Publl. RIMS, Kyoto Univ.
43 (2007), 525–533

Proper Stacks

By

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Abstract

We generalize the notion of proper stack introduced by Kashiwara and Schapira to the case of a general site, and we prove that a proper stack is a stack.

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Introduction

In [3] Kashiwara and Schapira defined the notion of proper stack on a locally compact topological space X. A proper stack is a separated prestack $\mathcal{S}$ satisfying suitable hypothesis. They proved that a proper stack is a stack. In this paper, we generalize the notion of proper stack to the case of a site X associated to a small category $\mathcal{C}_X$ and we prove that a proper stack is a stack.

Communicated by M. Kashiwara. Received June 27, 2006.
2000 Mathematics Subject Classification(s): 14A20, 18D30, 18F10.
Key words: Stacks, Grothendieck topologies
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§1. Review on Grothendieck Topologies and Sheaves

Let $\mathcal{C}$ be a category[1]. As usual we denote by $\mathcal{C}^\wedge$ the category of functors from $\mathcal{C}^{op}$ to **Set** and we identify $\mathcal{C}$ with its image in $\mathcal{C}^\wedge$ via the Yoneda embedding. If $A \in \mathcal{C}^\wedge$, we will denote by $\mathcal{C}_A$ the category of arrows $U \to A$ with $U \in \mathcal{C}$. When taking inductive and projective limits on a category I we will always assume that I is small.

We recall here some classical definitions (see [2]), following the presentation of [4].

Definition 1.1. A Grothendieck topology on a small category $\mathcal{C}_X$ is a collection of morphisms in $\mathcal{C}_X^\wedge$ called local epimorphisms, satysfying the following conditions:

LE1 For any $U \in \mathcal{C}_X$, $\mathrm{id}_U : U \to U$ is a local epimorphism.

LE2 Let $A_1 \stackrel{u}{\to} A_2 \stackrel{v}{\to} A_3$ be morphisms in $\mathcal{C}_X^\wedge$. If u and v are local epimorphisms, then $v \circ u$ is a local epimorphism.

LE3 Let $A_1 \stackrel{u}{\to} A_2 \stackrel{v}{\to} A_3$ be morphisms in $\mathcal{C}_X^\wedge$. If $v \circ u$ is a local epimorphism, then v is a local epimorphism.

LE4 A morphism $u : A \to B$ in $\mathcal{C}_X^\wedge$ is a local epimorphism if and only if for any $U \in \mathcal{C}_X$ and any morphism $U \to B$, the morphism $A \times_B U \to U$ is a local epimorphism.

Definition 1.2. A morphism $A \to B$ in $\mathcal{C}_X^\wedge$ is a local monomorphism if $A \to A \times_B A$ is a local epimorphism.

A morphism $A \to B$ in $\mathcal{C}_X^\wedge$ is a local isomorphism if it is both a local epimorphism and a local monomorphism.

Definition 1.3. A site X is a category $\mathcal{C}_X$ endowed with a Grothendieck topology.

Let $\mathcal{A}$ be a category admitting small inductive and projective limits.

Definition 1.4. An $\mathcal{A}$-valued presheaf on X is a functor $\mathcal{C}_X^{op} \to \mathcal{A}$. A morphism of presheaves is a morphism of such functors. One denotes by $\mathrm{Psh}(X, \mathcal{A})$ the category of $\mathcal{A}$-valued presheaves on X.

[1]We shall work in a given universe $\mathcal{U}$, small means $\mathcal{U}$-small (i.e. a set is $\mathcal{U}$-small if it is isomorphic to a set belonging to $\mathcal{U}$) and a category $\mathcal{C}$ means a $\mathcal{U}$-category (i.e. $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is $\mathcal{U}$-small for any $X, Y \in \mathcal{C}$).

If $F \in \mathrm{Psh}(X, \mathcal{A})$, it extends naturally to $\mathcal{C}_X^\wedge$ by setting

$$F(A) = \varprojlim_{(U \to A) \in \mathcal{C}_A} F(U),$$

where $A \in \mathcal{C}_X^\wedge$ and $U \in \mathcal{C}_X$.

Definition 1.5. Let X be a site.

- One says that $F \in \mathrm{Psh}(X, \mathcal{A})$ is separated, if for any local isomorphism $A \to U$ with $U \in \mathcal{C}_X$ and $A \in \mathcal{C}_X^\wedge$, $F(U) \to F(A)$ is a monomorphism.
- One says that $F \in \mathrm{Psh}(X, \mathcal{A})$ is a sheaf, if for any local isomorphism $A \to U$ with $U \in \mathcal{C}_X$ and $A \in \mathcal{C}_X^\wedge$, $F(U) \to F(A)$ is an isomorphism.

§2. Review on Stacks

Let $\mathcal{C}_X$ be a small category. We suppose that a Grothendieck topology on $\mathcal{C}_X$ is defined and we denote by X the associated site. We recall some classical definitions (see [1]), following the presentation of [4].

Definition 2.1. A prestack $\mathcal{S}$ on X is the data of:

- for each $U \in \mathcal{C}_X$, a category $\mathcal{S}(U)$,
- for each $V \to U \in \mathcal{C}_U$, a functor $j_{VU*} : \mathcal{S}(U) \to \mathcal{S}(V)$,
- given $U, V, W \in \mathcal{C}_X$ and $W \to V \to U$, an isomorphism of functors $\lambda_{WVU} : j_{WV*} \circ j_{VU*} \xrightarrow{\sim} j_{WU*}$,

such that

- $j_{UU*} = \mathrm{id}_{\mathcal{S}(U)}$,
- given $\{U_i\}_{i \in I} \in \mathcal{C}_X$, $i = 1, 2, 3, 4$ and $U_1 \to U_2 \to U_3 \to U_4$, the following diagram commutes:

$$\begin{array}{ccc} j_{12*} \circ j_{23*} \circ j_{34*} & \xrightarrow{\lambda_{234}} & j_{12*} \circ j_{24*} \\ \downarrow{\scriptstyle \lambda_{123}} & & \downarrow{\scriptstyle \lambda_{124}} \\ j_{13*} \circ j_{34*} & \xrightarrow{\lambda_{134}} & j_{14*} \end{array}$$

Let $\varprojlim_{U \in \mathcal{C}_X} \mathcal{S}(U)$ denote a category defined as follows. An object F of $\varprojlim_{U \in \mathcal{C}_X} \mathcal{S}(U)$ is a family $\{(F_U)_U, (\psi_u)_u\}$ where

- for any $U \in \mathcal{C}_X$, $F_U \in \mathrm{Ob}(\mathcal{S}(U))$,
- for any morphism $U_1 \to U_2$ in $\mathcal{C}_X$, $\psi_{12} : j_{12*}F_{U_2} \to F_{U_1}$ is an isomorphism, such that for any sequence $U_1 \to U_2 \to U_3$ the following diagram commutes

$$\begin{array}{ccc} j_{12*}j_{23*}F_{U_3} & \xrightarrow{\psi_{23}} & j_{12*}F_{U_2} \\ \downarrow{\scriptstyle \lambda_{123}} & & \downarrow{\scriptstyle \psi_{12}} \\ j_{13*}F_{U_3} & \xrightarrow{\psi_{13}} & F_{U_1}. \end{array}$$

Note that $\psi_{\mathrm{id}_U} = \mathrm{id}_{F_U}$ for any $U \in \mathcal{C}_X$.

The morphisms are defined in natural way. Let $F, G \in \varprojlim_{U \in \mathcal{C}_X} \mathcal{S}(U)$. Then

$$\mathrm{Hom}_{\varprojlim_{U\in\mathcal{C}_X} \mathcal{S}(U)}(F,G) \simeq \varprojlim_{U\in\mathcal{C}_X} \mathrm{Hom}_{\mathcal{S}(U)}(F_U, G_U).$$

For any $A \in \mathcal{C}_X^\wedge$, we set

$$\mathcal{S}(A) = \varprojlim_{(U\to A)\in\mathcal{C}_A} \mathcal{S}(U).$$

A morphism $\varphi : A \to B$ in $\mathcal{C}_X^\wedge$ defines a functor $j_{AB*} : \mathcal{S}(B) \to \mathcal{S}(A)$, therefore a prestack on $\mathcal{C}_X$ extends naturally to a prestack on $\mathcal{C}_X^\wedge$.

Definition 2.2. Let X be a site.

- A prestack $\mathcal{S}$ on X is called separated if for any $U \in \mathcal{C}_X$, and for any local isomorphism $A \to U$ in $\mathcal{C}_X^\wedge$, $j_{AU*} : \mathcal{S}(U) \to \mathcal{S}(A)$ is fully faithful.
- A prestack $\mathcal{S}$ on X is called a stack if for any $U \in \mathcal{C}_X$, and for any local isomorphism $A \to U$ in $\mathcal{C}_X^\wedge$, $j_{AU*} : \mathcal{S}(U) \to \mathcal{S}(A)$ is an equivalence.

Proposition 2.3. *Let $\mathcal{S}$ be a prestack on X. Then $\mathcal{S}$ is a stack if and only if $\mathcal{S}$ satisfies the following conditions*:

(i) *$\mathcal{S}$ is separated,*

(ii) *for any $U \in \mathcal{C}_X$ and for any local isomorphism $A \to U$ the restriction functor $j_{AU*} : \mathcal{S}(U) \to \mathcal{S}(A)$ admits a left adjoint j_{AU}^{-1} satisfying $j_{AU*} \circ j_{AU}^{-1} \simeq \mathrm{id}$* (*or, equivalently, the functor j_{AU}^{-1} is fully faithful*).

Proof. The result follows from the fact that two categories are equivalent if and only if they admit a pair of fully faithful adjoint functors. ☐

§3. Proper Stacks

Let $\mathcal{C}_X$ be a small category. In this section we extend a result of [3] to the case of a site X associated to a small category $\mathcal{C}_X$.

Let $\mathcal{S}$ be a prestack on X and assume the following hypothesis

$$(1)\quad \begin{cases} \text{- for any } U, V \in \mathcal{C}_X \text{ and any morphism } U \to V \text{ in } \mathcal{C}_X^\wedge, \text{ the functor} \\ \quad j_{UV*} : \mathcal{S}(V) \to \mathcal{S}(U) \text{ admits a left adjoint } j_{UV}^{-1} \text{ satisfying} \\ \quad \mathrm{id}_{\mathcal{S}(U)} \xrightarrow{\sim} j_{UV*} \circ j_{UV}^{-1} \text{ (or, equivalently, } j_{UV}^{-1} \text{ is fully faithful),} \\ \text{- for all } U \in \mathcal{C}_X \text{ the category } \mathcal{S}(U) \text{ admits small inductive limits.} \end{cases}$$

Lemma 3.1. *Let $\mathcal{S}$ be a prestack and assume* (1). *Let $A \in \mathcal{C}_X^\wedge$, $V \in \mathcal{C}_X$ and $A \to V$. Then the functor j_{AV*} admits a left adjoint, denoted by j_{AV}^{-1}.*

Proof. Let $F = \{F_U\}_{(U\to A)\in\mathcal{C}_A} \in \mathcal{S}(A)$, and let $j_{AV}^{-1}F := \varinjlim_{(U\to A)\in\mathcal{C}_A} j_{UV}^{-1}F_U$. This defines a functor $j_{AV}^{-1} : \mathcal{S}(A) \to \mathcal{S}(V)$. Let $G \in \mathcal{S}(V)$. We have the chain of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{S}(V)}(j_{AV}^{-1}F, G) &= \mathrm{Hom}_{\mathcal{S}(V)}(\varinjlim_{(U\to A)\in\mathcal{C}_A} j_{UV}^{-1}F_U, G) \\ &\simeq \varprojlim_{(U\to A)\in\mathcal{C}_A} \mathrm{Hom}_{\mathcal{S}(V)}(j_{UV}^{-1}F_U, G) \\ &\simeq \varprojlim_{(U\to A)\in\mathcal{C}_A} \mathrm{Hom}_{\mathcal{S}(U)}(F_U, j_{UV*}G) \\ &\simeq \mathrm{Hom}_{\mathcal{S}(A)}(F, j_{AV*}G). \end{aligned}$$

□

Lemma 3.2. *Let $\mathcal{S}$ be a prestack on X satisfying* (1), *let $U', U, V \in \mathcal{C}_X$ and $U' \to U \to V$. Then*

(i) *there exists a canonical morphism $j_{U'V}^{-1} \circ j_{U'V*} \to j_{UV}^{-1} \circ j_{UV*}$,*

(ii) *we have $j_{U'V}^{-1} \circ j_{U'V*} \simeq j_{U'V}^{-1} \circ j_{U'V*} \circ j_{UV}^{-1} \circ j_{UV*}$.*

Proof. (i) The adjunction morphism $j_{U'U}^{-1} \circ j_{U'U*} \to \mathrm{id}_{\mathcal{S}(U)}$ defines

$$j_{U'V}^{-1} \circ j_{U'V*} \simeq j_{UV}^{-1} \circ j_{U'U}^{-1} \circ j_{U'U*} \circ j_{UV*} \to j_{UV}^{-1} \circ j_{UV*}.$$

(ii) We have $j_{U'V*} \simeq j_{U'U*} \circ j_{UV*}$, and then

$$j_{U'V*} \circ j_{UV}^{-1} \simeq j_{U'U*} \circ j_{UV*} \circ j_{UV}^{-1} \simeq j_{U'U*}.$$

Hence we have the chain of isomorphisms

$$j^{-1}_{U'V} \circ j_{U'V*} \circ j^{-1}_{UV} \circ j_{UV*} \simeq j^{-1}_{U'V} \circ j_{U'U*} \circ j_{UV*} \simeq j^{-1}_{U'V} \circ j_{U'V*}.$$

□

Lemma 3.3. *Let $\mathcal{S}$ be a prestack on X satisfying* (1). *Let $U, V, W \in \mathcal{C}_X$ and let $U \to W$, $V \to W$ be morphisms. Consider the diagram*

$$\begin{array}{ccc} U \times_W V & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & W \end{array}$$

where $U \times_W V \in \mathcal{C}_X^\wedge$. Then there exists a canonical morphism

$$j^{-1}_{U\times_W VW} \circ j_{U\times_W VW*} \to j^{-1}_{UW} \circ j_{UW*} \circ j^{-1}_{VW} \circ j_{VW*}. \tag{2}$$

Proof. Since $U \times_W V \in \mathcal{C}_X^\wedge$ for each $F \in \mathcal{S}(W)$ we have

$$j_{U\times_W VW*}F = \{j_{W'W*}F\}_{(W'\to U\times_W V)\in\mathcal{C}_{U\times_W V}} \in \mathcal{S}(U \times_W V)$$

hence as in Lemma 3.1

$$j^{-1}_{U\times_W VW} j_{U\times_W VW*}F \simeq \varinjlim_{(W'\to U\times_W V)\in\mathcal{C}_{U\times_W V}} j^{-1}_{W'W} j_{W'W*}F.$$

By Lemma 3.2 we have $j^{-1}_{W'W} \circ j_{W'W*} \circ j^{-1}_{VW} \circ j_{VW*} \simeq j^{-1}_{W'W} \circ j_{W'W*}$ for each $(W' \to U \times_W V) \in \mathcal{C}_{U\times_W V}$. We have natural morphisms

$$\begin{aligned} j^{-1}_{U\times_W VW} \circ j_{U\times_W VW*} &\xrightarrow{\sim} j^{-1}_{U\times_W VW} \circ j_{U\times_W VW*} \circ j^{-1}_{VW} \circ j_{VW*} \\ &\to j^{-1}_{UW} \circ j_{UW*} \circ j^{-1}_{VW} \circ j_{VW*}. \end{aligned}$$

□

Let $U, V, W \in \mathcal{C}_X$ and let $U \to W$, $V \to W$ be morphisms. The morphism (2) induces a natural arrow

$$j^{-1}_{U\times_W VV} \circ j_{U\times_W VU*} \xrightarrow{\sim} j_{VW*} \circ j^{-1}_{U\times_W VW} \circ j_{U\times_W VW*} \circ j^{-1}_{UW} \to j_{VW*} \circ j^{-1}_{UW}.$$

Definition 3.4. A proper stack $\mathcal{S}$ on X is a prestack satisfying

PRS1 $\mathcal{S}$ is separated,

PRS2 for each $U \in \mathcal{C}_X$, $\mathcal{S}(U)$ admits small inductive limits,

PRS3 for all $U, V \in \mathcal{C}_X$ and $U \to V$ the functor $j_{UV*} : \mathcal{S}(V) \to \mathcal{S}(U)$ commutes with $\varinjlim$,

PRS4 for all $U, V \in \mathcal{C}_X$ and $U \to V$ the functor $j_{UV*} : \mathcal{S}(V) \to \mathcal{S}(U)$ admits a left adjoint j_{UV}^{-1}, satisfying $\mathrm{id}_{\mathcal{S}(U)} \xrightarrow{\sim} j_{UV*} \circ j_{UV}^{-1}$ (or, equivalently, the functor j_{UV}^{-1} is fully faithful),

PRS5 for all $V, U, W \in \mathcal{C}_X$, $U \to W$ and $V \to W$, the morphism

$$j_{U\times_W V V}^{-1} \circ j_{U\times_W V U*} \to j_{VW*} \circ j_{UW}^{-1}$$

is an isomorphism.

Remark 3.5. Here $U \times_W V \in \mathcal{C}_X^\wedge$, since we have not assumed that $\mathcal{C}_X$ admits fiber products.

Lemma 3.6. *Let us consider the following diagram*

$$\begin{array}{ccc} A \times_V U & \longrightarrow & A \\ \downarrow & & \downarrow \\ U & \longrightarrow & V \end{array}$$

where $U, V \in \mathcal{C}_X$ and $A \in \mathcal{C}_X^\wedge$. Let $\mathcal{S}$ be a proper stack on X. Then we have

$$j_{UV*} \circ j_{AV}^{-1} \simeq j_{A\times_V UU}^{-1} \circ j_{A\times_V UA*}.$$

Proof. Let $F = \{F_W\}_{(W\to A)\in\mathcal{C}_A} \in \mathcal{S}(A)$. We have the chain of isomorphisms

$$\begin{aligned} j_{UV*} \circ j_{AV}^{-1} F &\simeq j_{UV*} \varinjlim_{(W\to A)\in\mathcal{C}_A} j_{WV}^{-1} F_W \\ &\simeq \varinjlim_{(W\to A)\in\mathcal{C}_A} j_{UV*} j_{WV}^{-1} F_W \\ &\simeq \varinjlim_{(W\to A)\in\mathcal{C}_A} j_{U\times_V WU}^{-1} j_{U\times_V WW*} F_W \\ &\simeq \varinjlim_{(W\to A)\in\mathcal{C}_A} \varinjlim_{(W'\to W\times_V U)\in\mathcal{C}_{W\times_V U}} j_{W'U}^{-1} F_{W'} \\ &\simeq \varinjlim_{(W''\to A\times_V U)\in\mathcal{C}_{A\times_V U}} j_{W''U}^{-1} F_{W''}, \end{aligned}$$

where the second and the third isomorphism follow from PRS3 and PRS5 respectively. The fourth isomorphism follows since $W \times_V U \in \mathcal{C}_X^\wedge$ and we have

$$\begin{aligned} j_{U\times_V W W*} F_W &\simeq \{j_{W'W*} F_W\}_{(W'\to W\times_V U)\in \mathcal{C}_{W\times_V U}} \\ &\simeq \{F_{W'}\}_{(W'\to W\times_V U)\in \mathcal{C}_{W\times_V U}}. \end{aligned}$$

On the other hand we have $j_{A\times_V U A*} F \simeq \{F_{W''}\}_{(W''\to A\times_V U)\in \mathcal{C}_{A\times_V U}}$, hence

$$j^{-1}_{A\times_V U U} \circ j_{A\times_V U A*} F \simeq \varinjlim_{(W''\to A\times_V U)\in \mathcal{C}_{A\times_V U}} j^{-1}_{W''U} F_{W''}.$$

□

Theorem 3.7. *Let X be a site associated to a small category $\mathcal{C}_X$. Let $\mathcal{S}$ be a proper stack on X. Then $\mathcal{S}$ is a stack.*

Proof. Let $A \to V$ be a local isomorphism. By Proposition 2.3 it is enough to show that $j_{AV*} \circ j^{-1}_{AV} \simeq \mathrm{id}$. Let $F = \{F_{V_i}\}_{(V_i\to A)\in \mathcal{C}_A} \in \mathcal{S}(A)$. It satisfies, for each $V_i \to V_j$

$$j_{V_i V_j *} F_{V_j} \xrightarrow{\sim} F_{V_i}. \tag{3}$$

We have to show that $j_{V_i V*} j^{-1}_{AV} F \simeq F_{V_i}$ for each $V_i \to A$. Let us consider $V_{i_0} \to A$. By PRS5 and (3), for each $V_k \to A$ we have the chain of isomorphisms

$$j_{V_{i_0} V*} j^{-1}_{V_k V} F_{V_k} \simeq j^{-1}_{V_k\times_V V_{i_0} V_{i_0}} j_{V_k\times_V V_{i_0} V_k *} F_{V_k} \simeq j^{-1}_{V_k\times_V V_{i_0} V_{i_0}} j_{V_k\times_V V_{i_0} V_{i_0} *} F_{V_{i_0}}.$$

Hence we obtain the isomorphism

$$j_{V_{i_0} V*} j^{-1}_{AV} F \simeq j^{-1}_{A\times_V V_{i_0} V_{i_0}} j_{A\times_V V_{i_0} V_{i_0} *} F_{V_{i_0}},$$

and $j^{-1}_{A\times_V V_{i_0} V_{i_0}} j_{A\times_V V_{i_0} V_{i_0} *} F_{V_{i_0}} \simeq F_{V_{i_0}}$ since $\mathcal{S}$ is separated and $A \times_V V_{i_0} \to V_{i_0}$ is a local isomorphism. □

Example 3.8. Let k be a field, and X a topological space (or, more generally, let X be a site associated to an ordered-set category). The prestack associating to an open set U of X the category of sheaves of k-vector spaces[2] on U is a proper stack.

[2]More generally, one can consider sheaves with values in a category $\mathcal{A}$ admitting small inductive and projective limits, such that filtrant inductive limits are exact and satisfying the ICP property (see [4] for a detailed exposition).

Acknowledgements

We thank Pierre Schapira who encouraged us to generalize the notion of proper stack. We thank Stéphane Guillermou and Pietro Polesello for their useful remarks.

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