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Picard Groups of Some Local Categories

By

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Abstract

For each p-local spectrum E, $\mathcal{L}_E$ denotes the full subcategory consisting of E-local spectra of the category of p-local spectra. The Picard group $\mathrm{Pic}(\mathcal{L}_E)$ is the collection of isomorphism classes of invertible spectra in $\mathcal{L}_E$. If this is a set, it is a group with multiplication defined by the smash product. We show that if a spectrum E satisfies a relation $\langle E\rangle \geq \langle H\boldsymbol{Z}/p\rangle$ of the Bousfield classes, then $\mathrm{Pic}(\mathcal{L}_E) = \boldsymbol{Z}$. In particular, $\mathrm{Pic}(\mathcal{L}_E) = \boldsymbol{Z}$ if E is connective.

§1. Introduction

Throughout this paper, we work in the homotopy category $\mathcal{S}_p$ of p-local spectra for a fixed prime number p. For a spectrum $E \in \mathcal{S}_p$, a spectrum $X \in \mathcal{S}_p$ is called E-acyclic if $X \wedge E = pt$, and called E-local if $[C, X]_* = 0$ for any E-acyclic spectrum C. Let $\mathcal{L}_E$ denote the full subcategory of $\mathcal{S}_p$ consisting of E-local spectra. Then we have the Bousfield localization functor $L_E\colon \mathcal{S}_p \to \mathcal{L}_E$ [1]. We call a spectrum $X \in \mathcal{L}_E$ invertible if there is a spectrum Y such that $L_E(X \wedge Y) = L_E S^0$. Let $\mathrm{Pic}(\mathcal{L}_E)$ denote the collection of the isomorphism classes of invertible spectra. If $\mathrm{Pic}(\mathcal{L}_E)$ is a set, then it is a group whose multiplication is given by $[X]\cdot[Y] = [L_E(X \wedge Y)]$ and the unit element is $[L_E S^0]$. Here $[X]$ denotes the isomorphism class of X. In the following, we

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write $X \in \mathrm{Pic}(\mathcal{L}_E)$ instead of $[X] \in \mathrm{Pic}(\mathcal{L}_E)$. It is well known that $\mathrm{Pic}(\mathcal{L}_{S^0}) = \mathrm{Pic}(\mathcal{S}_p) = \{S^n \mid n \in \boldsymbol{Z}\} \cong \boldsymbol{Z}$. Consider the Morava K-theories $K(n)$ and the Johnson-Wilson spectra $E(n)$. In [5], Hopkins, Mahowald and Sadofsky showed that the Picard group $\mathrm{Pic}(\mathcal{L}_{K(n)})$ for each $K(n)$ contains $\boldsymbol{Z}_p$ and determined it for $n < 2$. Note that $\mathrm{Pic}(\mathcal{L}_{K(n)})$ is often written by Pic_n. For $E(n)$, Hovey and Sadofsky showed $\mathrm{Pic}(\mathcal{L}_{E(n)}) = \boldsymbol{Z}$ in [9] if $2p-2 > n^2+n$. In [9], they also determined $\mathrm{Pic}(\mathcal{L}_{E(1)})$ at the prime 2, which is the direct sum of $\boldsymbol{Z}$ and $\boldsymbol{Z}/2$. In [12], we gave an estimate of $\mathrm{Pic}(\mathcal{L}_{E(n)})$ by using E_r-term of the $E(n)$-based Adams spectral sequence converging to $\pi_*(L_{E(n)}S^0)$, and showed that $\mathrm{Pic}(\mathcal{L}_{E(2)})$ at the prime 3 is isomorphic to one of $\boldsymbol{Z} \oplus \boldsymbol{Z}/3$ and $\boldsymbol{Z} \oplus \boldsymbol{Z}/3 \oplus \boldsymbol{Z}/3$.

Recall that the Bousfield class $\langle E \rangle$ denotes the collection of E-acyclic spectra, and we write $\langle F \rangle \geq \langle E \rangle$ if $\langle F \rangle \subset \langle E \rangle$. Let HG denote the Eilenberg-MacLane spectrum for a group G. Then one of our results is the following:

Theorem 1.1. *If* $\langle E \rangle \geq \langle H\boldsymbol{Z}/p \rangle$, *then* $\mathrm{Pic}(\mathcal{L}_E) = \boldsymbol{Z}$.

It is shown in [1, Lemma 3.2] that if E is connective, then $\langle E \rangle \geq \langle H\boldsymbol{Z}/p \rangle$ or $\langle E \rangle = \langle H\boldsymbol{Q} \rangle$. It is well known (*cf.* [5]) that $\mathrm{Pic}(\mathcal{L}_{HQ}) = \boldsymbol{Z}$.

Corollary 1.2. *If E is connective, then* $\mathrm{Pic}(\mathcal{L}_E) = \boldsymbol{Z}$.

Let I denote the Brown-Comenetz dual of S^0. In [8] and [10], it is shown that $L_E S^0 = S^0$ or $S\boldsymbol{Z}_p$ if $\langle E \rangle \geq \langle I \rangle$. Then the above theorem follows from

Theorem 1.3. *If* $\langle F \rangle \geq \langle E \rangle \geq \langle I \rangle$, *then* $\mathrm{Pic}(\mathcal{L}_F) \subset \mathrm{Pic}(\mathcal{L}_E) \subset \mathrm{Pic}(\mathcal{L}_I)$.

Note that if $\mathrm{Pic}(\mathcal{L}_I) = \boldsymbol{Z}$, then $\mathrm{Pic}(\mathcal{L}_F) = \mathrm{Pic}(\mathcal{L}_E) = \boldsymbol{Z}$, though the assumption $\mathrm{Pic}(\mathcal{L}_I) = \boldsymbol{Z}$ seems strong.

In the next section, we study basic relations between the Bousfield classes and Picard groups, and then we consider the Picard groups for spectra with a finite local and prove Theorem 1.3 in the following section. In Section 4, we show $\mathrm{Pic}(\mathcal{L}_{HZ/p}) = \boldsymbol{Z}$, which together with results given in the previous section proves Theorem 1.1. We describe a filtration of $\mathrm{Pic}(\mathcal{L}_I)$ in the last section as well as the proofs of some facts. For instance, we prove that the only invertible spectrum which is also a ring spectrum is the E-local sphere $L_E S^0$. We also discuss there the small objects in some local stable homotopy categories.

§2. Relations between Picard Groups

The function spectrum $\boldsymbol{F}(X,Y)$ is defined to be a spectrum such that $[W\wedge X,Y]_* = [W,\boldsymbol{F}(X,Y)]_*$ for any spectra W. We define the Spanier-Whitehead dual $D_E(X)$ of X in $\mathcal{L}_E$ by $D_E(X) = \boldsymbol{F}(X, L_E S^0)$. In particular, we write $D(X) = D_{S^0}(X)$ for the ordinary Spanier-Whitehead dual. It is shown in [11, Proposition A.2.8] that if $X \in \mathcal{L}_E$ is invertible, then $D_E(X)$ is the inverse of it, that is, $L_E(X \wedge D_E(X)) = L_E S^0$.

Lemma 2.1. *For any spectrum $X \in \mathcal{S}_p$, the Spanier-Whitehead dual $D_E(X)$ is E-local.*

Proof. Since $C\wedge X$ is E-acyclic if so is C, $[C, D_E(X)]_* = [C\wedge X, L_E S^0]_* = 0$. □

Note that there is a relation $L_E(L_E(X) \wedge L_E(Y)) = L_E(X \wedge Y)$ of E-localization, which we often use. For example, we have $L_E(X \wedge L_E S^0) = X$ for an E-local spectrum X. By definition, if $\langle F\rangle \geq \langle E\rangle$, then $L_E L_F X = L_E X$.

Lemma 2.2. *Suppose that $\langle F\rangle \geq \langle E\rangle$. The localization functor L_E induces a correspondence $l\colon \mathrm{Pic}(\mathcal{L}_F) \to \mathrm{Pic}(\mathcal{L}_E)$.*

Proof. Define $l(X) = L_E X$. Then we see that $l(X) \in \mathrm{Pic}(\mathcal{L}_E)$. In fact, $L_E(l(X)\wedge l(D_F(X))) = L_E(X\wedge D_F(X)) = L_E(L_F(X\wedge D_F(X))) = L_E L_F S^0 = L_E S^0$, since $D_F(X)$ is the inverse of X in $\mathcal{L}_F$. □

We call a spectrum X *strongly dualizable* in $\mathcal{L}_E$ if $L_E(\boldsymbol{F}(X, L_E S^0) \wedge Y) = \boldsymbol{F}(X,Y)$ for any $Y \in \mathcal{L}_E$. In [13, Chapter III], Lewis, May and Steinberger studied properties of strongly dualizable spectra. Among them, we have isomorphisms

(2.3) $$X = D_E(D_E(X)) \quad \text{and} \quad L_E(D_E(X) \wedge D_E(Y)) = D_E(X \wedge Y)$$

for a strongly dualizable spectrum X and a spectrum Y.

(2.4) ([11, Proposition A.2.8]) $X \in \mathrm{Pic}(\mathcal{L}_E)$ *is strongly dualizable.*

Lemma 2.5. *If $\langle F\rangle \geq \langle E\rangle$ and $L_F S^0 = L_E S^0$, then $\mathrm{Pic}(\mathcal{L}_F) \subset \mathrm{Pic}(\mathcal{L}_E)$.*

Proof. Take any spectrum X in $\mathrm{Pic}(\mathcal{L}_F)$. Then $X = D_F(D_F(X))$ by (2.4) and (2.3), which is isomorphic to $D_E(D_E(X))$, since $D_E(-) = D_F(-)$ by the assumption $L_FS^0 = L_ES^0$. Thus, X is E-local by Lemma 2.1. Furthermore, $L_E(X \wedge D_E(X)) = L_EL_F(X \wedge D_F(X)) = L_EL_FS^0 = L_ES^0$, and hence $X \in \mathrm{Pic}(\mathcal{L}_E)$. □

§3. On Spectra with a Finite Local

We say that a spectrum E has a finite local if there exists a non-trivial finite spectrum X such that $L_EX = X$. It is shown in [8, Proposition 7.2] that E has a finite local if and only if $\langle E\rangle \geq \langle I\rangle$. Here I denotes the Brown-Comenetz dual of the sphere spectrum. Since $\langle I\rangle = \langle I \wedge X\rangle$ for any non-trivial finite spectrum X by [8, Proposition 7.1 (e)], we see that

(3.1) *If E has a finite local, then so does $E \wedge X$ for a non-trivial finite spectrum X.*

It is shown in [10, Theorem B.6(a)] that if E has a finite local and $L_EH\boldsymbol{Q} \neq pt$ (resp. $L_EH\boldsymbol{Q} = pt$), then $L_ES^0 = S^0$ (resp. $L_ES^0 = S\boldsymbol{Z}_p$) (*cf.* [7, Theorem 3.6]). Hereafter, SG for a group G denotes the Moore spectrum with $H\boldsymbol{Z}_*(SG) = G$. If $\langle F\rangle \geq \langle E\rangle \geq \langle I\rangle$, then there are two cases:

1) $L_FS^0 = L_ES^0$
2) $L_FS^0 = S^0$ and $L_ES^0 = S\boldsymbol{Z}_p$

In the first case, Lemma 2.5 shows that $\mathrm{Pic}(\mathcal{L}_F) \subset \mathrm{Pic}(\mathcal{L}_E)$. We study the second case. Put $E/p^i = E \wedge S\boldsymbol{Z}/p^i$.

Lemma 3.2. *Let E have a finite local and suppose $L_EH\boldsymbol{Q} = pt$. Then $\langle E/p\rangle = \langle E\rangle$.*

Proof. By [14, Lemma 1.34], we see that $\langle E\rangle = \langle E \wedge S\boldsymbol{Q}\rangle \vee \langle E/p\rangle$. Since $E \wedge S\boldsymbol{Q} = E \wedge L_ES\boldsymbol{Q} = E \wedge L_EH\boldsymbol{Q} = pt$ by assumption, we have the lemma. □

Lemma 3.3. *If X is E-local, then $L_{E/p}X = L_{SZ/p}X$.*

Proof. Consider the cofiber sequence $X \xrightarrow{\eta} L_{E/p}X \to C_{E/p}X$ of the localization. Then $S\boldsymbol{Z}/p \wedge C_{E/p}X = pt$, since $C_{E/p}X$ is E-local and $S\boldsymbol{Z}/p \wedge C_{E/p}X$

is E-acyclic. Applying now $L_{SZ/p}$ to the cofiber sequence, we see that $L_{SZ/p}\eta$ is the desired isomorphism since $C_{E/p}X$ is $S\boldsymbol{Z}/p$-acyclic. □

Lemma 3.4. *Let E be a spectrum with a finite local. Then the correspondence $l\colon \mathrm{Pic}(\mathcal{L}_E) \to \mathrm{Pic}(\mathcal{L}_{E/p})$ given in Lemma 2.2 is an injection.*

Proof. If $L_E H\boldsymbol{Q} = pt$, then $L_E S^0 = L_{E/p}S^0$ by Lemma 3.6, and the lemma follows from Lemma 2.5. Suppose that $L_E H\boldsymbol{Q} \neq pt$. Then $L_E S^0 = S^0$ and so $D_E(-) = D(-)$. Let $X \in \mathrm{Pic}(\mathcal{L}_E)$ be an element such that $l(X) = L_{E/p}S^0$. Since l satisfies $l([X]\cdot[Y]) = l([X])\cdot l([Y])$, it suffices to show that $X = S^0$.

Consider the cofiber sequence $X \wedge D(X) \xrightarrow{\eta} L_E(X \wedge D(X)) \to C_E(X \wedge D(X))$ of the localization, which is

$$X \wedge D(X) \xrightarrow{\eta} S^0 \longrightarrow C \tag{3.5}$$

for $C = C_E(X \wedge D(X))$. Since $L_E H\boldsymbol{Q} \neq pt$, $\langle E\rangle \geq \langle E\rangle \wedge \langle H\boldsymbol{Q}\rangle = \langle H\boldsymbol{Q}\rangle$ by [14, Theorem 2.1(h)]. Therefore, C is $H\boldsymbol{Q}$-acyclic.

Note that $D(X)$ is E-local by Lemma 2.1 and that $L_{SZ/p}X = L_{E/p}X = L_{SZ/p}S^0$ by Lemma 3.3 and the assumption on X. Then, $S\boldsymbol{Z}/p \wedge X = S\boldsymbol{Z}/p \wedge L_{SZ/p}X = S\boldsymbol{Z}/p \wedge L_{SZ/p}S^0 = S\boldsymbol{Z}/p$ and $S\boldsymbol{Z}/p \wedge D(X) = \Sigma D(S\boldsymbol{Z}/p \wedge X) = S\boldsymbol{Z}/p$ by (2.3), since $D(S\boldsymbol{Z}/p) = \Sigma^{-1}S\boldsymbol{Z}/p$. Smashing $S\boldsymbol{Z}/p$ with the cofiber sequence (3.5) yields the cofiber sequence

$$S\boldsymbol{Z}/p \xrightarrow{\eta} S\boldsymbol{Z}/p \longrightarrow S\boldsymbol{Z}/p \wedge C.$$

Applying L_E to this cofiber sequence, we have a commutative diagram with horizontal cofiber sequences:

$$\begin{array}{ccccc}
S\boldsymbol{Z}/p & \xrightarrow{\quad\eta\quad} & S\boldsymbol{Z}/p & \longrightarrow & S\boldsymbol{Z}/p \wedge C \\
\cong\downarrow & & \cong\downarrow & & \downarrow \\
L_E(S\boldsymbol{Z}/p) & \xrightarrow{L_E(\eta)} & L_E(S\boldsymbol{Z}/p) & \longrightarrow & L_E(S\boldsymbol{Z}/p \wedge C) = pt.
\end{array}$$

Here, the left and the middle vertical arrows are equivalences since $L_E(S^0) = S^0$, and so is the right vertical arrow. Thus, $S\boldsymbol{Z}/p \wedge C = L_E(S\boldsymbol{Z}/p \wedge C) = S\boldsymbol{Z}/p \wedge L_E(C) = S\boldsymbol{Z}/p \wedge pt = pt$, which shows C is $S\boldsymbol{Z}/p$-acyclic.

These show that $C = pt$ by [14, Theorem 2.1(i)] and $\eta\colon X \wedge D(X) = S^0$. Therefore, $X \in \mathrm{Pic}(\mathcal{L}_{S^0})$, that is, X is a suspension of S^0. Since $S\boldsymbol{Z}/p \wedge X = S\boldsymbol{Z}/p$, we see that $X = S^0$ as desired. □

Corollary 3.6. *Suppose that E has a finite local and $\langle F\rangle \geq \langle E\rangle$. Then* $\mathrm{Pic}(\mathcal{L}_F) \subset \mathrm{Pic}(\mathcal{L}_E)$. *In particular, if* $\mathrm{Pic}(\mathcal{L}_E) = \boldsymbol{Z}$, *then* $\mathrm{Pic}(\mathcal{L}_F) = \boldsymbol{Z}$.

Proof. If $L_F S^0 = L_E S^0$, then the corollary follows from Lemma 2.5. If $L_F S^0 = S^0$ and $L_E S^0 = S\boldsymbol{Z}_p$, then $\langle F\rangle \geq \langle F/p\rangle \geq \langle E/p\rangle = \langle E\rangle$ by Lemma 3.2. Therefore, Lemmas 2.5 and 3.4 imply the corollary. □

Corollary 3.7. *If E has a finite local, then* $\mathrm{Pic}(\mathcal{L}_E) \subset \mathrm{Pic}(\mathcal{L}_I)$.

§4. The Picard Group of $\mathcal{L}_{HZ/p}$

Put $D_p(X) = \boldsymbol{F}(X, S\boldsymbol{Z}_p) = D_{HZ/p}(X)$. Then, for $X \in \mathrm{Pic}(\mathcal{L}_{HZ/p})$, $L_{HZ/p}(X \wedge D_p(X)) = L_{HZ/p} S^0$ by [11, Proposition A.2.8], and so $H\boldsymbol{Z}/p_*(X) = \boldsymbol{Z}/p$. Let $\mathrm{Pic}(\mathcal{L}_{HZ/p})^0$ denote the subcollection consisting of isomorphism classes $X \in \mathrm{Pic}(\mathcal{L}_{HZ/p})$ such that $(H\boldsymbol{Z}/p)_0(X) = \boldsymbol{Z}/p$. Then any element of $\mathrm{Pic}(\mathcal{L}_{HZ/p})$ is a suspension of an element of $\mathrm{Pic}(\mathcal{L}_{HZ/p})^0$. Furthermore, note the relation $D_p D_p(X) = X$ by (2.3). We also consider the cofiber sequence

$$S\boldsymbol{Z}/p \xrightarrow{p^r} S\boldsymbol{Z}/p^{r+1} \xrightarrow{j_r} S\boldsymbol{Z}/p^r \xrightarrow{\delta_r} \Sigma S\boldsymbol{Z}/p. \tag{4.1}$$

Lemma 4.2. *Let $X \in \mathrm{Pic}(\mathcal{L}_{HZ/p})^0$. Then we have an equivalence $L_{HZ}(X \wedge S\boldsymbol{Z}/p^r) \xrightarrow{\simeq} S\boldsymbol{Z}/p^r$ for each $r > 0$ such that the diagram*

$$\begin{array}{ccc} L_{HZ}(X \wedge S\boldsymbol{Z}/p^{r+1}) & \xrightarrow{1\wedge j_r} & L_{HZ}(X \wedge S\boldsymbol{Z}/p^r) \\ \simeq\downarrow & & \downarrow\simeq \\ S\boldsymbol{Z}/p^{r+1} & \xrightarrow[j_r]{} & S\boldsymbol{Z}/p^r \end{array}$$

commutes for the map j_r of (4.1).

Proof. Put $Y = X \wedge S\boldsymbol{Z}/p$ and let g be a generator of $(H\boldsymbol{Z})_0(Y) = (H\boldsymbol{Z}/p)_0(X) = \boldsymbol{Z}/p$. Since $H\boldsymbol{Z}/p$ is a ring spectrum, $g\colon S^0 \to H\boldsymbol{Z} \wedge Y$ extends to a homotopy equivalence $\widetilde{g}\colon H\boldsymbol{Z}/p \to H\boldsymbol{Z} \wedge Y$. Then we have a map $f\colon Y \xrightarrow{\iota\wedge 1} H\boldsymbol{Z}\wedge Y \xleftarrow{\widetilde{g}} H\boldsymbol{Z}/p$ for the unit map ι of $H\boldsymbol{Z}$. Note that $[Y, \Sigma^k H\boldsymbol{Z}/p]_0$

$= (H\boldsymbol{Z}/p)^k(Y) = \mathrm{Hom}((H\boldsymbol{Z}/p)_k(Y), \boldsymbol{Z}/p) = \mathrm{Hom}((H\boldsymbol{Z}/p)_k(S\boldsymbol{Z}/p), \boldsymbol{Z}/p) = (H\boldsymbol{Z}/p)^k(S\boldsymbol{Z}/p) = 0$ unless $k = 0, 1$. Consider the Postnikov tower of $S\boldsymbol{Z}/p$:

$$\begin{array}{ccccccc} H\boldsymbol{Z}/p & \longleftarrow & X_1 & \longleftarrow & X_2 & \longleftarrow & \cdots \longleftarrow S\boldsymbol{Z}/p \\ \downarrow & & \downarrow & & \downarrow & & \\ \Sigma^2 H\pi_1(S\boldsymbol{Z}/p) & & \Sigma^3 H\pi_2(S\boldsymbol{Z}/p) & & \Sigma^4 H\pi_3(S\boldsymbol{Z}/p) & & \end{array}$$

Since $[Y, \Sigma^k H\pi_{k-1} S\boldsymbol{Z}/p]_0 = H^k(Y; \pi_{k-1}(S\boldsymbol{Z}/p)) = 0$ for $k > 1$, the map $f: Y \to H\boldsymbol{Z}/p$ is lifted up to an $H\boldsymbol{Z}/p$-equivalence $\widetilde{f}: Y \to S\boldsymbol{Z}/p$. The Spanier-Whitehead dual yields an $H\boldsymbol{Z}/p$-equivalence $D_p(\widetilde{f})$: $\Sigma^{-1} S\boldsymbol{Z}/p \to D_p(Y) = D_p(X) \wedge \Sigma^{-1} S\boldsymbol{Z}/p$. We do this for $D_p(X)$ instead of X, and obtain an $H\boldsymbol{Z}/p$-equivalence $f': S\boldsymbol{Z}/p \to Y$ since $D_p D_p(X) = X$. Since $\widetilde{f} f': S\boldsymbol{Z}/p \to S\boldsymbol{Z}/p$ is an $H\boldsymbol{Z}/p$-equivalence, it is an equivalence. Therefore, we have a decomposition $Y \overset{\simeq}{\to} S\boldsymbol{Z}/p \vee A$ for some spectrum A. Since $(H\boldsymbol{Z})_*(Y) = \boldsymbol{Z}/p$, we have $H\boldsymbol{Z}_*(A) = 0$ and so $L_{H\boldsymbol{Z}} Y \overset{\simeq}{\to} S\boldsymbol{Z}/p$.

Inductively suppose that we have the equivalences of the lemma up to r. Since $\delta_r \in [S\boldsymbol{Z}/p^r, S\boldsymbol{Z}/p]_{-1} = \boldsymbol{Z}/p$ is a generator, so is $1 \wedge \delta_r \in [L_{HZ}(X \wedge S\boldsymbol{Z}/p^r), L_{HZ}(X \wedge S\boldsymbol{Z}/p)]_{-1}$. Here $[L_{HZ}(X \wedge S\boldsymbol{Z}/p^r), L_{HZ}(X \wedge S\boldsymbol{Z}/p)]_{-1}$ is isomorphic to $[S\boldsymbol{Z}/p^r, S\boldsymbol{Z}/p]_{-1} = \boldsymbol{Z}/p$, since X is $H\boldsymbol{Z}/p$-invertible. Then the composite $\delta'_r: S\boldsymbol{Z}/p^r \overset{\simeq}{\leftarrow} L_{HZ}(X \wedge S\boldsymbol{Z}/p^r) \overset{1\wedge\delta_r}{\longrightarrow} L_{HZ}(X \wedge S\boldsymbol{Z}/p) \overset{\simeq}{\to} S\boldsymbol{Z}/p$ is not zero and so $\delta'_r = k\delta_r$ for some $k \neq 0 \in \boldsymbol{Z}/p$. Then we obtain the commutative diagram:

$$\begin{array}{ccccccc} L_{HZ}(X \wedge S\boldsymbol{Z}/p) & \overset{1\wedge p^r}{\longrightarrow} & L_{HZ}X \wedge S\boldsymbol{Z}/p^{r+1} & \overset{1\wedge j_r}{\longrightarrow} & L_{HZ}(X \wedge S\boldsymbol{Z}/p^r) & \overset{1\wedge \delta_r}{\longrightarrow} & \Sigma L_{HZ}X \wedge S\boldsymbol{Z}/p \\ \simeq\downarrow & & \vdots & & \downarrow\simeq & & \downarrow\simeq \\ S\boldsymbol{Z}/p & \longrightarrow & W_r & \longrightarrow & S\boldsymbol{Z}/p^r & \overset{\delta'_r}{\longrightarrow} & \Sigma S\boldsymbol{Z}/p \\ k^{-1}\downarrow & & \vdots & & \downarrow = & & \downarrow k^{-1} \\ S\boldsymbol{Z}/p & \overset{p^r}{\longrightarrow} & S\boldsymbol{Z}/p^{r+1} & \overset{j_r}{\longrightarrow} & S\boldsymbol{Z}/p^r & \overset{\delta_r}{\longrightarrow} & \Sigma S\boldsymbol{Z}/p \end{array}$$

Here W_r denotes a fiber of δ'_r. Now the dotted arrows give the desired equivalence. □

Theorem 4.3. $\mathrm{Pic}(\mathcal{L}_{HZ/p}) = \boldsymbol{Z}$.

Proof. Let $X \in \mathrm{Pic}(\mathcal{L}_{HZ/p})^0$. Then by [7, Corollary 2.2] with $n = 1$ and $E = H\boldsymbol{Z}$, we see that $X = \lim_r (L_{HZ}X \wedge S\boldsymbol{Z}/p^r)$, which is equivalent to $\lim_r S\boldsymbol{Z}/p^r = S\boldsymbol{Z}_p$ by Lemma 4.2. Since any element X of $\mathrm{Pic}(\mathcal{L}_{HZ/p})$ is a suspension of an element of $X \in \mathrm{Pic}(\mathcal{L}_{HZ/p})^0$, $X = \Sigma^t S\boldsymbol{Z}_p$ for some $t \in \boldsymbol{Z}$. Therefore, $\mathrm{Pic}(\mathcal{L}_{HZ/p}) = \boldsymbol{Z}$ whose generator is $\Sigma S\boldsymbol{Z}_p$. □

Proof of Theorem 1.1. Since $\langle H\boldsymbol{Z}/p\rangle > \langle I\rangle$, any spectrum E with $\langle E\rangle \geq \langle H\boldsymbol{Z}/p\rangle$ has a finite local. Theorem 1.1 is now a corollary of Corollary 3.6 and Theorem 4.3. □

§5. Some Remarks

Let $K(n)$ denote the n-th Morava K spectrum and consider the spectrum $H_n = \bigvee_{i\geq n} K(i)$. Then H_0-local (resp. H_0-acyclic) spectrum is called harmonic (resp. dissonant). A finite spectrum X is called a type n spectrum if $K(n-1)_*(X) = 0$ and $K(n)_*(X) \neq 0$. By [6] and [4], we have a finite ring spectrum MJ_n such that $BP_*(MJ_n) = BP_*/J_n$ for an invariant ideal $J_n = (p^{e_0}, v_1^{e_1}, \dots, v_{n-1}^{e_{n-1}})$ of BP_*. The spectrum MJ_n is a type n spectrum, that is $K(i) \wedge MJ_n = pt$ for $i < n$. Therefore, $H_0 \wedge MJ_n = \bigvee_{i\geq 0} K(i) \wedge MJ_n = H_n$.

Lemma 5.1. *For $n \geq 0$, $\langle H_n\rangle = \langle H_0 \wedge MJ_n\rangle$ for any MJ_n.*

Proposition 5.2. *H_n for $n \geq 0$ has a finite local.*

Proof. Since a finite spectrum is harmonic [14, Corollary 4.5], H_0 has a finite local. Therefore, H_n for $n > 0$ has a finite local by Lemma 5.1 and (3.1). □

Let $F(n)$ denote a finite spectrum of type n. Then it admits a self map $f\colon F(n) \to F(n)$ that induces the multiplication of v_n^a on BP_*-homology for some integer $a > 0$ by [6]. The cofiber of f yields $F(n+1)$, and so $\langle F(n)\rangle \geq \langle F(n+1)\rangle$ (*cf.* the class invariance theorem [15]). Furthermore, we denote a spectrum $f^{-1}F(n)$ as $\mathrm{Tel}(n)$. Note that the Bousfield class of $\mathrm{Tel}(n)$ depends only on n (*cf.* [7, Lemma 1.3]). Then the Bousfield class of $H_n^f = \bigvee_{i\geq n} \mathrm{Tel}(i)$ also depends only on n. Since $K(n)\wedge\mathrm{Tel}(n) \neq pt$, we see that $\langle\mathrm{Tel}(n)\rangle \geq \langle K(n)\rangle$ by [14, Theorem 2.1(h)], and so $\langle H_0^f\rangle \geq \langle E(n)\rangle$ for each $n \geq 0$. Indeed, $\langle E(n)\rangle = \langle\bigvee_{0\leq i\leq n} K(i)\rangle$.

These yield relations of Bousfield classes of spectra.

$$\langle S^0\rangle \geq \langle F(1)\rangle \geq \langle F(2)\rangle \geq \cdots \geq \langle H\boldsymbol{Z}/p\rangle \geq \langle I\rangle$$
$$\langle S^0\rangle \geq \langle X(1)\rangle \geq \langle X(2)\rangle \geq \cdots \geq \langle BP\rangle \geq \langle H\boldsymbol{Z}\rangle \geq \langle H\boldsymbol{Z}/p\rangle \geq \langle I\rangle$$
$$\langle S^0\rangle \geq \langle H_0^f\rangle \geq \langle H_1^f\rangle \geq \cdots \geq \langle I\rangle$$
$$\langle BP\rangle \geq \langle H_0\rangle \geq \langle H_1\rangle \geq \cdots \geq \langle I\rangle$$
$$\langle BP\rangle \geq \langle H_0\rangle \geq \cdots \geq \langle I \vee E(n)\rangle \geq \langle I \vee E(n-1)\rangle \geq \cdots$$
$$\cdots \geq \langle I \vee E(0)\rangle = \langle I \vee H\boldsymbol{Q}\rangle \geq \langle I\rangle$$
$$\langle H\boldsymbol{Z}/p\rangle \geq \langle IBP\rangle \geq \cdots \geq \langle IX(n)\rangle \geq \cdots \geq \langle IX(1)\rangle \geq \langle I\rangle$$

Here $X(i)$ is the spectrum given in [14], and IY is the Brown-Comenetz dual of Y. The second sequence is given in [14, p. 369] and the bottom sequence is given in [8, Theorem 8.4]. The Picard groups of the spectra appearing in the first two sequences except for I are all isomorphic to $\boldsymbol{Z}$ by Corollary 3.6 and Theorem 4.3. The other sequences yield the filtrations of $\mathrm{Pic}(\mathcal{L}_I)$

$$\begin{gathered}
\boldsymbol{Z} = \mathrm{Pic}(\mathcal{L}_{S^0}) \subset \mathrm{Pic}(\mathcal{L}_{H_0^f}) \subset \mathrm{Pic}(\mathcal{L}_{H_1^f}) \subset \cdots \subset \mathrm{Pic}(\mathcal{L}_I) \\
\boldsymbol{Z} = \mathrm{Pic}(\mathcal{L}_{BP}) \subset \mathrm{Pic}(\mathcal{L}_{H_0}) \subset \mathrm{Pic}(\mathcal{L}_{H_1}) \subset \cdots \subset \mathrm{Pic}(\mathcal{L}_I) \\
\boldsymbol{Z} = \mathrm{Pic}(\mathcal{L}_{BP}) \subset \mathrm{Pic}(\mathcal{L}_{H_0}) \subset \cdots \subset \mathrm{Pic}(\mathcal{L}_{I\vee E(n)}) \subset \mathrm{Pic}(\mathcal{L}_{I\vee E(n-1)}) \subset \cdots \\
\cdots \subset \mathrm{Pic}(\mathcal{L}_{I\vee E(0)}) \subset \mathrm{Pic}(\mathcal{L}_I) \\
\boldsymbol{Z} = \mathrm{Pic}(\mathcal{L}_{HZ/p}) \subset \mathrm{Pic}(\mathcal{L}_{IBP}) \subset \cdots \subset \mathrm{Pic}(\mathcal{L}_{IX(n)}) \subset \cdots \\
\cdots \subset \mathrm{Pic}(\mathcal{L}_{IX(1)}) \subset \mathrm{Pic}(\mathcal{L}_I)
\end{gathered}$$

by Corollary 3.6. We have a problem

Problem. Is $\mathrm{Pic}(\mathcal{L}_I) = \boldsymbol{Z}$?

If this has an affirmative answer, then for all spectra E with a finite local, $\mathrm{Pic}(\mathcal{L}_E) = \boldsymbol{Z}$ by Corollary 3.6. For each $X \in \mathrm{Pic}(\mathcal{L}_I)$, put $W_X = X \wedge D_p(X)$ and let F_X be the fiber of the localization map $W_X \to L_I W_X = S\boldsymbol{Z}_p$. For a spectrum E, let aE denote the spectrum given by Bousfield in [1, Lemma 1.13] (see also [2] for its proof). Then, we have $\langle aE\rangle \geq \langle C\rangle$ for any E-acyclic spectrum C.

Proposition 5.3. *The collection of $X \in \mathrm{Pic}(\mathcal{L}_I)$ such that $\mathrm{Pic}(\mathcal{L}_{aF_X}) = \boldsymbol{Z}$ consists of suspensions of $S\boldsymbol{Z}_p$. In particular, if $\mathrm{Pic}(\mathcal{L}_{aF_X}) = \boldsymbol{Z}$ for each $X \in \mathrm{Pic}(\mathcal{L}_I)$, then $\mathrm{Pic}(\mathcal{L}_I) = \boldsymbol{Z}$.*

Proof. By definition, F_X is I-acyclic, which implies that $\langle aF_X\rangle \geq \langle I\rangle$. Since F_X is also aF_X-acyclic, it follows that $L_{aF_X} W_X = S\boldsymbol{Z}_p$. Therefore, $X \in \mathrm{Pic}(\mathcal{L}_{aF_X}) = \boldsymbol{Z}$ and so X is a suspension of $S\boldsymbol{Z}_p$. □

Since $\langle C\rangle \leq \langle aE\rangle$ for an E-acyclic spectrum C as noted above, we see that $H\boldsymbol{Z}/p \wedge F_X = pt$ implies $\langle aF_X\rangle \geq \langle H\boldsymbol{Z}/p\rangle$, and so $\mathrm{Pic}(\mathcal{L}_{aF_X}) = \boldsymbol{Z}$ by Theorem 1.1.

Corollary 5.4. *Let X be an element of $\mathrm{Pic}(\mathcal{L}_I)$. If $(H\boldsymbol{Z}/p)_*(F_X) = 0$, then X is a suspension of $S\boldsymbol{Z}_p$.*

To study the filtration of $\mathrm{Pic}(\mathcal{L}_I)$, the next problem seems interesting.

Problem. Determine $\mathrm{Pic}(\mathcal{L}_{I\vee K(n)})$.

Next we consider an invertible spectrum that is also a ring spectrum. We deduce the following by the similar argument given in the previous sections.

Proposition 5.5. *An invertible ring spectrum is the sphere spectrum in $\mathcal{L}_E$ for any E.*

Proof. Let X be an invertible ring spectrum in $\mathcal{L}_E$. Then $X \wedge X = X \vee (X \wedge \overline{X})$ for a cofiber $\overline{X}$ of the unit map $S^0 \to X$. Let D_E denote the Spanier-Whitehead dual functor on $\mathcal{L}_E$. Since $D_E(X)$ is an inverse of X, we see that

$$\begin{aligned} X = L_E(D_E(X) \wedge X \wedge X) &= L_E(D_E(X) \wedge X) \vee L_E(D_E(X) \wedge X \wedge \overline{X}) \\ &= L_E S^0 \vee L_E \overline{X}. \end{aligned}$$

Thus $[S^0, L_E S^0]_*$ is a direct summand of $[X, X]_*$. On the other hand, $[X, X]_*$ is isomorphic to $[S^0, L_E S^0]_*$ under the map assigning $f\colon X \to X$ to $f \wedge 1\colon L_E(X \wedge D_E(X)) \to L_E(X \wedge D_E(X))$. Since the isomorphism is the one of $[S^0, L_E S^0]_*$-modules and it sends the identity to identity, the other direct summands of $[X, X]_*$ are zero. In particular, we obtain $L_E\overline{X} = pt$ as desired. □

In the category $\mathcal{L}_E$, a spectrum A is small if $[A, L_E(\bigvee_\alpha X_\alpha)]_* = \bigoplus_\alpha [A, X_\alpha]_*$ for any wedge sum $L_E(\bigvee_\alpha X_\alpha)$ in $\mathcal{L}_E$. In [10, Appendix B], Hovey and Strickland showed that if E has a finite local and is I-acyclic, then the category $\mathcal{L}_E$ does not have non-zero small objects, and that BP, H_0, $H\boldsymbol{Z}$, $H\boldsymbol{Z}/p$ and I are examples for E and $F(n)$ is not. Here, we give other examples. Since I and $K(n)$ are I-acyclic, we see the following:

Proposition 5.6. *For $n \geq 0$, $\mathcal{L}_{H_n}$, $\mathcal{L}_{I\vee E(n)}$ and $\mathcal{L}_{I\vee K(n)}$ do not have non-zero small objects.*

Since the sphere spectrum S^0 has a finite local, so does the finite spectrum $F(n)$ by (3.1). Furthermore, we see $\mathrm{Tel}(n) \wedge F(n+1) = pt$ by definition. Thus $\mathrm{Tel}(n)$ is $F(n+1)$-acyclic, and so is I-acyclic. Thus we have the following:

Proposition 5.7. *For $n \geq 0$, $\mathcal{L}_{H_n^f}$ does not have non-zero small objects.*

This proof indicates the following fact on spectra with a finite acyclic. Here a spectrum E has a finite acyclic if there is a finite spectrum $X \neq pt$ such that $L_E X = pt$.

Proposition 5.8. *If E has a finite acyclic, then E is I-acyclic.*

This follows from

Proposition 5.9. *If E admits an E-acyclic spectrum that has a finite local, then E is I-acyclic.*

Proof. Let X denote the E-acyclic spectrum of the proposition. Then $E \wedge X = pt$. Since $\langle X \rangle \geq \langle I \rangle$, we obtain $E \wedge I = pt$. □

Proposition 5.10. *Neither $\mathcal{L}_{X(n)}$ nor $\mathcal{L}_{IX(n)}$ has non-zero small objects.*

Proof. By [8, Theorem 8.4], $\langle I \rangle \leq \langle IX(n) \rangle \leq \langle X(n) \rangle$, and so both of $X(n)$ and $IX(n)$ have finite locals. It is also shown in [8, Lemma 7.1] that $X(n)$ is I-acyclic, and so is $IX(n)$. □

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