu^2}{8}\right) = \Psi(\theta) \end{cases}$$

and $\widehat{Q}_z^{\delta,\,\mu;\,\frac{\delta\mu}{2}\downarrow} = P_z^{\hat{\delta};\,\frac{\mu^2}{8}\downarrow}$.

Again, let us explain, very much in the same spirit, the first relation

$$\widehat{\varphi}_{\frac{\mu^2}{8}\downarrow}(z) = e^{-\frac{\mu}{2}z}\varphi_{\theta\downarrow}(z)$$

in (2.14). This translates as

$$(2.15) \qquad \widehat{E}_z\left(\exp\left(-\frac{\mu^2}{8}T_0(\widehat{Z})\right)\right) = e^{-\frac{\mu}{2}z}E_z\left(\exp\left(-\frac{\delta\mu}{2}T_0(Z)\right)\right),$$

where $\widehat{Z}$ simply denotes a BES process with dimension $\widehat{\delta}$, and Z a process with law $Q_z^{\delta,\,\mu}$. This may be well understood by considering the absolute continuity relationship (2.9), when we replace t by $T_0(Z)$. Then, from that relationship, it follows that

$$Q_z^{\delta,-\mu}\left(\exp\left(-\frac{\delta\mu}{2}T_0(Z)\right)\right) = e^{\frac{\mu z}{2}}Q_z^{\delta}\left(e^{-\frac{\mu^2}{2}\int_0^{T_0} ds\, Z_s}\right)$$

$$= e^{\frac{\mu z}{2}}P_z^{\widehat{\delta}}\left(e^{-\frac{\mu^2}{8}T_0(\widehat{Z})}\right),$$

which is precisely (2.15).

Now, it is well known (see [3], [4], [10]) that the Laplace transform of T_0, under the distribution $P_z^{\hat{\delta}}$ of a Bessel process, is given by

$$(2.16)\ \ \widehat{\varphi}_{\frac{\mu^2}{8}\downarrow}(z) := E_z^{\hat{\delta}}\left(\exp(-\frac{\mu^2}{8}T_0)\right) = 2^{1-\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)^{-1}\left(\frac{\mu z}{2}\right)^{\alpha/2}K_{\alpha/2}\left(\frac{\mu z}{2}\right).$$

Using (2.14), we can recover the expression of $\varphi_{\theta\downarrow}$ for $\theta = \delta\mu/2$ obtained in (??) using the identity (2.6).

§2.3

We now develop a discussion similar to the one which led to (2.8), but with the downwards arrows $\downarrow$ now changed into upwards arrows $\uparrow$; for the definition of these pushed upwards and downwards processes obtained from a diffusion, see Pitman-Yor [10]. The analogue of formula (2.10) is now

$$(2.17) \quad Q_z^{\delta,\,\mu;\,\theta\uparrow}|_{\mathcal{F}_t}$$

$$= \frac{\varphi_{\theta\uparrow}(Z_t)}{\varphi_{\theta\uparrow}(z)}\exp(-\theta t)\exp\left(-\frac{\mu}{2}(Z_t - \delta t - z) - \frac{\mu^2}{2}\int_0^t ds\, Z_s\right)\cdot Q_z^{\delta}|_{\mathcal{F}_t}$$

and we note again that, for $\theta = \delta\mu/2$, this relation simplifies as

$$(2.18) \qquad Q_z^{\delta,\,\mu;\,\frac{\delta\mu}{2}\uparrow}|_{\mathcal{F}_t} = \frac{\exp\left(-\frac{\mu}{2}Z_t\right)\varphi_{\theta\uparrow}(Z_t)}{\exp\left(-\frac{\mu}{2}z\right)\varphi_{\theta\uparrow}(z)}\exp\left(-\frac{\mu^2}{2}\int_0^t ds\, Z_s\right)\cdot Q_z^{\delta}|_{\mathcal{F}_t}.$$

We note that this formula is even simpler than (2.11), since here there is no local time contribution.

We now continue to develop an analogous discussion to that made in Subsection 2.2, formula (2.8). Thus, we time-change both sides of the absolute continuity relation (2.18) with the inverse of $(4 \int_0^t Z_u \, du, \ t \geq 0)$. Thus, with $\widehat{Z}$, as defined from Z in (2.12), we obtain

$$\widehat{Q}_z^{\delta, \, \mu; \, \frac{\delta\mu}{2}\uparrow}|_{\widehat{\mathcal{F}}_u} = \frac{\exp\left(-\frac{\mu}{2}\widehat{Z}_u\right) \varphi_{\theta\uparrow}(\widehat{Z}_u)}{\exp\left(-\frac{\mu}{2}z\right) \varphi_{\theta\uparrow}(z)} \, \exp\left(-\frac{\mu^2 u}{8}\right) \cdot \widehat{Q}_z^{\delta}|_{\widehat{\mathcal{F}}_u}.$$

With obvious notation, this RHS may be written

$$\frac{\widehat{\varphi}_{\frac{\mu^2}{8}\uparrow}(\widehat{Z}_u)}{\widehat{\varphi}_{\frac{\mu^2}{8}\uparrow}(z)} \, \exp\left(-\frac{\mu^2 u}{8}\right) \cdot \widehat{Q}_z^{\delta}|_{\widehat{\mathcal{F}}_u}$$

with

(2.19) $$\widehat{\varphi}_{\frac{\mu^2}{8}\uparrow}(z) = e^{-\frac{\mu}{2}z}\varphi_{\theta\uparrow}(z).$$

$\widehat{Q}_z^{\delta, \, \mu; \, \frac{\delta\mu}{2}\uparrow}$ is the distribution of a Bessel process of dimension $\hat{\delta} = 2 - \alpha$ with drift $\frac{\mu^2}{8}$, i.e.

$$\widehat{Q}_z^{\delta, \, \mu; \, \frac{\delta\mu}{2}\uparrow} = P_z^{\hat{\delta}; \frac{\mu^2}{8}\uparrow}.$$

The analytical counterpart of (2.19) is the companion formula of (2.6) (see Appendix)

$$M_{0, -\alpha/2}(\xi) = 4^{-\frac{\alpha}{2}}\Gamma(1 - \frac{\alpha}{2})\sqrt{\xi} \, I_{-\alpha/2}\left(\frac{\xi}{2}\right),$$

while the companion formula of (2.16) is

$$\widehat{\varphi}_{\frac{\mu^2}{8}\uparrow}(z) := \frac{1}{E_0^{\hat{\delta}}\left(\exp(-\frac{\mu^2}{8}T_z)\right)} = 2^{-\frac{\alpha}{2}}\Gamma(1 - \frac{\alpha}{2})\left(\frac{\mu z}{2}\right)^{\frac{\alpha}{2}} I_{-\alpha/2}\left(\frac{\mu z}{2}\right),$$

a well-known formula which goes back to Kent [4], Pitman-Yor [10].

§3.   Appendix : On the Whittaker and Bessel-Macdonald Functions

The following formulae involving these classical special functions are found in Lebedev [7], to which we refer with numberings such as : (N_*) ...

a) The Whittaker functions $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ are a pair of solutions of Whittaker's equation

$$u'' + \left(-\frac{1}{4} + \frac{k}{z} + \frac{(\frac{1}{4} - \mu^2)}{z^2}\right) u = 0$$

(p. 279$_*$).

b) $W_{k,\mu}$ admits the integral representation

$$W_{k,\mu}(z) = \frac{z^k\, e^{-\frac{z}{2}}}{\Gamma(\mu - k + \frac{1}{2})} \int_0^\infty e^{-t}\, t^{\mu - k - \frac{1}{2}} \left(1 + \frac{t}{z}\right)^{\mu - k + \frac{1}{2}} dt$$

(see Problem 17$_*$, p. 279$_*$).

c)
$$W_{0,\mu}(z) = \sqrt{\frac{z}{\pi}}\, K_\mu\left(\frac{z}{2}\right)$$

(see Problem 19$_*$, p. 279$_*$).

d) In terms of the confluent hypergeometric functions Ψ, there are the relations

$$W_{k,\mu}(z) = z^{\mu + \frac{1}{2}}\, e^{-\frac{z}{2}} \Psi\left(\frac{1}{2} - k + \mu,\ 2\mu + 1;\ z\right)$$

(see (9.13.16)$_*$, p. 274$_*$),

$$K_\mu(z) = \sqrt{\pi}(2z)^\mu\, e^{-z} \Psi\left(\mu + \frac{1}{2},\ 2\mu + 1;\ 2z\right)$$

(see (9.13.15)$_*$, p. 274$_*$).
Thus, taking $k = 0$ in the above formula for $W_{k,\mu}$, one recovers c).

e) In terms of the confluent hypergeometric functions Φ, there are the relations

$$M_{k,\mu}(z) = z^{\mu + \frac{1}{2}}\, e^{-\frac{z}{2}} \Phi\left(\frac{1}{2} - k + \mu,\ 2\mu + 1;\ z\right)$$

(see (9.13.16)$_*$, p. 274$_*$),

$$I_\mu(z) = \frac{(z/2)^\mu}{\Gamma(\mu + 1)}\, e^{-z} \Phi\left(\mu + \frac{1}{2},\ 2\mu + 1;\ 2z\right)$$

(see (9.13.14)$_*$, p. 274$_*$).
Thus, taking $k = 0$ in the above formula for $M_{k,\mu}$, we obtain

$$M_{0,\mu}(z) = 4^\mu \Gamma(\mu + 1)\sqrt{z}\, I_\mu\left(\frac{z}{2}\right).$$

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