Rend. Lincei Mat. Appl. 21 (2010), 99–113
DOI 10.4171/RLM/563

Partial Differential Equations — *Subelliptic Hamilton-Jacobi equations: the coercive stationary case*, by Marco Biroli, communicated on 15 January 2010.[1]

Dedicated to the memory of Renato Caccioppoli

Abstract. — We prove the existence uniqueness and comparison results for a (Lipschitz) viscosity solution for an Hamilton-Jacobi equation on a Carnot group.

Key words: Partial differential equations in Carnot groups, viscosity solutions of Hamilton-Jacobi equations, regularity.

Mathematics Subject Classification AMS: 35H20, 49L25, 35B65.

1. Introduction

We consider the Hamilton-Jacobi equation

$$u + \gamma H(x, \nabla u) = 0 \tag{1.1}$$

where the Hamiltonian $H(x, p) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is not coercive in p. In some cases the lack of coerciveness of the Hamiltonian can be overcome by chancing the underlying geometry with a suitable family of vector fields. More precisely we consider the case $H(x, t, q) = H(x, t, \sigma(x)p)$ where $\sigma(x)$ is a $m \times N$ matrix, $m < N$ and H is coercive in $q = \sigma(x)p$. Here the rows of the matrix $\sigma(x)$ are considered as coefficients of vector fields satisfying the Hörmander condition, which generate a Carnot group and are left translation invariant, therefore $\sigma(x)\nabla u$ is the horizontal gradient in the Carnot group denoted in the following by $D_h u$, see section 2.

Here we are interested in existence and uniqueness of viscosity solutions of the problem (1.1) and in their Lipschitz continuity (in the group). We recall that the Lipschitz continuity in the group is the Lipschitz continuity for the right translations with respect to the Carnot-Carathéodory (left translation invariant) distance on the group and that the Lipschitz continuity is equivalent to the boundness of the horizontal gradient. We observe that in the Euclidean case ($\sigma(x) = I$) with $H(x, \nabla u) = H(\nabla u) - f(x)$, where $f(x)$ is Lipschitz continuous, the Lipschitz continuity of u may easily be deduced from comparison results;

[1] The Author has been supported by the MIUR Research Project n. 2007WECYEA.

this proof can not be generalized to the case of Carnot groups, since the vector fields and the Carnot-Carathéodory distance are left translation invariant and we are interested in the Lipschitz continuity in the group (i.e. the Lipschitz continuity for the right translations with respect to the Carnot-Carathéodory distance).

We recall that the notion of viscosity solution has been introduced by M. G. Crandall, P. L. Lions, [8]; existence, uniqueness and comparison results for viscosity solutions of (1.1) in the coercive Euclidean case have been summarized in the paper [14] and in the books [1], [2]. We recall also the paper [9], where the existence of viscosity solutions is proved using the Perron's method.

Concerning the case of Carnot groups an existence and comparison result of (Lipschitz continuous in the group) viscosity solutions has been proved in the case of Heisenberg group by I. Birindelli, J. Wigniolle, [4], and in the general case by B. Stroffolini, [15], but, in the last paper, it seems that some commutativity conditions on the group are tacitly assumed. We recall also that in the evolution case Hopf-Lax-Oleinik type representation formulas has been proved in [12] for the case of the Heisenberg group and in [5] for general Hörmander's vector fields (in this last paper a review of the result of Hopf-Lax-Oleinik type representation formulas in the subelliptic framework is given).

The method used to prove our result is founded on a careful choice of the penalization functions in the doubling variables method, which take into account some suggestions of [4], and on the results on the Perron's method given [9].

In section 2, we give the fundamental definitions concerning Carnot groups and some examples of Carnot groups and in section 3 we give the notion of viscosity solution in the framework of Carnot groups. In sections 4 and 5 we give the results, which are the main goal of this paper, and their proofs.

2. Carnot groups

We consider $\mathbb{R}^N$ as a Carnot group with a group operation (translation) denoted by . and a family of dilations, compatible with the Lie structure.

A Carnot group G of step $r \geq 1$ is a simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ is stratified. This means that $\mathfrak{g}$ admits a decomposition as a vector space sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r$$

such that

$$[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}$$

for $j = 1, \dots, r$ where $\mathfrak{g}_k = \emptyset$ when $k > r$. Let $m_j = \dim \mathfrak{g}_j$ and denote by $X_{i,j}$ a basis of $\mathfrak{g}_j$ formed by left invariant vector fields. We observe that G considered as a manifold has dimension $N = m_1 + \cdots + m_r$.

For convenience we fix a Riemannian metric in $\mathfrak{g}$ so that $\mathbb{X} = \{X_{i,j}\}$ is an orthonormal frame and the Riemannian volume element coincides with the Haar measure on G and then with the Lebesgue measure on $\mathbb{R}^N$.

The horizontal tangent space at a point $\xi \in G$ is the m_1 dimensional subspace linearly spanned by $X_{1,1}(\xi), \dots, X_{m_1,1}(\xi)$. In the following we will denote by $X_1, \dots, X_m$ a frame of vector fields spanning the firs layer $\mathfrak{g}_1$. The exponential coordinates are given by the diffeomorphism $F : \mathbb{R}^N \to G$ defined by

$$F(x) = \exp\Big(\sum_{j=1}^{r}\sum_{i=1}^{m_j} x_{i,j}X_{i,j}\Big)$$

We denote $x_j = (x_{i,j})_{i=1,\dots,m_j}$, $j = 1, \dots, r$.

In the following we consider G endowed with the exponential coordinates; then the mapping $(\xi, \eta) \to \xi.\eta$ has polynomial entries.

If we use the exponential coordinates the following formula holds:

$$X_\alpha = \frac{\partial}{\partial x_{\alpha,1}} + \sum_{j=2}^{r}\sum_{i=1}^{m_s} b^i_{j,\alpha}(x_1(\xi), \dots, x_{s-1}(\xi))\frac{\partial}{\partial x_{i,j}} \tag{2.1}$$

$\alpha = 1, \dots, m$, where each $b^i_{j,\alpha}$ is a polynomial of weighted degree $j - 1$. By weighted degree we mean that the layer $\mathfrak{g}_j$ in the stratification of $\mathfrak{g}$ has degree j. Then each homogeneous monomial $x_1^{\alpha_1} \dots x_r^{\alpha_r}$ with multindices $\alpha_j = (\alpha_{1,j}, \dots, \alpha_{m_j,j})$, $j = 1, \dots r$ has a weighted degree p if

$$\sum_{j=1}^{r} j\Big(\sum_{i=1}^{m_j} \alpha_{i,j}\Big) = p$$

There is a family of dilations compatible with the group operation:

$$\delta_\lambda(\xi) = (\lambda x_{1,1}, \dots, \lambda x_{m,1}, \lambda^2 x_{1,2}, \dots, \lambda^2 x_{m_2,2}, \dots, \lambda^r x_{m_r,r})$$

With the above notations the horizontal subspace in the point ξ can be identified by the left translation by ξ of G_1 (G_1 denotes the subspace of $\mathbb{R}^N$ corresponding to $\mathfrak{g}_1$).

A horizontal curve $\gamma(t)$, $t \in [0, 1]$, is a piecewise smooth curve whose tangent vector $\gamma'(t)$, whenever it exists, is in the horizontal tangent space in $\gamma(t)$.

Given two points ξ and η we consider the set

$$\Gamma(\xi, \eta) = \{\gamma \text{ horizontal curve: } \gamma(0) = \xi, \gamma(1) = \eta\}$$

By Chow's accessibility theorem, [3], the above set is never empty.

The Carnot-Carathéodory distance is defined as the infimum of the length of horizontal curves of the set Γ:

$$d_{CC}(\xi, \eta) = \inf_{\Gamma(\xi,\eta)} \int_0^1 |\gamma'(t)|\, dt$$

The Carnot-Carathéodory ball of radius R centered at ξ is given by

$$B(\xi, R) = \{\eta \in G : d_{CC}(\xi, \eta) < R\}$$

The Carnot-Carathéodory gauge is defined by

$$|\xi|_{CC} = d_{CC}(0, \xi)$$

We recall that $d_{CC}(\xi, \eta)$ is left translation invariant and gives a topology on G, which is equivalent to the Euclidean one.

The important property that $|\xi|_{CC}$ is a viscosity and a.e. solution of the horizontal eikonal equation in $\mathbb{R}^N \backslash 0$ has been proved by Monti and Serra-Cassano, [13].

A smooth gauge in G is defined by

$$|\xi|_G = \Big(\sum_{j=1}^{r} \Big(\sum_{i=1}^{m_j} |x_{i,j}|^2\Big)^{r!/j}\Big)^{1/2r!}$$

The following result holds, [3]:

Theorem 2.1. *We have*

$$|\xi|_{CC} \simeq |\xi|_G \simeq \sum_{j=1}^{r} \sum_{i=1}^{m_j} |x_{i,j}|^{1/j}$$

$$meas(B(0, R)) \simeq R^Q$$

where $Q = \sum_{j=1}^{r} j m_j$ *is called the homogeneous dimension of* G.

As a consequence of Theorem 2.1 we have that the Lipschitz continuity (for the right translations) with respect to the Carnot-Carathéodory distance or with respect to the smooth gauge are equivalent.

Examples of Carnot groups are the Heisenberg group and the Engel group.

2.1. The Heisenberg group

The Heisenberg group can be identified with $\mathbb{R}^{2N+1}$ endowed with the non commutative group law

$$(x, y, t).(x', y', t') = \Big(x + x', y + y', t + t' - \frac{1}{2}(\langle x, y'\rangle - \langle x', y\rangle)\Big)$$

where $x, y \in \mathbb{R}^N$ and $t \in \mathbb{R}$. The Heisenberg algebra is splitted in $V_1 \oplus V_2$ where $V_1 = \mathbb{R}^{2N} \times \{0\}$ and $V_2 = \{0\} \times \mathbb{R}$ and it is generated by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}$$

$$Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}$$

The only non trivial commutator is

$$[X_j, Y_j] = \frac{\partial}{\partial t}$$

and the homogeneous dimension is $2N + 2$.

2.2. The Engel group

The Engel group can be identified with $\mathbb{R}^4$ endowed with the non commutative group law

$$(x, y, t, s).(x', y', t', s') = (x + x', y + y', t + t' + Q_2, s + s' + Q_3)$$

where

$$Q_2 = \frac{1}{2}(xy' - x'y)$$

$$Q_3 = \frac{1}{2}(xt' - x't) + \frac{1}{12}(x^2 y' - xx'(y + y') + y(x')^2)$$

The Engel algebra is splitted in $V_1 \oplus V_2 \oplus V_3$ where $V_1 = \mathbb{R}^2 \times \{0\} \times \{0\}$, $V_2 = \{0\} \times \{0\} \times \mathbb{R} \times \{0\}$, $V_3 = \{0\} \times \{0\} \times \{0\} \times \mathbb{R}$ and it is generated by the vector fields

$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial t} - \left(\frac{t}{2} + \frac{y}{12}(x + y)\right)\frac{\partial}{\partial s}$$

$$X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial t} + \left(\frac{t}{2} + \frac{x}{12}(x + y)\right)\frac{\partial}{\partial s}$$

The nontrivial commutators are

$$[X_1, X_2] = X_3 = \frac{\partial}{\partial t} + \frac{(x + y)}{2}\frac{\partial}{\partial s}$$

$$[X_1, X_3] = [X_1, [X_1, X_2]] = \frac{\partial}{\partial s}$$

and the homogeneous dimension is 7.

3. Viscosity solutions

In order to give a first definition of viscosity solution in G for the equation

$$H(\xi, u, D_h u) = 0 \tag{3.1}$$

where $\xi \in G$, and D_h is the horizontal gradient in G and $H : G \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function, we must identify the first order jets adapted to our framework.

Definition 3.1. *A function $u : G \to \mathbb{R}$ is of class C^1 (on G) if the horizontal derivatives $X_1 u, \ldots, X_m u$ are continuous (on G).*

If u is a locally bounded function we define

$$\begin{aligned} u^*(\xi) &= \inf\{v(\xi) \mid v \in C(G), v \geq u \text{ on } G\} \\ &= \limsup_{\eta \to \xi} u(\eta) \\ u_*(\xi) &= \sup\{v(\xi) \mid v \in C(G),\ v \leq u \text{ on } G\} \\ &= \liminf_{\eta \to \xi} u(\eta) \end{aligned}$$

We observe that u^* is upper semicontinuous and u_* is lower semicontinuous.

We recall that if u is in C^1 the following Taylor expansion holds

$$u(\xi) = u(\xi_0) + \langle D_h u(\xi_0), \overline{\xi_0^{-1}.\xi}\rangle + o(|\xi_0^{-1}.\xi|_G)$$

where $\bar{\xi}$ is the horizontal projection (of ξ).

If a function u is not necessarily smooth but merely upper semicontinuous (on G) we denote by $J_u^{1,+}(\xi_0)$, $\xi_0 \in G$, the collection of vectors $p^*(\xi_0) \in \mathbb{R}^m$ such that

$$u(\xi) \leq u(\xi_0) + \langle p^*(\xi_0), \overline{\xi_0^{-1}.\xi}\rangle + o(|\xi_0^{-1}.\xi|_G)$$

Definition 3.1. *A locally bounded function u is a viscosity subsolution of equation* (3.1) *if for every $\xi_0 \in G$ and for every $p^*(\xi_0) \in J_{u^*}^{1,+}(\xi_0)$ we have*

$$H(\xi_0, u^*(\xi_0), p^*(\xi_0)) \leq 0$$

If a function u is not necessarily smooth but merely lower semicontinuous (on G) we denote by $J_u^{1,-}(\xi_0)$, $\xi_0 \in G$ the collection of vectors $p_*(\xi_0) \in \mathbb{R}^m$ such that

$$u(\xi) \geq u(\xi_0) + \langle p_*(\xi_0), \overline{\xi_0^{-1}.\xi}\rangle + o(|\xi_0^{-1}.\xi|_G)$$

Definition 3.2. *A locally bounded function u is a viscosity supersolution of equation* (3.1) *if for every $\xi_0 \in G$ and for every $p_*(\xi_0) \in J_{u_*}^{1,-}(\xi_0)$ we have*

$$H(\xi_0, u_*(\xi_0), p_*(\xi_0)) \geq 0$$

We observe that $J_{u^*}^{1,+}(\xi_0)$ $(J_{u_*}^{1,-}(\xi_0))$ may be empty, but the set where $J_{u^*}^{1,+}(\xi_0)$ $(J_{u_*}^{1,-}(\xi_0))$ is not empty is dense in G. The proof of the property is the same as in Euclidean case, [2], replacing $\frac{|x-\bar{x}|^2}{\varepsilon}$ by $|\delta_{\varepsilon^{-1/r}}(\bar{\xi}^{-1}.\xi)|_G^{r!}$.

DEFINITION 3.3. *A locally bounded function u is a viscosity solution of equation* (3.1) *if u^* is a viscosity subsolution and u_* is a viscosity supersolution of equation* (3.1).

There are different but equivalent definitions for viscosity subsolutions, supersolutions and solutions of equation (3.1).

DEFINITION 3.4. *A locally bounded function u is a viscosity subsolution* (*supersolution*) *of equation* (3.1) *if for every function $\psi \in C^1(G)$ and every local maximum* (*minimum*) *point ξ_0 of $(u^* - \psi)$ $((u_* - \psi))$ on G we have*

$$H(\xi_0, u^*(\xi_0), D_h\psi(\xi_0)) \leq 0$$
$$(H(\xi_0, u_*(\xi_0), D_h\psi(\xi_0)) \geq 0)$$

A locally bounded function u is a viscosity solution of equation (3.1) *if u^* is a viscosity subsolution and u_* is a viscosity supersolution of equation* (3.1).

In the definition 3.4 the condition $\psi \in C^1(G)$ can be replaced by the condition $\psi \in C^\infty(G)$. The definitions $3.1, \ldots, 3.4$ has been given in Euclidean setting in [8] and [6]; the generalization of Definitions $3.1, \ldots, 3.4$ to the Carnot group setting has been given in [15]. The equivalence of the three notions has been proved in Euclidean setting by [8] and by [6]; in the Carnot groups setting the equivalence is founded on the observation that the definitions in Euclidean setting and Carnot group setting are equivalent.

REMARK 3.1. We observe that an upper (lower) semicontinuos function u is a viscosity sub- (super-) solution of the equation $H(\xi, u, D_h u) = 0$ if and only if $v = -u$ is viscosity super- (sub-) solution of the equation $-H(\xi, -v, -D_h v) = 0$.

PROPOSITION 3.1. *A bounded upper* (*lower*) *semicontinuous function is Lipschitz continuous on G if and only if $(J_u^{1,-}(\xi))$ $J_u^{1,+}(\xi)$ is bounded on G.*

PROOF. We prove the result in the case of upper semicontinuous functions; in the case of lower semicontinuous functions the proof is analogous.

Let u be a bounded upper semicontinuous functions such that $|p^*(\eta)| \leq L$ for $p^*(\eta) \in J_u^{1,+}(\xi)$. Let $w(\eta) = u(\eta) - u(\xi)$, where ξ is fixed. We have that u is a viscosity subsolution of the Hamilton-Jacobi equation

$$|D_h u| = L$$

Let M be such that $|u| \leq M$ then w is a viscosity subsolution of the Hamilton-Jacobi equation

$$w + |D_h w| = L + 2M \tag{3.3}$$

in $\mathbb{R}^N\backslash\xi$ with the condition $w(\xi)=0$. Let now be $v(\eta)=(L+2M)d_{CC}(\xi,\eta)$. Since $d_{CC}(\xi,\eta)$ is a viscosity solution of the eikonal equation in $\mathbb{R}^N\backslash\xi$, the function v is a Lipschitz continuous viscosity solution of the Hamilton-Jacobi equation

$$v+|D_h v|=(L+2M)+v \tag{3.4}$$

in $\mathbb{R}^N\backslash\xi$.

From (3.3) w is a viscosity subsolution of the equation (3.4) (with second term $(L+2M)+v$).

We now postpone the proof that $w\le v$ to Theorem 5.1 and Remark 5.1.

Then we have

$$w(\eta)\le(L+2M)d_{CC}(\xi,\eta) \tag{3.5}$$

Using the left translation invariance of D_h we obtain in the general case

$$u(\eta)-u(\xi)\le(L+2M)d_{CC}(\xi,\eta) \tag{3.6}$$

Interchanging the role of ξ and η we obtain also

$$u(\xi)-u(\eta)\le(L+2M)d_{CC}(\xi,\eta)$$

then

$$|u(\xi)-u(\eta)|\le(L+2M)d_{CC}(\xi,\eta)$$

and the result is proved.

The following result is proved in [4] for the Heisenberg group and in [15] for general Carnot groups in the case of continuous u. We need the result for upper semicontinuous u and we give here a different proof as an easy corollary of Proposition 3.1.

Corollary 3.2. *Let u be a bounded upper semicontinuous viscosity subsolution of $|D_h u|\le C$ then u is Lipschitz on G, i.e.*

$$|u(\eta)-u(\xi)|\le C d_{CC}(\eta,\xi)$$

Proof. Let be $p^*(\xi)\in J_u^{1,+}(\xi)$. Then we have $|p^*(\xi)|\le C$. The result follows from Proposition 3.1.

4. Existence of the viscosity solution

In the present section we consider the problem

$$u+\gamma H(\xi,D_h u)=f \tag{4.1}$$

where $H : G \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function, which is uniformly continuous on $G \times B$ for every ball B (for the Carnot-Carathéodory distance) in G, f is uniformly continuous on G and γ is a positive constant.

We assume:

$$f \text{ bounded in } G \tag{4.2}$$

$$|H(\xi, 0)| \leq C \text{ on } G \tag{4.3}$$

$$\lim_{|p| \to +\infty} H(\xi, p) = +\infty \text{ uniformly on } G \tag{4.4}$$

Theorem 4.1. *Let the assumptions* (4.2), (4.3) *and* (4.4) *hold. There exists a bounded Lipschitz continuous viscosity solution u of* (4.1). *Moreover every bounded upper semicontinuous solution of* (4.1) *is Lipschitz continuous on G.*

Proof. Corollary 3.2 gives easily the second part of Theorem 4.1.

For the first part of the result we consider our problem in the Euclidean framework then H is a continuous Hamiltonian. The constant $-M - \gamma C$ $(M + \gamma C)$ is a viscosity sub- (super-) solution of our problem, where $M = \sup |f|$ and C is as in (4.3). Then there exists a bounded viscosity solution of problem (4.1) u, which is the supremum of upper semicontinuous subsolutions v of (4.1) with $-M - \gamma C \leq v \leq M + \gamma C$, [9]. We recall that the notion of subsolution in the Euclidean and in Carnot groups setting are equivalent and that the topologies on $\mathbb{R}^N$ and on G are equivalent. From Corollary 3.2. and assumption (4.4) we have that all upper semicontinuous subsolutions v with $-M - \gamma C \leq v \leq M + \gamma C$ are uniformly Lipschitz continuous on G, then u is Lipschitz continuous on G.

5. Uniqueness and continuous dependence for the viscosity solution

The main difficulty in this section derives (as pointed out in the introduction) from the noncommutative nature of G. In fact, if we consider the function $|\eta^{-1}.\xi|_G^{2r}$, which is the natural substitute of $|x - y|^2$ in the Euclidean case, we have that

$$D_{h,\xi}(|\eta^{-1}.\xi|_G^{2r}) \neq D_{h,\eta}(|\eta^{-1}.\xi|_G^{2r})$$

This fact does not allow to use for the proof of the uniqueness and continuous dependence the method of [8] or its developments in [7], [10] (see also [1], [2]). We have to choose carefully the penalization functions in the doubling variables method and to take into account the Lipschitz continuity of upper semicontinuous viscosity subsolutions of our equation.

Theorem 5.1. *Let u* (*v*) *be an upper* (*lower*) *semicontinuos viscosity subsolution* (*supersolution*) *of the Hamilton-Jacobi equation* (4.1). *Assume that u* (*or v*) *is Lipschitz continuous on G and that v* (*or u*) *is bounded then*

$$u \leq v$$

PROOF. We assume that u is Lipschitz continuous and v is bounded; the proof in the other case (v Lipschitz continuous and u bounded) is analogous.

Suppose by contradiction that $\sup(u-v) \geq \delta > 0$; there exists a point ξ_0 such that $u(\xi_0) - v(\xi_0) = \tau > 0$, for τ suitable. Let $A_\varepsilon(\xi,\eta)$ and $\rho_\alpha(\xi,\eta)$ be defined as

$$A_\varepsilon(\xi,\eta) = \sum_{j=1}^{r}\sum_{i=1}^{m_j}\left|\frac{(\xi^{-1}.\eta)_{i,j}}{\varepsilon^{r/j}}\right|^{2r/j}$$

$$\rho_\alpha(\xi,\eta) = \sum_{j=1}^{r}\sum_{i=1}^{m_j}\alpha^{2r}(|(\xi_0^{-1}.\xi)_{i,j}|^{2r/j} + |(\xi_0^{-1}.\eta)_{i,j}|^{2r/j})$$

where $0 < \varepsilon,\ \alpha \leq \frac{1}{2}$.

Consider the function

$$\Phi_{\varepsilon,\alpha}(\xi,\eta) = u(\xi) - v(\eta) - A_\varepsilon(\xi,\eta) - \rho_\alpha(\xi,\eta)$$

Taking into account Theorem 2.1, we have that for fixed (ε,α) the function $\Phi_{\varepsilon,\alpha}(\xi,\eta)$ diverges to $-\infty$ as $(|\xi|_G + |\eta|_G) \to +\infty$.

Since $(u-v)$ is upper semicontinuous, for every ε, α there exists a maximum point $(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha})$ of $\Phi_{\varepsilon,\alpha}(\xi,\eta)$.

We have

$$\Phi_{\varepsilon,\alpha}(\xi_{\varepsilon,\alpha},\xi_{\varepsilon,\alpha}) + \Phi_{\varepsilon,\alpha}(\eta_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha}) \leq 2\Phi_{\varepsilon,\alpha}(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha}) \tag{5.1}$$

We recall that u is Lipschitz continuous on G and we denote by L his Lipschitz constant. We easily obtain from (5.1)

$$A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha}) \leq 2(M + CL|\xi_{\varepsilon,\alpha}^{-1}.\eta_{\varepsilon,\alpha}|_G) \tag{5.2}$$

Then from Theorem 2.1 we have

$$A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha}) \leq C_1 \tag{5.3}$$

From the inequality $\Phi_{\varepsilon,\alpha}(\xi_0,\xi_0) \leq \Phi_{\varepsilon,\alpha}(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha})$ we obtain

$$u(\xi_0) - v(\xi_0) \leq u(\xi_{\varepsilon,\alpha}) - v(\eta_{\varepsilon,\alpha}) - A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha}) - \rho_\alpha(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha})$$

Then from (5.3) we have

$$\sum_{j=1}^{r}\sum_{i=1}^{m_j}\alpha^{2r}(|(\xi_0^{-1}.\xi_{\varepsilon,\alpha})_{i,j}|^{2r/j} + |(\xi_0^{-1}.\eta_{\varepsilon,\alpha})_{i,j}|^{2r/j}) \leq C_2 + CL|\xi_0^{-1}.\xi_{\varepsilon,\alpha}|_G$$

Then we obtain

$$\sum_{j=1}^{r}\sum_{i=1}^{m_j}\alpha^{4r}(|(\xi_0^{-1}.\xi_{\varepsilon,\alpha})_{i,j}|^{2r/j}+|(\xi_0^{-1}.\eta_{\varepsilon,\alpha})_{i,j}|^{2r/j})$$

$$\leq \alpha^{2r}\Big(C_2+C_3L\Big(\sum_{j=1}^{r}\sum_{i=1}^{m_j}|(\xi_0^{-1}.\xi_{\varepsilon,\alpha})_{i,j}|^{1/j}\Big)\Big)$$

$$\leq \alpha^{2r}\Big(C_4+C_3L\Big(\sum_{j=1}^{r}\sum_{i=1}^{m_j}|(\xi_0^{-1}.\xi_{\varepsilon,\alpha})_{i,j}|^{r/j}\Big)\Big)$$

From the above inequality we deduce easily the following estimate

$$\rho_\alpha(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha})\leq C_5 \tag{5.4}$$

We consider now $D_{h,\xi}A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha})$ and $D_{h,\eta}A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha})$.

LEMMA 5.2. *We have*

$$D_{h,\xi}A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha})=\frac{1}{\varepsilon^{2r^2}}(|(\xi_{\varepsilon,\alpha}^{-1}.\eta_{\varepsilon,\alpha})_{i,1}|^{2r-2}(\eta_{\varepsilon,\alpha}^{-1}.\xi_{\varepsilon,\alpha})_{i,1}+o(\varepsilon^{1/r}))_{i=1,\dots,m}$$

and

$$D_{h,\eta}A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha})=\frac{1}{\varepsilon^{2r^2}}(|(\xi_{\varepsilon,\alpha}^{-1}.\eta_{\varepsilon,\alpha})_{i,1}|^{2r-2}(\xi_{\varepsilon,\alpha}^{-1}.\eta_{\varepsilon,\alpha})_{i,1}+o(\varepsilon^{1/r}))_{i=1,\dots,m}$$

PROOF. Since our vector fields are invariant for left translations in the group we have

$$D_{h,\xi}A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha})=\left(D_{h,\xi}\left(\sum_{j=1}^{r}\sum_{i=1}^{m_j}\left|\frac{(\xi)_{i,j}}{\varepsilon^{r/j}}\right|^{2r/j}\right)\right)(\eta_{\varepsilon,\alpha}^{-1}.\xi_{\varepsilon,\alpha})$$

Then

$$X_{1,\xi}A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha})=\frac{1}{\varepsilon^{2r^2}}|(\eta_{\varepsilon,\alpha}^{-1}.\xi_{\varepsilon,\alpha})_{1,1}|^{2r-2}(\eta_{\varepsilon,\alpha}^{-1}.\xi_{\varepsilon,\alpha})_{1,1}$$
$$+\sum_{j=2}^{r}\sum_{i=1}^{m_j}b_{j,1}^{i}(\eta_{\varepsilon,\alpha}^{-1}.\xi_{\varepsilon,\alpha})\Big(\frac{1}{\varepsilon^{2r^2/j^2}}|(\eta_{\varepsilon,\alpha}^{-1}.\xi_{\varepsilon,\alpha})_{i,j}|^{2r/j-2}\Big)(\eta_{\varepsilon,\alpha}^{-1}.\xi_{\varepsilon,\alpha})_{i,j}$$

where the $b_{j,1}^{i}$ are as in (2.1).

From (5.3) we have that

$$|(\eta_{\varepsilon,\alpha}^{-1}.\xi_{\varepsilon,\alpha})_{i,j}|\leq C\varepsilon^{r/j}$$

so

$$|b^i_{j,1}(\eta^{-1}_{\varepsilon,\alpha}.\xi_{\varepsilon,\alpha})| \le C\varepsilon^{r/(j-1)}$$

Then

$$(5.5)\qquad |b^i_{j,1}(\eta^{-1}_{\varepsilon,\alpha}.\xi_{\varepsilon,\alpha})\Big(\frac{1}{\varepsilon^{2r^2/j^2}}|(\eta^{-1}_{\varepsilon,\alpha}.\xi_{\varepsilon,\alpha})_{i,j}|^{2r/j-2}\Big)(\eta^{-1}_{\varepsilon,\alpha}.\xi_{\varepsilon,\alpha})_{i,j}| \le C\varepsilon^{r/(j-1)-r/j} = C\varepsilon^{r/(j(j-1))}$$

From (5.5) we obtain that

$$X_{1,\xi}A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha}) = \frac{1}{\varepsilon^{2r^2}}|(\eta^{-1}_{\varepsilon,\alpha}.\xi_{\varepsilon,\alpha})_{1,1}|^{2r-2}(\eta^{-1}_{\varepsilon,\alpha}.\xi_{\varepsilon,\alpha})_{1,1} + o(\varepsilon^{1/r})$$

By the same methods we prove

$$X_{i,\xi}A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha}) = \frac{1}{\varepsilon^{2r^2}}|(\eta^{-1}_{\varepsilon,\alpha}.\xi_{\varepsilon,\alpha})_{i,1}|^{2r-2}(\eta^{-1}_{\varepsilon,\alpha}.\xi_{\varepsilon,\alpha})_{i,1} + o(\varepsilon^{1/r})$$

for $s = 2,\dots,m$ and

$$X_{i,\eta}A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha}) = \frac{1}{\varepsilon^{2r^2}}|(\xi^{-1}_{\varepsilon,\alpha}.\eta_{\varepsilon,\alpha})_{i,1}|^{2r-2}(\xi^{-1}_{\varepsilon,\alpha}.\eta_{\varepsilon,\alpha})_{i,1} + o(\varepsilon^{1/r})$$

The result follows.

By the same methods we can prove:

Lemma 5.3. *We have*

$$\lim_{\alpha\to 0} D_{h,\xi}\rho_\alpha(\eta_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha}) = 0$$

$$\lim_{\alpha\to 0} D_{h,\eta}\rho_\alpha(\eta_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha}) = 0$$

uniformly in $\varepsilon > 0$.

We now end the proof of Theorem 5.1.

We have that $\xi_{\varepsilon,\alpha}$ is a maximum point for the function $\xi \to u(\xi) - A_\varepsilon(\xi,\eta_{\varepsilon,\alpha}) - \rho_\alpha(\xi,\eta_{\varepsilon,\alpha})$ and $\eta_{\varepsilon,\alpha}$ is a minimum point for the function $\eta \to v(\eta) + A_\varepsilon(\xi_{\varepsilon,\alpha},\eta) + \rho_\alpha(\xi_{\varepsilon,\alpha},\eta)$. Then we have

$$(5.6)\qquad u(\xi_{\varepsilon,\alpha}) + \gamma H(\xi_{\varepsilon,\alpha}, p_{\varepsilon,\alpha}) \le f(\xi_{\varepsilon,\alpha})$$

where $p_{\varepsilon,\alpha} = D_{h,\xi}A_\varepsilon(\xi_{\varepsilon,\alpha},\eta_{\varepsilon,\alpha}) + D_{h,\xi}\rho_\alpha(\xi,\eta_{\varepsilon,\alpha}) \in J^{1,+}_u(\xi_{\varepsilon,\alpha})$. Since u is Lipschitz continuous, $p_{\varepsilon,\alpha}$ is bounded in ε, α.

We have also

$$v(\eta_{\varepsilon,\alpha}) + \gamma H(\eta_{\varepsilon,\alpha}, q_{\varepsilon,\alpha}) \geq f(\eta_{\varepsilon,\alpha}) \tag{5.7}$$

where $q_{\varepsilon,\alpha} = -D_{h,\eta}A_\varepsilon(\xi_{\varepsilon,\alpha}, \eta_{\varepsilon,\alpha}) - D_{h,\eta}\rho_\alpha(\eta_{\varepsilon,\alpha}) \in J_v^{1,-}(\xi_{\varepsilon,\alpha})$.

From the lemma 5.2 we have

$$p_{\varepsilon,\alpha} = \left(\frac{1}{\varepsilon^{2r^2}}|(\xi_{\varepsilon,\alpha}^{-1}.\eta_{\varepsilon,\alpha})_{i,1}|^{2r-2}(\eta_{\varepsilon,\alpha}^{-1}.\xi_{\varepsilon,\alpha})_{i,1} + o(\varepsilon^{1/r})\right)_{i=1,\dots,m} + D_{h,\xi}\rho_\alpha(\xi, \eta_{\varepsilon,\alpha}) \tag{5.8}$$

$$q_{\varepsilon,\alpha} = \left(\frac{1}{\varepsilon^{2r^2}}|(\xi_{\varepsilon,\alpha}^{-1}.\eta_{\varepsilon,\alpha})_{i,1}|^{2r-2}(\eta_{\varepsilon,\alpha}^{-1}.\xi_{\varepsilon,\alpha})_{i,1} + o(\varepsilon^{1/r})\right)_{i=1,\dots,m_1} - D_{h,\eta}\rho_\alpha(\eta_{\varepsilon,\alpha}) \tag{5.9}$$

Since $p_{\varepsilon,\alpha}$ is bounded in ε, α (for ε, α small enough) we have that

$$\frac{1}{\varepsilon^{2r^2}}\sum_{i=1}^{m_1}|(\xi_{\varepsilon,\alpha}^{-1}\eta_{\varepsilon,\alpha})_{i,1}|^{2r-2}(\eta_{\varepsilon,\alpha}^{-1}\xi_{\varepsilon,\alpha})_{i,1}$$

is bounded in ε, α and then $q_{\varepsilon,\alpha}$ is bounded in ε, α.

From (5.1), (5.6) and (5.7) we have

$$\begin{aligned} \tau &\leq (u(\xi_{\varepsilon,\alpha}) - v(\eta_{\varepsilon,\alpha})) \\ &\leq \gamma(H(\eta_{\varepsilon,\alpha}, q_{\varepsilon,\alpha}) - H(\xi_{\varepsilon,\alpha}, p_{\varepsilon,\alpha})) + (f(\xi_{\varepsilon,\alpha}) - f(\eta_{\varepsilon,\alpha})) \\ &= \gamma(H(\eta_{\varepsilon,\alpha}, q_{\varepsilon,\alpha}) - H(\xi_{\varepsilon,\alpha}, q_{\varepsilon,\alpha})) \\ &\quad + \gamma(H(\xi_{\varepsilon,\alpha}, q_{\varepsilon,\alpha}) - H(\xi_{\varepsilon,\alpha}, p_{\varepsilon,\alpha})) + (f(\xi_{\varepsilon,\alpha}) - f(\eta_{\varepsilon,\alpha})) \end{aligned} \tag{5.10}$$

Since $|\xi_{\varepsilon,\alpha}^{-1}.\eta_{\varepsilon,\alpha}|_G$ converges to 0 as $\varepsilon \to 0$ (uniformly with respect to α) we have

$$\lim_{\varepsilon\to 0}(H(\eta_{\varepsilon,\alpha}, q_{\varepsilon,\alpha}) - H(\xi_{\varepsilon,\alpha}, q_{\varepsilon,\alpha})) = 0 \tag{5.11}$$

$$\lim_{\varepsilon\to 0}(f(\xi_{\varepsilon,\alpha}) - f(\eta_{\varepsilon,\alpha})) = 0 \tag{5.12}$$

uniformly with respect to α.

From (5.8) and (5.9) we have

$$\begin{aligned} &(H(\xi_{\varepsilon,\alpha}, q_{\varepsilon,\alpha}) - H(\xi_{\varepsilon,\alpha}, p_{\varepsilon,\alpha})) \\ &\quad = (H(\xi_{\varepsilon,\alpha}, (\tilde{p}_{\varepsilon,\alpha,i} + o(\varepsilon^{1/r}))_{i=1,\dots,m} - D_{h,\eta}\rho_\alpha(\xi_{\varepsilon,\alpha}, \eta_{\varepsilon,\alpha})) \\ &\qquad - H(\xi_{\varepsilon,\alpha}, (\tilde{p}_{\varepsilon,\alpha,i} + o(\varepsilon^{1/r}))_{i=1,\dots,m} + D_{h,\eta}\rho_\alpha(\xi_{\varepsilon,\alpha}, \eta_{\varepsilon,\alpha}))) \end{aligned}$$

where

$$\tilde{p}_{\varepsilon,\alpha,i} = \frac{1}{\varepsilon^{2r^2}}|(\xi_{\varepsilon,\alpha}^{-1}.\eta_{\varepsilon,\alpha})_{i,1}|^{2r-2}(\eta_{\varepsilon,\alpha}^{-1}.\xi_{\varepsilon,\alpha})_{i,1}$$

is bounded as $\alpha, \varepsilon \to 0$.

Then we obtain

$$\lim_{\alpha\to 0,\varepsilon\to 0}(H(\xi_{\varepsilon,\alpha}, q_{\varepsilon,\alpha}) - H(\xi_{\varepsilon,\alpha}, p_{\varepsilon,\alpha})) = 0 \tag{5.13}$$

From (5.10), (5.11), (5.12) and (5.13) we obtain a contradiction and the result is proved.

REMARK 5.1. The result of Theorem 5.1 holds again if we assume that u (v) are viscosity sub(super)-solutions of the equation (4.1) in $\mathbb{R}^N\backslash 0$ with $u(0) = v(0)$. The proof is the same of Theorem 5.1 if $\xi_{\varepsilon,\alpha}, \eta_{\varepsilon,\alpha} \in \mathbb{R}^N\backslash 0$ (as $\varepsilon, \alpha \to 0$). Assume that there is a sequence ε_k, α_k such that $\varepsilon_k, \alpha_k \to 0$ as $k \to +\infty$ and $\eta_{\varepsilon_k,\alpha_k} = 0$. Then we have $\xi_{\varepsilon_k,\alpha_k} \to 0$ as $\varepsilon_k \to 0$. We recall that $u - v$ is upper semicontinuous then

$$\frac{\delta}{2} \leq \lim_{k\to+\infty}(u(\xi_{\varepsilon_k,\alpha_k}, \eta_{\varepsilon_k,\alpha_k}) - v(\xi_{\varepsilon_k,\alpha_k}, \eta_{\varepsilon_k,\alpha_k})) \leq (u(0) - v(0)) = 0$$

Then we have a contradiction with the assumption $\sup(u - v) \geq \delta > 0$. By the same methods we prove that there is no subsequences ε_k, α_k such that $\xi_{\varepsilon_k,\alpha_k} = 0$ then $\xi_{\varepsilon,\alpha}, \eta_{\varepsilon,\alpha} \in \mathbb{R}^N\backslash 0$.

As easy consequences of the Theorem 5.1 we obtain:

COROLLARY 5.4. *Let the assumptions* (4.2), (4.3) *and* (4.4) *hold. Let* u (v) *be a bounded upper* (*lower*) *semicontinuos viscosity subsolution* (*supersolution*) *of the Hamilton-Jacobi equation* (4.1). *Then*

$$u \leq v$$

It is enough to observe that from Corollary 3.2 we have that u is Lipschitz continuous on G.

COROLLARY 5.5. *Let the assumptions* (4.3) *and* (4.4) *hold. Let* u_1 *and* u_2 *be upper semicontinuous solutions of the Hamilton-Jacobi equation* (4.1) *with* $f = f_1$ *and* $f = f_2$, *where* f_1, f_2 *are bounded uniformly continuous on* G. *Then*

$$\sup_G |u_1 - u_2| \leq \sup_G |f_1 - f_2|$$

PROOF. It is enough observe that $u_2 - M$ ($u_1 - M$) is a subsolution of (4.1) with $f = f_1$ ($f = f_2$), where $M = \sup_G |f_1 - f_2|$, and use Theorem 4.1.

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Received 30 April 2009,
and in revised form 22 December 2009.

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