Publ. RIMS, Kyoto Univ.
43 (2007), 471–504

Modified Elastic Wave Equations on Riemannian and Kähler Manifolds

By

Yoshiyasu YASUTOMI*

Abstract

We introduce some geometrically invariant systems of differential equations on any Riemannian manifolds and also on any Kähler manifolds, which are natural extensions of the elastic wave equations on $\mathbb{R}^3$. Further we prove the local decomposition theorems of distribution solutions for those systems. In particular, the solutions of our systems on Kähler manifolds are decomposed into 4 solutions with different propagation speeds.

§0 Introduction and Results

Introduction

The elastic wave equation on $\mathbb{R}^3$ is written as follows:

$$\begin{aligned} Pu &:= \rho \frac{\partial^2}{\partial t^2} u - (\lambda + \mu)\text{grad div } u - \mu \Delta u \\ &= \rho \frac{\partial^2}{\partial t^2} u - (\lambda + 2\mu)\text{grad div } u + \mu \text{ rot rot } u = f, \end{aligned}$$

where u is a 3-dimensional vector field of the displacement of an elastic body, ρ is the density constant and $\lambda,\ \mu$ are the Lamé constants. It is well-known that any distribution solution u of $Pu = 0$ is decomposed into a sum $u = u_1 + u_2$ of solutions u_1, u_2 satisfying the following additional equations:

$$\text{rot } u_1 = 0, \quad \text{div } u_2 = 0.$$

Communicated by H. Okamoto. Received May 9, 2003. Revised August 17, 2005, February 1, 2006.
2000 Mathematics Subject Classification(s): 58A10, 35L10, 32C38.
*Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-0041, Japan.

We call u_1, u_2 a longitudinal wave solution and a transverse wave solution, respectively.

The elastic wave equations on Euclidean space are well studied in the Scattering theorem and propagation problems by Kawashita [6], Shimizu [7], and so on.

Real elastic waves (earthquakes) propagate through many layers which do not necessarily lie in good order. Therefore, more general study, we extend the elastic wave equations on Euclidean space to ones on Riemannian manifolds. In a physical generalization of this system of equations to Riemannian manifolds we replace div, rot with some covariant differentiations. However covariant differentiations do not commute with each other in general. Hence we consider the new elastic wave equations which do not depend on the choice of coordinates, and find the natural model of hyperbolic equations with multi-values and multi-modes extended from the theory of the elastic wave equations as the model to apply the polarization theory.

To begin with, in Chapter 2, we consider the physical generalization $P^{\mathrm{org}}\ u$ =0 of the elastic wave equation on a Riemannian manifold. In Chapter 3, we introduce a new differential equation $P_{\mathrm{R}}\ u = 0$ which is a modification on the lower order term of the original equation $P_{\mathrm{org}}\ u = 0$. Then we show that; the new differential equation admits a decomposition of any solutions into longitudinal wave solutions and transverse wave solutions. However, the original equation does not admit any similar decompositions in general. Moreover we generalize P_{R} and P_{org} to operators on p-differential forms. In Chapter 4, we deal with the differential equations $P_{\mathrm{C}}\ u = 0$, $P^*_{\mathrm{C}}\ u = 0$ on complex manifolds and $P_{\mathrm{K}}\ u = 0$ on Kähler manifolds. We show any distribution solutions of the differential equations $P_{\mathrm{C}}\ u = 0$ and $P^*_{\mathrm{C}}\ u = 0$ admit some decompositions into 2 solutions with different propagation speeds. In the same way, we also show that any distribution solution of the equation $P_{\mathrm{K}}\ u = 0$ admits a decomposition into 4 solutions with 4 different propagation speeds.

Results

Definition 0.1. Let M be an n-dimensional Riemannian manifold and $\widetilde{M} = \mathbb{R}_t \times M$. Let $u = \sum u^i \partial_i$ be a contravariant vector field on M with parameter t; precisely, a contravariant vector field on $\widetilde{M}$ with $\langle dt, u \rangle = 0$. We assume the density constant ρ and the Làme constants λ, μ are positive. Because the Riemannian metric tensor g_{ij} (and the inverse metric tensor g^{ij} of g_{ij}) and the covariant differentiation ∇_j are commutative on the Riemannian manifolds (cf. [2] Section 15), we define the original elastic wave equation as follows:

$$\begin{aligned} P^{\text{org}}\ u^i &:= \rho\frac{\partial^2}{\partial t^2}u^i - \lambda g^{ij}\nabla_j\nabla_k u^k - \mu g^{ik}\nabla_j\nabla_k u^j - \mu g^{jk}\nabla_j\nabla_k u^i \\ &= \rho\frac{\partial^2}{\partial t^2}u^i - \lambda\nabla^i\nabla_k u^k - \mu\nabla_j\nabla^i u^j - \mu\nabla_j\nabla^j u^i \\ &= \rho\frac{\partial^2}{\partial t^2}u^i - \lambda\nabla^i\nabla_k u^k - \mu\nabla_k\nabla^i u^k - \mu\nabla_k\nabla^k u^i = f^i, \end{aligned}$$

where we denote $\nabla^i = g^{ij}\nabla_j$ according to the custom (cf. [1] Section 26).

In this paper, we often omit $\sum$ by Einstein's convention.

Because of the duality between contravariant vector fields and covariant forms, and the fact that differential operators attach covariant vector (tensor), we consider a new differential equation

$$\begin{aligned} P_{\text{org}}\ u_i :&= \rho\frac{\partial^2}{\partial t^2}u_i - \lambda\nabla_i\nabla^k u_k - \mu\nabla^k\nabla_i u_k - \mu\nabla^k\nabla_k u_i \\ &= \rho\frac{\partial^2}{\partial t^2}u_i - (\lambda+2\mu)\nabla_i\nabla^k u_k + \mu\nabla^k\nabla_i u_k - \mu\nabla^k\nabla_k u_i - 2\mu R^l{}_i u_l \\ &= f_i, \end{aligned}$$

where $R^l{}_i$ is the Ricci tensor (cf. [1] Section 26).

When we put

$$P_{\text{R}}\ u_i := \rho\frac{\partial^2}{\partial t^2}u_i - (\lambda+2\mu)\nabla_i\nabla^k u_k + \mu\nabla^k\nabla_i u_k - \mu\nabla^k\nabla_k u_i,$$

the differential equation $P_{\text{R}}\ u_i = f_i$ on $\widetilde{M}$ is a modification on the part of order 0 of the differential equation $P_{\text{org}}u_i = f_i$, and we can rewrite $P_{\text{R}}\ u = \rho\frac{\partial^2}{\partial t^2}u + (\lambda+2\mu)d\delta u + \mu\delta du = f$ for a 1-differential form $u = u_i dx^i$ (cf. [1] Section 26). Here, d, δ are the exterior differential operator and the associated exterior differential operator on M, respectively.

The differential operators d, δ operate on p-differential forms for all p, then we extend the equations naturally to equations for p-differential forms.

Let $\bigwedge^{(p)} T^*M$ be a vector bundle of p-differential forms on M. Let $\mathcal{E}_M^{(p)}$ be a sheaf of p-forms on M with C^∞ coefficients, and $\mathcal{D}b_M^{(p)}$ a sheaf of p-currents on M; that is, p-forms with distribution coefficients. In this article, we do not mean distributions the dual space of $C_0^\infty(M)$. Our distributions behave as "functions" for coordinate transformations. Further we define $\widetilde{\mathcal{E}}_M^{(p)}$ and $\widetilde{\mathcal{D}b}_M^{(p)}$.

Definition 0.2. We denote by $\widetilde{\mathcal{E}}_M^{(p)}$, $\widetilde{\mathcal{D}b}_M^{(p)}$ the sheaves of sections of $\mathcal{E}_{\widetilde{M}}^{(p)}$, $\mathcal{D}b_{\widetilde{M}}^{(p)}$ which do not include the covariant vector dt. That is, setting the projection $\pi : \mathbb{R}_t \times M \to M$, we define

$$\widetilde{\mathcal{E}}_M^{(p)} := \mathcal{E}_{\widetilde{M}}^{(0)} \underset{\pi^{-1}\mathcal{E}_M^{(0)}}{\otimes} \pi^{-1}\mathcal{E}_M^{(p)}, \qquad \widetilde{\mathcal{D}b}_M^{(p)} := \mathcal{D}b_{\widetilde{M}}^{(0)} \underset{\pi^{-1}\mathcal{E}_M^{(0)}}{\otimes} \pi^{-1}\mathcal{E}_M^{(p)}.$$

For

$$u = \sum_{1 \le i_1 < \cdots < i_p \le n} u_{i_1 \cdots i_p}(t,x) dx^{i_1} \wedge \cdots \wedge dx^{i_p} \in \widetilde{\mathcal{D}b}_M^{(p)},$$

we define an operator P_{R} for $\widetilde{\mathcal{D}b}_M^{(p)}$ on M $(1 \le p \le n-1)$, where the coefficients $\{u_{i_1 \cdots i_p}\}$ are supposed to be alternating with respect to $(i_1 \cdots i_p)$.

Definition 0.3. We define sheaf-morphisms $P_{\mathrm{R}} : \widetilde{\mathcal{D}b}_M^{(p)} \longrightarrow \widetilde{\mathcal{D}b}_M^{(p)}$ by

$$P_{\mathrm{R}}\, u := \rho \frac{\partial^2}{\partial t^2} u + (\lambda + 2\mu) d\delta u + \mu \delta d u.$$

For $p = 1$, this equation is the covariant form of $P_{\mathrm{R}}\, u^i$.

When $p = 0$ or n, $P_{\mathrm{R}}\, u = 0$ reduces to a wave equation. Therefore we suppose $1 \le p \le n-1$.

For $u \in \widetilde{\mathcal{D}b}_M^{(p)}$, we define equations $\mathfrak{M}^{\mathrm{R}}$, $\mathfrak{M}_1^{\mathrm{R}}$, $\mathfrak{M}_2^{\mathrm{R}}$, $\mathfrak{M}_0^{\mathrm{R}}$ as follows:

$$\mathfrak{M}^{\mathrm{R}} : \quad P_{\mathrm{R}}\, u = 0,$$

$$\mathfrak{M}_1^{\mathrm{R}} : \begin{cases} P_{\mathrm{R}}\, u = 0, \\ du = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha \varDelta) u = 0, \\ du = 0, \end{cases}$$

$$\mathfrak{M}_2^{\mathrm{R}} : \begin{cases} P_{\mathrm{R}}\, u = 0, \\ \delta u = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \beta \varDelta) u = 0, \\ \delta u = 0, \end{cases}$$

$$\mathfrak{M}_0^{\mathrm{R}} : \begin{cases} P_{\mathrm{R}}\, u = 0, \\ du = 0, \\ \delta u = 0, \end{cases} \iff \begin{cases} \partial_t^2 u = 0, \\ du = 0, \\ \delta u = 0. \end{cases}$$

Here, $\alpha = (\lambda + 2\mu)/\rho$, $\beta = \mu/\rho$ and $\varDelta = d\delta + \delta d : \widetilde{\mathcal{D}b}_M^{(p)} \to \widetilde{\mathcal{D}b}_M^{(p)}$ is the Laplacian on M.

Further we define subsheaves $\mathcal{S}ol(\mathfrak{M}^{\mathrm{R}}; p)$, $\mathcal{S}ol(\mathfrak{M}_j^{\mathrm{R}}; p)$, $(j = 0, 1, 2)$ of $\widetilde{\mathcal{D}b}_M^{(p)}$ as follows: For $\mathfrak{N}^{\mathrm{R}} = \mathfrak{M}^{\mathrm{R}}, \mathfrak{M}_j^{\mathrm{R}}$,

$$\mathcal{S}ol(\mathfrak{N}^{\mathrm{R}}; p) := \left\{ u \in \widetilde{\mathcal{D}b}_M^{(p)} \;\middle|\; u \text{ satisfies } \mathfrak{N}^{\mathrm{R}} \right\}.$$

Then, we have the following theorem.

Theorem A (Theorem 3.1). *For any germ* $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{R}};p)\Big|_{(\overset{\circ}{t},\overset{\circ}{x})}$, *there exist some germs* $u_j \in \mathcal{S}ol(\mathfrak{M}_j^{\mathrm{R}};p)\Big|_{(\overset{\circ}{t},\overset{\circ}{x})}$ $(j=1,2)$ *such that* $u = u_1 + u_2$.

Furthermore, the equation $u = u_1 + u_2 = 0$ *implies* $u_1, u_2 \in \mathcal{S}ol(\mathfrak{M}_0^{\mathrm{R}};p)\Big|_{(\overset{\circ}{t},\overset{\circ}{x})}$. *Equivalently, we have the following exact sequence*:

$$0 \longrightarrow \mathcal{S}ol(\mathfrak{M}_0^{\mathrm{R}};p) \xrightarrow{F} \mathcal{S}ol(\mathfrak{M}_1^{\mathrm{R}};p) \oplus \mathcal{S}ol(\mathfrak{M}_2^{\mathrm{R}};p) \xrightarrow{G} \mathcal{S}ol(\mathfrak{M}^{\mathrm{R}};p) \longrightarrow 0,$$

where $F(U) = U \oplus (-U)$, $G(U_1 \oplus U_2) = U_1 + U_2$.

Let X be an n-dimensional complex manifold with a Hermitian metric, and $\bigwedge^{(q,r)} T^*X$ a vector bundle of (q,r)-type differential forms on X. Let $\mathcal{E}_X^{(q,r)}$ be a sheaf of (q,r)-forms on X with C^∞ coefficients, and $\mathcal{D}b_X^{(q,r)}$ a sheaf of (q,r)-currents on X. Setting $\widetilde{X} = \mathbb{R}_t \times X$, we also define $\widetilde{\mathcal{E}}_X^{(q,r)}$, $\widetilde{\mathcal{D}b}_X^{(q,r)}$ similarly to $\widetilde{\mathcal{E}}_M^{(p)}$, $\widetilde{\mathcal{D}b}_M^{(p)}$.

Definition 0.4. We define sheaf-morphisms P_{C}, $P_{\mathrm{C}}^* : \widetilde{\mathcal{D}b}_X^{(q,r)} \longrightarrow \widetilde{\mathcal{D}b}_X^{(q,r)}$ on $\widetilde{X}$ which are similar to P_{R}:

$$P_{\mathrm{C}} = \frac{\partial^2}{\partial t^2} + \alpha_1 \partial \overline{\vartheta} + \alpha_2 \overline{\vartheta} \partial, \qquad P_{\mathrm{C}}^* = \frac{\partial^2}{\partial t^2} + \alpha_3 \overline{\partial} \vartheta + \alpha_4 \vartheta \overline{\partial},$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are positive constants. Here, ∂, $\overline{\partial}$ are the exterior differential operator, the conjugate exterior differential operator on X, and $\overline{\vartheta}$, ϑ are the associated operators of ∂, $\overline{\partial}$, respectively.

For $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}$, we define equations $\mathfrak{M}^{\mathrm{C}}$, $\mathfrak{M}_1^{\mathrm{C}}$, $\mathfrak{M}_2^{\mathrm{C}}$, $\mathfrak{M}^{\mathrm{C}*}$, $\mathfrak{M}_3^{\mathrm{C}*}$, $\mathfrak{M}_4^{\mathrm{C}*}$ as follows:

$$\mathfrak{M}^{\mathrm{C}} : \quad P_{\mathrm{C}}\, u = 0,$$

$$\mathfrak{M}_1^{\mathrm{C}} : \begin{cases} P_{\mathrm{C}}\, u = 0, \\ \partial u = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha_1 \Box) u = 0, \\ \partial u = 0, \end{cases}$$

$$\mathfrak{M}_2^{\mathrm{C}} : \begin{cases} P_{\mathrm{C}}\, u = 0, \\ \overline{\vartheta} u = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha_2 \Box) u = 0, \\ \overline{\vartheta} u = 0, \end{cases}$$

$$\mathfrak{M}^{\mathbb{C}*} \;:\quad P_{\mathbb{C}}^{*}\, u = 0,$$

$$\mathfrak{M}_3^{\mathbb{C}*} : \begin{cases} P_{\mathbb{C}}^{*}\, u = 0, \\ \overline{\partial} u = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha_3 \overline{\square}) u = 0, \\ \overline{\partial} u = 0, \end{cases}$$

$$\mathfrak{M}_4^{\mathbb{C}*} : \begin{cases} P_{\mathbb{C}}^{*}\, u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} (\partial_t^2 + \alpha_4 \overline{\square}) u = 0, \\ \vartheta u = 0. \end{cases}$$

Here, $\square = \partial\overline{\vartheta} + \overline{\vartheta}\partial$ and $\overline{\square} = \overline{\partial}\vartheta + \vartheta\overline{\partial}$ are the complex Laplace-Beltrami operators.

Further we define subsheaves $\mathcal{S}ol(\mathfrak{M}^{\mathbb{C}}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_j^{\mathbb{C}}; q, r)$ $(j = 1, 2)$, $\mathcal{S}ol(\mathfrak{M}^{\mathbb{C}*}; q, r)$, $\mathcal{S}ol(\mathfrak{M}_k^{\mathbb{C}*}; q, r)$ $(k = 3, 4)$ of $\widetilde{\mathcal{D}b}_X^{(q,r)}$ as follows: For $\mathfrak{N}^{\mathbb{C}} = \mathfrak{M}^{\mathbb{C}}, \mathfrak{M}_j^{\mathbb{C}}$, $\mathfrak{M}^{\mathbb{C}*}, \mathfrak{M}_k^{\mathbb{C}*}$,

$$\mathcal{S}ol(\mathfrak{N}^{\mathbb{C}}; q, r) := \Big\{ u \in \widetilde{\mathcal{D}b}_M^{(q,r)} \;\Big|\; u \text{ satisfies } \mathfrak{N}^{\mathbb{C}} \Big\}.$$

Then, we get the following theorems.

Theorem B (Theorem 4.1). *For any germ $u \in \mathcal{S}ol(\mathfrak{M}^{\mathbb{C}}; q, r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$, there exist some germs $u_j \in \mathcal{S}ol(\mathfrak{M}_j^{\mathbb{C}}; q, r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$ $(j = 1, 2)$ such that $u = u_1 + u_2$.*

Theorem B′ (Theorem 4.2). *For any germ $u \in \mathcal{S}ol(\mathfrak{M}^{\mathbb{C}*}; q, r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$, there exist some germs $u_k \in \mathcal{S}ol(\mathfrak{M}_k^{\mathbb{C}*}; q, r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$ $(k = 3, 4)$ such that $u = u_3 + u_4$.*

Now we assume that X is a Kähler manifold; that is, for the Hermitian metric h, we have the equation $d\left(\sum h_{j\overline{k}}(z) dz^j \wedge d\overline{z}^k\right) = 0$, and we know that $h_{j\overline{k}}$ can be described as $h_{j\overline{k}} = \partial_j \overline{\partial}_k \phi$ with a smooth real function ϕ locally (cf. [3] Chapter 1, Section 7). Then the following equations for operators on $\widetilde{\mathcal{D}b}_X^{(q,r)}$ are well-known (cf. [4] Chapter 3, Section 2):

$$\text{(0.1)} \qquad \begin{cases} \square = \overline{\square} = \frac{1}{2}\Delta, \\ \partial\vartheta + \vartheta\partial = 0, \quad \overline{\partial}\,\overline{\vartheta} + \overline{\vartheta}\,\overline{\partial} = 0, \\ \partial\overline{\partial} + \overline{\partial}\partial = 0, \quad \vartheta\overline{\vartheta} + \overline{\vartheta}\vartheta = 0. \end{cases}$$

As for the relationship between the conditions (0.1) and the Kähler condition, we give a brief introduction and a proof of the equivalency in Appendix.

Definition 0.5. We define sheaf-morphisms $P_{\mathrm{K}} : \widetilde{\mathcal{D}b}_X^{(q,r)} \longrightarrow \widetilde{\mathcal{D}b}_X^{(q,r)}$ on $\widetilde{X}$ by

$$P_{\mathrm{K}} = \frac{\partial^2}{\partial t^2} + \alpha_1 \partial\overline{\vartheta} + \alpha_2 \overline{\vartheta}\partial + \alpha_3 \overline{\partial}\vartheta + \alpha_4 \vartheta\overline{\partial}.$$

Here, $\alpha_1, \alpha_2, \alpha_3$ and α_4 are positive constants.

When $q, r = 0$ or n, $P_{\mathrm{K}}\ u = 0$ reduces to a wave equation. When $q = 0, n$ or $r = 0, n$, P_{K} stands for P_{C}^* or P_{C}, respectively. Therefore, we suppose $1 \leq q, r \leq n-1$.

For $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}$, we define equations $\mathfrak{M}^{\mathrm{K}}$, $\mathfrak{M}_i^{\mathrm{K}}$ $(i = 1,2,3,4)$, $\mathfrak{M}_{jk}^{\mathrm{K}}$, $\mathfrak{M}_{jk0}^{\mathrm{K}}$ $\big((jk) = (13), (23), (14), (24)\big)$ as follows:

$$\mathfrak{M}^{\mathrm{K}} \ : \quad P_{\mathrm{K}}\ u = 0,$$

$$\mathfrak{M}_1^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\ u = 0, \\ \partial u = 0, \end{cases} \qquad \mathfrak{M}_2^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\ u = 0, \\ \overline{\vartheta} u = 0, \end{cases}$$

$$\mathfrak{M}_3^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\ u = 0, \\ \overline{\partial} u = 0, \end{cases} \qquad \mathfrak{M}_4^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\ u = 0, \\ \vartheta u = 0, \end{cases}$$

$$\mathfrak{M}_{13}^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\ u = 0, \\ \partial u = 0, \\ \overline{\partial} u = 0, \end{cases} \iff \begin{cases} \Big(\partial_t^2 + \dfrac{\alpha_1 + \alpha_3}{2} \Delta\Big) u = 0, \\ \partial u = 0, \\ \overline{\partial} u = 0, \end{cases}$$

$$\mathfrak{M}_{23}^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\ u = 0, \\ \overline{\vartheta} u = 0, \\ \overline{\partial} u = 0, \end{cases} \iff \begin{cases} \Big(\partial_t^2 + \dfrac{\alpha_2 + \alpha_3}{2} \Delta\Big) u = 0, \\ \overline{\vartheta} u = 0, \\ \overline{\partial} u = 0, \end{cases}$$

$$\mathfrak{M}_{14}^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\ u = 0, \\ \partial u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} \Big(\partial_t^2 + \dfrac{\alpha_1 + \alpha_4}{2} \Delta\Big) u = 0, \\ \partial u = 0, \\ \vartheta u = 0, \end{cases}$$

$$\mathfrak{M}_{24}^{\mathrm{K}} : \begin{cases} P_{\mathrm{K}}\ u = 0, \\ \overline{\vartheta} u = 0, \\ \vartheta u = 0, \end{cases} \iff \begin{cases} \Big(\partial_t^2 + \dfrac{\alpha_2 + \alpha_4}{2} \Delta\Big) u = 0, \\ \overline{\vartheta} u = 0, \\ \vartheta u = 0, \end{cases}$$

$$\mathfrak{M}_{130}^{\mathrm{K}}:\begin{cases}P_{\mathrm{K}}\ u=0,\\ \partial u=0,\\ \overline{\partial}u=0,\\ \Delta^2u=0,\end{cases}\iff\begin{cases}\partial_t^4u=\Delta^2u=0,\\ \Big(\partial_t^2+\dfrac{\alpha_1+\alpha_3}{2}\Delta\Big)u=0,\\ \partial u=0,\\ \overline{\partial}u=0,\end{cases}$$

$$\mathfrak{M}_{230}^{\mathrm{K}}:\begin{cases}P_{\mathrm{K}}\ u=0,\\ \overline{\vartheta}u=0,\\ \overline{\partial}u=0,\\ \Delta^2u=0,\end{cases}\iff\begin{cases}\partial_t^4u=\Delta^2u=0,\\ \Big(\partial_t^2+\dfrac{\alpha_2+\alpha_3}{2}\Delta\Big)u=0,\\ \overline{\vartheta}u=0,\\ \overline{\partial}u=0,\end{cases}$$

$$\mathfrak{M}_{140}^{\mathrm{K}}:\begin{cases}P_{\mathrm{K}}\ u=0,\\ \partial u=0,\\ \vartheta u=0,\\ \Delta^2u=0,\end{cases}\iff\begin{cases}\partial_t^4u=\Delta^2u=0,\\ \Big(\partial_t^2+\dfrac{\alpha_1+\alpha_4}{2}\Delta\Big)u=0,\\ \partial u=0,\\ \vartheta u=0,\end{cases}$$

$$\mathfrak{M}_{240}^{\mathrm{K}}:\begin{cases}P_{\mathrm{K}}\ u=0,\\ \overline{\vartheta}u=0,\\ \vartheta u=0,\\ \Delta^2u=0,\end{cases}\iff\begin{cases}\partial_t^4u=\Delta^2u=0,\\ \Big(\partial_t^2+\dfrac{\alpha_2+\alpha_4}{2}\Delta\Big)u=0,\\ \overline{\vartheta}u=0,\\ \vartheta u=0.\end{cases}$$

Further we define subsheaves $\mathcal{S}ol(\mathfrak{M}^{\mathrm{K}};q,r)$, $\mathcal{S}ol(\mathfrak{M}_i^{\mathrm{K}};q,r)$ $\big(i=1,2,3,4\big)$, $\mathcal{S}ol(\mathfrak{M}_{jk}^{\mathrm{K}};q,r)$, $\mathcal{S}ol(\mathfrak{M}_{jk0}^{\mathrm{K}};q,r)$ $\big((jk)=(13),(23),(14),(24)\big)$ of $\widetilde{\mathcal{D}b}_X^{(q,r)}$ as the sheaves of $\widetilde{\mathcal{D}b}_X^{(q,r)}$-solutions, respectively.

Then, we have the following theorem.

Theorem C (Theorem 4.3). *For any germ $u\in\mathcal{S}ol(\mathfrak{M}^{\mathrm{K}};q,r)\Big|_{(\mathring{t},\mathring{z})}$, there exist some germs $u_{ij}\in\mathcal{S}ol(\mathfrak{M}_{ij}^{\mathrm{K}};q,r)\Big|_{(\mathring{t},\mathring{z})}$ $((ij)=(13),(23),\ (14),(24))$ such that $u=u_{13}+u_{23}+u_{14}+u_{24}$.*

Further, we find that $u=u_{13}+u_{23}+u_{14}+u_{24}=0$ implies

$$u_{jk}\in\mathcal{S}ol(\mathfrak{M}_{jk0}^{\mathrm{K}};q,r)\quad\big((jk)=(13),(23),(14),(24)\big).$$

Equivalently, we have the following exact sequence:

$$0\longrightarrow{\bigoplus_{(ij)}}'\mathcal{S}ol(\mathfrak{M}_{ij0}^{\mathrm{K}};q,r)\xrightarrow{G}\bigoplus_{(ij)}\mathcal{S}ol(\mathfrak{M}_{ij}^{\mathrm{K}};q,r)\xrightarrow{H}\mathcal{S}ol(\mathfrak{M}^{\mathrm{K}};q,r)\longrightarrow0.$$

Here,

$$\bigoplus_{(ij)}{}' \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}_{ij0}; q, r) := \Big\{ (u_{ij}) \in \bigoplus_{(ij)} \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}_{ij0}; q, r) \;\Big|\; \sum_{(ij)} u_{ij} = 0 \Big\},$$

$G(U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}) = U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}$, $H(U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}) = U_{13} + U_{23} + U_{14} + U_{24}$.

§1. Preparation from Riemannian Geometry

In this section, we recall some notations and terminologies in Riemannian geometry used in this paper according to [1] (Chapter 2,5), [2] (Chapter 3), and [5] (Chapter 1,4).

We assume that M is oriented. Then, there is a global section Ω of $\mathcal{E}^{(n)}_M$ on M, which never vanishes on M.

Definition 1.1. The inner products $\langle \cdot, \cdot \rangle : \bigwedge^{(1)} T^*_x M \times \bigwedge^{(1)} T_x M \to \mathbb{R}$, $\langle \cdot, \cdot \rangle^* : \bigwedge^{(p)} T^*_x M \times \bigwedge^{(p)} T^*_x M \to \mathbb{R}$, are defined as follows. We choose a local positive orthonormal system $(\omega^1, \ldots, \omega^n)$ of C^∞ sections of T^*M concerning the Riemannian metric; that is, there is a positive valued C^∞ function α such that $\omega^1 \wedge \cdots \wedge \omega^n = \alpha\Omega > 0$, and for $\omega^i = \sum_{j=1}^n a_{ij} dx^j (i = 1, 2, \ldots, n)$ with a local coordinate system $(x^1, \ldots, x^n)$, we have $g_{ij} = \sum_{k=1}^n a_{ki} a_{kj}$. Then for

$$\sigma = \sum_{1 \le i \le n} \sigma_i dx^i, \qquad \tau = \sum_{1 \le i \le n} \tau^i \partial_i \ ,$$

we define

$$\langle \sigma, \tau \rangle := \sum_{1 \le i \le n} \sigma_i \tau^i,$$

and for

$$\phi = \sum_{1 \le i_1 < \cdots < i_p \le n} \phi_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

$$\psi = \sum_{1 \le i_1 < \cdots < i_p \le n} \psi_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

we define

$$\begin{aligned} \langle \phi, \psi \rangle^* &:= \sum_{1 \le i_1 < \cdots < i_p \le n} \phi_{i_1 \cdots i_p} \psi^{i_1 \cdots i_p} \\ &:= \sum_{\substack{1 \le i_1 < \cdots < i_p \le n \\ 1 \le j_1 < \cdots < j_p \le n}} \phi_{i_1 \cdots i_p} g^{i_1 j_1} \cdots g^{i_p j_p} \psi_{j_1 \cdots j_p}. \end{aligned}$$

Definition 1.2. We denote by $d : \mathcal{D}b_M^{(p)} \to \mathcal{D}b_M^{(p+1)}$ the exterior differential operator which acts on $\mathcal{D}b_M^{(p)}$ as a sheaf-morphism. Then the following formulas are well-known:

$$\begin{cases} d(\phi \pm \psi) = d\phi \pm d\psi & (\phi, \psi \in \mathcal{D}b_M^{(p)}), \\ d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi & (\phi \in \mathcal{D}b_M^{(p)}, \psi \in \mathcal{D}b_M^{(q)}), \\ d(d\phi) = 0 & (\phi \in \mathcal{D}b_M^{(p)}), \\ \text{for } f \in \mathcal{D}b_M^{(0)},\ df := \sum \dfrac{\partial f}{\partial x_j} dx^j \in \mathcal{D}b_M^{(1)}. \end{cases}$$

Here $0 \leq p \leq n$. If $p = n$, $d\phi = 0$ holds.

Definition 1.3. The isomorphism $* : \bigwedge T^*M \to \bigwedge T^*M$ of vector bundle is defined as follows:

$$\begin{cases} * : \bigwedge^{(p)} T_x^*M \mapsto \bigwedge^{(n-p)} T_x^*M \quad \text{is a linear map,} \\ *(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}) = \delta\binom{1 \cdots\cdots\cdots\cdots\cdots n}{i_1 \cdots i_p j_1 \cdots j_{n-p}} \omega^{j_1} \wedge \cdots \wedge \omega^{j_{n-p}} \\ \qquad\qquad = (-1)^{(i_1-1)+\cdots+(i_p-p)} \omega^{j_1} \wedge \cdots \wedge \omega^{j_{n-p}}, \\ \qquad \text{for any permutation } (i_1, \ldots, i_p, j_1, \ldots, j_{n-p}) \text{ of } (1, \ldots, n). \end{cases}$$

Here $(i_1 \cdots i_p)$ and $(j_1 \cdots j_{n-p})$ are indices satisfying

$$\begin{cases} (i_1 \cdots i_p j_1 \cdots j_{n-p}) \text{ is a permutation of } (1 \cdots n), \\ 1 \leq i_1 < \cdots < i_p \leq n,\ 1 \leq j_1 < \cdots < j_{n-p} \leq n. \end{cases}$$

Remark. The definition above does not depend on the choice of the positive orthonormal system $\{\omega^1, \ldots, \omega^n\}$.

Proposition 1.1. *We set $\phi, \psi \in \bigwedge^{(p)} T_x^*M$. Then we obtain*

$$\begin{cases} \phi \wedge *\psi = (*\phi) \wedge \psi = \langle \phi, \psi \rangle^*\ \omega^1 \wedge \cdots \wedge \omega^n, \\ *1 = \omega^1 \wedge \cdots \wedge \omega^n = \sqrt{g}\ dx^1 \wedge \cdots \wedge dx^n, \\ *\phi = (-1)^{(i_1-1)+\cdots+(i_p-p)} \sqrt{g}\ g^{i_1 j_1} \cdots g^{i_p j_p} \phi_{i_1 \cdots i_p}\ dx^{j_1} \wedge \cdots \wedge dx^{j_{n-p}} \\ \qquad \in \bigwedge^{(n-p)} T_x^*M. \end{cases}$$

Here $g = \det(g_{kl})$.

Let $U \subset M$ be an open subset. Let $\alpha^{(p)} \in \mathcal{D}b_M^{(p)}(U)$, $\beta^{(p)} \in \mathcal{E}_M^{(p)}(U)$ be sections. We suppose that $\beta^{(p)}$ has a compact support in U. Then the following integral is well-defined:

$$(\alpha^{(p)}, \beta^{(p)}) := \int_M \langle \alpha^{(p)}, \beta^{(p)} \rangle^*\ \omega^1 \wedge \cdots \wedge \omega^n.$$

Definition 1.4. Let $\alpha^{(p)} \in \mathcal{D}b_M^{(p)}$, $\beta^{(p-1)} \in \mathcal{E}_M^{(p-1)}$ be sections. We suppose $\beta^{(p-1)}$ has a compact support. Then the sheaf-morphism $\delta : \mathcal{D}b_M^{(p)} \to \mathcal{D}b_M^{(p-1)}$ is defined as

$$(\delta\alpha^{(p)}, \beta^{(p-1)}) = (\alpha^{(p)}, d\beta^{(p-1)}).$$

Hence we have

$$\delta = (-1)^{n(p-1)+1} * d * .$$

Definition 1.5. Let $\mathfrak{X}_s^r$ be the sheaf of $\bigotimes^r T_x M \otimes \bigotimes^s T_x^* M$-valued C^∞ functions, and $\mathcal{D}b_s^r$ the sheaf of $\bigotimes^r T_x M \otimes \bigotimes^s T_x^* M$-valued distributions. Then, the sheaf-morphisms $\nabla : \mathfrak{X}_s^r \to \mathfrak{X}_{s+1}^r$, $\mathcal{D}b_s^r \to \mathcal{D}b_{s+1}^r$ are defined as follows:

$$\begin{cases} \text{for } a(x) \in \mathfrak{X}_0^0, & \text{we have} \quad \nabla a(x) = \dfrac{\partial a}{\partial x^j} dx^j, \\ \text{for } \dfrac{\partial}{\partial x^j} \in \mathfrak{X}_0^1, & \text{we have} \quad \nabla \left(\dfrac{\partial}{\partial x^j}\right) = \mathit{\Gamma}_{j\ k}^{\ i} \dfrac{\partial}{\partial x^i} \otimes dx^k, \\ \text{for } dx^j \in \mathfrak{X}_1^0, & \text{we have} \quad \nabla\left(dx^j\right) = -\mathit{\Gamma}_{i\ k}^{\ j} dx^i \otimes dx^k, \\ \text{for } e \in \mathfrak{X}_s^r, f \in \mathfrak{X}_{s'}^{r'}, & \text{we have} \quad \nabla(e \otimes f) = (\nabla e) \otimes f + e \otimes \nabla f. \end{cases}$$

Here,

$$\left\{ \mathit{\Gamma}_{i\ k}^{\ j} = g^{jl} \mathit{\Gamma}_{ilk} = g^{jl} \cdot \frac{1}{2} \left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^l} \right) \right\}$$

are the Riemann-Christoffel symbols.

Proposition 1.2. *We set*

$$e = e_{i_1 \cdots i_s}^{j_1 \cdots j_r} dx^{i_1} \otimes \cdots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}} \in \mathfrak{X}_s^r.$$

Then we have

$$\begin{aligned} \nabla e = &\left(\partial_k e_{i_1 \cdots i_s}^{j_1 \cdots j_r} + e_{i_1 \cdots i_s}^{j_1 \cdots j_{p-1} q j_{p+1} \cdots j_r} \mathit{\Gamma}_{q\ k}^{\ j_p} - e_{i_1 \cdots i_{p-1} q i_{p+1} \cdots i_s}^{j_1 \cdots j_r} \mathit{\Gamma}_{i_p\ k}^{\ q} \right) \\ &\times dx^{i_1} \otimes \cdots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}} \otimes dx^k. \end{aligned}$$

Hence we call the following *the covariant differentiation*:

$$\begin{aligned} \nabla_k e = &\left(\partial_k e_{i_1 \cdots i_s}^{j_1 \cdots j_r} + e_{i_1 \cdots i_s}^{j_1 \cdots j_{p-1} q j_{p+1} \cdots j_r} \mathit{\Gamma}_{q\ k}^{\ j_p} - e_{i_1 \cdots i_{p-1} q i_{p+1} \cdots i_s}^{j_1 \cdots j_r} \mathit{\Gamma}_{i_p\ k}^{\ q} \right) \\ &\times dx^{i_1} \otimes \cdots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}}. \end{aligned}$$

§2. Elastic Mechaniques on Riemannian Manifolds

Let M be an n-dimensional Riemannian manifold with metric g. We consider an elastic body G in M. A motion of G is identified with an open subset $\widetilde{G}$ of $\widetilde{M}$ with one parameter family of diffeomorphisms $h_t(\cdot)$ $(t \in \mathbb{R})$:

$$h_s : \widetilde{G} \cap \{t = 0\} \xrightarrow{\sim} \widetilde{G} \cap \{t = s\}.$$

Then the elastic wave equation for G is formulated as a time development equation for small motions of G; that is, h_t is close to the identity map and it is expressed as

$$h_t^i(x) = x^i + u^i(x,t)$$

in a local coordinate system $(x^1, \dots, x^n)$, where $u = \sum u^i(x,t)\partial_i$ is the small displacement vector field.

Then, the differential dh of the map h and its dual map dh^* are given as follows:

$$dh : T_xM \ni \xi^i \longmapsto \eta^i = \xi^j \left(\frac{\partial h^i}{\partial x^j}\right) = \xi^i + \frac{\partial u^i}{\partial x^j}\xi^j \in T_{h(x)}M,$$

$$dh^* : T^*_{h(x)}M \ni \eta_i \longmapsto \xi_j = \eta_i \left(\frac{\partial h^i}{\partial x^j}\right) = \eta_j + \frac{\partial u^i}{\partial x^j}\eta_i \in T^*_xM.$$

Let us calculate the difference between the line element of M

$$ds(x)^2 = g_{ij}(x)\ dx^i \otimes dx^j$$

and its pull-back ${}^*ds(x)^2$ by h:

$$\begin{aligned}\varepsilon_{ij}(x)\ dx^i \otimes dx^j :=& \frac{1}{2}\left\{{}^*ds(x)^2 - ds(x)^2\right\}\\ =& \frac{1}{2}\left\{g_{kl}(h(x))\frac{\partial h^k}{\partial x^i}\frac{\partial h^l}{\partial x^j}dx^i \otimes dx^j - g_{ij}(x)dx^i \otimes dx^j\right\}.\end{aligned}$$

Here ε_{ij} is called the strain tensor. By ignoring the non-linear terms of u^m, $\partial_k u^m$, $\partial_l u^m$, we have

$$\varepsilon_{kl} = \frac{1}{2}\left(g_{km}\partial_l u^m + g_{ml}\partial_k u^m + u^m \partial_m g_{kl}\right).$$

On Riemannian spaces, equations

$$\nabla_m g_{kl} = \partial_m g_{kl} - \Gamma^n_{ml} g_{kn} - \Gamma^n_{mk} g_{nl} = 0$$

hold (cf. [2] Section 15), so we have

$$
\begin{aligned}
\varepsilon_{kl} &= \frac{1}{2}\left\{g_{km}\partial_l u^m + g_{ml}\partial_k u^m + u^m(\Gamma^n_{ml}g_{kn} + \Gamma^n_{mk}g_{nl})\right\} \\
&= \frac{1}{2}\left\{g_{km}(\partial_l u^m + u^n\Gamma^m_{nl}) + g_{ml}(\partial_k u^m + u^n\Gamma^m_{nk})\right\} \\
&= \frac{1}{2}\left(g_{km}\nabla_l u^m + g_{ml}\nabla_k u^m\right).
\end{aligned} \tag{2.1}
$$

In physics, we assume that the stress tensor σ^{ji} has a linear relationship with the strain tensor at each point. Hence there exists the elastic coefficient tensor E^{jilk} such that

$$
\frac{\sigma^{ji}}{\sqrt{g}} = E^{jilk}\varepsilon_{kl}. \tag{2.2}
$$

As a physical assumption for E^{ijkl}, we have the equations:

$$
E^{ijkl} = E^{jikl} = E^{ijlk}.
$$

In particular, it is well-known that the elastic coefficient tensor of an isotropic elastic body has the following form:

$$
E^{ijkl} = \lambda g^{ij}g^{kl} + \mu g^{ik}g^{jl} + \mu g^{il}g^{jk}, \tag{2.3}
$$

where λ and μ are the two Láme constants. Therefore, from (2.1), (2.2) and (2.3), we get

$$
\begin{aligned}
\frac{\sigma^{ji}}{\sqrt{g}} &= E^{ijkl}\varepsilon_{kl} \\
&= (\lambda g^{ij}g^{kl} + \mu g^{ik}g^{jl} + \mu g^{il}g^{jk})\left(\frac{1}{2}g_{mk}\nabla_l u^m + \frac{1}{2}g_{ml}\nabla_k u^m\right) \\
&= \lambda g^{ij}\nabla_l u^l + \mu g^{jl}\nabla_l u^i + \mu g^{il}\nabla_l u^j.
\end{aligned} \tag{2.4}
$$

Hence the equation of power-balance between the stress of the elastic body and the external force is written as follows:

$$
df^i = \frac{\sigma^{ji}}{\sqrt{g}}dS_j,
$$

where df^i is the external force vector for the surface element dS_j.

In order to introduce the elastic wave equation, we consider a small neighborhood V of a point x in M, whose boundary is given by a smooth closed surface S. Then the equation of motion for this small part V of the elastic body is written as follows:

$$\int_V \rho \frac{\partial^2 u^i}{\partial t^2} dV = -\int_S df^i = -\int_S \frac{\sigma^{ji}}{\sqrt{g}} dS_j = \int_V \nabla_j \left(\frac{\sigma^{ji}}{\sqrt{g}}\right) dV.$$

Here $dV = \sqrt{g}\ dx^1 \cdots dx^k$ with $\sqrt{g} = \sqrt{\det(g_{kl})}$. We divide both sides by the volume $|V|$ and shrink V to $\{x\}$. Then from (2.4), we have the elastic equation on Riemannian manifolds:

$$\rho \frac{\partial^2 u^i}{\partial t^2} = \nabla_j \left(\frac{\sigma^{ji}}{\sqrt{g}}\right) = \lambda g^{ij} \nabla_j \nabla_k u^k + \mu g^{jk} \nabla_j \nabla_k u^i + \mu g^{ik} \nabla_j \nabla_k u^j. \tag{2.5}$$

§3. Decomposition of Modified Elastic Wave Equations on Riemannian Manifolds

We set $\alpha = (\lambda + 2\mu)/\rho$, $\beta = \mu/\rho$. Then, we have the next lemmas.

Lemma 3.1. *For any $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{R}}; p)$ and the variety*

$$\begin{aligned} V_{\mathrm{R}} &:= \bigcup_{k=1,2} \left\{ (t, x; \tau, \xi) \;\middle|\; \tau^2 - c_k |\xi|^2 = 0 \right\} \\ &= \bigcup_{k=1,2} \left\{ (t, x; \tau, \xi) \;\middle|\; \tau^2 - c_k g^{ij}(x) \xi_i \xi_j = 0 \right\}, \end{aligned}$$

we have $\mathrm{WF}(u) \subset V_{\mathrm{R}}$*, where* $c_1 = \alpha$, $c_2 = \beta$.

Proof. The symbol of the second-order operator $\rho^{-1} P_{\mathrm{R}}\ (t, x, \partial_t, \partial_x)$ at $\overset{\circ}{pt} = (\overset{\circ}{x}; \overset{\circ}{\xi}) \in T^*M$ is defined as a linear operator

$$\sigma_2(\rho^{-1} P_{\mathrm{R}}\ )(\overset{\circ}{pt}) : \bigwedge\nolimits^{(p)} T^*_{\overset{\circ}{x}} M \to \bigwedge\nolimits^{(p)} T^*_{\overset{\circ}{x}} M$$

given by the following:

$$\sigma_2(\rho^{-1} P_{\mathrm{R}}\ )(\overset{\circ}{pt}) U := \lim_{\lambda \to \infty} \frac{1}{\lambda^2} e^{-\lambda i([x, \overset{\circ}{\xi}] + t \overset{\circ}{\tau})} \rho^{-1} P_{\mathrm{R}}\ (\overset{\circ}{t}, \overset{\circ}{x}, \partial_t, \partial_x) \left(e^{\lambda i([x, \overset{\circ}{\xi}] + t \overset{\circ}{\tau})} U \right),$$

where we consider the coefficients of $U \in \bigwedge^{(p)} T^*_{\overset{\circ}{x}} M$ as constants in a local coordinate system. Here $[x, \overset{\circ}{\xi}] = x_l \overset{\circ}{\xi}_l$. We suppose $(\overset{\circ}{\tau}, \overset{\circ}{\xi}) \neq (0, 0)$. Therefore we get

$$\begin{aligned} \sigma_2(\rho^{-1} P_{\mathrm{R}}\ )(\overset{\circ}{pt}) U =& \sigma_2(\partial_t^2 + \alpha d\delta + \beta \delta d) U \\ =& -\overset{\circ}{\tau}{}^2 U + (-1)^{n(p-1)} \alpha (\tilde{\xi} \wedge (*(\tilde{\xi} \wedge *U))) \\ &+ (-1)^{np} \beta (*(\tilde{\xi} \wedge *(\tilde{\xi} \wedge U))). \end{aligned}$$

Here $\tilde{\xi} := \sum \overset{\circ}{\xi}_j dx^j$. Let $\omega = (\omega_1, \dots, \omega_n)$ be a positive orthonormal system at $T^*_{\overset{\circ}{x}} M$ satisfying $\omega_1 = \tilde{\xi}/|\tilde{\xi}|$. We set $U = U_1 + U_2$ where $U_1 = \sum_{1 \in I} U_{1I}\omega^I$, $U_2 = \sum_{1 \notin I} U_{2I}\omega^I$, then we get $\tilde{\xi} \wedge U_1 = 0$, $\tilde{\xi} \wedge * U_2 = 0$. Therefore we have

$$\begin{aligned}
&*(\tilde{\xi} \wedge *(\tilde{\xi} \wedge U)) = * \left(\tilde{\xi} \wedge * \left(|\tilde{\xi}| \sum_{1 \notin I} U_{2I} \omega_1 \wedge \omega^I \right) \right) = (-1)^{np} |\tilde{\xi}|^2 U_2, \\
&\tilde{\xi} \wedge (*(\tilde{\xi} \wedge *U)) = (-1)^{n(p-1)} |\tilde{\xi}|^2 U_1, \\
&\sigma_2(\rho^{-1} P_{\mathrm{R}})(\overset{\circ}{pt}) U = (-\overset{\circ}{\tau}{}^2 + \alpha |\tilde{\xi}|^2) U_1 + (-\overset{\circ}{\tau}{}^2 + \beta |\tilde{\xi}|^2) U_2.
\end{aligned}$$

Thus we find that $\sigma_2(\rho^{-1} P_{\mathrm{R}})(\overset{\circ}{pt})$ has 2 eigenvalues $(-\overset{\circ}{\tau}{}^2 + \alpha|\tilde{\xi}|^2)$, $(-\overset{\circ}{\tau}{}^2 + \beta|\tilde{\xi}|^2)$, and their multiplicities are ${}_{n-1}C_{p-1}$, ${}_{n-1}C_p$, respectively. Hence we get

$$\det(\sigma_2(\rho^{-1} P_{\mathrm{R}})(\overset{\circ}{pt})) = (-\overset{\circ}{\tau}{}^2 + \alpha|\tilde{\xi}|^2)^{{}_{n-1}C_{p-1}} (-\overset{\circ}{\tau}{}^2 + \beta|\tilde{\xi}|^2)^{{}_{n-1}C_p}.$$

If $\sigma_2(\rho^{-1} P_{\mathrm{R}})(\overset{\circ}{pt})$ is an isomorphism, $\overset{\circ}{pt}$ does not belong to the characteristic variety of $P_{\mathrm{R}} u = 0$. Therefore, for a characteristic point $\overset{\circ}{pt}$, we have $\det(\sigma_2(\rho^{-1} P_{\mathrm{R}})(\overset{\circ}{pt})) = 0$. Hence,

$$\begin{aligned}
\mathrm{WF}(u) \subset V_{\mathrm{R}} &= \bigcup_{k=1,2} \left\{ (t, x; \tau, \xi) \;\middle|\; \tau^2 - c_k |\xi|^2 = 0 \right\} \\
&= \bigcup_{k=1,2} \left\{ (t, x; \tau, \xi) \;\middle|\; \tau^2 - c_k g^{ij}(x) \xi_i \xi_j = 0 \right\}.
\end{aligned}$$

□

Lemma 3.2. *For a germ $u \in \widetilde{\mathcal{D}b}{}^{(p)}_M$ at $(\overset{\circ}{t}, \overset{\circ}{x})$, we assume* $\mathrm{WF}(u) \not\ni (\overset{\circ}{t}, \overset{\circ}{x}; \pm dt)$. *Then, there exists a germ $w \in \widetilde{\mathcal{D}b}{}^{(p)}_M$ at $(\overset{\circ}{t}, \overset{\circ}{x})$, which satisfies $\Delta w = u$ and* $\mathrm{WF}(w) \not\ni (\overset{\circ}{t}, \overset{\circ}{x}; \pm dt)$.

Proof. For a differential form $u = \sum u_J(t,x) dx^J$, we write $\Delta u = \sum \{ P_{JK}(x, \partial_x) u_K(t,x) \} dx^J$. Δ is an elliptic operator on $\mathcal{D}b^{(p)}_M$. Therefore, in a neighborhood of $(\overset{\circ}{x}, \overset{\circ}{x}) \in M \times M$, there exist integral kernels $\{G_{KL}(x,y)\}_{KL}$, which satisfy

$$\begin{cases}
\sum P_{JK}(x, \partial_x) G_{KL}(x, y) = \delta_{J,L} \cdot \delta(x - y), \\
\mathrm{WF}(G_{KL}) \subset \left\{ (x, y; \xi, \eta) \;\middle|\; x = y, \xi = -\eta \right\}.
\end{cases}$$

Here, $\delta_{J,L}$ is a Kronecker's delta. Hence,

$$w \equiv \sum \Big(\int G_{KL}(x,y) u_L(t,y) \psi(y) dy \Big) dx^K$$

satisfies $\Delta w = u$ at $(\overset{\circ}{t}, \overset{\circ}{x})$ and $\mathrm{WF}(w) \not\ni (\overset{\circ}{t}, \overset{\circ}{x}; \pm dt)$. Here $\psi \in C_0^\infty(M)$ has a compact support in a small neighborhood of $\overset{\circ}{x}$ and $\psi(y) \equiv 1$ near $\overset{\circ}{x}$. □

Then, we get the following theorem.

Theorem 3.1. *For any germ* $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{R}}; p) \Big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$*, there exist some germs* $u_j \in \mathcal{S}ol(\mathfrak{M}_j^{\mathrm{R}}; p) \Big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$ $(j = 1, 2)$ *such that* $u = u_1 + u_2$*.*

Furthermore, the equation $u = u_1 + u_2 = 0$ *implies* $u_1, u_2 \in \mathcal{S}ol(\mathfrak{M}_0^{\mathrm{R}}; p) \Big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$*. Equivalently, we have the following exact sequence:*

$$0 \longrightarrow \mathcal{S}ol(\mathfrak{M}_0^{\mathrm{R}}; p) \xrightarrow{F} \mathcal{S}ol(\mathfrak{M}_1^{\mathrm{R}}; p) \oplus \mathcal{S}ol(\mathfrak{M}_2^{\mathrm{R}}; p) \xrightarrow{G} \mathcal{S}ol(\mathfrak{M}^{\mathrm{R}}; p) \longrightarrow 0,$$

where $F(U) = U \oplus (-U)$, $G(U_1 \oplus U_2) = U_1 + U_2$*.*

Proof. For $u \in \widetilde{\mathcal{D}b}_M^{(p)} \Big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$, we suppose u_1, u_2 are of the form;

$$u_2 = \delta v, \quad u_1 = u - \delta v.$$

Here $v \in \widetilde{\mathcal{D}b}_M^{(p+1)} \Big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$. Hence we have only to impose the following conditions on v:

$$\begin{cases} P_{\mathrm{R}}\,(\delta v) = 0, \\ d(u - \delta v) = 0, \\ dv = 0. \end{cases}$$

Since the equation $P_{\mathrm{R}}\,(\delta v) = \left(\partial_t^2 + \beta \delta d\right) \delta v = \delta \left(\partial_t^2 + \beta \Delta\right) v$ holds, it is sufficient to impose the next conditions:

$$\begin{cases} \partial_t^2 v + \beta \Delta v = 0, \\ d\delta v = du, \\ dv = 0, \end{cases} \iff \begin{cases} \partial_t^2 v = -\beta du, \\ \Delta v = du, \\ dv = 0. \end{cases}$$

From Lemma 3.1 and Lemma 3.2, we have a w satisfying $\Delta w = u$ and $\mathrm{WF}(w) \not\ni (\overset{\circ}{t}, \overset{\circ}{x}; \pm dt)$. Then it is sufficient to impose the following equations on v:

$$\begin{cases} \partial_t^2 (v - dw) = -d(\beta u + \partial_t^2 w), \\ \Delta(v - dw) = 0, \\ d(v - dw) = 0. \end{cases}$$

Therefore we can take v as follows:

$$v - dw = -d \int_{\overset{\circ}{t}}^{t} ds \int_{\overset{\circ}{t}}^{s} \Big(\beta u(\tau, x) + \partial_t{}^2 w(\tau, x)\Big) d\tau.$$

Then u_2 satisfies the following wave equation:

$$\rho \frac{\partial^2}{\partial t^2} u_2 + \beta \Delta u_2 = 0.$$

Hence $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{R}}; p) \Big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$ can be decomposed as a sum $u = u_1 + u_2$, where $u_j \in \mathcal{S}ol(\mathfrak{M}_j^{\mathrm{R}}; p) \Big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$ (j=1,2).

When $u = u_1 + u_2 = 0$ holds, the equations $\delta u_1 = 0$ and $du_2 = 0$ imply $u_1, u_2 \in \mathcal{S}ol(\mathfrak{M}_0^{\mathrm{R}}; p) \Big|_{(\overset{\circ}{t}, \overset{\circ}{x})}$.

Therefore we have an exact sequence

$$0 \longrightarrow \mathcal{S}ol(\mathfrak{M}_0^{\mathrm{R}}; p) \longrightarrow \mathcal{S}ol(\mathfrak{M}_1^{\mathrm{R}}; p) \oplus \mathcal{S}ol(\mathfrak{M}_2^{\mathrm{R}}; p) \longrightarrow \mathcal{S}ol(\mathfrak{M}^{\mathrm{R}}; p) \longrightarrow 0.$$

□

Remark. For the case $p = 1$, the contravariant form of this decomposition means the decomposition $u^i = u_1^i + u_2^i \in \widetilde{\mathcal{D}b}_0^1$ satisfying the next conditions:

$$\nabla_i \, u_1{}^i = 0, \quad \nabla^i u_2{}^j - \nabla^j u_2{}^i = 0.$$

Remark. In Einstein spaces satisfying $R_{ij} = \lambda g_{ij}$, a distribution solution u of $P_{\mathrm{org}} \, u = 0$ has a similar decomposition $u = u_1 + u_2$. However, if M is not an Einstein space, a distribution solution u of $P_{\mathrm{org}} \, u = 0$ does not necessarily admit any decomposition of solutions above.

§4. Decomposition of Modified Elastic Wave Equations on Complex Manifolds

We extend the results on a Riemannian manifold to ones in a complex manifold X with a Hermitian metric h. Firstly, we recall operators on complex manifolds according to [3] (Chapter 5), [4] (Chapter 3), and [5] (Chapter 5).

Definition 4.1. We denote by $\partial : \mathcal{D}b_X^{(q,r)} \to \mathcal{D}b_X^{(q+1,r)}$ the exterior differential operator which acts on $\mathcal{D}b_X^{(q,r)}$ as a sheaf-morphism and $\overline{\partial} : \mathcal{D}b_X^{(q,r)} \to \mathcal{D}b_X^{(q,r+1)}$ the conjugate exterior differential operator. For a section

$$\phi = \phi_{i_1 \cdots i_q \overline{j}_1 \cdots \overline{j}_r} \, dz^{i_1} \wedge \cdots \wedge dz^{i_q} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_r} \quad \text{of} \;\; \mathcal{D}b_X^{(q,r)},$$

the following formulas are well-known:

$$\begin{cases} d\phi = (\partial + \overline{\partial})\phi, \\ \partial\phi = \dfrac{\partial\phi}{\partial z^k}\, dz^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_q} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_r} \in \mathcal{D}b_X^{(q+1,r)}, \\ \overline{\partial}\phi = \dfrac{\partial\phi}{\partial \overline{z}^k}\, d\overline{z}^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_q} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_r} \in \mathcal{D}b_X^{(q,r+1)}. \end{cases}$$

Definition 4.2. The linear operator $*$ on X induces isomorphisms $\bigwedge^{(q,r)} T^*X \longrightarrow \bigwedge^{(n-r,n-q)} T^*X$ of vector bundle. Hence we have sheaf-morphisms $* : \mathcal{D}b_X^{(q,r)} \longrightarrow \mathcal{D}b_X^{(n-r,n-q)}$ on X as follows: For

$$\psi = \psi_{I\overline{J}}\, \omega^I \wedge \overline{\omega}^J \in \mathcal{D}b_X^{(q,r)},$$

we have

$$*\psi = \delta\begin{pmatrix} 1\cdots n\overline{1}\cdots\overline{n} \\ I\ \ \overline{J}\ \ \overline{J}^C\ \ I^C \end{pmatrix} \psi_{I\overline{J}}\, \omega^{J^C} \wedge \overline{\omega}^{I^C} \in \mathcal{D}b_X^{(n-r,n-q)},$$

where $\{\omega^1, \ldots, \omega^n\}$ is a local orthonormal system of C^∞ sections of T^*X concerning the Hermitian metric and $I^C := \{1, \ldots, n\}\backslash I$. Here $\delta(\cdot) = \pm 1$ is the signature of the permutation $(I\overline{J}\,\overline{J}^C I^C)$ of $(1\cdots n\overline{1}\cdots\overline{n})$.

Let $U \subset X$ be an open subset. Let $\alpha^{(q,r)} = \alpha_{I\overline{J}}\, \omega^I \wedge \overline{\omega}^J \in \mathcal{D}b_X^{(q,r)}(U)$, $\beta^{(q,r)} = \beta_{I\overline{J}}\, \omega^I \wedge \overline{\omega}^J \in \mathcal{E}_X^{(q,r)}(U)$ be sections. We suppose that $\beta^{(q,r)}$ has a compact support in U. Then the following integral is well-defined:

$$(\alpha^{(q,r)}, \beta^{(q,r)}) := \int_X \langle \alpha^{(q,r)}, \beta^{(q,r)} \rangle^*\, \omega^1 \wedge \cdots \wedge \omega^n \wedge \overline{\omega}^1 \wedge \cdots \wedge \overline{\omega}^n,$$

where, $\langle \alpha^{(q,r)}, \beta^{(q,r)} \rangle^* = \sum_{I,J} \alpha_{I\overline{J}}\, \overline{\beta^{I\overline{J}}}$.

Definition 4.3. Let $\alpha^{(q,r)} \in \mathcal{D}b_X^{(q,r)}$, $\beta^{(q-1,r)} \in \mathcal{E}_X^{(q-1,r)}$, and $\gamma^{(q,r-1)} \in \mathcal{E}_X^{(q,r-1)}$ be sections. We suppose $\beta^{(q-1,r)}$ and $\gamma^{(q,r-1)}$ have compact supports. Then sheaf-morphisms $\overline{\vartheta} : \mathcal{D}b_X^{(q,r)} \to \mathcal{D}b_X^{(q-1,r)}$ and $\vartheta : \mathcal{D}b_X^{(q,r)} \to \mathcal{D}b_X^{(q,r-1)}$ are defined as

$$(\overline{\vartheta}\alpha^{(q,r)}, \beta^{(q-1,r)}) = (\alpha^{(q,r)}, \partial\beta^{(q-1,r)}),$$
$$(\vartheta\alpha^{(q,r)}, \gamma^{(q,r-1)}) = (\alpha^{(q,r)}, \overline{\partial}\gamma^{(q,r-1)}).$$

Further they satisfy the following equations:

$$\begin{cases} \delta = \overline{\vartheta} + \vartheta, \\ \overline{\vartheta} = - * \overline{\partial} *, \\ \vartheta = - * \partial * . \end{cases}$$

Then, we get the lemmas as follows.

Lemma 4.1. *For any* $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{C}}; q, r)$ *and the variety*

$$\begin{aligned} V_{\mathrm{C}} &:= \bigcup_{k=1,2} \left\{ (t, z; \tau, \zeta) \;\middle|\; \tau^2 - \alpha_k |\zeta|^2 = 0 \right\} \\ &= \bigcup_{k=1,2} \left\{ (t, z; \tau, \zeta) \;\middle|\; \tau^2 - \alpha_k h^{ij}(z) \zeta_i \overline{\zeta_j} = 0 \right\}, \end{aligned}$$

we have $\mathrm{WF}(u) \subset V_{\mathrm{C}}$*, where* (τ, ζ) *is identified with the real cotangent vector* $\tau dt + 2\mathrm{Re}(\zeta dz)$*.*

Proof. The symbol of the second-order operator $P_{\mathrm{C}}\ (t, z, \partial_t, \partial_z, \overline{\partial_z})$ at $\overset{\circ}{pt} = (\overset{\circ}{z}; \overset{\circ}{\zeta}) \in T^*X$ is defined as a linear operator

$$\sigma_2(P_{\mathrm{C}}\ )(\overset{\circ}{pt}) : \bigwedge^{(q,r)} T^*_{\overset{\circ}{z}} X \to \bigwedge^{(q,r)} T^*_{\overset{\circ}{z}} X$$

given by the following:

$$\begin{aligned} &\sigma_2(P_{\mathrm{C}}\ )(\overset{\circ}{pt}) U \\ &\quad := \lim_{\lambda \to \infty} \frac{1}{\lambda^2} e^{-\lambda i (2\mathrm{Re}[z, \overset{\circ}{\zeta}] + t \overset{\circ}{\tau})} P_{\mathrm{C}}\ (\overset{\circ}{t}, \overset{\circ}{z}, \partial_t, \partial_z, \overline{\partial_z}) \left(e^{\lambda i (2\mathrm{Re}[z, \overset{\circ}{\zeta}] + t \overset{\circ}{\tau})} U \right), \end{aligned}$$

where we consider the coefficients of $U \in \bigwedge^{(q,r)} T^*_{\overset{\circ}{z}} X$ as constants in a local coordinate system. Here, $[z, \overset{\circ}{\zeta}] = z_l \overline{\overset{\circ}{\zeta}_l}$. We suppose $(\overset{\circ}{\tau}, \overset{\circ}{\zeta}) \neq (0, 0)$. Therefore we get

$$\begin{aligned} \sigma_2(P_{\mathrm{C}}\ )(\overset{\circ}{pt}) U &= \sigma_2(\partial_t^2 + \alpha_1 \partial \overline{\vartheta} + \alpha_2 \overline{\vartheta} \partial) U \\ &= -\overset{\circ}{\tau}{}^2 U - \alpha_1 (\Xi \wedge (*(\overline{\Xi} \wedge *U))) - \alpha_2 (*(\overline{\Xi} \wedge *(\Xi \wedge U))). \end{aligned}$$

Here $\Xi := \sum \overset{\circ}{\zeta}_j dz^j$. Let $\omega = (\omega_1, \ldots, \omega_n)$ be an orthonormal system of $T^*_{\overset{\circ}{z}} X$ satisfying $\omega_1 = \Xi / |\Xi|$. When we set $U = U_1 + U_2$ where $U_1 = \sum_{1 \in I} U_{1I\overline{J}} \omega^I \wedge \overline{\omega}^J$,

$U_2 = \sum_{1\notin I} U_{2I\overline{J}}\omega^I \wedge \overline{\omega}^J$, we get $\Xi \wedge U_1 = 0$, $\overline{\Xi} \wedge *U_2 = 0$. Therefore we have

$$\begin{aligned}
*\Big(\overline{\Xi} \wedge *(\Xi \wedge U)\Big) &= *\Big(\overline{\Xi} \wedge *\big(|\Xi| \sum_{1\notin I} U_{2I\overline{J}}\ \omega_1 \wedge \omega^I \wedge \overline{\omega}^J\big)\Big) \\
&= \delta\begin{pmatrix} 1\cdots n\overline{1}\cdots\overline{n} \\ I\ \ \overline{J}\ \ \overline{J}^C\ \ I^C \end{pmatrix} |\Xi| \left\{ *\Big(\overline{\Xi} \wedge \sum_{1\notin I} U_{2I\overline{J}}\ \omega^{J^C} \wedge \overline{\omega}^{\{1,I\}^C}\Big)\right\} \\
&= -|\Xi|^2 U_2, \\
\Xi \wedge (*(\overline{\Xi} \wedge *U)) &= -|\Xi|^2 U_1, \\
\sigma_2(P_{\mathrm{C}}\ )(\overset{\circ}{pt})U &= (-\overset{\circ}{\tau}{}^2 + \alpha_1|\Xi|^2)U_1 + (-\overset{\circ}{\tau}{}^2 + \alpha_2|\Xi|^2)U_2.
\end{aligned}$$

Thus we find that $\sigma_2(P_{\mathrm{C}}\ )(\overset{\circ}{pt})$ has 2 eigenvalues $(-\overset{\circ}{\tau}{}^2 + \alpha_1|\Xi|^2)$, $(-\overset{\circ}{\tau}{}^2 + \alpha_2|\Xi|^2)$, and their multiplicities are ${}_{n-1}C_{q-1} \cdot {}_nC_r$, ${}_{n-1}C_q \cdot {}_nC_r$, respectively. Hence we get

$$\det(\sigma_2(P_{\mathrm{C}}\ )(\overset{\circ}{pt})) = (-\overset{\circ}{\tau}{}^2 + \alpha_1|\Xi|^2)^{{}_{n-1}C_{q-1}\cdot{}_nC_r}(-\overset{\circ}{\tau}{}^2 + \alpha_2|\Xi|^2)^{{}_{n-1}C_q\cdot{}_nC_r}.$$

If $\sigma_2(P_{\mathrm{C}}\ )(\overset{\circ}{pt})$ is an isomorphism, $\overset{\circ}{pt}$ does not belong to the characteristic variety of $P_{\mathrm{C}}\ u = 0$. Therefore, for a characteristic point $\overset{\circ}{pt}$, we have $\det(\sigma_2(P_{\mathrm{C}}\ )(\overset{\circ}{pt})) = 0$. Hence,

$$\begin{aligned}
\mathrm{WF}(u) \subset V_{\mathrm{C}} &= \bigcup_{k=1,2} \Big\{(t,z;\tau,\zeta)\ \Big|\ \tau^2 - \alpha_k|\zeta|^2 = 0\Big\} \\
&= \bigcup_{k=1,2} \Big\{(t,z;\tau,\zeta)\ \Big|\ \tau^2 - \alpha_k h^{ij}(z)\zeta_i\overline{\zeta_j} = 0\Big\}.
\end{aligned}$$

□

Lemma 4.2. *For a germ $u \in \widetilde{\mathcal{D}b}{}_X^{(q,r)}$ at $(\overset{\circ}{t}, \overset{\circ}{z})$, we assume* $\mathrm{WF}(u) \not\ni (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$. *Then, there exists a germ $w \in \widetilde{\mathcal{D}b}{}_X^{(q,r)}$ at $(\overset{\circ}{t}, \overset{\circ}{z})$, which satisfies $\Box w = u$ and* $\mathrm{WF}(w) \not\ni (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$.

Proof. For $u = \sum u_{J\overline{L}}(t,z)dz^J \wedge d\overline{z}^L$, we write

$$\Box u = \sum \big\{P_{JK}(z, \partial_z, \overline{\partial_z})u_{K\overline{L}}(t,z)\big\}dz^J \wedge d\overline{z}^L.$$

$\Box$ is an elliptic operator on $\mathcal{D}b_X^{(q,r)}$. Therefore, in a neighborhood of $(\overset{\circ}{z}, \overset{\circ}{z}) \in X \times X$, there exist integral kernels $\{G_{KM}(z,z')\}_{KM}$, which satisfy

$$\begin{aligned}
&\sum P_{JK}(z, \partial_z, \overline{\partial_z})G_{KM}(z,z') = \delta_{J,M} \cdot \delta(\mathrm{Re}\, z - \mathrm{Re}\, z') \cdot \delta(\mathrm{Im}\, z - \mathrm{Im}\, z'), \\
&\mathrm{WF}(G_{KM}) \subset \big\{(z,z'; \zeta dz + \overline{\zeta} d\overline{z} + \zeta' dz' + \overline{\zeta'} d\overline{z'})\ \big|\ z = z', \zeta = -\zeta'\big\}.
\end{aligned}$$

Here, $\delta_{J,M}$ is a Kronecker's delta. Hence,

$$w \equiv \sum\Big(\int G_{KM}(z,z')u_{M\overline{L}}(t,z')\psi(z')d(\mathrm{Re}\,z')d(\mathrm{Im}\,z')\Big)dz^K \wedge d\overline{z}^L$$

satisfies $\Box w = u$ at $(\overset{\circ}{t},\overset{\circ}{z})$ and $\mathrm{WF}(w) \not\ni (\overset{\circ}{t},\overset{\circ}{z};\pm dt)$. Here $\psi \in C_0^\infty(X)$ has a compact support in a small neighborhood of $\overset{\circ}{z}$ and $\psi(z') \equiv 1$ near $\overset{\circ}{z}$. □

Then, we get the following theorem.

Theorem 4.1. *For any germ $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{C}};q,r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$, there exist some germs $u_j \in \mathcal{S}ol(\mathfrak{M}_j^{\mathrm{C}};q,r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$ $(j=1,2)$ such that $u = u_1 + u_2$.*

Proof. For $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$, we suppose u_1, u_2 are of the form;

$$u_1 = \partial v, \quad u_2 = u - \partial v.$$

Here $v \in \widetilde{\mathcal{D}b}_X^{(q-1,r)}\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$. Hence we have only to impose the following conditions on v:

$$\begin{cases} P_{\mathrm{C}}\ (\partial v) = 0, \\ \overline{\vartheta}(u - \partial v) = 0, \\ \overline{\vartheta} v = 0. \end{cases}$$

Since the equation $P_{\mathrm{C}}\ (\partial v) = \left(\partial_t^2 + \alpha_1 \partial\overline{\vartheta}\right)\partial v = \partial\left(\partial_t^2 + \alpha_1 \Box\right) v = 0$ holds, it is sufficient to impose the next conditions:

$$\begin{cases} \partial_t^2 v + \alpha_1 \Box v = 0, \\ \overline{\vartheta}\partial v = \overline{\vartheta} u, \\ \overline{\vartheta} v = 0, \end{cases} \quad\Longleftrightarrow\quad \begin{cases} \partial_t^2 v = -\alpha_1 \overline{\vartheta} u, \\ \Box v = \overline{\vartheta} u, \\ \overline{\vartheta} v = 0. \end{cases}$$

From Lemma 4.1 and Lemma 4.2, we have a w satisfying $\Box w = u$ and $\mathrm{WF}(w) \not\ni (\overset{\circ}{t},\overset{\circ}{z};\pm dt)$. Then it is sufficient to impose the following equations on v:

$$\begin{cases} \partial_t^2 (v - \overline{\vartheta} w) = -\overline{\vartheta}(\alpha_1 u + \partial_t^2 w), \\ \Box(v - \overline{\vartheta} w) = 0, \\ \overline{\vartheta}(v - \overline{\vartheta} w) = 0. \end{cases}$$

Therefore we can take v as follows:

$$v - \overline{\vartheta} w = -\overline{\vartheta}\int_{\overset{\circ}{t}}^{t} ds \int_{\overset{\circ}{t}}^{s} \Big(\alpha_1 u(\tau,z) + {\partial_t}^2 w(\tau,z)\Big)d\tau.$$

Then u_1 satisfies the following wave equation:

$$\frac{\partial^2}{\partial t^2}u_1 + \alpha_1 \Box u_1 = 0.$$

Hence $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{C}}; q, r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$ can be decomposed as a sum $u = u_1 + u_2$, where $u_j \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{C}}_j; q, r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$ (j=1,2). □

In the same way, we obtain the following similar lemmas and theorem as adjoint versions of Lemma 4.1, Lemma 4.2, and Theorem 4.1. We omit the proofs because the arguments go in a similar way.

Lemma 4.3. *For any $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{C}*}; q, r)$ and the variety*

$$\begin{aligned} V^*_{\mathrm{C}} &:= \bigcup_{k=3,4} \left\{(t, z; \tau, \zeta) \,\Big|\, \tau^2 - \alpha_k |\zeta|^2 = 0\right\} \\ &= \bigcup_{k=3,4} \left\{(t, z; \tau, \zeta) \,\Big|\, \tau^2 - \alpha_k h^{ij}(z)\zeta_i \overline{\zeta_j} = 0\right\}, \end{aligned}$$

we have $\mathrm{WF}(u) \subset V^*_{\mathrm{C}}$, *where* (τ, ζ) *is identified with the real cotangent vector* $\tau dt + 2\mathrm{Re}(\zeta dz)$.

Lemma 4.4. *For a germ $u \in \widetilde{\mathcal{D}b}^{(q,r)}_X$ at $(\overset{\circ}{t}, \overset{\circ}{z})$, we assume* $\mathrm{WF}(u) \not\ni (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$. *Then, there exists a germ $w \in \widetilde{\mathcal{D}b}^{(q,r)}_X$ at $(\overset{\circ}{t}, \overset{\circ}{z})$, which satisfies $\overline{\Box} w = u$ and* $\mathrm{WF}(w) \not\ni (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$.

Theorem 4.2. *For any germ $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{C}*}; q, r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$, there exist some germs $u_k \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{C}*}_k; q, r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$ $(k = 3, 4)$ such that $u = u_3 + u_4$.*

Now we assume that X is a Kähler manifold. For the sheaf-morphisms P_{K}, we have the following lemmas.

Lemma 4.5. *For any $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}; q, r)$ and the variety*

$$\begin{aligned} V_{\mathrm{K}} &:= \bigcup_{k=1,2,3,4} \left\{(t, z; \tau, \zeta) \,\Big|\, \left(\tau^2 - \beta_k |\zeta|^2 = 0\right)\right\} \\ &= \bigcup_{k=1,2,3,4} \left\{(t, z; \tau, \zeta) \,\Big|\, \left(\tau^2 - \beta_k h^{ij}(z)\zeta_i \overline{\zeta_j} = 0\right)\right\}, \end{aligned}$$

we have $\mathrm{WF}(u) \subset V_{\mathrm{K}}$, *where* $\beta_1 = \alpha_1 + \alpha_3$, $\beta_2 = \alpha_2 + \alpha_3$, $\beta_3 = \alpha_1 + \alpha_4$, $\beta_4 = \alpha_2 + \alpha_4$ *and* (τ, ζ) *is identified with the real cotangent vector* $\tau dt + 2\mathrm{Re}(\zeta dz)$.

Proof. Similarly to Lemma 4.1, we get the symbol of the second-order operator

$$\begin{aligned}\sigma_2(P_{\text{K}})(\overset{\circ}{pt})U =&\sigma_2(\partial_t^2+\alpha_1\partial\overline{\vartheta}+\alpha_2\overline{\vartheta}\partial+\alpha_3\overline{\partial}\vartheta+\alpha_4\vartheta\overline{\partial})U\\ =&-\overset{\circ}{\tau}{}^2U-\alpha_1\Big(\Xi\wedge\Big(*\Big(\overline{\Xi}\wedge *U\Big)\Big)\Big)-\alpha_2\Big(*\Big(\overline{\Xi}\wedge *\Big(\Xi\wedge U\Big)\Big)\Big)\\ &-\alpha_3\Big(\overline{\Xi}\wedge\Big(*\Big(\Xi\wedge *U\Big)\Big)\Big)-\alpha_4\Big(*\Big(\Xi\wedge *\Big(\overline{\Xi}\wedge U\Big)\Big)\Big).\end{aligned}$$

Here $\Xi := \sum \overset{\circ}{\zeta}_j dz^j$. Let $\omega = (\omega_1, \dots, \omega_n)$ be an orthonormal system of $T^*_{\overset{\circ}{z}}X$ satisfying $\omega_1 = \Xi/|\Xi|$. We set $U = U_1 + U_2 + U_3 + U_4$ where

$$U_1=\sum_{1\in I,1\in J}U_{1I\overline{J}}\omega^I\wedge\overline{\omega}^J,\qquad U_2=\sum_{1\notin I,1\in J}U_{2I\overline{J}}\omega^I\wedge\overline{\omega}^J,$$

$$U_3=\sum_{1\in I,1\notin J}U_{3I\overline{J}}\omega^I\wedge\overline{\omega}^J,\qquad U_4=\sum_{1\notin I,1\notin J}U_{4I\overline{J}}\omega^I\wedge\overline{\omega}^J.$$

Then we get

$$\begin{cases}\Xi\wedge U_1=0,\\ \overline{\Xi}\wedge U_1=0,\end{cases}\quad \begin{cases}\overline{\Xi}\wedge U_2=0,\\ \overline{\Xi}\wedge *U_2=0,\end{cases}\quad \begin{cases}\Xi\wedge U_3=0,\\ \Xi\wedge *U_3=0,\end{cases}\quad \begin{cases}\overline{\Xi}\wedge *U_4=0,\\ \Xi\wedge *U_4=0.\end{cases}$$

Therefore we have

$$\begin{aligned}*\Big(\overline{\Xi}\wedge *\Big(\Xi\wedge U\Big)\Big)&=*\Big(\overline{\Xi}\wedge *\Big(|\Xi|\Big(\sum_{1\notin I}(U_{2I\overline{J}}+U_{4I\overline{J}})\omega_1\wedge\omega^I\wedge\overline{\omega}^J\Big)\Big)\Big)\\ &=-|\Xi|^2(U_2+U_4),\\ \Xi\wedge\Big(*\Big(\overline{\Xi}\wedge *U\Big)\Big)&=-|\Xi|^2(U_1+U_3),\\ *\Big(\Xi\wedge *\Big(\overline{\Xi}\wedge U\Big)\Big)&=-|\Xi|^2(U_3+U_4),\\ \overline{\Xi}\wedge\Big(*\Big(\Xi\wedge *U\Big)\Big)&=-|\Xi|^2(U_1+U_2).\end{aligned}$$

Hence,

$$\sigma_2(P_{\text{K}})(\overset{\circ}{pt})U=\sum_{k=1,2,3,4}\Big(-\overset{\circ}{\tau}{}^2+\beta_k|\Xi|^2\Big)U_k.$$

Thus we find that $\sigma_2(P_{\text{K}})(\overset{\circ}{pt})$ has 4 eigenvalues $-\overset{\circ}{\tau}{}^2 + \beta_k|\Xi|^2$ $(k = 1, 2, 3, 4)$, and their multiplicities are γ_k, respectively. Here $\gamma_1 = {}_{n-1}C_{q-1} \cdot {}_{n-1}C_{r-1}$, $\gamma_2 = {}_{n-1}C_q \cdot {}_{n-1}C_{r-1}$, $\gamma_3 = {}_{n-1}C_{q-1} \cdot {}_{n-1}C_r$, and $\gamma_4 = {}_{n-1}C_q \cdot {}_{n-1}C_r$. Hence we get

$$\det(\sigma_2(P_{\text{K}})(\overset{\circ}{pt}))=\prod_{k=1}^{4}\Big(-\overset{\circ}{\tau}{}^2+\beta_k|\Xi|^2\Big)^{\gamma_k}.$$

If $\sigma_2(P_{\text{K}})(\overset{\circ}{pt})$ is an isomorphism, $\overset{\circ}{pt}$ does not belong to the characteristic variety of $P_{\text{K}}\, u = 0$. Therefore, for a characteristic point $\overset{\circ}{pt}$, we have $\det(\sigma_2(P_{\text{K}})(\overset{\circ}{pt})) = 0$. Hence,

$$\begin{aligned}\mathrm{WF}(u) \subset V_{\text{K}} &= \bigcup_{k=1,2,3,4} \Big\{(t,z;\tau,\zeta) \,\Big|\, \big(\tau^2 - \beta_k|\zeta|^2 = 0\big)\Big\} \\ &= \bigcup_{k=1,2,3,4} \Big\{(t,z;\tau,\zeta) \,\Big|\, \big(\tau^2 - \beta_k h^{ij}(z)\zeta_i\overline{\zeta_j} = 0\big)\Big\}.\end{aligned}$$

□

Lemma 4.6. *For any germ $u \in \mathcal{S}ol(\mathfrak{M}^{\text{K}}; q, r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$, there exist some germs $u_j \in \mathcal{S}ol(\mathfrak{M}^{\text{K}}_j; q, r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$ $(j = 1, 2)$ such that $u = u_1 + u_2$.*

Proof. For $u \in \widetilde{\mathcal{D}b}_X^{(q,r)}\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$, we suppose u_1, u_2 are of the form;

$$u_1 = \partial v_1, \quad u_2 = u - \partial v_1.$$

Here $v_1 \in \widetilde{\mathcal{D}b}_X^{(q-1,r)}\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$. Hence we have only to impose the following conditions on v_1:

$$\begin{cases} P_{\text{K}}\,(\partial v_1) = 0, \\ \overline{\vartheta}(u - \partial v_1) = 0, \\ \overline{\vartheta} v_1 = 0. \end{cases}$$

Since the equation

$$\begin{aligned} P_{\text{K}}\,(\partial v_1) &= \left(\partial_t^2 + \alpha_1\partial\overline{\vartheta} + \alpha_3\overline{\partial}\vartheta + \alpha_4\vartheta\overline{\partial}\right)\partial v_1 \\ &= \partial\left(\partial_t^2 + \alpha_1\Box + \alpha_3\overline{\partial}\vartheta + \alpha_4\vartheta\overline{\partial}\right)v_1 = 0 \end{aligned}$$

holds, it is sufficient to impose the next conditions:

$$\begin{cases} \left(\partial_t^2 + \alpha_1\Box + \alpha_3\overline{\partial}\vartheta + \alpha_4\vartheta\overline{\partial}\right)v_1 = 0, \\ \overline{\vartheta}(u - \partial v_1) = 0, \\ \overline{\vartheta} v_1 = 0, \end{cases} \iff \begin{cases} P_{\text{C}}^*\, v_1 = -\alpha_1\overline{\vartheta} u, \\ \Box v_1 = \overline{\vartheta} u, \\ \overline{\vartheta} v_1 = 0. \end{cases}$$

From Lemma 4.5 and Lemma 4.2, we have a w satisfying $\Box w = u$ and $\mathrm{WF}(w) \not\ni (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$. Then, when we put

$$v_1 - \overline{\vartheta} w =: \varphi, \quad -\alpha_1\overline{\vartheta} u - \overline{\vartheta}\, P_{\text{C}}^*\, w =: \psi,$$

and use $\square = \overline{\square}$, it is sufficient to impose the following equations on φ:

$$\begin{cases} P^*_{\mathrm{C}}\, \varphi = \Big(\partial_t^2 + \alpha_3 \overline{\partial}\vartheta + \alpha_4 \vartheta\overline{\partial}\Big)\varphi = \psi, \\ \square\varphi = 0, \\ \overline{\vartheta}\varphi = 0, \end{cases} \iff \begin{cases} \Big(\partial_t^2 + (\alpha_4 - \alpha_3)\vartheta\overline{\partial}\Big)\varphi = \psi, \\ \square\varphi = 0, \\ \overline{\vartheta}\varphi = 0. \end{cases}$$

Setting $\mathcal{F} := \Big\{\Psi \in \widetilde{\mathcal{D}b}_X^{(q,r)}\Big|_{(\overset{\circ}{t},\overset{\circ}{z})} ;\ \square\Psi = 0, \overline{\vartheta}\Psi = 0\Big\}$, we note that

$$\vartheta\overline{\partial}(\mathcal{F}) \subset \mathcal{F}, \qquad (\vartheta\overline{\partial})^2\mathcal{F} = 0.$$

Since $\psi \in \mathcal{F}$ and $\mathrm{WF}(\psi) \not\ni (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$, then we can take φ as follows:

$$\begin{aligned} \varphi = &\int_{\overset{\circ}{t}}^{t} ds \int_{\overset{\circ}{t}}^{s} \psi(s', z) ds' \\ &+ (\alpha_3 - \alpha_4)\int_{\overset{\circ}{t}}^{t} ds \int_{\overset{\circ}{t}}^{s} ds' \int_{\overset{\circ}{t}}^{s'} d\tau \int_{\overset{\circ}{t}}^{\tau} \vartheta\overline{\partial}\psi(\tau', z) d\tau'. \end{aligned}$$

Hence $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}; q, r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$ can be decomposed into a sum $u = u_1 + u_2$, where $u_j \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}_j; q, r)\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$ (j=1,2). □

By the proof above, we get the following more precise form of u_1, u_2.

Lemma 4.7. *Let $u = u_1 + u_2$ be a decomposition. We can write u_1, u_2 in the preceding lemma as follows*:

$$u_1 = \partial v_1, \qquad u_2 = \overline{\vartheta} v_2,$$

where $v_1 \in \widetilde{\mathcal{D}b}_X^{(q-1,r)}\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$, $v_2 \in \widetilde{\mathcal{D}b}_X^{(q+1,r)}\Big|_{(\overset{\circ}{t},\overset{\circ}{z})}$ satisfying $\mathrm{WF}(v_j) \not\ni (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$ ($j = 1, 2$) *and the equations*

$$\text{(4.1)} \qquad \begin{cases} \partial_t^2 v_1 + \alpha_1 \square v_1 + \alpha_3 \overline{\partial}\vartheta v_1 + \alpha_4 \vartheta\overline{\partial} v_1 = 0, \\ \partial_t^2 v_2 + \alpha_2 \square v_2 + \alpha_3 \overline{\partial}\vartheta v_2 + \alpha_4 \vartheta\overline{\partial} v_2 = 0. \end{cases}$$

Proof. We define $D_t^{-l} := \int_{\overset{\circ}{t}}^{t} dt_1 \int_{\overset{\circ}{t}}^{t_1} dt_2 \cdots \int_{\overset{\circ}{t}}^{t_{l-1}} dt_l$. Putting $v_1 = \overline{\vartheta} w + D_t^{-2}\psi + (\alpha_3 - \alpha_4)\vartheta\overline{\partial} D_t^{-4}\psi$, we calculate the difference:

$$\begin{aligned} u_2 &\equiv u - u_1 = u - \partial v_1 \\ &= \overline{\vartheta}\partial w - D_t^{-2}\partial\psi + (\alpha_4 - \alpha_3)\vartheta\overline{\partial} D_t^{-4}\partial\psi \\ &= \overline{\vartheta}\partial w - \big(D_t^{-2} - (\alpha_4 - \alpha_3)\vartheta\overline{\partial} D_t^{-4}\big)\partial\overline{\vartheta}(-\alpha_1 u - P^*_{\mathrm{C}} w) \\ &= \overline{\vartheta}\partial w - \big(D_t^{-2} - (\alpha_4 - \alpha_3)\vartheta\overline{\partial} D_t^{-4}\big)\big(\overline{\vartheta}\partial(\alpha_1 u + P^*_{\mathrm{C}} w) + (\alpha_2 - \alpha_1)\overline{\vartheta}\partial u\big) \\ &= \overline{\vartheta}\partial\Big(w - \big(D_t^{-2} - (\alpha_4 - \alpha_3)\vartheta\overline{\partial} D_t^{-4}\big)(\alpha_2 u + P^*_{\mathrm{C}} w)\Big) . \end{aligned}$$

Hence, we can get

$$v_1 = \overline{\vartheta}\Big(w - \big(D_t^{-2} - (\alpha_4 - \alpha_3)\vartheta\overline{\partial}D_t^{-4}\big)(\alpha_1 u + P_{\mathrm{C}}^* w)\Big) \ ,$$
$$v_2 = \partial\Big(w - \big(D_t^{-2} - (\alpha_4 - \alpha_3)\vartheta\overline{\partial}D_t^{-4}\big)(\alpha_2 u + P_{\mathrm{C}}^* w)\Big) \ .$$

In particular, $\mathrm{WF}(v_j) =\not\ni (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$ $(j = 1, 2)$. The first equation of (4.1) is already obtained in the proof of Lemma 4.6. The second equation of (4.1) is also obtained in a similar way as follows: Set $\Phi = v_2 - \partial w$. Then Φ satisfies

$$\begin{cases} \Big(\partial_t^2 + (\alpha_4 - \alpha_3)\vartheta\overline{\partial}\Big)\Phi = \Psi := -\partial(\alpha_2 u + P_{\mathrm{C}}^* w), \\ \Box\Phi = 0, \\ \partial\Phi = 0. \end{cases}$$

Hence,

$$\begin{cases} P_{\mathrm{C}}^*\Phi = \Psi, \\ \Box\Phi = 0, \\ \partial\Phi = 0, \end{cases} \iff \begin{cases} P_{\mathrm{C}}^* \, v_2 = -\alpha_2\partial u, \\ \Box v_2 = \partial u, \\ \partial v_2 = 0. \end{cases}$$

Thus we obtain

$$(P_{\mathrm{C}}^* \ + \alpha_2\Box)v_2 = 0.$$

This is just the second equation of (4.1). □

Then we have the following lemma.

Lemma 4.8. *For any germ* $u_1 = \partial v_1 \in \mathcal{S}ol(\mathfrak{M}_1^{\mathrm{K}}; q, r)\Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$ *with a* $v_1 \in \widetilde{\mathcal{D}b}_X^{(q-1,r)}\Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$ *satisfying* (4.1) *and* $\mathrm{WF}(v_1) \not\ni (\overset{\circ}{t}, \overset{\circ}{z}; \pm dt)$, *there exist some germs* $u_{1j} \in \mathcal{S}ol(\mathfrak{M}_{1j}^{\mathrm{K}}; q, r)\Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$ $(j = 3, 4)$ *such that* $u_1 = u_{13} + u_{14}$.

Proof. For $u_1 = \partial v_1 \in \widetilde{\mathcal{D}b}_X^{(q,r)}\Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$, we suppose u_{13}, u_{14} are of the form;

$$u_{13} = \partial\overline{\partial}V, \quad u_{14} = u_1 - \partial\overline{\partial}V = \partial(v_1 - \overline{\partial}V).$$

Here $V \in \widetilde{\mathcal{D}b}_X^{(q-1,r-1)}\Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$. Hence we have only to impose the following conditions on V:

$$\begin{cases} P_{\mathrm{K}}(\partial\overline{\partial}V) = 0, \\ \vartheta(v_1 - \overline{\partial}V) = 0, \\ \vartheta V = 0. \end{cases}$$

Since the equation

$$\begin{aligned} P_{\kappa}(\partial\overline{\partial}V) &= \left(\partial_t^2 + \alpha_1\partial\overline{\vartheta} + \alpha_2\overline{\vartheta}\partial + \alpha_3\overline{\partial}\vartheta + \alpha_4\vartheta\overline{\partial}\right)\partial\overline{\partial}V \\ &= \partial\overline{\partial}\left(\partial_t^2 + \alpha_1\Box + \alpha_3\overline{\Box}\right)V = \partial\overline{\partial}\Big(\partial_t^2 + (\alpha_1+\alpha_3)\overline{\Box}\Big)V \end{aligned}$$

holds, it is sufficient to impose the next conditions:

$$\begin{cases} \Big(\partial_t^2 + (\alpha_1+\alpha_3)\overline{\Box}\Big)V = 0, \\ \vartheta(v_1 - \overline{\partial}V) = 0, \\ \vartheta V = 0, \end{cases} \iff \begin{cases} \Big(\partial_t^2 + (\alpha_1+\alpha_3)\overline{\Box}\Big)V = 0, \\ \overline{\Box}V = \vartheta v_1, \\ \vartheta V = 0. \end{cases}$$

From Lemma 4.5 and Lemma 4.4, we have a w' satisfying $\overline{\Box}w' = v_1$ and $\mathrm{WF}(w') \not\ni (\mathring{t}, \mathring{z}; \pm dt)$. Then it is sufficient to impose the following equations on V:

$$\begin{cases} \partial_t^2(V - \vartheta w') = -\vartheta\Big((\alpha_1+\alpha_3)v_1 + \partial_t^2 w'\Big), \\ \overline{\Box}(V - \vartheta w') = 0, \\ \vartheta(V - \vartheta w') = 0. \end{cases}$$

Therefore we get

$$V - \vartheta w' = -\vartheta D_t^{-2}\Big((\alpha_1+\alpha_3)v_1(t,z) + \partial_t{}^2 w'(t,z)\Big). \tag{4.2}$$

Indeed the $\overline{\Box}$-closedness of right side of (4.2) follows from (4.1). Hence $u_1 \in \mathcal{S}ol(\mathfrak{M}_1^{\kappa}; q, r)\Big|_{(\mathring{t},\mathring{z})}$ can be decomposed as a sum $u_1 = u_{13} + u_{14}$, where $u_{1j} \in \mathcal{S}ol(\mathfrak{M}_{1j}^{\kappa}; q, r)\Big|_{(\mathring{t},\mathring{z})}$ (j=3,4). □

In the same way, we obtain the following similar lemma as adjoint version of Lemma 4.8. We omit the proof because the arguments go in a similar way.

Lemma 4.9. *For any germ $u_2 = \overline{\vartheta}v_2 \in \mathcal{S}ol(\mathfrak{M}_2^{\kappa}; q, r)\Big|_{(\mathring{t},\mathring{z})}$ with a $v_2 \in \widetilde{\mathcal{D}b}_X^{(q+1,r)}\Big|_{(\mathring{t},\mathring{z})}$ satisfying (4.1) and $\mathrm{WF}(v_2) \not\ni (\mathring{t}, \mathring{z}; \pm dt)$, there exist some germs $u_{2k} \in \mathcal{S}ol(\mathfrak{M}_{2k}^{\kappa}; q, r)\Big|_{(\mathring{t},\mathring{z})}$ $(k = 3, 4)$ such that $u_2 = u_{23} + u_{24}$.*

Then we have the following theorem.

Theorem 4.3. *For any germ $u \in \mathcal{S}ol(\mathfrak{M}^{\kappa}; q, r)\Big|_{(\mathring{t},\mathring{z})}$, there exist some germs $u_{ij} \in \mathcal{S}ol(\mathfrak{M}_{ij}^{\kappa}; q, r)\Big|_{(\mathring{t},\mathring{z})}$ $\big((ij) = (13), (23), (14), (24)\big)$ such that $u = u_{13} + u_{23} + u_{14} + u_{24}$.*

Further, we find that $u = u_{13} + u_{23} + u_{14} + u_{24} = 0$ *implies*

$$u_{jk} \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}_{jk0}; q, r) \quad \big((jk) = (13), (23), (14), (24)\big).$$

Equivalently, we have the following exact sequence:

$$0 \longrightarrow {\bigoplus_{(ij)}}' \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}_{ij0}; q, r) \xrightarrow{G} \bigoplus_{(ij)} \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}_{ij}; q, r) \xrightarrow{H} \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}; q, r) \longrightarrow 0.$$

Here,

$${\bigoplus_{(ij)}}' \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}_{ij0}; q, r) := \Big\{ (u_{ij}) \in \bigoplus_{(ij)} \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}_{ij0}; q, r) \ \Big| \ \sum_{(ij)} u_{ij} = 0 \Big\},$$

$G(U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}) = U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}$, $H(U_{13} \oplus U_{23} \oplus U_{14} \oplus U_{24}) = U_{13} + U_{23} + U_{14} + U_{24}$.

Proof. By virtue of Lemma 4.6, 4.7, 4.8, and 4.9, we find that $u \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}; q, r) \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$ is decomposed as a sum $u = u_{13} + u_{23} + u_{14} + u_{24}$ by using $u_{ij} \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}_{ij}; q, r) \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$ $\big((ij) = (13), (23), (14), (24)\big)$.

When $u = u_{13}+u_{23}+u_{14}+u_{24} = 0$ holds, we set $w = u_{13}+u_{23} = -u_{14}-u_{24}$. Then w satisfies $\overline{\partial} w = 0$, $\vartheta w = 0$. Hence we have $\Delta w = 2\overline{\Box} w = 0$, and so $\Delta u_{13} = -\Delta u_{23}$. By a similar argument we obtain $\Delta^2 u_{13} = 0$. Therefore we have $u_{13} \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}_{130}; q, r) \Big|_{(\overset{\circ}{t}, \overset{\circ}{z})}$. In a similar way we conclude that

$$u_{jk} \in \mathcal{S}ol(\mathfrak{M}^{\mathrm{K}}_{jk0}; q, r) \quad \big((jk) = (23), (14), (24)\big).$$

This completes the proof of Theorem 4.3. □

Example 1. We assume $X = \mathbb{C}^2$. Then, X is a Kähler manifold with the complex Euclidean metric. We find a solution $u \in \widetilde{\mathcal{D}b}^{(1,1)}_X$ of the form with $\zeta \equiv \zeta_1 dz^1 + \zeta_2 dz^2$ where $(\zeta_1, \zeta_2) \in \mathbb{C}^2 \backslash \{0\}$;

$$u(t, z) = U(t) e^{i(z \cdot \zeta + \overline{z} \cdot \overline{\zeta})}.$$

Then,

$$\begin{aligned} P_{\mathrm{K}}\, u = \Big\{ & U'' + (\alpha_1 - \alpha_2)\, \zeta \wedge \Big(* \big(\overline{\zeta} \wedge *U \big) \Big) + \alpha_2 |\zeta|^2 U \\ & + (\alpha_3 - \alpha_4)\, \overline{\zeta} \wedge \Big(* \big(\zeta \wedge *U \big) \Big) + \alpha_4 |\zeta|^2 U \Big\} e^{i(z \cdot \zeta + \overline{z} \cdot \overline{\zeta})} = 0. \end{aligned}$$

We put

$$U(t) = c_1(t)\, \zeta \wedge \overline{\zeta} + c_2(t)\, \zeta \wedge \overline{\zeta}^{\perp} + c_3(t)\, \zeta^{\perp} \wedge \overline{\zeta} + c_4(t)\, \zeta^{\perp} \wedge \overline{\zeta}^{\perp},$$

where $\zeta^\perp = \overline{\zeta}_2 dz^1 - \overline{\zeta}_1 dz^2$, $|\zeta| = |\zeta^\perp|$ hold. Then, we get

$$\Big(c_1'' + (\alpha_1+\alpha_3)|\zeta|^2 c_1\Big)\zeta\wedge\overline{\zeta} + \Big(c_2'' + (\alpha_1+\alpha_4)|\zeta|^2 c_2\Big)\zeta\wedge\overline{\zeta}^\perp$$
$$\Big(c_3'' + (\alpha_2+\alpha_3)|\zeta|^2 c_3\Big)\zeta^\perp\wedge\overline{\zeta} + \Big(c_4'' + (\alpha_2+\alpha_4)|\zeta|^2 c_4\Big)\zeta^\perp\wedge\overline{\zeta}^\perp = 0.$$

Hence, we obtain

$$\begin{aligned}
c_1(t) &= A_{13}^+ \exp\left(i\sqrt{\alpha_1+\alpha_3}|\zeta|t\right) + A_{13}^- \exp\left(-i\sqrt{\alpha_1+\alpha_3}|\zeta|t\right),\\
c_2(t) &= A_{14}^+ \exp\left(i\sqrt{\alpha_1+\alpha_4}|\zeta|t\right) + A_{14}^- \exp\left(-i\sqrt{\alpha_1+\alpha_4}|\zeta|t\right),\\
c_3(t) &= A_{23}^+ \exp\left(i\sqrt{\alpha_2+\alpha_3}|\zeta|t\right) + A_{23}^- \exp\left(-i\sqrt{\alpha_2+\alpha_3}|\zeta|t\right),\\
c_4(t) &= A_{24}^+ \exp\left(i\sqrt{\alpha_2+\alpha_4}|\zeta|t\right) + A_{24}^- \exp\left(-i\sqrt{\alpha_2+\alpha_4}|\zeta|t\right).
\end{aligned}$$

Since

$$\begin{aligned}
U(0) &= (A_{13}^+ + A_{13}^-)\ \zeta\wedge\overline{\zeta} + (A_{14}^+ + A_{14}^-)\ \zeta\wedge\overline{\zeta}^\perp\\
&\quad + (A_{23}^+ + A_{23}^-)\ \zeta^\perp\wedge\overline{\zeta} + (A_{24}^+ + A_{24}^-)\ \zeta^\perp\wedge\overline{\zeta}^\perp,\\
\frac{\partial}{\partial t}U(0) =& i\sqrt{\alpha_1+\alpha_3}|\zeta|(A_{13}^+ - A_{13}^-)\ \zeta\wedge\overline{\zeta}\\
&\quad + i\sqrt{\alpha_1+\alpha_4}|\zeta|(A_{14}^+ - A_{14}^-)\ \zeta\wedge\overline{\zeta}^\perp\\
&\quad + i\sqrt{\alpha_2+\alpha_3}|\zeta|(A_{23}^+ - A_{23}^-)\ \zeta^\perp\wedge\overline{\zeta}\\
&\quad + i\sqrt{\alpha_2+\alpha_4}|\zeta|(A_{24}^+ - A_{24}^-)\ \zeta^\perp\wedge\overline{\zeta}^\perp,
\end{aligned}$$

we get

$$\begin{aligned}
A_{13}^+ &= \frac{\langle U(0), \zeta\wedge\overline{\zeta}\rangle^*}{2|\zeta|^4} - i\ \frac{\langle \frac{\partial}{\partial t}U(0), \zeta\wedge\overline{\zeta}\rangle^*}{2\sqrt{\alpha_1+\alpha_3}|\zeta|^5}\ ,\\
A_{13}^- &= \frac{\langle U(0), \zeta\wedge\overline{\zeta}\rangle^*}{2|\zeta|^4} + i\ \frac{\langle \frac{\partial}{\partial t}U(0), \zeta\wedge\overline{\zeta}\rangle^*}{2\sqrt{\alpha_1+\alpha_3}|\zeta|^5}\ ,\\
A_{14}^+ &= \frac{\langle U(0), \zeta\wedge\overline{\zeta}^\perp\rangle^*}{2|\zeta|^4} - i\ \frac{\langle \frac{\partial}{\partial t}U(0), \zeta\wedge\overline{\zeta}^\perp\rangle^*}{2\sqrt{\alpha_1+\alpha_4}|\zeta|^5}\ ,\\
A_{14}^- &= \frac{\langle U(0), \zeta\wedge\overline{\zeta}^\perp\rangle^*}{2|\zeta|^4} + i\ \frac{\langle \frac{\partial}{\partial t}U(0), \zeta\wedge\overline{\zeta}^\perp\rangle^*}{2\sqrt{\alpha_1+\alpha_4}|\zeta|^5}\ ,\\
A_{23}^+ &= \frac{\langle U(0), \zeta^\perp\wedge\overline{\zeta}\rangle^*}{2|\zeta|^4} - i\ \frac{\langle \frac{\partial}{\partial t}U(0), \zeta^\perp\wedge\overline{\zeta}\rangle^*}{2\sqrt{\alpha_2+\alpha_3}|\zeta|^5}\ ,\\
A_{23}^- &= \frac{\langle U(0), \zeta^\perp\wedge\overline{\zeta}\rangle^*}{2|\zeta|^4} + i\ \frac{\langle \frac{\partial}{\partial t}U(0), \zeta^\perp\wedge\overline{\zeta}\rangle^*}{2\sqrt{\alpha_2+\alpha_3}|\zeta|^5}\ ,\\
A_{24}^+ &= \frac{\langle U(0), \zeta^\perp\wedge\overline{\zeta}^\perp\rangle^*}{2|\zeta|^4} - i\ \frac{\langle \frac{\partial}{\partial t}U(0), \zeta^\perp\wedge\overline{\zeta}^\perp\rangle^*}{2\sqrt{\alpha_2+\alpha_4}|\zeta|^5}\ ,
\end{aligned}$$

$$A_{24}^{-} = \frac{\langle U(0), \zeta^{\perp} \wedge \overline{\zeta}^{\perp} \rangle^{*}}{2|\zeta|^{4}} + i \; \frac{\langle \frac{\partial}{\partial t} U(0), \zeta^{\perp} \wedge \overline{\zeta}^{\perp} \rangle^{*}}{2\sqrt{\alpha_2 + \alpha_4}|\zeta|^{5}} \; .$$

Appendix

As an appendix, we give a brief introduction to Kähler manifolds. Further we give a proof of the equivalency of the conditions (0.1) and the Kähler condition in A.2.

A.1. The Kähler Condition on Hermitian Manifolds

Let X be an n-dimensional Hermitian manifold. Taking a holomorphic local coordinates $(z^1, \ldots, z^n)$, we define a positive definite Hermitian matrix $h_{j\overline{k}}(z) = \langle \partial_{z^j}, \partial_{z^k} \rangle_z$, when $\langle \cdot, \cdot \rangle$ is the Hermitian metric. Then the $(1,1)$-form

$$\Phi = h_{j\overline{k}} \; dz^j \wedge d\overline{z}^k$$

does not depend on the choice of the coordinates.

Definition A.1. We call a Hermitian manifold (X, h) a Kähler manifold when the equation

$$d\Phi = 0$$

is satisfied. We call this a Kähler condition.

To explain the meaning of $d\Phi = 0$, we introduce the canonical connection $\mathcal{D}$ induced by Φ. Let $\mathfrak{X}^{(q,r)}_{(s,t)}$ be a sheaf of $\bigotimes^{(q,r)} T_z X \otimes \bigwedge^{(s,t)} T^*_z X$-valued C^∞ functions.

Definition A.2. A holomorphic connection $\mathcal{D} : \mathfrak{X}^{(1,0)}_{(s,t)} \to \mathfrak{X}^{(1,0)}_{(s+1,t)}$ is defined as follows:

For $a \in \mathcal{E}^{(s,t)}, f \in \mathfrak{X}^{(1,0)}_{(s',t')}$, we have $\mathcal{D}(a \wedge f) = (\partial a) \wedge f + (-1)^{s+t} a \wedge \mathcal{D} f.$

To find the holomorphic connection induced by the Hermitian metric, we put

$$\mathcal{D}\left(\frac{\partial}{\partial z^j} \right) = \omega_j{}^i{}_k \; \frac{\partial}{\partial z^i} \otimes dz^k.$$

Since $\mathcal{D}$ induced by Φ satisfy the following commutative diagram

$$\begin{array}{ccc} \mathfrak{X}^{(1,0)}_{(0,0)} \ni \dfrac{\partial}{\partial z^j} & \xrightarrow{\Phi} & h_{j\bar{l}}\, d\bar{z}^l \in \mathcal{E}^{(0,1)}_X \\ {\scriptstyle \mathcal{D}}\big\downarrow & & \big\downarrow{\scriptstyle \partial} \\ \mathfrak{X}^{(1,0)}_{(1,0)} \ni \omega_j{}^i{}_k \dfrac{\partial}{\partial z^i} \otimes dz^k & \xrightarrow[\Phi]{} & \dfrac{\partial h_{j\bar{l}}}{\partial z^k}\, dz^k \wedge d\bar{z}^l \in \mathcal{E}^{(1,1)}_X , \end{array}$$

we get

$$\omega_j{}^i{}_k = h^{i\bar{l}} \partial_k h_{j\bar{l}} \, .$$

Hence we have

$$\mathcal{D}\left(\alpha^j \frac{\partial}{\partial z^j}\right) = \left(\frac{\partial \alpha^j}{\partial z^k} + \omega_i{}^j{}_k \alpha^i\right) \frac{\partial}{\partial z^j} \otimes dz^k.$$

By the canonical duality between ∂_{z^j} and dz^j, $\mathcal{D}$ induces a holomorphic connection on the cotangent bundle. Indeed we have

$$\mathcal{D}\left(dz^j\right) = -\omega_i{}^j{}_k dz^i \otimes dz^k.$$

Then, if this $\mathcal{D}$ is compatible with the exterior differential (torsion-free condition), we must impose the condition $\mathcal{D}(dz^j) = 0$ in $\bigwedge^{(2,0)} T^*X$; that is,

$$\omega_i{}^j{}_k = \omega_k{}^j{}_i \, .$$

This condition is just the Kähler condition.

A.2. Necessity of Kähler Conditions

We know that the Kähler condition leads the conditions (0.1). Conversely, we show that the conditions (0.1) imply the Kähler condition.

Theorem A.1. *If* $\Box = \overline{\Box}$ *as operators on* $\mathcal{E}^{(1,1)}_X$ *on a Hermitian manifold* X*, then* X *is a Kähler manifold.*

Proof. We fix any point p on X. Then by using a suitable complex linear transformation we can take a local holomorphic coordinate system $(z^1, \ldots, z^n)$ around p satisfying the following:

$$\begin{cases} z^1(p) = \cdots = z^n(p) = 0, \\ h_{i\bar{j}}(z) = \delta_{ij} + \omega_{ijl} z^l + \overline{\omega_{jil}}\, \bar{z}^l + O(|z|^2). \end{cases}$$

Then we get

$$\begin{cases} h^{j\overline{k}}(z) & = \delta_{jk} - \omega_{kjl} z^l - \overline{\omega_{jkl}}\ \overline{z}^l + O(|z|^2), \\ \det(h_{jk}) & = 1 + \omega_{jjl} z^l + \overline{\omega_{jjl}}\ \overline{z}^l + O(|z|^2), \\ |\det(h_{jk})| & = 1 + \omega_{jjl} z^l + \overline{\omega_{jjl}}\ \overline{z}^l + O(|z|^2). \end{cases}$$

Let $f = f_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and $g = g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ be C^∞ $(1,1)$-forms with compact supports. Therefore we have

$$\begin{aligned}
(\overline{\vartheta}\partial f, g) &= (\partial f, \partial g) \\
&= \int \frac{\partial f_{i\bar{j}}}{\partial z^k} \overline{\left(\frac{\partial g_{\alpha\overline{\beta}}}{\partial z^l}\right)} (h^{k\bar{l}} h^{i\overline{\alpha}} - h^{i\bar{l}} h^{k\overline{\alpha}}) h^{\beta\bar{j}} |\det h|\ dz^{\{\mathrm{all}\}} \wedge d\overline{z}^{\ \{\mathrm{all}\}} \\
&= -\int \frac{\partial}{\partial \overline{z}^l} \left(\frac{\partial f_{i\bar{j}}}{\partial z^k} \ (h^{k\bar{l}} h^{i\overline{\alpha}} - h^{i\bar{l}} h^{k\overline{\alpha}}) h^{\beta\bar{j}} |\det h| \right) |\det h|^{-1} \overline{g_{\alpha\overline{\beta}}}\ dV.
\end{aligned}$$

Here, $\{\mathrm{all}\} = \{1, 2, \ldots, n\}$, $dV = |\det h|\ dz^{\{\mathrm{all}\}} \wedge d\overline{z}^{\ \{\mathrm{all}\}}$. Hence we have

$$\begin{aligned}
&(\overline{\vartheta}\partial f)_{i\bar{j}} h^{i\overline{\alpha}} h^{\beta\bar{j}} \\
&\quad = -\frac{\partial}{\partial \overline{z}^l} \left(\frac{\partial f_{i\bar{j}}}{\partial z^k} \ (h^{k\bar{l}} h^{i\overline{\alpha}} - h^{i\bar{l}} h^{k\overline{\alpha}}) h^{\beta\bar{j}} |\det h| \right) |\det h|^{-1}.
\end{aligned}$$

Taking the values at $z = 0$, we obtain the following:

$$\begin{aligned}
(\overline{\vartheta}\partial f)_{\alpha\overline{\beta}} &= -\frac{\partial}{\partial \overline{z}^l} \left(\frac{\partial f_{i\bar{j}}}{\partial z^k} \ (h^{k\bar{l}} h^{i\overline{\alpha}} - h^{i\bar{l}} h^{k\overline{\alpha}}) h^{\beta\bar{j}} |\det h| \right) |\det h|^{-1} \\
&= -\frac{\partial^2 f_{\alpha\overline{\beta}}}{\partial z^k \partial \overline{z}^k} + \frac{\partial^2 f_{i\overline{\beta}}}{\partial z^\alpha \partial \overline{z}^i} + \overline{\omega_{kll}} \frac{\partial f_{\alpha\overline{\beta}}}{\partial z^k} + \overline{\omega_{i\alpha k}} \frac{\partial f_{i\overline{\beta}}}{\partial z^k} - \overline{\omega_{ill}} \frac{\partial f_{i\overline{\beta}}}{\partial z^\alpha} - \overline{\omega_{k\alpha l}} \frac{\partial f_{l\overline{\beta}}}{\partial z^k} \\
&\quad + \overline{\omega_{\beta jk}} \frac{\partial f_{\alpha\bar{j}}}{\partial z^k} - \overline{\omega_{\beta jl}} \frac{\partial f_{l\bar{j}}}{\partial z^\alpha} - \overline{\omega_{kkl}} \frac{\partial f_{\alpha\overline{\beta}}}{\partial z^l} + \overline{\omega_{kkl}} \frac{\partial f_{l\overline{\beta}}}{\partial z^\alpha} \\
&= -\frac{\partial^2 f_{\alpha\overline{\beta}}}{\partial z^k \partial \overline{z}^k} + \frac{\partial^2 f_{i\overline{\beta}}}{\partial z^\alpha \partial \overline{z}^i} + (\overline{\omega_{llk}} - \overline{\omega_{kll}}) \left(\frac{\partial f_{k\overline{\beta}}}{\partial z^\alpha} - \frac{\partial f_{\alpha\overline{\beta}}}{\partial z^k} \right) \\
&\quad + (\overline{\omega_{i\alpha k}} - \overline{\omega_{k\alpha i}}) \frac{\partial f_{i\overline{\beta}}}{\partial z^k} + \overline{\omega_{\beta jk}} \left(\frac{\partial f_{\alpha\bar{j}}}{\partial z^k} - \frac{\partial f_{k\bar{j}}}{\partial z^\alpha} \right).
\end{aligned}$$

On the other hand, let $f = f_{i\bar{j}} dz^i \wedge d\bar{z}^j$ and $g = g_{\bar{j}} d\bar{z}^j$ be $C^\infty(1,1)$- and $C^\infty(0,1)$-forms with compact supports, respectively. Then we have

$$\begin{aligned}
(\overline{\vartheta} f, g) = (f, \partial g) &= \int f_{i\overline{k}}\ h^{i\bar{l}} h^{j\overline{k}} \ \overline{\left(\frac{\partial g_{\bar{j}}}{\partial z^l}\right)}\ |\det h|\ dz^{\{\mathrm{all}\}} \wedge d\overline{z}^{\ \{\mathrm{all}\}} \\
&= -\int |\det h|^{-1} \frac{\partial}{\partial \overline{z}^l} \left(f_{i\overline{k}} h^{i\bar{l}} h^{j\overline{k}} |\det h| \right) \overline{g_{\bar{j}}}\ dV.
\end{aligned}$$

Therefore,

$$(\overline{\vartheta} f)_{\overline{\beta}} = -|\det h|^{-1} h_{j\overline{\beta}} \frac{\partial}{\partial \overline{z}^l} \left(f_{i\overline{k}} h^{i\overline{l}} h^{j\overline{k}} |\det h| \right).$$

Consequently we obtain the following:

$$\begin{aligned}(\partial\overline{\vartheta} f)_{\alpha\overline{\beta}} &= -\frac{\partial}{\partial z^\alpha}\left(|\det h|^{-1} h_{j\overline{\beta}} \frac{\partial}{\partial \overline{z}^l} \left(f_{i\overline{k}} h^{i\overline{l}} h^{j\overline{k}} |\det h| \right)\right) \\ &= -\frac{\partial}{\partial z^\alpha}\left(h^{i\overline{l}} \frac{\partial f_{i\overline{\beta}}}{\partial \overline{z}^l} + |\det h|^{-1} h_{j\overline{\beta}} f_{i\overline{k}} \frac{\partial}{\partial \overline{z}^l}\left(h^{i\overline{l}} h^{j\overline{k}} |\det h| \right)\right).\end{aligned}$$

Taking the values at $z = 0$ again, we get

$$\begin{aligned}(\partial\overline{\vartheta} f)_{\alpha\overline{\beta}} = &- \frac{\partial^2 f_{i\overline{\beta}}}{\partial z^\alpha \partial \overline{z}^i} + \omega_{li\alpha} \frac{\partial f_{i\overline{\beta}}}{\partial \overline{z}^l} + \overline{\omega_{ill}} \frac{\partial f_{i\overline{\beta}}}{\partial z^\alpha} + \overline{\omega_{\beta ki}} \frac{\partial f_{i\overline{k}}}{\partial z^\alpha} - \overline{\omega_{kki}} \frac{\partial f_{i\overline{\beta}}}{\partial z^\alpha} \\ &+ C_{\alpha\beta jk} f_{j\overline{k}}.\end{aligned}$$

Here $C_{\alpha\beta jk}$ are some constants which are independent of f and df. Therefore we have the following equalities at $z = 0$:

$$\begin{aligned}(\Box f)_{\alpha\overline{\beta}} &= (\partial\overline{\vartheta} f)_{\alpha\overline{\beta}} + (\overline{\vartheta}\partial f)_{\alpha\overline{\beta}} \\ &= -\frac{\partial^2 f_{\alpha\overline{\beta}}}{\partial z^k \partial \overline{z}^k} + (\overline{\omega_{ill}} - \overline{\omega_{lli}}) \frac{\partial f_{\alpha\overline{\beta}}}{\partial z^i} + (\overline{\omega_{j\alpha k}} - \overline{\omega_{k\alpha j}}) \frac{\partial f_{j\overline{\beta}}}{\partial z^k} \\ &\quad + \overline{\omega_{\beta jk}} \frac{\partial f_{\alpha\overline{j}}}{\partial z^k} + \omega_{li\alpha} \frac{\partial f_{i\overline{\beta}}}{\partial \overline{z}^l} + C_{\alpha\beta jk} f_{j\overline{k}}.\end{aligned}$$

Since $\overline{\Box}$ is the complex conjugate to $\Box$, we also have

$$\begin{aligned}(\overline{\Box} f)_{\alpha\overline{\beta}} &= -\frac{\partial^2 f_{\alpha\overline{\beta}}}{\partial z^k \partial \overline{z}^k} + (\omega_{ill} - \omega_{lli}) \frac{\partial f_{\alpha\overline{\beta}}}{\partial \overline{z}^i} + (\omega_{j\beta k} - \omega_{k\beta j}) \frac{\partial f_{\alpha\overline{j}}}{\partial \overline{z}^k} \\ &\quad + \omega_{\alpha jk} \frac{\partial f_{j\overline{\beta}}}{\partial \overline{z}^k} + \overline{\omega_{li\beta}} \frac{\partial f_{\alpha\overline{i}}}{\partial z^l} + \overline{C}_{\beta\alpha kj} f_{j\overline{k}}.\end{aligned}$$

When the equation

$$\begin{aligned}0 &= (\Box f - \overline{\Box} f)_{\alpha\overline{\beta}} \\ &= (\overline{\omega_{ill}} - \overline{\omega_{lli}}) \frac{\partial f_{\alpha\overline{\beta}}}{\partial z^i} + (\overline{\omega_{j\alpha k}} - \overline{\omega_{k\alpha j}}) \frac{\partial f_{j\overline{\beta}}}{\partial z^k} + (\overline{\omega_{\beta jk}} - \overline{\omega_{kj\beta}}) \frac{\partial f_{\alpha\overline{j}}}{\partial z^k} \\ &\quad - (\omega_{ill} - \omega_{lli}) \frac{\partial f_{\alpha\overline{\beta}}}{\partial \overline{z}^i} - (\omega_{j\beta k} - \omega_{k\beta j}) \frac{\partial f_{\alpha\overline{j}}}{\partial \overline{z}^k} - (\omega_{\alpha jk} - \omega_{kj\alpha}) \frac{\partial f_{j\overline{\beta}}}{\partial \overline{z}^k} \\ &\quad + (C_{\alpha\beta jk} - \overline{C}_{\beta\alpha kj}) f_{j\overline{k}}\end{aligned}$$

holds, for any given $\mu, \nu, \kappa \in \{1, \ldots, n\}$ we can choose f such that

$$\partial f_{j\overline{k}}(0)/\partial z^i = \delta_{i\mu}\delta_{j\nu}\delta_{k\kappa}, \quad \partial f_{j'\overline{k'}}(0)/\partial \overline{z}^{i'} = 0, \quad f_{j''\overline{k''}}(0) = 0,$$

and we have

$$0 = (\overline{\omega_{\mu ll}} - \overline{\omega_{ll\mu}})\delta_{\alpha\nu}\delta_{\beta\kappa} + (\overline{\omega_{\nu\alpha\mu}} - \overline{\omega_{\mu\alpha\nu}})\delta_{\beta\kappa} + (\overline{\omega_{\beta\kappa\mu}} - \overline{\omega_{\mu\kappa\beta}})\delta_{\alpha\nu}.$$

If we choose $\alpha = \nu$ and $\beta \neq \kappa$, then we see that

$$\overline{\omega_{\beta\kappa\mu}} - \overline{\omega_{\mu\kappa\beta}} = 0$$

holds for any β, κ, μ with $\beta \neq \kappa$. Further if we take $\mu = \kappa \neq \beta$, we have $\overline{\omega_{\beta\kappa\kappa}} - \overline{\omega_{\kappa\kappa\beta}} = 0$ (not the summation in κ) for any $\beta \neq \kappa$. Consequently we get

$$\omega_{ijk} = \omega_{kji} \quad \text{for any} \quad i,\ j,\ k.$$

This is equivalent to the Kähler condition at p. That is, $d\Phi(p) = 0$. Since p is an arbitrary point of X, so X is a Kähler manifold. □

References

[1] G. de Rham, Variétés Différentiables, HERMANN Paris, 1955.

[2] K. Yano and Y. Muto, Tensor analysis ("Tensor kaiseki" in Japanese), Hirokawa-shoten, 1968.

[3] S. Kobayashi, *Differential geometry of complex vector bundles*, Publ. Math. Soc. Japan, No.15, Iwanami-Princeton, Univ. Press, 1987.

[4] S. Murakami, *Manifolds* ("Tayōtai" in Japanese), Kyōritsu-sūgaku-kōza **19**, Kyōritsu-shuppan, 1969.

[5] Y. Akizuki, Harmonic integral theory ("Chōwa-Sekibunron" in Japanese), Gendai-sūgaku **16**, Iwanami-shoten, 1956.

[6] M. Kawashita, On a region free from the poles of the resolvent and decay rate of the local energy for the elastic wave equation, Indiana Univ. Math. J. **43** (1994), no. 3, 1013–1043.

[7] S. Shimizu, Eigenfunction expansions for elastic wave propagation problems in stratified media $\mathbb{R}^3$, Tsukuba J. Math. **19**, (1995). Math. Methods Appl. Sci. **19** (1996), no. 3, 187–215.