Structure of Quasi-analytic Ultradistributions

Dedicated to Professor Akira Kaneko on his sixtieth birthday

By

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Abstract

We study the structure of functions between distributions and hyperfunctions. The structure theorem is known for distributions, non-quasi-analytic ultradistributions and hyperfunctions. In this paper, we try to fill the gap among them. We prove the structure theorem for quasi-analytic ultradistributions.

§1.   Introduction

In this paper, we discuss the structure of generalized functions. It is well-known that any distribution f is locally represented as $f = P(D)g$, where $P(D)$ is a finite order differential operator with constant coefficients and g is a continuous function, which is the structure theorem for distributions. The structure theorems for non-quasi-analytic ultradistributions ([1, 5]) and hyperfunctions ([3]) are also known. In this paper, we study the structure of functions between them, namely, the structure of quasi-analytic ultradistributions. We prove the structure theorem for non-analytic ultradistributions which includes both non-quasi-analytic and quasi-analytic ones. It is our main theorem to prove that any non-analytic ultradistribution f of the class $*$ is locally represented as $f = P(D)g$, where $P(D)$ is an ultradifferential operator of the class $*$ and g is an ultradifferentiable function of the class $\dagger > *$. We also claim that this

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ultradifferentiable function g can be taken from any class † satisfying † $> *$. Our main theorem gives the structure theorem for quasi-analytic ultradistributions and the proof of our main theorem gives another proof of the structure theorem for non-quasi-analytic ultradistributions. In the proof of our main theorem, it is essentially important to construct ultradifferential operators of the given non-analytic class. In [5], H. Komatsu applied an infinite product $P(\xi) := \prod_{p=1}^{\infty} \left(1 + \frac{l_p^2 \xi^2}{m_p^2}\right)$, where $m_p := \frac{M_p}{M_{p-1}}$, $\xi^2 := \xi_1^2 + \cdots + \xi_n^2$ and l_p is some sequence of positive numbers, to construct the symbol of an ultradifferential operator in the given non-quasi-analytic class, which does not converge in quasi-analytic classes. On the other hand, it is not easy to modify A. Kaneko's method in [3] to construct an ultradifferential operator suitable for our purpose, since the class of non-analyticity is strictly given in our theory. Therefore we apply our original method to construct the symbols of ultradifferential operators. Before proving our main theorem, we prepare some elementary properties of quasi-analytic ultradistributions.

§2.   Ultradistributions

In this section, we review the definition of ultradistributions. Let $\Omega \subset \mathbb{R}^n$ be an open subset and M_p, $p = 0, 1, \ldots$, be a sequence of positive numbers. For non-quasi-analytic classes, we impose the following conditions on M_p.
$(M.0)$ (normalization)
$$M_0 = M_1 = 1.$$

$(M.1)$ (logarithmic convexity)

$$M_p^2 \le M_{p-1} M_{p+1}, \quad p = 1, 2, \ldots .$$

$(M.2)$ (stability under ultradifferential operators)

$$\exists G, \quad \exists H \text{ such that } M_p \le GH^p \min_{0 \le q \le p} M_q M_{p-q}, \quad p = 0, 1, \ldots .$$

$(M.3)$ (strong non-quasi-analyticity)

$$\exists G \text{ such that } \sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \le Gp \frac{M_p}{M_{p+1}}, \quad p = 1, 2, \ldots .$$

$(M.2)$ and $(M.3)$ are often replaced by the following weaker conditions respectively;
$(M.2)'$ (stability under differential operators)

$$\exists G, \quad \exists H \text{ such that } M_{p+1} \le GH^p M_p, \quad p = 0, 1, \ldots .$$

$(M.3)'$ (non-quasi-analyticity)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$$

For two sequences M_p and N_p of positive numbers we define their *orders*.

Definition 2.1. Let M_p and N_p be the sequences of positive numbers.

(i) $M_p \subset N_p$ if and only if there exist such constants $L > 0$ and $C > 0$ that $M_p \leq CL^p N_p$ for any p.

(ii) $M_p \prec N_p$ if and only if for any $L > 0$ there exists such a constant $C > 0$ that $M_p \leq CL^p N_p$ for any p.

In order to define quasi-analytic classes, we impose the following conditions, (QA) and (NA), instead of $(M.3)$ or $(M.3)'$.
(QA) (quasi-analyticity)

$$p! \subset M_p, \quad \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} = \infty.$$

Let M_p be a sequence of positive numbers satisfying (QA). If

$$\liminf_{p \to \infty} \sqrt[p]{\frac{p!}{M_p}} > 0$$

then $\mathcal{E}^{\{M_p\}}$ is the class of analytic functions. We impose the condition that $\{M_p\}$ would not define the analytic class, namely,
(NA) (non-analyticity)

$$\lim_{p \to \infty} \sqrt[p]{\frac{p!}{M_p}} = 0.$$

Definition 2.2. Let M_p be a sequence of positive numbers and $\Omega \subset \mathbb{R}^n$ be an open subset. A function $f \in \mathcal{E}(\Omega) = C^{\infty}(\Omega)$ is called an *ultradifferentiable function* of the class (M_p) (resp. $\{M_p\}$) if and only if for any compact subset $K \subset \Omega$ and for any $h > 0$ there exists such a constant C (resp. for any compact subset $K \subset \Omega$ there exist such constants h and C) that

(1) $$\sup_{x \in K} |D^\alpha \varphi(x)| \leq Ch^{|\alpha|} M_{|\alpha|} \quad \text{for all } \alpha$$

holds. Denote the set of the ultradifferentiable functions of the class (M_p) (resp. $\{M_p\}$) on Ω by $\mathcal{E}^{(M_p)}(\Omega)$ (resp. $\mathcal{E}^{\{M_p\}}(\Omega)$) and denote by $\mathcal{D}^*(\Omega)$ the set of all functions in $\mathcal{E}^*(\Omega)$ with their supports compact in Ω, where $* = (M_p)$ or $\{M_p\}$.

Let $K \subset \mathbb{R}^n$ be a compact set, M_p satisfy $(M.1)$ and (NA). Denote by $\mathcal{E}^*[K]$ the set of the ultradifferentiable functions of the class $* = (M_p)$ or $\{M_p\}$ defined on some neighborhood of K. We define $\varphi \in \mathcal{E}^{\{M_p\},h}[K]$ if and only if $\varphi \in \mathcal{E}^{\{M_p\}}[K]$ and (1) holds for given $h > 0$.

For M_p satisfying $(M.3)'$ and a compact subset $K \subset \Omega$ let

$$(2) \qquad\qquad \mathcal{D}_K^* = \{\varphi \in \mathcal{D}^*(\mathbb{R}^n) \; ; \; \operatorname{supp} f \subset K\},$$

where $* = (M_p)$ or $\{M_p\}$ and we define

$$(3) \quad \mathcal{D}_K^{\{M_p\},h} = \{\varphi \in \mathcal{D}_K^{\{M_p\}} \; ; \; \exists C \text{ such that } \sup_{x \in K} |D^\alpha \varphi(x)| \le C h^{|\alpha|} M_{|\alpha|}\}.$$

Let M_p satisfy $(M.1)$ and $(M.3)'$. We define $\mathcal{D}^{*'}(\Omega)$ as the strong dual of $\mathcal{D}^*(\Omega)$ for any open set Ω and call it *the set of ultradistributions of the class $*$* defined on Ω. These spaces are endowed with natural structure of locally convex spaces.

For non-quasi-analytic ultradifferentiable functions and non-quasi-analytic ultradistributions confer [5] and [6].

Definition 2.3. Let $K \subset \mathbb{R}^n$ be a compact set, M_p satisfy $(M.1)$ and (NA). For $f \in \mathcal{E}^{\{M_p\},h}[K]$ we define its norm by

$$(4) \qquad\qquad \|f\|_{\mathcal{E}^{\{M_p\},h}[K]} := \sup_{x \in K, \alpha} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}}.$$

Let Ω be an open set and K be a compact set. Topologies of ultradifferentiable classes are defined as follows.

$$(5) \quad
\begin{aligned}
\mathcal{E}^{\{M_p\}}[K] &= \varinjlim_{h \to \infty} \mathcal{E}^{\{M_p\},h}[K], \\
\mathcal{E}^{\{M_p\}}(\Omega) &= \varprojlim_{K \Subset \Omega} \mathcal{E}^{\{M_p\}}[K], \\
\mathcal{E}^{(M_p)}[K] &= \varprojlim_{h \to 0} \mathcal{E}^{\{M_p\},h}[K], \\
\mathcal{E}^{(M_p)}(\Omega) &= \varprojlim_{K \Subset \Omega} \mathcal{E}^{(M_p)}[K].
\end{aligned}$$

We define $\mathcal{E}_K^{*\,\prime}$ as the strong dual of $\mathcal{E}^*[K]$ and call it *the set of ultradistributions of the class $*$ supported by K*. We also define $\mathcal{E}^{*\prime}(\Omega) := \cup_{K \subset \Omega} \mathcal{E}_K^{*\,\prime}$.

Let us define the sheaf of ultradistributions.

Definition 2.4. Let a sequence M_p of positive numbers satisfy $(M.0)$, $(M.1)$, $(M.2)'$, and

$$(6) \qquad \limsup_{p \to \infty} \sqrt[p]{\frac{p!}{M_p}} < \infty.$$

We define a presheaf F^* on $\mathbb{R}^n$ by

$$(7) \qquad F^*(\Omega) := \mathcal{E}^{*'}(\mathbb{R}^n)/\mathcal{E}^{*'}(\mathbb{R}^n \setminus \Omega),$$

where Ω is any open set in $\mathbb{R}^n$ and $* = (M_p)$ or $\{M_p\}$. We denote the corresponding sheaf by $\mathcal{F}^*$. If M_p satisfies $(M.3)'$, $\mathcal{F}^* = \mathcal{D}^{*'}$. If M_p satisfies (QA) and (NA), then we call $\mathcal{F}^*$ *the sheaf of quasi-analytic ultradistributions of the class* $*$.

Definition 2.5. For two classes $*$ and $\dagger$ we define their inclusion relations.

$$(8) \qquad \begin{aligned} \dagger \leq * &\Leftrightarrow \mathcal{E}^\dagger \subset \mathcal{E}^*, \\ \dagger < * &\Leftrightarrow \mathcal{E}^\dagger \subsetneq \mathcal{E}^*. \end{aligned}$$

Definition 2.6. A differential operator $P(D) := \sum_\alpha a_\alpha D^\alpha$ of infinite order is defined to belong to the class (M_p) (resp. $\{M_p\}$), if and only if there exist such constants L and C (resp. for any $L > 0$ there exists such a constant C) that $|a_\alpha| \leq (CL^{|\alpha|})/M_{|\alpha|}$ holds for any α. We call this operator *an ultradifferential operator* of the class (M_p) (resp. $\{M_p\}$).

Definition 2.7. For a positive sequence M_p satisfying (NA), define its *associated function* by

$$(9) \qquad \widetilde{M}(t) := \sup_p \frac{t^p}{M_p},$$

for $t > 0$.

§3. Known Results

In this section, we review the known results on the structure theorems. The structure theorem for distributions was proved by L.Schwartz.

Theorem 3.1 (cf. [7]). *Any distribution f is locally represented as*

$$(10) \qquad f = P(D)g,$$

where $P(D)$ is a differential operator of finite order with constant coefficients and g is a continuous function.

Extensions of this theorem for non-quasi-analytic ultradistributions and for hyperfunctions are known. H. Komatsu [5] proved the structure theorem for strongly non-quasi-analytic ultradistributions.

Theorem 3.2 (cf. [5]).   *Let the sequence M_p satisfy the conditions, $(M.1)$, $(M.2)$ and $(M.3)$. $f \in \mathcal{D}^{*\prime}$, where $*$ is (M_p) or $\{M_p\}$, is locally represented in the form* (10), *where $P(D)$ is an ultradifferential operator of the class $*$ with constant coefficients and g is a continuous function.*

This theorem was extended by R. W. Braun [1] for non-quasi-analytic ultradistributions.

Theorem 3.3 (cf. [1]).   *Let the sequence M_p satisfy the conditions, $(M.1)$, $(M.2)$ and $(M.3)'$. For $f \in \mathcal{D}^{*\prime}$, where $*$ is (M_p) or $\{M_p\}$, and for any class $\dagger$ satisfying $* < \dagger$, there exist an ultradifferential operator $P(D)$ of the class $*$ with constant coefficients and an ultradifferential function g of the class $\dagger$ such that the representation* (10) *locally holds.*

In [3], A. Kaneko proved the structure theorem for hyperfunctions.

Theorem 3.4 (cf. [3]).   *Any hyperfunction f is locally represented as*

$$(11) \qquad\qquad\qquad f = J(D)g,$$

where $J(D)$ is a local operator with constant coefficients, that is, $J(D)$ is an infinite order differential operator $J(D) = \sum_\alpha a_\alpha D^\alpha$ with the coefficients satisfying $\lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{|a_\alpha| \alpha!} = 0$, and g is an infinitely differentiable function.

The structure theorem for quasi-analytic ultradistributions is left open, to prove which is our main purpose in this paper.

§4. Fourier Transform of Non-analytic Functions and Non-analytic Ultradistributions

In this section, we study the Fourier transform of non-analytic functions and non-analytic ultradistributions. The properties proved in this section take important roles to prove our main theorem. For a function f defined on $\mathbb{R}^n$, we define its Fourier-Laplace transform $\widehat{f}(\zeta)$, $\zeta \in \mathbb{C}^n$ by

$$(12) \qquad\qquad \widehat{f}(\zeta) := \int_{\mathbb{R}^n} e^{-ix \cdot \zeta} f(x) dx$$

when it is well-defined.

Definition 4.1.   Let M_p be a sequence of positive numbers. A function $f \in \mathcal{E}^{(M_p)}(\mathbb{R}^n)$ (resp. $f \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^n)$) belongs to $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) if and only if for any $k > 0$ and $h > 0$ there exists a constant $C = C_{h,k} > 0$ (resp. there exists a constant $h > 0$ and for any $k > 0$ there exists a constant $C = C_k > 0$) such that

$$(13) \qquad |D^\alpha f(x)| \leq C|h|^{|\alpha|} M_{|\alpha|}(1 + |x|)^{-k},$$

for any multi-index α. Let us define

$$(14) \quad \mathcal{S}^{\{M_p\},h} = \{\varphi \in \mathcal{S}^{\{M_p\}} \; ; \; \forall \, k, \; \exists C, \; |D^\alpha \varphi(x)| \leq Ch^{|\alpha|} M_{|\alpha|}(1 + |x|)^{-k}\}.$$

For $f \in \mathcal{S}^{\{M_p\},h}$, we define its norm by

$$(15) \qquad \|f\|_{\mathcal{S}^{\{M_p\},h}} := \sup_{x \in \mathbb{R}^n, \alpha, k} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}(1 + |x|)^k}.$$

For topologies of ultradifferentiable classes, the following relations hold.

$$(16) \qquad \begin{aligned} \mathcal{S}^{\{M_p\}} &= \varinjlim_{h \to \infty} \mathcal{S}^{\{M_p\},h}, \\ \mathcal{S}^{(M_p)} &= \varprojlim_{h \to 0} \mathcal{S}^{\{M_p\},h}. \end{aligned}$$

The set $\mathcal{S}^{*\prime}$ is defined as the strong dual of $\mathcal{S}^*$, where $* = (M_p)$ or $\{M_p\}$.

Lemma 4.1 (cf. Proposition 3.2 in [5]).   *A sequence M_p satisfies the condition $(M.1)$ if and only if*

$$(17) \qquad M_p = M_0 \sup_{t>0} \frac{t^p}{\widetilde{M}(t)},$$

for $t > 0$.

Proposition 4.1.   *Assume that a sequence M_p, $p = 0, 1, 2, \ldots$, of positive numbers satisfies the conditions $(M.0)$, $(M.1)$, $(M.2)'$ and (NA). Then the following conditions are equivalent.*

(i) *The function $\widehat{f}$ is the Fourier-Laplace transform of $f \in \mathcal{S}^{(M_p)}$ (resp. $f \in \mathcal{S}^{\{M_p\}}$).*

(ii) *For any multi-index α and $h > 0$ there exists a constant $C = C_{\alpha,h}$ (resp. there exists a constant $h > 0$ and for any multi-index α there exists a constant $C = C_\alpha > 0$) such that*

(18)
$$|D^\alpha \widehat{f}(\xi)| \leq \frac{C}{\widetilde{M}(h|\xi|)}, \quad for\ \xi \in \mathbb{R}^n.$$

Proof. Let us treat both (M_p) and $\{M_p\}$ classes simultaneously. (ii) $\Rightarrow$ (i); By virtue of Lemma 4.1, the following estimate holds.

(19)
$$
\begin{aligned}
\left|x^\beta D_x^\alpha f(x)\right| &= \left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} D_\xi^\beta(\widehat{f}(\xi)\xi^\alpha)e^{ix\cdot\xi}d\xi\right| \\
&\leq \frac{1}{(2\pi)^n}\int_{|\xi|\leq 1}\left|D^\beta(\widehat{f}(\xi)\xi^\alpha)\right|d\xi + \frac{1}{\pi^n}\int_{|\xi|>1}\frac{|\xi|^{n+1}}{(1+|\xi|)^{n+1}}\left|D^\beta(\widehat{f}(\xi)\xi^\alpha)\right|d\xi \\
&\leq \frac{C_{\beta,1}}{M(h|\xi|)} + C_{\beta,2}\int_{|\xi|>1}\frac{1}{(1+|\xi|)^{n+1}}d\xi\sum_{j=0}^{\min(|\alpha|+n+1,|\beta|)}\frac{|\xi|^{|\alpha|+n+1-j}}{M(h|\xi|)} \\
&\leq \frac{C_{\beta,1}}{M(h|\xi|)} + C_{\beta,3}\frac{|\xi|^{|\alpha|+n+1}}{M(h|\xi|)} \leq C_\beta\frac{M_{|\alpha|+n+1}}{h^{|\alpha|+n+1}},
\end{aligned}
$$

where $C_{\beta,i}$, $i=1,2,3$, and C_β are suitable constants. The condition $(M.2)'$ yields that

(20)
$$\frac{M_{|\alpha|+n+1}}{h^{|\alpha|+n+1}} \leq \frac{GH^{n+1}}{h^{n+1}}\left(\frac{H}{h}\right)^{|\alpha|}M_{|\alpha|},$$

for some constants G and H. Therefore, (i) is obtained if the conditions on the constants are properly interpreted according to the class (M_p) or $\{M_p\}$.

(i) $\Rightarrow$ (ii);

(21)
$$\xi^\beta D^\alpha \widehat{f}(\xi) = \int e^{-ix\cdot\xi}\left(D^\beta(x^\alpha f(x))\right)dx.$$

Note that $x^\alpha f(x) \in \mathcal{S}^*$, $* = (M_p)$ or $\{M_p\}$. Then we have

(22)
$$\left|\xi^\beta D^\alpha \widehat{f}(\xi)\right| \leq C_\alpha h^{|\beta|}M_{|\beta|}\int\frac{1}{(1+|x|)^{(n+1)}}dx.$$

Therefore

(23)
$$\left|D^\alpha \widehat{f}(\xi)\right| \leq C_\alpha \inf_{|\beta|}\frac{M_{|\beta|}}{(|\xi|/h)^{|\beta|}} \leq \frac{C_\alpha}{\widetilde{M}(|\xi|/h)},$$

which proves (ii) with an appropriate interpretation on the constants in accordance with the class. $\qquad\square$

Proposition 4.2.   *Assume that a sequence M_p, $p = 0, 1, 2, \ldots$, of positive numbers satisfies the conditions $(M.0)$, $(M.1)$, $(M.2)'$ and (NA). If for any $h > 0$ there exists such a constant $C = C_h$ (resp. there exist such constants $h > 0$ and $C > 0$) that*

$$(24) \qquad\qquad |f(x)| \leq \frac{C}{\widetilde{M}(h|x|)},$$

then $\widehat{f} \in \mathcal{E}^{(M_p)}$ (resp. $\widehat{f} \in \mathcal{E}^{\{M_p\}}$).

   Proof.

(25)

$$\left| D^\alpha \widehat{f}(\xi) \right| = \left| \int_{\mathbb{R}^n} f(x) x^\alpha e^{-ix\cdot\xi} dx \right|$$

$$\leq \int_{|x|\leq 1} |f(x)||x|^{|\alpha|} dx + 2^n \int_{|x|>1} \frac{|f(x)||x|^{|\alpha|+n+1}}{(1+|x|)^{n+1}} dx$$

$$\leq \left(\int_{|x|\leq 1} dx \right) \sup_{x\in\mathbb{R}^n} \frac{C|x|^{|\alpha|}}{\widetilde{M}(h|x|)} + \left(\int_{|x|>1} \frac{1}{(1+|x|)^{n+1}} dx \right) \sup_{x\in\mathbb{R}^n} \frac{C|x|^{|\alpha|+n+1}}{\widetilde{M}(h|x|)}$$

$$\leq C_1 \frac{M_{|\alpha|}}{|h|^{|\alpha|}} + C_2 \frac{M_{|\alpha|+n+1}}{|h|^{|\alpha|+n+1}},$$

where we have applied the fact that for a positive sequence M_p, $(M.1)$ is equivalent to

$$(26) \qquad\qquad M_p = \sup_{t>0} \frac{t^p}{\widetilde{M}(t)}.$$

By virtue of $(M.2)'$, there exist such constants C_3 and H that

$$(27) \qquad\qquad M_{|\alpha|+n+1} \leq C_3 H^{|\alpha|} M_{|\alpha|}.$$

By (25) and (27), we have

$$(28) \qquad\qquad \left| D^\alpha \widehat{f}(\xi) \right| \leq C H^{|\alpha|} \max\left\{ \frac{1}{h^{|\alpha|}}, \frac{1}{h^{|\alpha|+n+1}} \right\} M_{|\alpha|},$$

which implies $\widehat{f} \in \mathcal{E}^{(M_p)}$ (resp. $\widehat{f} \in \mathcal{E}^{\{M_p\}}$). $\square$

   Theorem 4.1.   (The Paley-Wiener theorem for non-analytic ultradistributions)   *Let M_p satisfy $(M.0)$, $(M.1)$, $(M.2)'$ and (NA). For a compact convex set $K \subset \mathbb{R}^n$, the following conditions are equivalent.*

(i) $\widehat{f}$ is the Fourier-Laplace transform of $f \in \mathcal{E}_K^{(M_p)'}$ (resp. $f \in \mathcal{E}_K^{\{M_p\}'}$).

(ii) There exist such constants $L > 0$ and $C > 0$ (resp. for any $L > 0$, there exists such a constant $C > 0$) that

$$|\widehat{f}(\xi)| \leq C\widetilde{M}(L|\xi|),$$

and $\widehat{f}(\zeta)$ is an entire function in $\zeta \in \mathbb{C}^n$ which satisfies that for any $\varepsilon > 0$ there exists such a constant C_ε that

$$(29) \qquad |\widehat{f}(\zeta)| \leq C_\varepsilon \exp(H_K(\mathrm{Im}\zeta) + \varepsilon|\zeta|), \qquad \zeta \in \mathbb{C}^n,$$

where $H_K(y) := \sup_{x \in K} x \cdot y$, $y \in \mathbb{R}^n$, is the supporting function of K.

(iii) $\widehat{f}(\zeta)$ is an entire function in $\zeta \in \mathbb{C}^n$ which satisfies that there exist such constants $L > 0$ and $C > 0$ (resp. for any $L > 0$, there exists such a constant $C > 0$) that

$$(30) \qquad |\widehat{f}(\zeta)| \leq C\widetilde{M}(L|\zeta|)e^{H_K(\mathrm{Im}\zeta)}, \qquad \zeta \in \mathbb{C}^n.$$

This theorem seems to be known, however, it seems difficult to find the proof of this theorem in our form. Hence we shall give its proof.

Proof. (i) $\Rightarrow$ (iii); Since $f \in \mathcal{E}_K^{*'}(\mathbb{R}^n) \subset \mathcal{E}^{*'}(\mathbb{R}^n)$, there exist constants h and C (resp. for any $h > 0$, there is a constant C) such that

$$(31) \qquad |\langle \varphi, f \rangle| \leq C \sup_{x \in K, \alpha} \frac{D^\alpha \varphi(x)}{h^{|\alpha|} M_{|\alpha|}}, \qquad \varphi \in \mathcal{E}^*(\mathbb{R}^n).$$

Let

$$(32) \qquad \varphi(x) = \exp(-ix \cdot \zeta), \qquad \zeta = \xi + i\eta \in \mathbb{C}^n.$$

Then there holds

$$(33) \qquad |\widehat{f}(\zeta)| \leq C \sup_{x \in K, \alpha} \frac{|\zeta|^{|\alpha|}}{h^{|\alpha|} M_{|\alpha|}} |e^{-ix \cdot \xi + x \cdot \eta}| \leq C e^{H_K(\mathrm{Im}\zeta)} \widetilde{M}\left(\frac{|\zeta|}{h}\right).$$

(iii) $\Rightarrow$ (ii); Let any $L > 0$ be fixed. For any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$(34) \qquad k! \leq M_k \left(\frac{\varepsilon}{L}\right)^k$$

for $k > m$ by virtue of (NA). Therefore

$$(35) \qquad \widetilde{M}(L|\zeta|) = \sup_k \frac{L^k |\zeta|^k}{M_k} \leq C \sup_k \frac{(\varepsilon|\zeta|)^k}{k!} \leq C e^{\varepsilon|\zeta|}.$$

(35) and the Paley-Wiener theorem for hyperfunctions (cf. [4]) give (ii).

(ii) $\Rightarrow$ (i); Let $* = (M_p)$ or $\{M_p\}$. We first prove that $f \in \mathcal{S}^{*\prime}$. By virtue of Proposition 4.1 and the assumption ii), there holds for $\varphi \in \mathcal{S}^*$,

$$
\begin{aligned}
|\langle \varphi, f \rangle| := \left| \int \varphi(x) f(x) dx \right| &= \frac{1}{(2\pi)^n} \left| \left\langle \widehat{\varphi}, \widehat{f} \right\rangle \right| \\
&\leq \frac{1}{(2\pi)^n} \int \left| \widehat{\varphi}(\xi) \widehat{f}(\xi) \right| d\xi \\
&\leq \frac{1}{(2\pi)^n} \int \frac{C_1}{\widetilde{M}(h|\xi|)} C_2 \widetilde{M}(L|\xi|) d\xi.
\end{aligned}
$$

(36)

For $* = (M_p)$, there exist some L, C_2, and for any $h > 0$, there exists some C_1 such that (36) holds. Hence take $h > 0$ such that $h > L$ then (36) converges. For $* = \{M_p\}$, there exist some h, C_1, and for any $L > 0$, there exists some C_2 such that (36) holds. Hence take $L > 0$ such that $h > L$ then (36) converges. The function f is then proved to be a linear map from $\mathcal{S}^*$ to $\mathbb{C}$. Take a sequence $\varphi_n \to \varphi$ in $\mathcal{S}^*$ and replace φ in (36) by φ_n, then the Lebesgue dominated convergence theorem proves the continuity of f. Therefore, it is proved that $f \in \mathcal{S}^{*\prime}$.

The estimate (29) and the Paley-Wiener Theorem for hyperfunctions yield that f is a hyperfunction with its support contained in K. The fact that the space of the analytic functions $\mathcal{A}[K]$ is dense in $\mathcal{E}^*[K]$ implies that $f \in \mathcal{E}_K^{*\prime}$. Therefore (i) is obtained. $\qquad\square$

The following proposition follows almost directly from Theorem 4.1.

Proposition 4.3. *For the positive sequences M_p and N_p satisfying the conditions $(M.1)$ and (NA), the following conditions are equivalent.*

(i) $M_p \prec N_p$.

(ii) $\lim_{p \to \infty} (M_p/N_p)^{1/p} = 0$.

If M_p and N_p satisfies $(M.2)'$ in addition, then above two condition are equivalent to the following one.

(iii) $\{M_p\} < \{N_p\} \quad and \quad (M_p) < (N_p)$.

By this proposition, we have $\{M_p\} \leq \{N_p\}$ for $M_p \subset N_p$.

Definition 4.2. A function $\varepsilon(t) > 0$ defined for $t > 0$ is called *sub-ordinate* if and only if it is continuous, monotonously increasing and $\varepsilon(t)/t$ is

monotonously decreasing to zero as $t \to \infty$, i.e.,

$$(37) \qquad \lim_{t \to \infty} \frac{\varepsilon(t)}{t} = 0.$$

Proposition 4.4 (cf. Lemma 3.10 in [5]). *For positive sequences M_p and N_p satisfying $(M.1)$, the following conditions are equivalent.*

(i) *$M_p \prec N_p$.*

(ii) *For any $L > 0$, there exists such a constant $C > 0$ that*

$$\widetilde{N}(t) \le C\widetilde{M}(Lt), \quad \text{for } 0 < t < \infty.$$

(iii) *There exists such a subordinate function $\varepsilon(t)$ that*

$$\widetilde{N}(t) \equiv \widetilde{M}(\varepsilon(t)).$$

By virtue of Proposition 4.4, we obtain the following equivalent conditions.

Proposition 4.5. *Let M_p satisfy $(M.1)$ and (NA). For a function f defined on $\mathbb{R}^n$, the following conditions are equivalent.*

(i) *For any $L > 0$ there exists such a constant $C > 0$ that*

$$(38) \qquad |f(x)| \le C\widetilde{M}(L|x|), \quad \text{for } \forall x \in \mathbb{R}^n.$$

(ii) *There exists such a subordinate function $\varepsilon(t)$ that*

$$(39) \qquad |f(x)| \le \widetilde{M}(\varepsilon(|x|)), \quad \text{for } \forall x \in \mathbb{R}^n.$$

Proof. The proof of "(ii) $\Rightarrow$ (i)" is clear.

Let us prove "(i) $\Rightarrow$ (ii)". We define $\varepsilon(t)$, $t > 0$ by

$$(40) \qquad \sup_{|x| \le t} |f(x)| = \widetilde{M}(\varepsilon(t)).$$

By the definition, (39) holds. It is also trivial that $\varepsilon(t)$ is monotonously increasing. What is left to prove is that

$$(41) \qquad \lim_{t \to \infty} \frac{\varepsilon(t)}{t} = 0.$$

Let us assume the contrary to (41), that is, there exist a constant $L > 0$ and a sequence t_j of positive numbers satisfying $t_1 < t_2 < \cdots < t_j < \cdots \to \infty$

such that $\varepsilon(t_j) \geq 2Lt_j$. Then for this constant L, there exists a constant C such that

$$(42) \qquad \widetilde{M}(2Lt_j) \leq \widetilde{M}(\varepsilon(t_j)) = \sup_{|x| \leq t_j} |f(x)| \leq C\widetilde{M}(Lt_j).$$

Hence we obtain $\widetilde{M}(2Lt_j) \leq C\widetilde{M}(Lt_j)$, which contradicts to the fact that

$$(43) \qquad \lim_{t \to \infty} \frac{\widetilde{M}(t)}{t^k} = \infty,$$

for any positive integer k. $\qquad\qquad\square$

§5. Main Theorem

In this section, we prove our main theorem. As an preparation, we construct the symbols of ultradifferential operators in the non-analytic classes, which serves as a key lemma to prove our main theorem.

Lemma 5.1. *Let N_p satisfy $(M.0)$, $(M.1)$, $(M.2)'$, (QA) and (NA). For any subordinate function $\varepsilon(t)$ there exist such a monotonously decreasing positive sequence l_p with $\lim_{p \to \infty} l_p = 0$ and such a constant $A > 0$ that*

$$(44) \qquad |P(\xi)| \geq A\widetilde{M}(\varepsilon(|\xi|)),$$

where

$$(45) \qquad P(\xi) := \sum_{p=0}^{\infty} \frac{(l_{2p}|\xi|)^{2p}}{M_{2p}}.$$

Proof. If the subordinate function $\varepsilon(t)$ satisfies $\lim_{t \to \infty} \varepsilon(t) < \infty$, then the lemma is easily obtained, for example, by letting $l_p = 1/p$. Therefore what is left to prove is the case where $\lim_{t \to \infty} \varepsilon(t) = \infty$. Let us represent the associate function $\widetilde{M}$ by

$$(46) \qquad \widetilde{M}(t) = \sup_p \frac{t^p}{M_p} = \sup_p \prod_{q=1}^{p} \frac{t}{m_q},$$

where $m_p := \frac{M_p}{M_{p-1}}$ is an increasing sequence satisfying

$$(47) \qquad \lim_{p \to \infty} m_p = \infty.$$

We define the sequence $\tilde{l}_p$, $p = 1, 2, \ldots$, by

$$(48) \qquad \varepsilon\left(\frac{m_p}{\tilde{l}_p}\right) = m_p.$$

Monotonous increase of ε and m_p together with (47) yields that the sequence $m_p/\tilde{l}_p$ is monotonously increasing and

$$(49) \qquad \lim_{p\to\infty} \frac{m_p}{\tilde{l}_p} = \infty.$$

Therefore the sequence

$$(50) \qquad \tilde{l}_p = \frac{\varepsilon(m_p/\tilde{l}_p)}{m_p/\tilde{l}_p}$$

is monotonously decreasing and satisfies $\lim_{p\to\infty} \tilde{l}_p = 0$.

If $m_p \le t < m_{p+1}$ then

$$(51) \qquad \frac{t^k}{M_k} = \prod_{q=1}^{k} \frac{t}{m_q}$$

attains its supremum at $k = p$. Therefore for

$$(52) \qquad m_p = \varepsilon\left(\frac{m_p}{\tilde{l}_p}\right) \le \varepsilon(t) \le \varepsilon\left(\frac{m_{p+1}}{\tilde{l}_{p+1}}\right) = m_{p+1},$$

there holds that

$$(53) \qquad \widetilde{M}(\varepsilon(t)) = \prod_{q=1}^{p} \frac{\varepsilon(t)}{m_q} \le \prod_{q=1}^{p} \frac{\tilde{l}_q t}{m_q} = \frac{\tilde{l}_1 \cdots \tilde{l}_p t^p}{M_p} = \frac{(\hat{l}_p t)^p}{M_p},$$

where $\hat{l}_p := \sqrt[p]{\tilde{l}_1 \cdots \tilde{l}_p}$ and we have applied the estimate

$$(54) \qquad \varepsilon(t) \le \tilde{l}_q t,$$

for $t \ge m_p/\tilde{l}_p$ and $q \le p$. Note that by virtue of (52), we obtain $\frac{m_p}{\tilde{l}_p} \le t \le \frac{m_{p+1}}{\tilde{l}_{p+1}}$. By the definition, $\hat{l}_p > 0$ is a decreasing sequence and satisfies $\lim_{p\to\infty} \hat{l}_p = 0$.

For $0 < \hat{l}_p t < 1$,

$$(55) \qquad \frac{(\hat{l}_{p-1} t)^{p-1}}{M_{p-1}} \ge \frac{(\hat{l}_p t)^p}{M_p}.$$

For $\hat{l}_p t \geq 1$, $(M.2)'$ implies

$$(56) \qquad \frac{1}{M_{p-1}} \leq \frac{GH^p}{M_p},$$

for some G and H, which yields

$$(57) \qquad \frac{(\hat{l}_{p-1}t)^{p-1}}{M_{p-1}} \leq \frac{(\hat{l}_p t)^p}{M_p} GH^p.$$

We have, by virtue of (55) and (57),

$$(58) \qquad \frac{(\hat{l}_p t)^p}{M_p} \leq G \sum_{p=0}^{\infty} \frac{(H'\hat{l}_{2p}t)^{2p}}{M_{2p}},$$

for any $p = 0, 1, 2, \cdots$, where $H' := \max\{1, H\}$. By (53) and (58), letting $l_p := H'\hat{l}_p$ yields

$$(59) \qquad A\widetilde{M}(\varepsilon(|\xi|)) \leq P(\xi) := G' \sum_{p=0}^{\infty} \frac{(l_{2p}|\xi|)^{2p}}{M_{2p}},$$

where $G' := \max\{1, G\}$, which proves the lemma. $\qquad\square$

Theorem 5.1. *Let N_p satisfy $(M.0)$, $(M.1)$, $(M.2)$, (QA) and (NA). Assume that $f \in \mathcal{F}^{*'}$, where $* = (M_p)$ or $\{M_p\}$. Then for any class $\dagger$ satisfying $* < \dagger$ there exist $g \in \mathcal{E}^{\dagger}$ and an ultradifferential operator $P(D)$ of the class $*$ such that the representation*

$$(60) \qquad f = P(D)g,$$

locally holds.

Proof. It is sufficient to prove that the representation (60) holds in some neighborhood of the origin. By the definition, for any $f \in \mathcal{F}^{*'}$ there exists $f_1 \in \mathcal{E}_K^{*'}$ such that $f = f_1$ in some neighborhood of the origin, where $K \subset \mathbb{R}^n$ is some compact set containing the origin in its inside. In the following proof, it is capable of assuming that $f \in \mathcal{E}_K^{*'}$ without loss of generality.

I. *The proof for (M_p) class.*
The ultradifferential operator

$$(61) \qquad \widetilde{P}(D) := \sum_{p=0}^{\infty} \frac{(-\Delta)^p}{M_{2p}},$$

belongs to (M_p) class and satisfies that there exist such constants $C_1 > 0$ and $C_2 > 0$ that

$$\text{(62)} \qquad |\widetilde{P}(\xi)| \geq C_1 \widetilde{M}(C_2|\xi|),$$

for any $\xi \in \mathbb{R}^n$. Theorem 4.1 yields that for $f \in \mathcal{E}^{(M_p)'}$ there exist such constants C and L that

$$\text{(63)} \qquad |\widehat{f}(\xi)| \leq C\widetilde{M}(L|\xi|), \quad \text{for } \forall \xi \in \mathbb{R}^n.$$

Define the ultradifferential operator

$$\text{(64)} \qquad P(D) := \sum_{p=0}^{\infty} \frac{(-\frac{L}{C_2}\Delta)^p}{M_{2p}}.$$

Then we have by (62) that there exists such a constant $C' > 0$ that

$$\text{(65)} \qquad |P(\xi)| \geq C'\widetilde{M}(L|\xi|),$$

for any $\xi \in \mathbb{R}^n$. (63) and (65) yield that there exists such a constant $C > 0$ that

$$\text{(66)} \qquad \left| \frac{\widehat{f}(\xi)}{P(\xi)^2} \right| \leq \frac{C}{\widetilde{M}(L|\xi|)}.$$

for any $\xi \in \mathbb{R}^n$. By Proposition 4.2, there holds

$$\text{(67)} \qquad g := \mathcal{F}^{-1}\left(\frac{\widehat{f}(\xi)}{P(\xi)^2} \right) \in \mathcal{E}^{\{M_p\}},$$

where $\mathcal{F}^{-1}$ is the inverse Fourier-Laplace transform operator. We have

$$\text{(68)} \qquad f(x) \equiv (P(D))^2 g(x).$$

By virtue of $(M.2)$, we have $(P(D))^2$ is an ultradifferential operator of the class (M_p). Therefore the theorem is proved for (M_p) class.

    II. *The proof for $\{M_p\}$ class.*
Let $\{M_p\} \prec \dagger = (N_p)$ or $\{N_p\}$. $L_p := \sqrt{M_p N_p}$ yields $M_p \prec L_p \prec N_p$. By Proposition 4.4, there exists such a subordinate function ε_1 that $\widetilde{L}(t) = \widetilde{M}(\varepsilon_1(t))$, hence by Lemma 5.1, there exist such a positive decreasing sequence $l_p^{(1)}$ with $\lim_{p\to\infty} l_p^{(1)} = 0$ and a constant $A_1 > 0$ that

$$\text{(69)} \qquad P_1(\xi) := \sum_{p=0}^{\infty} \frac{(l_{2p}^{(1)}|\xi|)^{2p}}{M_{2p}} \geq A_1 \widetilde{M}(\varepsilon_1(|\xi|)),$$

for any $\xi \in \mathbb{R}^n$. By the definition, $P_1(D)$ is an ultradifferential operator of the class $\{M_p\}$. By virtue of Theorem 4.1 and Proposition 4.4, there exists such a subordinate function ε_2 that

$$(70) \qquad\qquad |\widehat{f}(\xi)| \leq \widetilde{M}(\varepsilon_2(|\xi|)),$$

for any $\xi \in \mathbb{R}^n$. In view of Lemma 5.1, there exist such a positive decreasing sequence $l_p^{(2)}$ satisfying $\lim_{p\to\infty} l_p^{(2)} = 0$ and a constant $A_2 > 0$ that

$$(71) \qquad\qquad P_2(\xi) := \sum_{p=0}^{\infty} \frac{(l_{2p}^{(2)}|\xi|)^{2p}}{M_{2p}} \geq A_2 \widetilde{M}(\varepsilon_2(|\xi|)),$$

for any $\xi \in \mathbb{R}^n$. Therefore, we have

$$(72) \qquad\qquad \left| \frac{\widehat{f}(\xi)}{P_1(\xi)P_2(\xi)} \right| \leq \frac{1}{A_1 A_2 \widetilde{M}(\varepsilon_1(|\xi|))} = \frac{1}{A_1 A_2 \widetilde{L}(|\xi|)},$$

for any $\xi \in \mathbb{R}^n$. Let us define

$$(73) \qquad\qquad g := \mathcal{F}^{-1}\left(\frac{\widehat{f}(\xi)}{P_1(\xi)P_2(\xi)} \right),$$

then Proposition 4.2 and (72) imply that $g \in \mathcal{E}^{\{L_p\}} \subset \mathcal{E}^{\dagger}$. We have

$$(74) \qquad\qquad P_1(D)P_2(D)g(x) = f(x).$$

By $(M.2)$, the ultradifferential operator $P_1(D)P_2(D)$ belongs to the $\{M_p\}$ class. Therefore the theorem is also proved for $\{M_p\}$ class. □

As an application of the proof of Theorem 5.1, we obtain a modification of Theorem 3.4.

Theorem 5.2. *For any hyperfunction f and for any non-analytic class $*$, there exist such a local operator $J(D)$ and $g \in \mathcal{E}^*$ that the representation (11) holds locally.*

In order to prove this theorem, we modify A. Kaneko's proof of Theorem 3.4 in [3] applying the fact that any ultradifferential operator of non-analytic classes is a local operator, that is, we take

$$(75) \qquad\qquad g := \mathcal{F}^{-1}\left(\frac{\widehat{f}(\xi)}{J(\xi)P(\xi)} \right),$$

where $J(D)$ is the local operator constructed in Lemma 1.2 in [3] and $P(D)$ is the ultradifferential operator constructed in (64) for (M_p) class (resp. in (71) for $\{M_p\}$ class).

In the proofs of both Theorems 5.1 and 5.2, it is essentially important to construct ultradifferential operators of non-analytic classes (Lemma 5.1 and (64)).

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