Publ. RIMS, Kyoto Univ.
43 (2007), 585–623

Magnetic Pseudodifferential Operators

By

Viorel Iftimie, Marius Măntoiu and Radu Purice*

Abstract

In previous papers, a generalization of the Weyl calculus was introduced in connection with the quantization of a particle moving in $\mathbb{R}^n$ under the influence of a variable magnetic field B. It incorporates phase factors defined by B and reproduces the usual Weyl calculus for $B=0$. In the present article we develop the classical pseudodifferential theory of this formalism for the standard symbol classes $S^m_{\rho,\delta}$. Among others, we obtain properties and asymptotic developments for the magnetic symbol multiplication, existence of parametrices, boundedness and positivity results, properties of the magnetic Sobolev spaces. In the case when the vector potential A has all the derivatives of order ≥ 1 bounded, we show that the resolvent and the fractional powers of an elliptic magnetic pseudodifferential operator are also pseudodifferential. As an application, we get a limiting absorption principle and detailed spectral results for self-adjoint operators of the form $H = h(Q, \Pi^A)$, where h is an elliptic symbol, Q denotes multiplication with the variables $\Pi^A = D - A$, D is the operator of derivation and A is the vector potential corresponding to a short-range magnetic field.

Introduction

One of the different, but related, points of view on the usual Weyl calculus says that the correspondence symbol ↦ operator, $f \mapsto \mathfrak{Op}(f)$, is a functional calculus for the family of operators $Q_1, \dots, Q_n; D_1, \dots, D_n$ on $L^2(\mathbb{R}^n)$, where Q_j is the multiplication with the variable x_j and $D_j = -i\partial_j$. The familiar

---

Communicated by T. Kawai. Received September 21, 2006.
2000 Mathematics Subject Classification(s): 35S05, 47A60, 81Q10.
Key words and phrases: Magnetic field, gauge invariance, quantization, pseudodifferential operator, Weyl calculus, Moyal product, Sobolev space, Gårding inequality, limiting absorption principle.

*Institute of Mathematics “Simion Stoilow” of the Romanian Academy, P.O. Box 1-764, Bucharest, RO-70700, Romania.
e-mail: `viftimie@math.math.unibuc.ro, mantoiu@imar.ro, purice@imar.ro`

notation $\mathfrak{Op}(f) = f(Q, D)$ keeps track of this fact. The relative sophistication of this formalism has its roots in the non-commutativity of the basic operators:

$$i[Q_j, Q_k] = 0 = i[D_j, D_k], \qquad i[D_j, Q_k] = \delta_{j,k}. \tag{0.1}$$

For a nonrelativistic quantum particle in $\mathbb{R}^n$ placed in a magnetic field B deriving from a vector potential A, the basic self-adjoint operators (quantum observables) are the positions $Q_1, \dots, Q_n$ and the magnetic momenta $\Pi_1^A := D_1 - A_1, \dots, \Pi_n^A := D_1 - A_n$, satisfying the commutation relations

$$i[Q_j, Q_k] = 0, \qquad i[\Pi_j^A, Q_k] = \delta_{jk}, \qquad i[\Pi_j^A, \Pi_k^A] = B_{jk}, \tag{0.2}$$

where B_{jk} is (the operator of multiplication by) the component (jk) of the magnetic field. These relations are a representation by unbounded operators of a Lie algebra that has infinite dimension if B_{jk} are not all polynomial functions. For the case of a constant magnetic field such a calculus has been developped in [BMGH], or in [B] for the case of a lattice.

It is natural to look for a pseudodifferential calculus adapted to such a situation. At first sight, a procedure could be to replace in the explicit formula for $\mathfrak{Op}(f)$ the symbol $f(x, \xi)$ by $f(x, \xi - A(x))$, obtaining an operator $\mathfrak{Op}_A(f)$. Although largely used in the literature (see [GMS], [Ic1], [Ic2], [II], [IT1], [IT2], [ITs1], [ITs2], [NU1], [NU2], [Pa], [Um]), this point of view does not seem adequate, due to the fact that the operators $\mathfrak{Op}_A(f)$, although representing physical observables, are not gauge covariant. Two vector potentials A and A' connected by $A' = A + d\varphi$ for some smooth real function φ, being assigned to the same magnetic field $B = dA = dA'$, should produce unitarily equivalent operators $\mathfrak{Op}_A(f)$ and $\mathfrak{Op}_{A'}(f)$ for all reasonable f. In Section 6 we are going to exibit large classes of symbols f (including the third order monomial $f(x, \xi) = \xi_j \xi_k \xi_l$) for which the expected equality $\mathfrak{Op}_{A+d\varphi}(f) = e^{i\varphi} \mathfrak{Op}_A(f) e^{-i\varphi}$ fails. The pseudodifferential calculus with a magnetic field has been used in several papers dealing with the Peierls substitution ([DS], [HS1], [HS2], [N], [PST], [T]). Although gauge covariance is not essential for the technical arguments used in this context, it is possible that our formalism may bring some new insight and even technical advantages.

The right formalism was proposed independently and with different emphases in [KO1], [KO2] and [MP1], [MP2] where a gauge covariant functional calculus $\mathfrak{Op}^A(f)$ is defined. It was generalized and related to a C^*-algebraic formalism in [MPR1], and applied to the strict deformation quantization in the sense of Rieffel for systems in a magnetic field in [MP3] and [MP4]. We shall remind very briefly this pseudodifferential point of view in Section 1, while the other sections will be dedicated to our actual purposes: a development of the calculus for Hörmander symbol classes $S^m_{\rho,\delta}$ and applications. For the time being, *we have only succeeded to treat smooth, bounded magnetic fields having bounded derivatives of all orders.* Since, however, no decay at infinity is requested, we belive that the theory we develop is general enough to support nontrivial applications.

In Section 2 we show first that for $f \in S^m_{\rho,\delta}$, $(\rho \geq 0,\ \delta < 1)$, $\mathfrak{Op}^A(f)$ leaves the Schwartz space invariant. Afterwards, we study the product $f \circ^B g$ for f, g belonging to Hörmander's classes of symbols $S^m_{\rho,\delta}$, $(0 \leq \delta < \rho \leq 1)$. This is basic for the rest of the article. We give an asymptotic series for $f \circ^B g$, that shows that the first term of the commutator $f \circ^B g - g \circ^B f$ is the Poisson bracket $\{f, g\}_B$ assigned to the magnetic symplectic form σ_B. Another consequence is the existence of a parametrix for elliptic magnetic pseudodifferential operators.

In Section 3 we prove that $\mathfrak{Op}^A(f)$ is bounded in $L^2(\mathbb{R}^n)$ if $f \in S^0_{\rho,\delta}$ and $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$. The case $\delta = \rho$ is a magnetic version of the Calderon-Vaillancourt theorem transcripted for the Weyl calculus (cf. [Fo]).

Section 4 is dedicated to the study of magnetic Sobolev spaces. Previously, they were considered only in situations when a vector potential can be chosen with bounded derivatives of strictly positive order, cf. [GMS] and [Pa].

In Section 5 we show that an elliptic magnetic pseudodifferential operator is self-adjoint on the corresponding Sobolev spaces. For convenient vector potentials, the Schwartz space is a core. As a consequence of a Gårding-type inequality, we also treat semiboundedness.

Throughout the paper we suppose the magnetic field B to have bounded components together with all their derivatives. In Section 6 we shall moreover assume that $B = dA$ for some smooth vector potential A having bounded derivatives of any strictly positive order. This facilitates certain arguments; in particular it leads to a connection between our magnetic calculus and the Weyl calculus for a certain A-dependent Hörmander-type metric, and this allows the transcription of certain classical results ([Ho1], [Ho2], [Bo3]) to our framework.

As said before, many authors use for a symbol $p(x, \xi)$ the magnetic quantization $\mathfrak{Op}_A(p)$ that does not provide a gauge covariant calculus. In Section 6, as a continuation of the analysis in Subsection IV D of [MP2], we shall compare this procedure with our gauge covariant quantization and prove that $\mathfrak{Op}^A(p) - \mathfrak{Op}_A(p)$ is a pseudodifferential operator of strictly smaller order. In fact we prove that under the above hypothesis on the magnetic field B, one can pass from the functional calculus $\mathfrak{Op}^A$ to the functional calculus $\mathfrak{Op}_A$ and in the opposite direction. Using the Weyl-Hörmander-Bony calculus we obtain a Fefferman-Phong type theorem and prove that the resolvent and the powers of a magnetic self-adjoint elliptic pseudo-differential operator are also pseudodifferential. In particular we are able to compare three candidates for the relativistic Schrödinger Hamiltonian with magnetic field: $\sqrt{(D-A)^2+1}$, $\mathfrak{Op}_A(\langle\xi\rangle)$ and $\mathfrak{Op}^A(\langle\xi\rangle)$.

The last section is devoted to the spectral analysis (obtaining a limiting absorption principle) for a class of elliptic pseudodifferential operators obtained through a quantization (either by Weyl calculus, or by $\mathfrak{Op}^A$, or $\mathfrak{Op}_A$), for a symbol of the form $p = p_0 + p_S + p_L$, where p_0 does not depend on x, p_S is a symbol with "short range" behaviour and p_L is a symbol with "long range" behaviour. We assume that all the derivatives of the magnetic field B verify conditions of type "short range" at infinity, and our example 3 shows that these hypothesis are in some sense optimal. The spectral analysis of $\mathfrak{Op}_A(\langle\xi\rangle)$ has been done in [Um] but without considering the problem of a limiting absorption principle; moreover, as shown in Example 2, our hypothesis are more general.

An earlier version of this paper, with some proofs given in extenso, may be found on the ArXiv electronic archieve [IMP].

Notations and conventions

We denote: $\mathcal{X} \equiv \mathbb{R}^n$ with elements x, y, z, $\mathcal{X}^*$ the dual of $\mathcal{X}$ with elements ξ, η, ζ, $\langle\xi, x\rangle$ the duality form, $\Xi = \mathbb{R}^{2n} = \mathcal{X} \oplus \mathcal{X}^*$ the phase space with elements $X = (x, \xi)$, $Y = (y, \eta)$, $Z = (z, \zeta)$, it is a symplectic space with the canonical symplectic form $[\![Y, Z]\!] = \langle\eta, z\rangle - \langle\zeta, y\rangle$. On $\mathcal{X}$ we consider the usual Lebesgue measure, but on $\mathcal{X}^*$ and Ξ, respectively, it will be convenient to use $\,\bar{d}\xi = (2\pi)^{-n} d\xi$ and $\,\bar{d}X = \pi^{-n} dX$. If $\mathcal{Y} \cong \mathbb{R}^m$ we set $BC^\infty(\mathcal{Y})$ the subspace of bounded functions in $C^\infty(\mathcal{Y})$ having bounded derivatives of any order, $C^\infty_{\text{pol}}(\mathcal{Y})$ is the space of all C^∞ functions on $\mathcal{Y}$ with the absolute value of each derivative dominated by an (arbitrary) polynomial and $C^\infty_{\text{pol,u}}(\mathcal{Y})$ the subspace of $C^\infty_{\text{pol}}(\mathcal{Y})$ consisting of elements whose all derivatives are dominated by a fixed polynomial of given (arbitrary) degree. $\mathcal{S}(\mathcal{Y})$ will be the Schwartz space on $\mathcal{Y}$, with dual $\mathcal{S}'(\mathcal{Y})$ and antidual $\mathcal{S}^*(\mathcal{Y})$; we denote by (u, v), $(u \in \mathcal{S}^*(\mathcal{Y}),\ v \in \mathcal{S}(\mathcal{Y}))$ the application of anti-duality. We use standard multi-index notations: $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$, $|\alpha| = \alpha_1 + \cdots + \alpha_m$, $\alpha! = \alpha_1! \dots \alpha_m!$, $\partial^\alpha_y = \partial^{\alpha_1}_{y_1} \dots \partial^{\alpha_m}_{y_m}$ or $\partial^\alpha = \partial^{\alpha_1}_1 \dots \partial^{\alpha_m}_m$, $D^\alpha = i^{-|\alpha|}\partial^\alpha$, where $m = \dim \mathcal{Y}$.

We frequently consider integrals as converging in $\mathcal{S}^*$, in particular as oscillatory integrals.

We denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the Banach space of all linear bounded operators $T : \mathcal{H}_1 \to \mathcal{H}_2$, with $\mathcal{H}_1$, $\mathcal{H}_2$ Hilbert (or Banach) spaces. In fact we preserve this notation even if $\mathcal{H}_1$, $\mathcal{H}_2$ are topological vector spaces, to signify continuous, linear operators. For $\mathcal{B}(\mathcal{H},\mathcal{H})$ we abbreviate $\mathcal{B}(\mathcal{H})$. $\mathcal{K}(\mathcal{H}_1,\mathcal{H}_2)$ will denote compact operators. $\mathcal{B}_p(\mathcal{H})$ will be the Schatten-von Neumann class of order $p\in[1,\infty]$ on $\mathcal{H}$.

Given a Riemannian metric g_X on Ξ and a positive function $M:\Xi\to\mathbb{R}_+^*$, we define the *symbol space* $S(M,g)$ to be the space of C^∞ functions $f:\Xi\to\mathbb{C}$ such that

$$\sup_{\substack{X,T_j\in\Xi\\ g_X(T_j)\le 1}}\left(M(X)^{-1}\left|\partial_{T_1}\dots\partial_{T_k}f(X)\right|\right)<\infty,\ \forall k\in\mathbb{N},$$

where we denote by $\partial_T f$ the derivative of f with respect to the direction $T\in\Xi$. We denote by $\boldsymbol{\Psi}(M,g)$ the family of Weyl operators $\mathfrak{Op}(f)$ with $f\in S(M,g)$.

If $M(X)=\langle\xi\rangle^m$ and the metric has the form

$$g_X=\langle\xi\rangle^{2\delta}|dx|^2+\langle\xi\rangle^{-2\rho}|d\xi|^2$$

for ρ, δ and m real numbers and $\langle\xi\rangle:=(1+|\xi|^2)^{1/2}$, we denote $S(M,g)$ by $S^m_{\rho,\delta}(\Xi)$. We still use the notations

$$S^m(\Xi):=S^m_{1,0}(\Xi),\qquad S^{-\infty}(\Xi)=\bigcap_{m\in\mathbb{R}}S^m_{\rho,\delta}(\Xi).$$

Explicitly, a function $f\in C^\infty(\Xi)$ belongs to $S^m_{\rho,\delta}(\Xi)$ if for any multi-indices α and β in $\mathbb{N}^n$ there exists a finite constant $C_{\alpha\beta}$ such that

$$\left|\left(\partial_x^\alpha\partial_\xi^\beta f\right)(X)\right|\le C_{\alpha\beta}\,\langle\xi\rangle^{m-\rho|\beta|+\delta|\alpha|},\quad\forall X=(x,\xi)\in\Xi.$$

§1. The Magnetic Functional Calculus

The mathematical framework that we consider is supposed to modelize a quantum particle without internal structure moving in $\mathcal{X}=\mathbb{R}^n$, in the presence of a non-uniform magnetic field. The *magnetic field* is described by a closed 2-form B on $\mathcal{X}\equiv\mathbb{R}^n$. In the standard coordinate system on $\mathbb{R}^n$, it is represented by a function taking real antisymmetric matrix values $B=(B_{jk})$ with $1\le j\le n$, $1\le k\le n$ and verifying the relation $\partial_jB_{kl}+\partial_kB_{lj}+\partial_lB_{jk}=0$. We shall always assume that $B_{jk}\in C^\infty_{\text{pol}}(\mathcal{X})$, although this is not necessary for all constructions or assertions. Anyhow, later on, even stronger assumptions on B will be imposed.

Any such field B may be written as the exterior differential dA of a 1-form A, *the vector potential*; by using coordinates, one has $B_{jk}=\partial_jA_k-\partial_kA_j$ for each $j,k=1,\dots,N$. The components of the vector potential will always be taken of class $C^\infty_{\rm pol}(\mathcal{X})$, in order to define multipliers for $\mathcal{S}(\mathcal{X})$ and $\mathcal{S}^*(\mathcal{X})$. This is, indeed, always possible, as can be seen by considering *the transversal gauge*

$$A_j(x) = -\sum_{k=1}^n \int_0^1 ds\, B_{jk}(sx) s x_k. \tag{1.1}$$

In the magnetic pseudodifferential calculus that we shall develop there are two phase factors that play an important role, one defined by B and the other by A. Given a k-form C on X and a compact piecewise smooth k-surface $\gamma \subset X$, we denote by $\Gamma^C(\gamma)$ the usual invariant integral of C on γ. We shall encounter circulations of the 1-form A along linear segments $\gamma = [x, y]$ defined by its ends x, y and fluxes of the 2-form B through triangles $\gamma = \langle x, y, z\rangle$ defined by its vertices x, y, z. By Stokes' theorem, one has

$$\Gamma^B(\langle x,y,z\rangle) = \Gamma^A([x,y]) + \Gamma^A([y,z]) + \Gamma^A([z,x]). \tag{1.2}$$

We shall constantly use the notations:

$$\Lambda^A(x,y) := \exp\{-i\Gamma^A([x,y])\}, \tag{1.3}$$

$$\Omega^B(x,y,z) := \exp\{-i\Gamma^B(\langle x,y,z\rangle)\}, \tag{1.4}$$

$$\begin{aligned}\omega_B(x,y,z) &:= \exp\{-4iF_B(x,y,z)\}\\ &= \Omega^B(x-y+z, x-y-z, x+y-z).\end{aligned} \tag{1.5}$$

Lemma 1.1. *If the components of B are of class $C^\infty_{\rm pol}(\mathcal{X})$, $\forall \gamma \in \mathbb{N}^{3n}$ $\exists p(\gamma) \in \mathbb{N}$ such that $\left|\partial^\gamma_{x,y,z}\omega_B(x,y,z)\}\right| \le C_\gamma(1+|x|+|y|+|z|)^{p(\gamma)}$.*

If the components of the magnetic field B are of class $BC^\infty(\mathcal{X})$, then

1. $\partial_{x_j} F_B = \sum_{k=1}^n (D_{jk}y_k + E_{jk}z_k)$, $\partial_{y_j} F_B = \sum_{k=1}^n \left(D'_{jk}y_k + E'_{jk}z_k\right)$, $\partial_{z_j} F_B = \sum_{k=1}^n \left(D''_{jk}y_k + E''_{jk}z_k\right)$, *where the coefficients $D_{jk}, \ldots, E''_{jk}$ are of class $BC^\infty(\mathcal{X}^3)$.*
2. $\left|\left(\partial^\alpha_x\partial^\beta_y\partial^\gamma_z\omega_B\right)(x,y,z)\right| \le C_{\alpha,\beta,\gamma}(\langle y\rangle + \langle z\rangle)^{|\alpha|+|\beta|+|\gamma|}$, $\forall(\alpha,\beta,\gamma) \in [\mathbb{N}^n]^3$, *where $C_{\alpha,\beta,\gamma}$ are positive constants.*
3. $|(\partial^\alpha_z\omega_B)(x,y,z)| \le C_\alpha(\langle x-z\rangle + \langle y-z\rangle)^{|\alpha|}$, $\forall\alpha \in \mathbb{N}^n$, *where C_α are positive constants.*

Proof. By straightforward computation. □

In a former paper [MP2] we have shown that for $f \in \mathcal{S}^*(\Xi)$ and $u \in \mathcal{S}(\mathcal{X})$, the formula (properly interpreted)

$$\left[\mathfrak{Op}^A(f)u\right](x) = \iint_{\mathcal{X}\times\mathcal{X}^*} dy\, đ\eta\, e^{i\langle x-y,\eta\rangle}\Lambda^A(x,y) f\left(\frac{x+y}{2},\eta\right) u(y) \tag{1.6}$$

defines an integral operator $\mathfrak{Op}^A(f)\in\mathcal{B}(\mathcal{S}(\mathcal{X}),\mathcal{S}^*(\mathcal{X}))$, and in fact $\mathfrak{Op}^A$ gives an isomorphism between $\mathcal{S}^*(\Xi)$ and $\mathcal{B}(\mathcal{S}(\mathcal{X}),\mathcal{S}^*(\mathcal{X}))$ (as linear topological spaces) that restricts to an isomorphism between $\mathcal{S}(\Xi)$ and $\mathcal{B}(\mathcal{S}^*(\mathcal{X}),\mathcal{S}(\mathcal{X}))$. Let us remark here that for any test functions u and v in $\mathcal{S}(\mathcal{X})$ and any distribution $f\in\mathcal{S}^*(\mathcal{X})$, we have the relation $\left(\mathfrak{Op}^A(f)u\,,\,v\right) = \left(u\,,\,\mathfrak{Op}^A(\overline{f})v\right)$, where $(\overline{f},u):=\overline{(f,\overline{u})}$. In particular, if f is a real distribution, then $\mathfrak{Op}^A(f)$ is a symetric operator in $\mathcal{B}(\mathcal{S}(\mathcal{X}),\mathcal{S}^*(\mathcal{X}))$. In [MP2] we show that $\mathfrak{Op}^A$ induces a unitary map from $L^2(\Xi)$ to $\mathcal{B}_2(L^2(\mathcal{X}))$, the ideal of all Hilbert-Schmidt operators. The family $\mathfrak{Op}^A(f)$ with f being the Fourier tranform of an arbitrary function in $L^1(\Xi)$, is dense in the closed ideal $\mathcal{K}(L^2(\mathcal{X}))$ of all compact operators.

An important property is *gauge covariance.* Let A and A' be two vector potentials of class C^∞_{pol}, defining the same magnetic field, $dA=B=dA'$. Then there exists a real function $\varphi\in C^\infty_{\text{pol}}(X)$ such that $A'=A+\nabla\varphi$ and $e^{i\varphi(Q)}\mathfrak{Op}^A(f)e^{-i\varphi(Q)}=\mathfrak{Op}^{A+\nabla\varphi}(f)$ for any $f\in\mathcal{S}'(\Xi)$ and all such functions φ; this second identity is valid in $\mathcal{B}\left[\mathcal{S}(\mathcal{X}),\mathcal{S}^*(\mathcal{X})\right]$.

If $\tilde{f}$ is the Fourier transform of $f\in\mathcal{S}^*(\Xi)$ in the second variable we can write

$$\left[\mathfrak{Op}^A(f)u\right](x)=(2\pi)^{-n}\int_{\mathcal{X}}dy\,\Lambda^A(x,y)\,\tilde{f}\left(\frac{x+y}{2},y-x\right)u(y).$$

For $f,g\in\mathcal{S}(\Xi)$, the product $\mathfrak{Op}^A(f)\mathfrak{Op}^A(g)$ is smoothing, consequently of the form $\mathfrak{Op}^A(f\circ^B g)$, defining *the Magnetic Moyal product* $f\circ^B g$ in $\mathcal{S}(\Xi)$. We use the above formula twice and (1.2), compute its partial Fourier transform (in the second variable) by the usual integral formula and obtain again an element in $\mathcal{S}(\Xi)$. After some changes of variables and using Fubini Theorem we get:

$$(1.7)\quad (f\circ^B g)(X) \;=\; \int\limits_{\Xi}\int\limits_{\Xi} đY\,đZ\; e^{-2i[\![Y,Z]\!]}\,\omega^B(x,y,z)\,f(X-Y)\,g(X-Z).$$

One can extend the validity of (1.7) by duality, using the fact that for any functions f and g in $\mathcal{S}(\Xi)$ we have

$$\int_{\Xi}dX\;(f\circ^B g)(X)=\int_{\Xi}dX\;f(X)g(X)=\langle f,g\rangle\equiv(f,\overline{g}).$$

Thus for $f,g,h\in\mathcal{S}(\Xi)$ we have $\langle f\circ^B g,h\rangle=\langle f,g\circ^B h\rangle=\langle g,h\circ^B f\rangle$. Considering $\langle\cdot,\cdot\rangle$ as duality between $\mathcal{S}'(\Xi)$ and $\mathcal{S}(\Xi)$, we define for $F\in\mathcal{S}'(\Xi)$ and $f\in\mathcal{S}(\Xi)$: $\langle F\circ^B f,h\rangle:=\langle F,f\circ^B h\rangle$, $\langle f\circ^B F,h\rangle:=\langle F,h\circ^B f\rangle$, $\forall h\in\mathcal{S}(\Xi)$, getting two bilinear continuous mappings with good associativity properties. A substantial extension of the magnetic Moyal product is obtained in [MP2] on the following class of distributions

$$\mathcal{M}^B(\Xi) := \left\{F \in \mathcal{S}'(\Xi) \ \mid \ F \circ^B f \in \mathcal{S}(\Xi), \ f \circ^B F \in \mathcal{S}(\Xi), \ \forall f \in \mathcal{S}(\Xi)\right\},$$

called *the magnetic Moyal algebra.*

For F, G in $\mathcal{M}^B(\Xi)$, we define $\langle F \circ^B G, h\rangle := \langle F, G \circ^B h\rangle$, $\forall h \in \mathcal{S}(\Xi)$. The set $\mathcal{M}^B(\Xi)$ together with the composition law $\circ^B$ defined above and the complex conjugation $F \mapsto \overline{F}$ is an unital *-algebra, containing $\mathcal{S}(\Xi)$ as a self-adjoint two-sided ideal. Maybe the most important fact is that $\mathfrak{Op}^A$ is an isomorphism of *-algebras betweeen $\mathcal{M}^B(\Xi)$ and $\mathcal{B}[\mathcal{S}(\mathcal{X})] \cap \mathcal{B}[\mathcal{S}'(\mathcal{X})]$. We have that $C^\infty_{\mathrm{pol,u}}(\Xi) \subset \mathcal{M}^B(\Xi)$ ([MP2]). We also have the following result.

Lemma 1.2. *If the components of B are of class $C^\infty_{\mathrm{pol}}(\mathcal{X})$, then for any $m \in \mathbb{R}$, any $\rho \geq 0$ and any $\delta < 1$ we have $S^m_{\rho,\delta}(\Xi) \subset \mathcal{M}^B(\Xi)$.*

Proof. We must prove that, for any couple $(f, \varphi) \in S^m_{\rho,\delta}(\Xi) \times \mathcal{S}(\Xi)$, we have $f \circ^B \varphi \in \mathcal{S}(\Xi)$. We have to study the oscillatory integral in (1.7) for $(f, \varphi) \in S^m_{\rho,\delta}(\Xi) \times \mathcal{S}(\Xi)$. We choose $\chi \in C^\infty_0(\Xi)$ with $\chi(0) = 1$, and for any $\epsilon > 0$ we define $f_\epsilon(X) := \chi(\epsilon X) f(X)$. We show that the limit $\lim_{\epsilon \to 0}(f_\epsilon \circ^B \varphi)$ exists pointwise and is independent of the choice of χ. Integrating by parts we get

$$\begin{aligned}(f_\epsilon \circ^B \varphi)(X) = \int_\Xi \int_\Xi & đY \, đZ \, e^{-2i[\![Y,Z]\!]} \langle y\rangle^{-2N_\zeta} \langle \eta\rangle^{-2N_z} \, f_\epsilon(X - Y) \\ & \times \mathfrak{L}_z^{N_z} \left[\omega^B(x, y, z) \mathfrak{L}_\zeta^{N_\zeta} \varphi(X - Z)\right],\end{aligned} \tag{1.8}$$

with $\mathfrak{L}_z = 1 - (1/4)\Delta_z$ and $\mathfrak{L}_\zeta = 1 - (1/4)\Delta_\zeta$. The integrals are well defined, due to the decay assumptions on φ. We choose first $N_z \geq (1/2)(m + n + 1)$ and then $N_\zeta \geq (1/2)(q(2N_z) + n + 1)$, where $q(N) := \max\{p(\gamma) \mid |\gamma| \leq N\}$ and $p(\gamma)$ as in the first statement of Lemma 1.1. Thus we can take the limit $\epsilon \to 0$ and obtain for $(f \circ^B \varphi)(X)$ an identity similar to (1.8). This equation is clearly independent on the choice of χ and of the exact choices of N_z and N_ζ (by integration by parts). For any $k \in \mathbb{N}$ we may choose $N_z \geq (1/2)(m + k(|\delta| + |\rho|) + n + 1)$ and $N_\zeta \geq (1/2)(q(2N_z + k) + n + 1)$ in order to prove (by further integration by parts with respect to y and η) that $f \circ^B \varphi \in C^k(\Xi)$. In conclusion $f \circ^B \varphi \in C^\infty(\Xi)$. If we consider a multiindex $\alpha = (\alpha_x, \alpha_\xi) \in \mathbb{N}^{2n}$ and integrate by parts with respect to y and η, we prove that $\partial^\alpha_X(f \circ^B \varphi)$ is a finite linear combination of terms of the form

$$
\begin{aligned}
I(X) = \int_\Xi \int_\Xi & \text{đ}Y \, \text{đ}Z \, e^{-2i[\![Y,Z]\!]} \langle z\rangle^{-2N_\eta} \langle\zeta\rangle^{-2N_y} \\
& \times \left(\partial_\eta^{\beta'} \langle\eta\rangle^{-2N_z}\right) \left(\partial_y^{\gamma'} \langle y\rangle^{-2N_\zeta}\right) \left\{\partial_y^{\gamma''} \partial_z^{\delta'} \partial_x^{\alpha'_x} \omega^B(x,y,z)\right\} \\
& \times \left\{\partial_x^{\alpha''_x} \partial_\xi^{\alpha'_\xi} \partial_z^{\delta''} \partial_\zeta^{\lambda} \varphi(X-Z)\right\} \left\{\partial_x^{\alpha'''_x} \partial_\xi^{\alpha''_\xi} \partial_y^{\gamma'''} \partial_\eta^{\beta''} \, f(X-Y)\right\},
\end{aligned}
$$

with $\alpha'_x + \alpha''_x + \alpha'''_x = \alpha_x$, $\alpha'_\xi + \alpha''_\xi = \alpha_\xi$, $|\alpha| = k$, $|\beta'| + |\beta''| \le 2N_\eta$, $|\gamma'| + |\gamma''| + |\gamma'''| \le 2N_y$, $|\delta'| + |\delta''| \le 2N_z$ and $|\lambda| \le 2N_\zeta$. Then taking into account that $\varphi \in \mathcal{S}(\Xi)$, we can choose $N_y, N_\eta, N_z, N_\zeta$ as functions of k and l so that $|I(X)| \le C_l \langle X\rangle^{-l}$. We may suppose that $\delta \in (0,1)$. Let us remark that

$$
\begin{aligned}
\langle\xi-\eta\rangle^{m-\rho|\alpha''_\xi+\beta''|+\delta|\alpha'''_x+\gamma'''|} & \le C\langle\xi\rangle^{m+\delta(k+2N_y)} \langle\eta\rangle^{|m|+\delta(k+2N_y)}, \\
\langle x-z\rangle^{-t} \ \le \ C\langle x\rangle^{-t}\langle z\rangle^{t}, & \quad \langle\xi-\zeta\rangle^{-s} \ \le \ C\langle\xi\rangle^{-s}\langle\zeta\rangle^{s}.
\end{aligned}
$$

We choose first $N_y \ge (1/2)(1-\delta)^{-1}(n+1+l+m+k\delta)$ in order to verify the integrability condition with respect to $\zeta \in \mathcal{X}$. Then we choose $s = l+m+\delta(k+2N_y)$, and obtain a factor $\langle\xi\rangle^{-l}$. We can also choose $N_z \ge (1/2)(n+1+|m|+(k+2N_y)\delta)$ in order to verify the integrability condition with respect to $\eta \in \mathcal{X}$ and $t = l + q(2N_y + 2N_z + k)$ to obtain a factor $\langle x\rangle^{-l}$. We end up by choosing $N_\eta \ge (1/2)(n+1+l+2q(2N_y+2N_z+k))$ and $N_\zeta \ge (1/2)(n+1+q(2N_y+2N_z+k))$ in order to get the integrability with respect to $(y,z) \in \mathcal{X} \times \mathcal{X}$. ☐

§2. The Magnetic Composition of Symbols

Our first task is to extend the magnetic composition law to classes of symbols and obtain a precise asymptotic development for this composition, generalizing the well known formulae from usual pseudodifferential calculus. This will be a key technical ingredient in the following developments, in particular leading to the existence of parametrices for elliptic operators.

If the magnetic field B has components of class $BC^\infty(\mathcal{X})$, by arguments similar to those above, for any $f \in S^{m_1}(\Xi)$ and $g \in S^{m_2}(\Xi)$, the magnetic Moyal product $f \circ^B g$ belongs to $S^{m_1+m_2}(\Xi)$. Our sharp estimations on the flux of the magnetic field (Lemma 1.1) will make possible a precise result concerning the asymptotic development of $f \circ^B g$.

An important ingredient for the estimation of the integral appearing in (1.7) is a 'stationary phase' result, for which we introduce some notations. For any $\varphi \in C^\infty(\mathcal{X}^3)$ we define the following first order differential operator (with respect to the variables $U = (u,\mu) \in \Xi$ and $V = (v,\nu) \in \Xi$), having coefficients that only depend on (x,y,z):

$$
M_B(\varphi) := [\omega_B(x,y,z)]^{-1} \sum_{j=1}^{n} \left[\partial_{y_j}\left(\omega_B\varphi\right)\partial_{\nu_j} - \partial_{z_j}\left(\omega_B\varphi\right)\partial_{\mu_j}\right].
$$

We shall denote

$$[\![\partial_U,\partial_V]\!] := \langle\partial_\mu,\partial_v\rangle - \langle\partial_u,\partial_\nu\rangle \equiv \sum_{j=1}^{n}\left(\partial_{\mu_j}\partial_{v_j} - \partial_{u_j}\partial_{\nu_j}\right)$$

and for $t\in\mathbb{R}\setminus\{0\}$ and $\varphi\in C^\infty_{\text{pol}}(\mathcal{X}^3)$ we define:

$$L_\varphi(t) := \frac{1}{2i}\left\{2\varphi[\![\partial_U,\partial_V]\!] + t^{-1}M_0(\varphi)\right\}.$$

Lemma 2.1. *Let* $\varphi\in C^\infty_{\text{pol}}(\mathcal{X}^3)$.

1. *We have the following equality*:

$$\int_\Xi\int_\Xi \bar{d}Y\,\bar{d}Z\;\exp\{-2i[\![Y,Z]\!]\}\varphi(x,y,z) = \varphi(x,0,0)$$

(*the integral being interpreted as an oscillatory integral*).

2. *If* $h\in\mathcal{S}(\Xi\times\Xi)$, *for any* $t\in\mathbb{R}^*$ *we have*

$$\int_\Xi\int_\Xi \bar{d}Y\,\bar{d}Z\;e^{-2i[\![Y,Z]\!]}\varphi(x,y,z)h(X-tY,X-tZ) \;=\; \varphi(x,0,0)h(X,X)$$
$$+t^2\int_0^1 s\,ds\int_\Xi\int_\Xi \bar{d}Y\,\bar{d}Z\;e^{-2i[\![Y,Z]\!]}\left[L_\varphi(st)h\right](X-stY,X-stZ).$$

Proof. (1) Let us fix $\chi\in C_0^\infty(\mathcal{X})$ such that $\chi(0)=1$. For any $\epsilon>0$ we define

$$\begin{aligned}I_\epsilon :&= \int_\Xi\int_\Xi \bar{d}Y\,\bar{d}Z\;\chi(\epsilon y)\chi(\epsilon\eta)\chi(\epsilon z)\chi(\epsilon\zeta)\exp\{-2i[\![Y,Z]\!]\}\varphi(x,y,z)\\ &= (2\pi)^{-2n}\int_{\mathcal{X}}\int_{\mathcal{X}} dy\,dz\;\chi(-(\epsilon^2/2)y)\chi((\epsilon^2/2)z)\hat{\chi}(y)\hat{\chi}(z)\varphi(x,-(\epsilon/2)y,(\epsilon/2)z).\end{aligned}$$

By the dominated convergence theorem, it goes to $\varphi(x,0,0)$ for $\epsilon\to 0$.

(2) We perform a first order Taylor expansion of $h(X-tY,X-tZ)$ with respect to t. The first term on the right-hand side is given by 1. For the second term we integrate by parts, using the identity $y_j e^{-2i[\![Y,Z]\!]} = \frac{1}{2i}\partial_{\zeta_j}e^{-2i[\![Y,Z]\!]}$ and similar ones for y replaced with η, z and ζ. ☐

To any differential operator $P := \sum\, c_{\alpha\beta}(x,y,z)\partial_U^\alpha\partial_V^\beta$, of order m with respect to the variables U and V, we associate another differential operator $M_B(P)$ defined by $M_B(P) := \sum\, M_B(c_{\alpha\beta})\partial_U^\alpha\partial_V^\beta$. This operator will evidently have the same form, but will be of order $m+1$.

For any sequence of positive numbers $\{t_j \in \mathbb{R}^* \mid j \in \mathbb{N}^*\}$, let $t_{(k)} := t_1 \cdot \ldots \cdot t_k$ (for $k \in \mathbb{N}^*$) and let us define by recurrence the sequence of differential operators:

$$\begin{gathered} L_0 := \mathbf{1}, \\ L_1(t_1) := \omega_B^{-1} L_{\omega^B}(t_1), \\ L_{j+1}(t_1, \ldots, t_{j+1}) := L_1(t_{(j+1)}) L_j(t_1, \ldots, t_j) + \frac{M_0\left(L_j(t_1, \ldots, t_j)\right)}{2it_{(j+1)}}. \end{gathered}$$

Theorem 2.2. *Let us assume that all the components of the magnetic field B are of class $BC^\infty(\mathcal{X})$. If $f \in S^{m_1}_{\rho,\delta}(\Xi)$ and $g \in S^{m_2}_{\rho,\delta}(\Xi)$, with $m_1 \in \mathbb{R}$, $m_2 \in \mathbb{R}$, $0 \leq \delta < \rho \leq 1$, then $f \circ^B g \in S^{m_1+m_2}_{\rho,\delta}(\Xi)$ and we have the following asymptotic development*:

$$f \circ^B g \sim \sum_{j=0}^{\infty} h_j, \qquad h_j \in S^{m_1+m_2-j(\rho-\delta)}_{\rho,\delta}(\Xi), \qquad h_0(X) = f(X)g(X),$$

$$\begin{aligned} h_j(X) = \int_0^1 \int_0^1 \cdots \int_0^1 dt_1 dt_2 \ldots dt_j \; t_1^{2j-1} t_2^{2j-3} \cdot \ldots \cdot t_j \\ \times [L_j(t_1, \ldots, t_j)(f(U)g(V))] \Big|_{\substack{U=V=X \\ y=z=0}}. \end{aligned}$$

Proof. We shall verify by induction that for any $k \geq 1$ we have

$$f \circ^B g = \sum_{j=0}^{k-1} h_j + R_k, \tag{2.1}$$

where

$$\begin{aligned} R_k(X) = \int_0^1 \int_0^1 \cdots \int_0^1 dt_1 dt_2 \ldots dt_k \; t_1^{2k-1} t_2^{2k-3} \cdot \ldots \cdot t_k \int_\Xi \int_\Xi \text{đ}Y \; \text{đ}Z \; e^{-2i[\![Y,Z]\!]} \\ \times \omega_B(x,y,z) \left[L_k(t_1, \ldots, t_k) \left(f(U) \otimes g(V)\right)\right] \Big|_{\substack{U=X-t_{(k)}Y \\ V=X-t_{(k)}Z}}. \end{aligned}$$

For $k = 1$ we apply Lemma 2.1, taking $\varphi = \omega_B$, $h = f \otimes g$ and $t = 1$, $s = t_1$. We just have to remark that in this case $L_{\omega_B}(t_1) = \omega_B L_1(t_1)$. Regarding the integrals as oscillatory integrals, we may assume that f and g belong to $\mathcal{S}(\Xi)$.

Let us suppose now that formula (2.1) is verified for some $k \geq 1$. In order to prove it for $k+1$ we shall rewrite the integral defining the rest R_k. We notice that

$$[L_k(t_1, \ldots, t_k)(f \otimes g)](U,V) = \sum_{\alpha,\beta} a_{\alpha\beta}(x,y,z,t_1,\ldots,t_k) \left(\partial^\alpha f\right)(U) \left(\partial^\beta g\right)(V).$$

For each term we apply Lemma 2.1, taking $\varphi = \omega_B a_{\alpha\beta}$, $h = (\partial^\alpha f) \otimes (\partial^\beta g)$ and $t = t_1 \cdots t_k$, $s = t_{k+1}$. We remark first that

$$\sum_{\alpha,\beta} a_{\alpha\beta}(x, 0, 0, t_1, \dots, t_k) \left(\partial^\alpha f\right)(X) \left(\partial^\beta g\right)(X)$$
$$= \left[L_k(t_1, \dots, t_k)(f(U)g(V))\right]\Big|_{\substack{U=V=X \\ y=z=0}}.$$

Then

$$L_{(\omega_B a_{\alpha\beta})}(t_{(k+1)}) = \frac{1}{2i}\omega_B \left\{2a_{\alpha\beta} [\![\partial_U, \partial_V]\!] + (t_{(k+1)})^{-1} M_B(a_{\alpha\beta})\right\},$$

and moreover

$$M_B(a_{\alpha\beta}) = \omega_B^{-1} \sum_{j=1}^{n} \left[\partial_{y_j}(\omega_B a_{\alpha\beta})\partial_{\nu_j} - \partial_{z_j}(\omega_B a_{\alpha\beta})\partial_{\mu_j}\right] = a_{\alpha\beta} M_B(1) + M_0(a_{\alpha\beta}).$$

Putting all these together we get

$$L_{(\omega_B a_{\alpha\beta})}(t_{(k+1)}) = \omega_B \left\{a_{\alpha\beta} L_1(t_{(k+1)}) + \frac{1}{2i}(t_{(k+1)})^{-1} M_0(a_{\alpha\beta})\right\}.$$

We remark further that $L_1(t_{(k+1)})$ is a differential operator with respect to the variables U and V and thus commutes with multiplication with the function $a_{\alpha\beta}$; thus $\sum M_0(a_{\alpha\beta})\partial_U^\alpha \partial_V^\beta = M_0(L_k(t_1, \dots, t_k))$. Finally we get $R_k(X) = h_k(X) + R_{k+1}(X)$ and the proof of (2.1) is finished.

Let us show now that, for each $j \in \mathbb{N}$, we have $h_j \in S_{\rho,\delta}^{m_1+m_2-j(\rho-\delta)}(\Xi)$. This is evident for $j = 0$. For $j \geq 1$ notice that

$$L_j(t_1, \dots, t_j) = \sum a_{\alpha',\alpha'',\beta',\beta''}(x, y, z; t_1, \dots, t_j)\partial_u^{\alpha'} \partial_\mu^{\alpha''} \partial_v^{\beta'} \partial_\nu^{\beta''},$$

where $|\alpha''| + |\beta''| = j$, $|\alpha'| + |\beta'| \leq j$ and $a_{\alpha',\alpha'',\beta',\beta''}$ are linear combinations of products of derivatives of F_B with respect to y and z and monomials in t_1^{-1}, …, t_j^{-1}, with exponents that do not exceed those of t_1, …, t_j appearing in the integrals. Lemma 1.1 shows that for $y = z = 0$ the derivatives of F_B are either vanishing or at least bounded functions of x. We conclude that h_j is a linear combination of terms of the type

$$b(x) \left[\left(\partial_x^{\alpha'} \partial_\xi^{\alpha''} f\right)(X)\right] \left[\left(\partial_x^{\beta'} \partial_\xi^{\beta''} g\right)(X)\right] \int_0^1 t_1^{p_1}\, dt_1 \cdots \int_0^1 t_j^{p_j}\, dt_j,$$

where $b \in BC^\infty(\mathcal{X})$, $p_l \geq 0$ for any $l \in \{1, \dots, j\}$ and $|\alpha''| + |\beta''| = j$, $|\alpha'| + |\beta'| \leq j$. It follows immediatly, from the hypothesis on f and g, that $h_j \in S_{\rho,\delta}^{m_1+m_2-j(\rho-\delta)}(\Xi)$.

We shall end the proof of the Theorem by checking that for any $k \geq 1$ one has $R_k \in S^{m_1+m_2+2n-k(\rho-\delta)}_{\rho,\delta}(\Xi)$. We shall use once again the structure of the operator $L_j(t_1, \dots, t_j)$ that was described above. It follows that R_k can be written as a linear combination of terms of the form

$$\begin{aligned} I(X) = C \int_0^1 \cdots \int_0^1 dt_1 \dots dt_k \, t_1^{p_1} \cdots t_k^{p_k} \int_\Xi \int_\Xi \bar{d}Y \, \bar{d}Z e^{-2i[\![Y,Z]\!]} \omega_B(x,y,z) \\ \times a_{\alpha',\alpha'',\beta',\beta''}(x,y,z;t_1,\dots,t_k) \\ \times \left[\left(\partial_x^{\alpha'}\partial_\xi^{\alpha''} f\right)(X - t_1\cdots t_k Y)\right] \left[\left(\partial_x^{\beta'}\partial_\xi^{\beta''} g\right)(X - t_1\cdots t_k Z)\right]. \end{aligned}$$

Now we no longer restrict to $y = z = 0$ and thus factors of the type $y^\beta z^\gamma$ may appear in the functions $a_{\alpha',\alpha'',\beta',\beta''}$ (that contain derivatives of F_B). These factors may be handled by integration by parts, using the exponential $e^{-2i[\![Y,Z]\!]}$, and will generate operators of the form $\partial_\eta^\gamma \partial_\zeta^\beta$ applied to the functions $\partial_x^{\alpha'}\partial_\xi^{\alpha''} f$ and $\partial_x^{\beta'}\partial_\xi^{\beta''} g$, but this will not alter their decay. We proceed as in the proof of Lemma 1.2, and after a number of integrations by parts we write $I(X)$ as a linear combination of terms of the type

$$\begin{aligned} J(X) = \int_0^1 \cdots \int_0^1 dt_1 \cdots dt_k \, t_1^{q_1} \cdots t_k^{q_k} \int_\Xi \int_\Xi \bar{d}Y \, \bar{d}Z e^{-2i[\![Y,Z]\!]} \\ \times \langle z\rangle^{-2N_\eta} \langle\zeta\rangle^{-2N_y} \left(\partial_\eta^{\gamma'} \langle\eta\rangle^{-2N_z}\right) \left(\partial_y^{\delta'} \langle y\rangle^{-2N_\zeta}\right) \\ \times \partial_y^{\delta''} \partial_z^{\epsilon'} \left(\omega^B a_{\alpha',\alpha'',\beta',\beta''}\right)(x,y,z) \\ \times \left(\partial_x^{\alpha'+\delta'''} \partial_\xi^{\alpha''+\gamma''} f\right)(X - t_{(k)}Y) \left(\partial_x^{\beta'+\epsilon''} \partial_\xi^{\beta''+\lambda} g\right)(X - t_{(k)}Z), \end{aligned}$$

where $|\gamma'| + |\gamma''| \leq 2N_\eta$, $|\delta'| + |\delta''| + |\delta'''| \leq 2N_y$, $|\epsilon'| + |\epsilon''| \leq 2N_z$, $|\lambda| \leq 2N_\zeta$ and $q_j \geq 0$ for any $j \in \{1, \dots, k\}$.

We fix $N_\eta = N_\zeta = n$ in order to have integrability in the variables y and z. Then we decompose the η-integral with respect to the two domains $\{|\eta| \leq \kappa\langle\xi\rangle\}$ and $\{|\eta| \geq \kappa\langle\xi\rangle\}$ for some small fixed $\kappa > 0$, and similarly for the ζ-integration, and thus write $J(X)$ as a sum of 4 terms $J_a(X)$ (with $a = 1,2,3,4$). In order to estimate each of these 4 terms separately we remark that we may choose different values for the pair (N_y, N_z) in each of these terms, due to the fact that these choices are made by integration by parts in the variables y and z. Moreover, for any $r \in \mathbb{R}$ and for $\forall N_y \geq 0$, $\forall N_z \geq 0$

$$\int_{\{|\eta| \leq \kappa\langle\xi\rangle\}} d\eta \langle\eta\rangle^{-2N_z} \langle\xi - t\eta\rangle^{r+2\delta N_y} \ \leq \ C\langle\xi\rangle^{r+2\delta N_y+n}.$$

Taking $N_y \geq 0$ and $2N_z > |r| + 2\delta N_y + n$, we have

$$\int_{\{|\eta|\geq\kappa\langle\xi\rangle\}} d\eta \langle\eta\rangle^{-2N_z} \langle\xi - t\eta\rangle^{r+2\delta N_y}$$
$$\leq \; C \int_{\{|\eta|\geq\kappa\langle\xi\rangle\}} d\eta \, \langle\eta\rangle^{|r|+2\delta N_y - 2N_z} \; \leq \; C\langle\xi\rangle^{|r|+2\delta N_y - 2N_z + n}.$$

Let us set $r_1 := m_1 - \rho|\alpha''| + \delta|\alpha'|$ and $r_2 := m_2 - \rho|\beta''| + \delta|\beta'|$. We denote $\Psi_{N_1,N_2}(t;\xi,\eta) := \frac{\langle\xi - t\eta\rangle^{N_1}}{\langle\eta\rangle^{N_2}}$ and we get the following upper bound for $|J_1(X)|$

$$\sup_{0\leq t\leq 1} \int_{\{|\eta|\leq\kappa\langle\xi\rangle\}} d\eta \Psi_{r_1+2\delta N_y, 2N_z}(t;\xi,\eta) \int_{\{|\zeta|\leq\kappa\langle\xi\rangle\}} d\zeta \Psi_{r_2+2\delta N_z, 2N_y}(t;\xi,\zeta)$$
$$\leq \; C\langle\xi\rangle^{m_1+m_2-k(\rho-\delta)+2n},$$

by choosing for this domain $N_y = N_z = 0$. For $|J_2(X)|$ we get the upper bound:

$$\sup_{0\leq t\leq 1} \int_{\{|\eta|\leq\kappa\langle\xi\rangle\}} d\eta \Psi_{r_1+2\delta N_y, 2N_z}(t;\xi,\eta) \int_{\{|\zeta|\geq\kappa\langle\xi\rangle\}} d\zeta \Psi_{r_2+2\delta N_z, 2N_y}(t;\xi,\zeta)$$
$$\leq \; C\langle\xi\rangle^{r_1+2\delta N_y+n} \langle\xi\rangle^{|r_2|-2N_y+n} \; = \; C\langle\xi\rangle^{r_1+|r_2|+2n-2(1-\delta)N_y},$$

and we have to choose on this domain $N_z = 0$ and N_y large enough. On the similar domain with η and ζ interchanged we have to choose $N_y = 0$ and N_z large enough. For $|J_4(X)|$ we obtain the upper bound

$$\sup_{0\leq t\leq 1} \int_{\{|\eta|\geq\kappa\langle\xi\rangle\}} d\eta \Psi_{r_1+2\delta N_y, 2N_z}(t;\xi,\eta) \int_{\{|\zeta|\geq\kappa\langle\xi\rangle\}} d\zeta \Psi_{r_2+2\delta N_z, 2N_y}(t;\xi,\zeta)$$
$$\leq \; C\langle\xi\rangle^{|r_1|+|r_2|+2n-2(1-\delta)(N_y+N_z)}$$

and thus we have to choose both N_y and N_z large enough. We conclude that

$$|R_k(X)| \; \leq \; C\langle\xi\rangle^{m_1+m_2+2n-k(\rho-\delta)}.$$

For the derivatives of R_k we may proceed in a similar way, since all the terms obtained by differentiating the factor ω_B with respect to x are bounded by monomials in y and z and thus can be dealt with by integration by parts in y and z. We get

$$\left| \left(\partial_x^\alpha \partial_\xi^\beta R_k \right)(X) \right| \; \leq \; C\langle\xi\rangle^{m_1+m_2+2n-k(\rho-\delta)-\rho|\beta|+\delta|\alpha|}, \;\; \forall\alpha\in\mathbb{N}^n, \; \forall\beta\in\mathbb{N}^n.$$

In order to end our proof we fix some $p \in \mathbb{N}$ and write the identity $f \circ^B g - \sum_{j=0}^{p} h_j = \sum_{j=p+1}^{k} h_j + R_k$, where $k \geq p+1$ is chosen large enough to have $2n - k(\rho - \delta) \leq -(p+1)(\rho - \delta)$. This gives $f \circ^B g - \sum_{j=0}^{p} h_j \in S^{m_1+m_2-(p+1)(\rho-\delta)}(\Xi)$, and thus $f \circ^B g \sim \sum_j h_j$. ☐

Let us explicitly compute the first terms of the asymptotic development of the magnetic Moyal product.

1. We know from the statement of Theorem 2.2 that $h_0 = fg$.

2. In order to compute h_1, we remark that for $y = z = 0$ the first order derivatives of F_B vanish and thus

$$[L_1(t_1)(f \otimes g)] \Big|_{y=z=0} (X, X) \\ = -i \sum_{j=1}^{n} \left[(\partial_{\xi_j} f)(X)(\partial_{x_j} g)(X) - (\partial_{x_j} f)(X)(\partial_{\xi_j} g)(X) \right].$$

In conclusion we have $h_1 = -\frac{i}{2}\{f, g\}$.

3. For h_2 we need to compute the explicit form of the operator $L_2(t_1, t_2)$. Using Lemma 1.1 we obtain:

$$[M_0(L_1(t_1))(f \otimes g)] \Big|_{y=z=0} (X, X) = -\frac{2}{t_1} \sum_{j,k=1}^{n} B_{jk}(x)[(\partial_{\xi_j} f) \otimes (\partial_{\xi_k} g)](X, X).$$

$$L_1(t_1 \cdot t_2)L_1(t_1)(f \otimes g) \\ = -\sum_{j,k=1}^{n} \left(\partial_{\mu_j}\partial_{\mu_k}\partial_{v_j}\partial_{v_k} + \partial_{u_j}\partial_{u_k}\partial_{\nu_j}\partial_{\nu_k} - 2\partial_{\mu_j}\partial_{u_k}\partial_{v_j}\partial_{\nu_k} \right)(f \otimes g) \\ + \{\textit{terms vanishing for } y = z = 0\}.$$

Finally, we put everything together to obtain

$$h_2(X) = \frac{1}{8} \sum_{j,k=1}^{n} \left[2\partial_{\xi_j}\partial_{x_k} f \otimes \partial_{x_j}\partial_{\xi_k} g - \partial_{\xi_j}\partial_{\xi_k} f \otimes \partial_{x_j}\partial_{x_k} g \right. \\ \left. - \partial_{x_j}\partial_{x_k} f \otimes \partial_{\xi_j}\partial_{\xi_k} g \right](X, X) - \frac{1}{2i} \sum_{j,k=1}^{n} B_{jk}(x)[(\partial_{\xi_j} f) \otimes (\partial_{\xi_k} g)](X, X).$$

In particular we have

$$f \circ^B g - g \circ^B f \cong \frac{1}{i}\{f, g\} - \frac{1}{i} \sum_{j,k=1}^{n} B_{jk}(\partial_{\xi_j} f)(\partial_{\xi_k} g) = \frac{1}{i}\{f, g\}_B \tag{2.2}$$

(*mod.* $S^{m_1+m_2-3(\rho-\delta)}_{\rho,\delta}(\Xi)$), with $\{\cdot, \cdot\}_B$ the Poisson bracket associated to the symplectic form σ_B defined by $\sigma_{B,(x,\xi)}[(y, \eta), (z, \zeta)] = \sum_{j=1}^{n}(z_j\eta_j - y_j\zeta_j) + \sum_{j,k} B_{jk}(x)y_j z_k$.

One of the main tools in pseudodifferential theory is the parametrix.

Definition 2.3. A symbol $a \in S^m_{\rho,\delta}(\Xi)$ is called elliptic when there exists two positive constants R and C such that

$$C\langle\xi\rangle^m \le |a(x,\xi)|, \quad \forall x \in \mathcal{X}, \quad \forall \xi \in \mathcal{X}^* \text{ with } |\xi| \ge R.$$

The operator $\mathfrak{Op}^A(a)$ will also be called elliptic.

Theorem 2.4. *Let $a \in S^m_{\rho,\delta}(\Xi)$, $0 \le \delta < \rho \le 1$ be an elliptic symbol. Then there exists $b \in S^{-m}_{\rho,\delta}(\Xi)$ such that $a \circ^B b - 1$, $b \circ^B a - 1 \in S^{-\infty}(\Xi)$. Thus for any vector potential A with components of class $C^\infty_{\rm pol}(\mathcal{X})$ associated to B, $\mathfrak{Op}^A(b)$ is a parametrix for $\mathfrak{Op}^A(a)$.*

The proof is quite standard.

§3. Sobolev Spaces

Theorem 3.1. *Suppose that the magnetic field B has components of class BC^∞. Let $f \in S^0_{\rho,\rho}(\Xi)$ for some $\rho \in [0,1)$. Then $\mathfrak{Op}^A(f) \in \mathcal{B}(L^2(\mathcal{X}))$ and we have the inequality*

$$\left\|\mathfrak{Op}^A(f)\right\|_{\mathcal{B}(L^2(\mathcal{X}))} \le c(n) \sup_{|\alpha|\le p(n)} \sup_{|\beta|\le p(n)} \sup_{X\in\Xi} \langle\xi\rangle^{\rho(|\beta|-|\alpha|)} \left|\partial^\alpha_x \partial^\beta_\xi f(X)\right|, \tag{3.1}$$

where $c(n), p(n)$ are constants depending only on n, that can be determined explicitly.

The proof is quite standard, making use of the Cotlar-Knapp-Stein lemma and an idea of L. Boutet de Monvel ([BM]).

Remark 3.2. Theorem 3.1 remains true also for symbols of class $S^0_{\rho,\delta}(\Xi)$ with $0 \le \delta < \rho \le 1$, due to the obvious inclusion $S^0_{\rho,\delta}(\Xi) \subset S^0_{\delta,\delta}(\Xi)$

In this section we shall suppose that the components of the magnetic field B are of class $BC^\infty(\mathcal{X})$; we shall work in a Schrödinger representation $\mathfrak{Op}^A$ associated to a vector potential A (such that $B = dA$) with components of class $C^\infty_{\rm pol}(\Xi)$.

Definition 3.3.

$$\mathbf{\Psi}^{A,m}_{\rho,\delta}(\mathcal{X}) := \left\{\mathfrak{Op}^A(a) \mid a \in S^m_{\rho,\delta}(\Xi)\right\}, \ \mathbf{\Psi}^{A,m}(\mathcal{X}) := \mathbf{\Psi}^{A,m}_{1,0}(\mathcal{X}).$$

Definition 3.4. For any $s \in \mathbb{R}_+$ we set

$$p_s(\xi) := \langle\xi\rangle^s \in S^s(\Xi), \qquad \mathfrak{P}_s := \mathfrak{Op}^A(p_s) \in \mathbf{\Psi}^{A,s}(\Xi),$$
$$\boldsymbol{H}^s_A(\mathcal{X}) := \left\{u \in L^2(\mathcal{X}) \mid \mathfrak{P}_s u \in L^2(\mathcal{X})\right\}.$$

As a consequence of the Proposition below, *the magnetic Sobolev space* $\boldsymbol{H}^s_A(\mathcal{X})$ (for $s \in \mathbb{R}_+$) may be defined using any elliptic operator of order s.

Proposition 3.5. *For any $s \in \mathbb{R}_+$ we have*:

1. $\boldsymbol{H}^s_A(\mathcal{X})$ *is a Hilbert space for the scalar product*

$$(u,v)_{s,A} := (\mathfrak{P}_s u, \mathfrak{P}_s v)_{L^2} + (u,v)_{L^2}, \quad \forall u,v \in \boldsymbol{H}^s_A(\mathcal{X}).$$

2. *If $0 \leq \delta < \rho \leq 1$, then $T \in \boldsymbol{\Psi}^{A,s}_{\rho,\delta}(\mathcal{X})$ is bounded from $\boldsymbol{H}^s_A(\mathcal{X})$ to $L^2(\mathcal{X})$.*
3. *For any elliptic operator $T \in \boldsymbol{\Psi}^{A,s}_{\rho,\delta}(\mathcal{X})$, the map*

$$\boldsymbol{H}^s_A(\mathcal{X}) \ni u \mapsto \|u\|^T_{s,A} := \left\{\|Tu\|^2_{L^2} + \|u\|^2_{L^2}\right\}^{1/2}$$

defines an equivalent norm on $\boldsymbol{H}^s_A(\mathcal{X})$.

Proof. The first conclusion is evident, because $\mathfrak{P}_s \in \mathcal{B}[\mathcal{S}^*(\mathcal{X})]$ (Lemma 1.2). Let us fix now an operator $T \in \boldsymbol{\Psi}^{A,s}_{\rho,\delta}(\mathcal{X})$. We notice that, being elliptic, $\mathfrak{P}_s \in \boldsymbol{\Psi}^{A,s}(\mathcal{X})$ has a parametrix $\mathfrak{Q}_s \in \boldsymbol{\Psi}^{A,-s}(\mathcal{X})$ (Theorem 2.4), i.e. there exists $\mathfrak{R}_s \in \boldsymbol{\Psi}^{A,-\infty}(\mathcal{X})$ such that $\mathfrak{Q}_s\mathfrak{P}_s = \mathbf{1} + \mathfrak{R}_s$. Thus, for any $u \in \boldsymbol{H}^s_A(\mathcal{X})$ we have $Tu = (T\mathfrak{Q}_s)\mathfrak{P}_s u - (T\mathfrak{R}_s)u$. Theorem 2.2 implies that $T\mathfrak{Q}_s$ and $T\mathfrak{R}_s$ belong to $\boldsymbol{\Psi}^{A,0}_{\rho,\delta}(\mathcal{X})$. We use Remark 3.2 to deduce that $Tu \in L^2(\mathcal{X})$ and $\|Tu\|^2_{L^2} \leq C_0\left(\|\mathfrak{P}_s u\|^2_{L^2} + \|u\|^2_{L^2}\right)$. Thus we get the second conclusion of the Proposition and the inequality $\|u\|^T_{s,A} \leq C\|u\|_{s,A}$ for any $u \in \boldsymbol{H}^s_A(\mathcal{X})$. In order to get the reversed inequality and thus the third conclusion of the Lemma, we have to suppose T elliptic. Then there is a parametrix $S \in \boldsymbol{\Psi}^{A,-s}_{\rho,\delta}(\mathcal{X})$ for T such that $ST = \mathbf{1} + R$, with $R \in \boldsymbol{\Psi}^{A,-\infty}(\mathcal{X})$. For $u \in L^2(\mathcal{X})$ such that $Tu \in L^2(\mathcal{X})$, we obtain $\mathfrak{P}_s u = (\mathfrak{P}_s S)Tu - (\mathfrak{P}_s R)u \in L^2(\mathcal{X})$ and also $\|u\|_{s,A} \leq C\|u\|'_{s,A}$. ☐

Lemma 3.6. *If $0 \leq s \leq t$, we have a continuous embedding $\boldsymbol{H}^t_A(\mathcal{X}) \hookrightarrow \boldsymbol{H}^s_A(\mathcal{X})$.*

Proof. Assume that $u \in \boldsymbol{H}^t_A(\mathcal{X})$. By definition of the Sobolev space, it follows that $u \in L^2(\mathcal{X})$ and $\mathfrak{P}_t u \in L^2(\mathcal{X})$. Making once again use of the parametrix $\mathfrak{Q}_t \in \boldsymbol{\Psi}^{A,-t}(\mathcal{X})$ of $\mathfrak{P}_t$, we deduce that there exists some $\mathfrak{R}_t \in \boldsymbol{\Psi}^{A,-\infty}(\mathcal{X})$ such that $u = \mathfrak{Q}_t\mathfrak{P}_t u + \mathfrak{R}_t u$. Thus $\mathfrak{P}_s u = \mathfrak{P}_s\mathfrak{Q}_t\mathfrak{P}_t u + \mathfrak{P}_s\mathfrak{R}_t u$. Using Theorem 2.2 we get that $\mathfrak{P}_s\mathfrak{Q}_t \in \boldsymbol{\Psi}^{A,-(t-s)}(\mathcal{X})$ and $\mathfrak{P}_s\mathfrak{R}_t \in \boldsymbol{\Psi}^{A,-\infty}(\mathcal{X})$, so that by Remark 3.2 we deduce that $\|\mathfrak{P}_s u\|_{L^2} \leq C(\|\mathfrak{P}_t u\|_{L^2} + \|u\|_{L^2})$. ☐

Lemma 3.7. *Suppose given $s \in \mathbb{R}_+$, $m \leq s$ and $T \in \boldsymbol{\Psi}^{A,m}_{\rho,\delta}(\mathcal{X})$. Then T is a bounded operator from $\boldsymbol{H}^s_A(\mathcal{X})$ to $\boldsymbol{H}^{s-m}_A(\mathcal{X})$.*

Proof. Consider $u \in \boldsymbol{H}^s_A(\mathcal{X})$. Since $m \leq s$ we also have $T \in \boldsymbol{\Psi}^{A,s}_{\rho,\delta}(\mathcal{X})$, thus $Tu \in L^2(\mathcal{X})$ and $\|Tu\|_{L^2(\mathcal{X})} \leq C\|u\|_{s,A}$. Moreover, due to our Theorem 2.2, $\mathfrak{P}_{s-m}T \in \boldsymbol{\Psi}^{A,s}_{\rho,\delta}(\mathcal{X})$, so that we have $\mathfrak{P}_{s-m}Tu \in L^2(\mathcal{X})$ and $\|\mathfrak{P}_{s-m}Tu\|_{L^2(\mathcal{X})} \leq C\|u\|_{s,A}$. We conclude that $Tu \in \boldsymbol{H}^{s-m}_A(\mathcal{X})$ and $\|Tu\|_{s-m,A} \leq C\|u\|_{s,A}$. ☐

Remark 3.8. If the vector potential A has components of class $C^\infty_{\text{pol}}(\Xi)$, then $R \in \boldsymbol{\Psi}^{A,-\infty}(\mathcal{X})$ defines a linear continuous operator from $\mathcal{S}^*(\mathcal{X})$ into $C^\infty(\mathcal{X})$.

Lemma 3.9. *For any $s \in \mathbb{R}_+$ we have the continuous, dense embeddings $\mathcal{S}(\mathcal{X}) \hookrightarrow \boldsymbol{H}^s_A(\mathcal{X}) \hookrightarrow \mathcal{S}^*(\mathcal{X})$.*

Proof. The existence and continuity of the two embeddings is evident. Let us prove the density of $\mathcal{S}(\mathcal{X})$ in $\boldsymbol{H}^s_A(\mathcal{X})$. Take $u \in \boldsymbol{H}^s_A(\mathcal{X})$. Let us choose a sequence $\{v_j\}_{j\in\mathbb{N}} \subset \mathcal{S}(\mathcal{X})$ that converges to $\mathfrak{P}_s u$ in $L^2(\mathcal{X})$. We consider once again the parametrix $\mathfrak{Q}_s$ of $\mathfrak{P}_s$, fix a cut-off function $\chi \in C^\infty_0(\mathcal{X})$ with $\chi(x) = 1$ for $|x| \leq 1$ and set $\chi_j(x) := \chi(x/j)$. We define $u_j := \mathfrak{Q}_s v_j - \chi_j \mathfrak{R}_s u$. Evidently $u_j \in \mathcal{S}(\mathcal{X})$ and $\mathfrak{Q}_s v_j$ converges to $\mathfrak{Q}_s \mathfrak{P}_s u$ in $\boldsymbol{H}^s_A(\mathcal{X})$. But $\mathfrak{Q}_s\mathfrak{P}_s u = u + \mathfrak{R}_s u$, so that the density conclusion will follow if we prove that $\chi_j \mathfrak{R}_s u$ converges to $\mathfrak{R}_s u$ in $\boldsymbol{H}^s_A(\mathcal{X})$. Let us put $\nu_t(\xi) := \sum_{j=1}^n \xi_j^{2t}$ for $t \in \mathbb{N}$, $2t \geq s$ and $T := \mathfrak{Op}^A(\nu_t) = \sum_{j=1}^n (D_j - A_j)^{2t}$. Then T is a differential operator of order $2t$ and an elliptic operator in $\boldsymbol{\Psi}^{A,2t}(\mathcal{X})$. A simple computation shows that $T\chi_j\mathfrak{R}_s u - \chi_j T \mathfrak{R}_s u$ is a finite sum of terms of the form $(D_k - A_k)^{m_k}\mathfrak{R}_s u$, each one multiplied by a bounded function of x (containing derivatives of χ) and a strictly negative power of j. Thus $T\chi_j\mathfrak{R}_s u$ converges to $T\mathfrak{R}_s u$ in $L^2(\mathcal{X})$. Using Proposition 3.5(3), we deduce the convergence of $\chi_j\mathfrak{R}_s u$ to $\mathfrak{R}_s u$ in $\boldsymbol{H}^{2t}_A(\mathcal{X})$, and thus also in $\boldsymbol{H}^s_A(\mathcal{X})$. ☐

Definition 3.10. For $s \in \mathbb{R}_+$, we denote by $\boldsymbol{H}^{-s}_A(\mathcal{X})$ the anti-dual of $\boldsymbol{H}^s_A(\mathcal{X})$ endowed with the natural norm (that induces a scalar product):

$$\|u\|_{-s,A} := \sup_{\varphi \in \boldsymbol{H}^s_A \setminus \{0\}} \frac{|(u,\varphi)|}{\|\varphi\|_{s,A}}.$$

Proposition 3.11. *If $s_1 \leq s_2$ are two real numbers, then we have a continuous embedding $\boldsymbol{H}^{s_2}_A(\mathcal{X}) \hookrightarrow \boldsymbol{H}^{s_1}_A(\mathcal{X})$.*

Proof. Just use Lemma 3.6 and a duality argument. ☐

Proposition 3.12. *Let us fix $s \in \mathbb{R}_+ \setminus \{0\}$.*

1. *If* $u \in \mathcal{S}^*(\mathcal{X})$ *is of the form* $u = \mathfrak{P}_s v + w$, *with* v *and* w *from* $L^2(\mathcal{X})$, *then* $u \in \boldsymbol{H}^{-s}_A(\mathcal{X})$ *and* $\|u\|_{-s,A} \le \left(\|v\|^2_{L^2} + \|w\|^2_{L^2}\right)^{1/2}$.
2. *Reciprocally, if* $u \in \boldsymbol{H}^{-s}_A(\mathcal{X})$, *then there exists* v *and* w *in* $L^2(\mathcal{X})$ *such that* $u = \mathfrak{P}_s v + w$ *and* $\left(\|v\|^2_{L^2} + \|w\|^2_{L^2}\right)^{1/2} \le \|u\|_{-s,A}$.

In conclusion we have

$$\boldsymbol{H}^{-s}_A(\mathcal{X}) = \left\{u \in \mathcal{S}^*(\mathcal{X}) \ \mid \ \exists\, v, w \in L^2(\mathcal{X}) \text{ such that } u = \mathfrak{P}_s v + w\right\},$$

and for $u \in \boldsymbol{H}^{-s}_A(\mathcal{X})$ *and* v, w *as above,* $\|u\|_{-s,A} = \left(\|v\|^2_{L^2} + \|w\|^2_{L^2}\right)^{1/2}$.

The proof uses a standard argument (see [Bo3]).

Lemma 3.13. *For any* $s \in \mathbb{R}_+$ *we have the continuous embeddings* $\mathcal{S}(\mathcal{X}) \hookrightarrow \boldsymbol{H}^{-s}_A(\mathcal{X}) \hookrightarrow \mathcal{S}^*(\mathcal{X})$, *the space* $\mathcal{S}(\mathcal{X})$ *being dense in* $\boldsymbol{H}^{-s}_A(\mathcal{X})$.

Proof. The continuous embeddings follow from Lemma 3.9 and the definition of $\boldsymbol{H}^{-s}_A(\mathcal{X})$ as anti-dual of $\boldsymbol{H}^{s}_A(\mathcal{X})$. Let us fix now $u \in \boldsymbol{H}^{-s}_A(\mathcal{X})$. There exists a pair $\{v, w\} \in L^2(\mathcal{X}) \times L^2(\mathcal{X})$ such that $u = \mathfrak{P}_s v + w$. Moreover, we may approach v and w in L^2-norm by sequences $\{v_j\}_{j\in\mathbb{N}}$ and $\{w_j\}_{j\in\mathbb{N}}$ from $\mathcal{S}(\mathcal{X})$. If we put $u_j := \mathfrak{P}_s v_j + w_j$ and use Proposition 3.12, we deduce that

$$\|u_j - u\|_{-s,A} \le \left(\|v_j - v\|^2_{L^2} + \|w_j - w\|^2_{L^2}\right)^{1/2} \underset{j\to\infty}{\longrightarrow} 0.$$

□

Proposition 3.14. *For* s *and* m *real numbers, any* $T \in \boldsymbol{\Psi}^{A,m}_{\rho,\delta}(\mathcal{X})$ *is bounded as operator from* $\boldsymbol{H}^{s}_A(\mathcal{X})$ *to* $\boldsymbol{H}^{s-m}_A(\mathcal{X})$.

Proof. The case $s \ge 0$ and $m \le s$ is the content of Lemma 3.7. By duality we obtain also the case $s \le 0$ and $m \ge s$. If $s \ge 0$ and $m > s$, let us choose $u \in \boldsymbol{H}^{s}_A(\mathcal{X})$ and write once again $u = \mathfrak{Q}_s \mathfrak{P}_s u + \mathfrak{R}_s u$, with $\mathfrak{Q}_s \in \boldsymbol{\Psi}^{A,-s}$ and $\mathfrak{R}_s \in \boldsymbol{\Psi}^{A,-\infty}$. Thus $Tu = (T\mathfrak{Q}_s)\mathfrak{P}_s u + T\mathfrak{R}_s u$. We have $\mathfrak{P}_s u \in L^2(\mathcal{X})$, $T\mathfrak{Q}_s \in \boldsymbol{\Psi}^{A,m-s}_{\rho,\delta}$ and $m - s \ge 0$, so that we conclude that $(T\mathfrak{Q}_s)\mathfrak{P}_s u \in \boldsymbol{H}^{s-m}_A(\mathcal{X})$. We also remark that $T\mathfrak{R}_s \in \boldsymbol{\Psi}^{A,-\infty}$, so that $T\mathfrak{R}_s u \in L^2(\mathcal{X}) \subset \boldsymbol{H}^{s-m}_A(\mathcal{X})$. The case $s \le 0$ and $m < s$ follows from Proposition 3.12. □

Definition 3.15. We define $\boldsymbol{H}^{-\infty}_A(\mathcal{X}) := \cup_{s\in\mathbb{R}} \boldsymbol{H}^{s}_A(\mathcal{X})$ endowed with the inductive limit topology and $\boldsymbol{H}^{\infty}_A(\mathcal{X}) := \cap_{s\in\mathbb{R}} \boldsymbol{H}^{s}_A(\mathcal{X})$ endowed with the projective limit topology.

Proposition 3.16. *Let* $T \in \boldsymbol{\Psi}^{A,m}_{\rho,\delta}(\mathcal{X})$*; then*

1. *T induces linear continuous maps* $\boldsymbol{H}^{-\infty}_A(\mathcal{X}) \to \boldsymbol{H}^{-\infty}_A(\mathcal{X})$ *and* $\boldsymbol{H}^{\infty}_A(\mathcal{X}) \to \boldsymbol{H}^{\infty}_A(\mathcal{X})$.
2. *If* $m = -\infty$*, T induces a linear continuous map* $\boldsymbol{H}^{-\infty}_A(\mathcal{X}) \to \boldsymbol{H}^{\infty}_A(\mathcal{X})$.
3. *If T is an elliptic operator and we have* $u \in \boldsymbol{H}^{-\infty}_A(\mathcal{X})$ *and* $Tu \in \boldsymbol{H}^{s}_A(\mathcal{X})$*, then* $u \in \boldsymbol{H}^{s+m}_A(\mathcal{X})$.

The proof is straightforward.

Remark 3.17. The property (3) of Proposition 3.16 may be completed as follows: If T is an elliptic operator and $u \in \mathcal{S}^*(\mathcal{X})$, then we have $\operatorname{sing\,supp} Tu = \operatorname{sing\,supp} u$; in particular, if $Tu \in C^\infty(\mathcal{X})$ then $u \in C^\infty(\mathcal{X})$.

The above statement follows from Lemma 3.8 and from the fact that any operator $T \in \boldsymbol{\Psi}^{A,m}_{\rho,\delta}(\mathcal{X})$ is *pseudo-local* (i.e. $\operatorname{sing\,supp} Tu \subset \operatorname{sing\,supp} u$). In fact, the integral kernel of T is the product of the C^∞ function $\exp\{-i\Gamma^A([x,y])\}$ and the distribution defined by the oscillatory integral $\int_{\mathcal{X}} đ\xi\, e^{i\langle x-y,\xi\rangle} t\left(\frac{x+y}{2},\xi\right)$, that is a C^∞ function outside the diagonal of $\mathcal{X} \times \mathcal{X}$.

Lemma 3.18. *For any* $m \in \mathbb{N}$ *we have the equality*

$$\boldsymbol{H}^m_A(\mathcal{X}) = \left\{u \in L^2(\mathcal{X}) \mid (D-A)^\alpha u \in L^2(\mathcal{X}),\ \forall \alpha \in \mathbb{N}^n \text{ with} |\alpha| \le m\right\}, \tag{3.2}$$

where $(D-A)^\alpha = (D_1-A_1)^{\alpha_1}\cdots(D_n-A_n)^{\alpha_n}$*. Moreover, we have the following equivalent norm on* $\boldsymbol{H}^m_A(\mathcal{X})$: $\|u\|_{m,A} \sim \left(\sum_{|\alpha|\le m}\|(D-A)^\alpha u\|^2_{L^2}\right)^{1/2}$.

Proof. Let us denote by $\mathcal{M}$ the linear space defined in (3.2) endowed with the norm $\|\cdot\|_{\mathcal{M}}$ defined by the formula above. Remark that $D_j - A_j = \mathfrak{Op}^A(\xi_j) \in \boldsymbol{\Psi}^{A,1}(\mathcal{X})$, so that $(D-A)^\alpha \in \boldsymbol{\Psi}^{A,m}(\mathcal{X})$ for $|\alpha| \le m$. In conclusion, for $u \in \boldsymbol{H}^m_A(\mathcal{X})$ we have $(D-A)^\alpha u \in L^2(\mathcal{X})$, and $\|(D-A)^\alpha u\|_{L^2} \le C\|u\|_{m,A}$. Reciprocally, let $u \in \mathcal{M}$. We consider the operator $E = \mathfrak{Op}^A(\nu_m) \in \boldsymbol{\Psi}^{A,2m}(\mathcal{X})$ (with the notation introduced in the proof of Lemma 3.9), that is elliptic. Thus we can find $F \in \boldsymbol{\Psi}^{A,-2m}(\mathcal{X})$ and $R \in \boldsymbol{\Psi}^{A,-\infty}(\mathcal{X})$ such that $FE - \mathbf{1} = R$. Due to our choice of u and the definition of $\mathcal{M}$, we have $(D_j - A_j)^m u \in L^2(\mathcal{X})$ for any $j \in \{1,\dots,n\}$ and thus $Eu \in \boldsymbol{H}^{-m}_A(\mathcal{X})$. We get $u = F(Eu) - Ru \in \boldsymbol{H}^m_A(\mathcal{X})$, and we finish the proof by the closed graph theorem, due to the fact that $\mathcal{M}$ is a Hilbert space. $\square$

Remark 3.19. Let us point out here that for any real function $\phi \in C^\infty_{\text{pol}}(\mathcal{X})$, the multiplication with $e^{i\phi}$ defines a unitary operator intertwining the Sobolev spaces $\boldsymbol{H}^s_{A+\nabla\phi}(\mathcal{X})$ and $\boldsymbol{H}^s_A(\mathcal{X})$.

§4. Self-Adjointness and Semiboundedness

Theorem 4.1. *Suppose given a magnetic field B with components of class $BC^\infty(\mathcal{X})$. Let $p \in S^m_{\rho,\delta}(\Xi)$ be real with $m \geq 0$, $0 \leq \rho < \delta \leq 1$; if $m > 0$ we also assume that p is elliptic. Let us set $P := \mathfrak{Op}^A(p)$ in a Schrödinger representation defined by a vector potential A associated to B (i.e. $B = dA$), having components of class $C^\infty_{\text{pol}}(\mathcal{X})$. Then P defines a self-adjoint operator $\tilde{P}$ on the domain $\mathcal{D}(\tilde{P}) := \boldsymbol{H}^m_A(\mathcal{X})$ and $\mathcal{S}(\mathcal{X})$ is a core for $\tilde{P}$.*

Proof. The operator P is symmetric on $\mathcal{S}(\mathcal{X})$, which is dense in $\boldsymbol{H}^m_A(\mathcal{X})$. The case $m = 0$ is clear, because P is a bounded operator. For $m > 0$ we can define $\tilde{P}$ on $\mathcal{D}(\tilde{P}) := \boldsymbol{H}^m_A(\mathcal{X})$ using Proposition 3.14 and obtain a symmetric operator. If $v \in \mathcal{D}(\tilde{P}^*)$, there exists $f \in L^2(\mathcal{X})$ such that $(\tilde{P}u, v)_{L^2} = (u, f)_{L^2}$ for any $u \in \mathcal{S}(\mathcal{X})$. Thus $Pv = f$ (as distributions) and $v \in \boldsymbol{H}^m_A(\mathcal{X}) = \mathcal{D}(P)$, proving self-adjointness. The last statement follows from the fact that the topology of $\boldsymbol{H}^m_A(\mathcal{X})$ may be defined by the graph norm of P (see Lemma 3.5(3)) and from the density of $\mathcal{S}(\mathcal{X})$ in $\boldsymbol{H}^m_A(\mathcal{X})$ (Lemma 3.9). ☐

We intend to prove a magnetic version of *the Gårding inequality.* The following Lemma is an adaptation to the case $B \neq 0$ of Lemma 2.2.2 in [Ho].

Lemma 4.2. *Let $0 \leq \delta < \rho \leq 1$ and suppose that f is a real valued symbol of class $S^0_{\rho,\delta}(\Xi)$ such that there exists a real valued symbol $f_0 \in S^0_{\rho,\delta}(\Xi)$ satisfying $\inf_{X\in\Xi} f_0(X) > 0$ and $f - f_0 \in S^{-(\rho-\delta)}_{\rho,\delta}(\Xi)$. Then there exists a real valued symbol $g \in S^0_{\rho,\delta}(\Xi)$ such that $f - g \circ^B g \in S^{-\infty}(\Xi)$.*

Theorem 4.3. *Let B be a magnetic field with components of class $BC^\infty(\mathcal{X})$. Let $m \in \mathbb{R}$, $0 \leq \delta < \rho \leq 1$, $p \in S^m_{\rho,\delta}(\Xi)$. Suppose that there exist two constants R and C such that* $\operatorname{Re} p(x,\xi) \geq C|\xi|^m$ *for $|\xi| \geq R$. Let us set $P := \mathfrak{Op}^A(p)$ in any Schrödinger representation defined by a vector potential A associated to B, whose components are of class $C^\infty_{pol}(\mathcal{X})$. Then $\forall s \in \mathbb{R}$ there exist two finite positive constants C_0 and C_1 such that*

$$\operatorname{Re}(Pu,u)_{L^2} \;\geq\; C_0\|u\|^2_{m/2,A} - C_1\|u\|^2_{s,A}, \quad \forall u \in \boldsymbol{H}^\infty_A(\mathcal{X}).$$

Proof. We assume first that $m = 0$. We can choose a positive constant d and a cut-off function $\chi \in C_0^\infty(\mathcal{X}^*)$ such that $\chi(\xi) \geq 0$ and $\chi(\xi) = 1$ on a given neighbourhood (large enough) of the origin $0 \in \mathcal{X}^*$, so that for $\tilde{p} := p + d\chi \in S^0_{\rho,\delta}(\Xi)$ we have $\mathrm{Re}(\tilde{p}(x,\xi)) \geq c > 0$. Hence it is evident that we can take from the begining $\mathrm{Re}\, p(x,\xi) \geq c > 0$. Using Lemma 4.2, we deduce the existence of $g \in S^0_{\rho,\delta}(\Xi)_{\mathbb{R}}$ and real $r_0 \in S^{-\infty}(\Xi)$ such that $\mathrm{Re}\, p - \frac{c}{2} - g \circ^B g = r_0$. We get

$$\mathrm{Re}\,(Pu,u)_{L^2} = \frac{c}{2}\|u\|^2_{L^2} + \left\|\mathfrak{Op}^A(g)u\right\|^2_{L^2} + \left(\mathfrak{Op}^A(r_0)u,u\right)_{L^2}, \quad \forall u \in \boldsymbol{H}^\infty_A(\mathcal{X}),$$

and thus the inequality for $m = 0$.

For the case $m \neq 0$, notice that the operator $Q := \mathfrak{P}_{-m/2} P \mathfrak{P}_{-m/2}$ satisfies the conditions of the case $m = 0$. Thus $\forall s' \in \mathbb{R}$

$$\mathrm{Re}(Qv,v)_{L^2} \ \geq\ C_0'\|v\|^2_{L^2} \ -\ C_1'\|v\|^2_{s',A}, \quad \forall v \in \boldsymbol{H}^\infty_A(\mathcal{X}).$$

For $u \in \boldsymbol{H}^\infty_A(\mathcal{X})$ we denote $v = \mathfrak{Q}_{-m/2}u \in \boldsymbol{H}^\infty_A(\mathcal{X})$ where $\mathfrak{Q}_{-m/2} \in \boldsymbol{\Psi}^{A,m/2}(\mathcal{X})$ and $\mathfrak{P}_{-m/2}\mathfrak{Q}_{-m/2} = \mathbf{1} + R'$, $\mathfrak{Q}_{-m/2}\mathfrak{P}_{-m/2} = \mathbf{1} + R''$, with R' and R'' belonging to $\boldsymbol{\Psi}^{A,-\infty}(\mathcal{X})$. We conclude that

$$\mathrm{Re}\,(P(\mathbf{1}+R')u, (\mathbf{1}+R')u)_{L^2} \ \geq\ C_0'\left\|\mathfrak{Q}_{-m/2}u\right\|^2_{L^2} \ -\ C_1'\left\|\mathfrak{Q}_{-m/2}u\right\|^2_{s',A}.$$

To obtain the stated inequality, we remark that for functions f and g in $\boldsymbol{H}^\infty_A(\mathcal{X})$ and $t \in \mathbb{R}$ we have $|(f,g)_{L^2}| \leq C\|f\|_{t,A}\,\|g\|_{-t,A}$, and that $\forall s \in \mathbb{R}$

$$\begin{aligned} \|u\|_{m/2,A} &\leq \left\|\mathfrak{P}_{-m/2}\mathfrak{Q}_{-m/2}u\right\|_{m/2,A} + \|R'u\|_{m/2,A} \\ &\leq C\left(\left\|\mathfrak{Q}_{-m/2}u\right\|_{L^2} + \|u\|_{s,A}\right). \end{aligned}$$

□

Corollary 4.4. *Under the hypothesis of Theorem* 4.1*, if* $p \geq 0$ *for* $|\xi| \geq R$*, the self-adjoint operator* P *is lower semibounded.*

Proof. The single non-trivial case is $m > 0$. Taking $s = 0$ in Theorem 4.3 one gets $(Pu,u)_{L^2} \ \geq\ C_0\|u\|^2_{m/2,A} - C_1\|u\|^2_{L^2} \ \geq\ -C_1\|u\|^2_{L^2}$, $\forall u \in \boldsymbol{H}^\infty_A(\mathcal{X})$. □

§5. Vector Potentials with Bounded Derivatives of Strictly Positive Order

In this Section we are going to assume (even when it is not stated explicitly) that the magnetic field B can be deduced from a vector potential A satisfying

$$|\left(\partial^\alpha A_j\right)(x)| \leq C_\alpha, \quad \forall j = 1,\dots,n, \quad \forall \alpha \in \mathbb{N}^n, \ |\alpha| \geq 1,$$

that implies evidently that all the components of B are of class BC^∞.

§5.1. General facts

We illustrate our assumption on B by two examples.

Example 1. Assume that the components $\{B_{jk}\}_{1\leq j<k\leq n}$ of the magnetic field are C^∞ real valued functions, periodic with respect to a lattice $\Gamma\subset\mathcal{X}$. It has been proved (see [HH] and [If]) that there exists a constant magnetic field $B^\circ=\{B^\circ_{jk}\}_{1\leq j<k\leq n}$ and a potential vector $\tilde{A}$ of class C^∞ and Γ-periodic, such that $B-B^\circ=d\tilde{A}$. If A° is a linear vector potential defining the magnetic field B°, then $A:=A^\circ+\tilde{A}$ has all the derivatives (of strictly positive order) bounded and $B=dA$.

Example 2. Let us assume $B_{jk}\in C^\infty\cap L^\infty$, and $|(\partial^\alpha B_{jk})(x)|\leq c_\alpha\langle x\rangle^{-1}$ for all multiindices α with $|\alpha|\geq 1$. We define the associated transversal gauge vector potential (1.1). Then for any α with $|\alpha|\geq 1$ we get that outside the ball of radius 1

$$|(\partial^\alpha A_j)(x)|\;\leq\;C|x|\int_0^1 ds\,s^{1+|\alpha|}\,(1+s|x|)^{-1}\;+\;C_1\;\leq\;C_2.$$

Definition 5.1. For $0\leq\delta<\rho\leq 1$ we consider the following metric on Ξ:

$$g^A_X\;\equiv\;g^A_{(x,\xi)}\;:=\;\langle\xi-A(x)\rangle^{2\delta}|dx|^2\;+\;\langle\xi-A(x)\rangle^{-2\rho}|d\xi|^2,$$

and its symplectic inverse (with respect to the canonical symplectic form $[\![.,.]\!]$)

$$g^{A,\sigma}_X\;\equiv\;g^{A,\sigma}_{(x,\xi)}\;:=\;\langle\xi-A(x)\rangle^{2\rho}|dx|^2\;+\;\langle\xi-A(x)\rangle^{-2\delta}|d\xi|^2.$$

Let $\mu^A(X):=\langle\xi-A(x)\rangle\;\geq 1$, $\nu^A(X):=\xi-A(x)\in\mathcal{X}^*$.

The following facts follow by straightforward arguments.

Lemma 5.2. *The metric defined in Definition* 5.1 *has the following properties*:

a) *It is a Hörmander metric, i.e.*:

- (*slow variation*) *there exists* $C>0$ *such that* $g^A_X(X-Y)\leq C^{-1}$ *implies* $(g^A_X/g^A_Y)^{\pm1}\leq C$
- (*temperedness condition*) *there exist* $C>0$ *and* $N\in\mathbb{N}$ *such that* $(g^A_X/g^A_Y)^{\pm1}\leq C\left(1+g^{A,\sigma}_X(X-Y)\right)^N$ *for any* X,Y *in* Ξ,
- (*uncertainty condition*) $g^A_X\;\leq\;g^{A,\sigma}_X$.

b) *It is a conformal metric, i.e.* $g_X^{A,\sigma} = \lambda^A(X)^2 g_X^A$, *where we have defined* $\lambda^A(X)^2 := \inf\left\{g_X^{A,\sigma}(T) \mid g_X^A(T) = 1\right\}$.

c) *It is geodesically tempered, i.e. it verifies the temperedness condition with respect to the geodesic distance associated to* $g_X^{A,\sigma}$.

Lemma 5.3. *For any* $m \in \mathbb{R}$ *the function* $M_m^A(X) := \langle \xi - A(x)\rangle^m$ *is a* g^A*-weight for the metric* g^A *defined in Definition* 5.1, *i.e. it is*

- (g^A*-continuous*) *There exists* $C > 0$ *such that* $g_X^A(X - Y) \leq C^{-1}$ *implies* $(M_m^A(X)/M_m^A(Y))^{\pm 1} \leq C$,
- (g^A*-tempered*) *There exists* $C > 0$ *and* $N \in \mathbb{N}$ *such that* $\left[M_m^A(X)/M_m^A(Y)\right]^{\pm 1} \leq C\left[1 + g_X^{A,\sigma}(X - Y)\right]^N$.

Definition 5.4. We consider the following spaces of symbols associated to the metric g^A of Definition 5.1 and a g^A-weight M:

- $S_{\rho,\delta}^A(M) \equiv S(M, g^A)$ the symbols $q \in C^\infty(\Xi)$ such that $\forall(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^n$, $\left|(\partial_x^\alpha \partial_\xi^\beta q)(x,\xi)\right| \leq C_{\alpha\beta}\, M(X)\mu^A(X)^{-\rho|\beta| + \delta|\alpha|}$.
- $S_{\rho,\delta}^{A,+}$ the symbols $q \in C^\infty(\Xi)$ such that $\forall \alpha,\beta \in \mathbb{N}^n$, with $|\alpha| + |\beta| \geq 1$, we have $\left|(\partial_x^\alpha \partial_\xi^\beta q)(x,\xi)\right| \leq C_{\alpha\beta}\, \mu^A(X)^{\rho - \delta - \rho|\beta| + \delta|\alpha|}$,
- $S_{\rho,\delta}^{A,m} := S_{\rho,\delta}^A(\mu^m)$ for $m \in \mathbb{R}$ (we call this m the order of the Weyl operator associated to a symbol of this class).

If $\rho = 1$ and $\delta = 0$ the indices ρ and δ will be omitted from the above notations. By a slight abuse, for any $p \in C^\infty(\Xi)$ we set $(p \circ \nu^A)(X) := p(x, \nu^A(X))$.

Remark 5.5. For $p \in C^\infty(\Xi)$ it is clear that $p \in S_{\rho,\delta}^m(\Xi)$ if and only if $p \circ \nu^A \in S_{\rho,\delta}^{A,m}$. This allows us to define asymptotic sums of symbols from $S_{\rho,\delta}^{A,m}$. In fact, for a sequence $\{q_j\}_{j\in\mathbb{N}}$ with $q_j \in S_{\rho,\delta}^{A,m_j}$ and $\{m_j\}_{j\in\mathbb{N}}$ decreasing, with $\lim\limits_{j\to\infty} m_j = -\infty$, there exists $q \in S_{\rho,\delta}^{A,m_0}$, uniquely defined *modulo* $S^{A,-\infty} := \bigcap\limits_{m\in\mathbb{R}} S_{\rho,\delta}^{A,m}$, such that $q - \sum_{j=0}^{k-1} q_j \in S_{\rho,\delta}^{A,m_k}$, $\forall k \geq 1$. We shall write $q \sim \sum_{j=0}^\infty q_j$.

Remark 5.6. The symbol $p \in S_{\rho,\delta}^m(\Xi)$ is elliptic if and only if $p \circ \nu^A$ is elliptic for the metric g^A (i.e. $1 + |(p \circ \nu^A)(x,\xi)| \geq c[\mu^A(x,\xi)]^m$).

§5.2. Comparison of two quantizations

We shall use the notation $\mathfrak{Op}(p) \equiv \mathfrak{Op}^0(p)$ for the usual Weyl quantization. As mentioned in the Introduction, $\mathfrak{Op}_A(p) := \mathfrak{Op}(p \circ \nu^A)$ is sometimes used as the quantization of the symbol f. We show now explicitly that it lacks gauge covariance, completing the discussion in [MP2].

For $f \in S^m(\Xi)$, $A \in C^\infty_{\mathrm{pol}}(\mathcal{X}, \mathcal{X}^*)$ and $\varphi \in C^\infty_{\mathrm{pol}}(\mathcal{X})$ real valued, set

$$F(f, A, \varphi) := e^{i\varphi}\mathfrak{Op}(f \circ \nu^A)e^{-i\varphi} - \mathfrak{Op}(f \circ \nu^{A+\nabla\varphi}).$$

It is an operator with distribution kernel

$$[K(f, A, \varphi)](x, y) = \exp\left\{i\left\langle x - y, A\left(\frac{x + y}{2}\right)\right\rangle\right\}\Phi(x, y)\tilde{f}\left(\frac{x + y}{2}, x - y\right),$$

where $\tilde{f}$ is the Fourier transform of f in the second variable and

$$\Phi(x, y) := \exp\{i[\varphi(x) - \varphi(y)]\} - \exp\left\{i\left\langle x - y, (\nabla\varphi)\left(\frac{x + y}{2}\right)\right\rangle\right\}.$$

Thus gauge covariance is equivalent with the vanishing of the tempered distribution $\exp\{i\langle y, A(x)\rangle\}\Phi(x + y/2, x - y/2)\tilde{f}(x, y)$. An easy argument proves that ϕ vanishes identically if and only if φ is a polynomial of degree ≤ 2. This can easily be used to prove the lack of gauge covariance for an enormous class of symbols f. Let us consider the monomial $f(x, \xi) = (\xi + A(x))^\alpha$, $\alpha \in \mathbb{N}$; one has $F(f, A, \varphi) = 0$ if and only if $\partial_y^\beta\left[\Phi(x + y/2, x - y/2)\right]|_{y=0} = 0$ for any $\beta \leq \alpha$. Simple calculations show that this holds if $|\beta| \leq 2$ but is no longer true for $|\beta| \geq 3$, (one checks easily that $f(x, \xi) = (\xi_j + A_j(x))(\xi_k + A_k(x))(\xi_l + A_l(x))$ is indeed a counterexample). Let us also notice that the Fourier transform of $f(x, \xi) = \langle\xi\rangle$ is a distribution $\tilde{f}$ with singular support $\mathcal{X} \times \{0\}$ and analytic outside. In fact some straightforward computation proves that $\tilde{f}$ is rotation invariant and verifies an ordinary differential equation (in the radial variable) with analytic coefficients. Thus it is nonzero on a dense set in $\mathcal{X} \times \mathcal{X}$ and in order to have gauge covariance, the function Φ should be identically zero and this is not the case if φ is not a second order polynomial. Thus we conclud that $\mathfrak{Op}_A(\langle\xi\rangle)$ does not provide a gauge covariant quantization.

In spite of all these, it is useful to express $\mathfrak{Op}^A(p)$ as $\mathfrak{Op}_A(q)$ for some symbol q, but keeping in mind that this operator is the magnetic quantization of p and not of q. We are going to explore this in the sequel. We define $\Gamma^A([x, y]) := \langle x - y, \Gamma^A(x, y)\rangle$ so that $\Gamma^A(x, y) = \int_0^1 ds\, A((1 - s)x + sy)$.

Proposition 5.7. *For any $p \in S^m_{\rho,\delta}(\Xi)$ there exists a unique $q \in S^m_{\rho,\delta}(\Xi)$ such that $\mathfrak{Op}^A(p) = \mathfrak{Op}(q \circ \nu^A)$. Besides, we have $q \circ \nu^A \sim \sum_{j=0}^\infty q_j^A$, where*

$$q_j^A(X) := \sum_{|\alpha|=j} \frac{1}{\alpha!} \left\{(-D_y)^\alpha \partial_\xi^\alpha \left[p(x, \xi - \Gamma^A(x + y/2, x - y/2)\right]\right\}\Big|_{y=0} .$$

In particular $q_0^A = p \circ \nu^A$, $q_1^A = 0$, $q_j^A \in S^{A,m-(j+1)\rho}_{\rho,\delta}$ $(\forall j \geq 1)$, $q - p \in S^{m-3\rho}_{\rho,\delta}(\Xi)$.

Proof. It is easy to see that the usual Weyl symbol of the operator $\mathfrak{Op}^A(p)$ is

$$\tilde{q}^A(X) = \iint_{\mathcal{X}\times\mathcal{X}^*} dy\,đ\eta\; e^{i\langle y,\eta\rangle}\, p\left(x, \xi + \eta - \Gamma^A(x + (y/2), x - (y/2))\right).$$

We use the Taylor expansion of $p(x, \zeta + \eta)$ and some standard estimations for the integrals in order to obtain the formulae of the Proposition. □

Remark 5.8. If $p \in S^2(\Xi)$ is a polynomial of degree less then or equal to 2 in the variable $\xi \in \mathcal{X}^*$ (with coefficients depending on the variable $x \in \mathcal{X}$), we have $q = p$ in the above Proposition and thus we get $\mathfrak{Op}^A(p) = \mathfrak{Op}(p \circ \nu^A)$.

Proposition 5.9 (Converse of Proposition 5.7)**.** *For any $q \in S^m_{\rho,\delta}(\Xi)$, there exists a unique $p \in S^m_{\rho,\delta}(\Xi)$ such that $\mathfrak{Op}^A(p) = \mathfrak{Op}(q \circ \nu^A)$.*

Proof. We proceed as in the proof of Proposition 5.7 using the formula:

$$p(X) = \iint_{\mathcal{X}\times\mathcal{X}^*} dy\,đ\eta\; e^{i\langle y,\eta\rangle} q\left(x, \xi + \eta + \Gamma^A(x + y/2, x - y/2) - A(x)\right).$$

. □

Remark 5.10. The Propositions 5.7 and 5.9 imply that, under the hypothesis of Section 5, the properties of the magnetic pseudodifferential operators may be obtained through the usual Weyl functional calculus associated to the metric g^A ([Bo3], [Ho1], [Ho2]). An example is the following Fefferman-Phong theorem:

Corollary 5.11. *Let us choose $p \in S^{2(\rho-\delta)}_{\rho,\delta}(\Xi)$ with $p \geq 0$. Then there exists a constant $C > 0$ such that $\left(\mathfrak{Op}^A(p)u, u\right)_{L^2} \geq -C\|u\|^2_{L^2}$, $\forall u \in \mathcal{S}(\mathcal{X})$.*

Proof. Choosing p as in the statement of the Corollary and using Proposition 5.7, we conclude that there exist $q \in S^{2(\rho-\delta)}_{\rho,\delta}(\Xi)$ and $r \in S^{-\rho-2\delta}_{\rho,\delta}(\Xi)$ such that $q = p + r$ and $\mathfrak{Op}^A(p) = \mathfrak{Op}(q \circ \nu)$. The condition $p \geq 0$ implies that $\mathfrak{Op}^A(p)$ is symetric and thus q and r will be real. Thus we can write $\mathfrak{Op}^A(p) = \mathfrak{Op}(p \circ \nu^A) + \mathfrak{Op}(r \circ \nu^A)$ and $p \circ \nu^A \in S^{A,2(\rho-\delta)}_{\rho,\delta}$. As a consequence of the Fefferman-Phong inequality ([Ho2], T.18.6.8), there exists a constant $C_0 > 0$ such that $\big(\mathfrak{Op}(p \circ \nu^A)u, u\big)_{L^2} \geq -C_0 \|u\|^2_{L^2}$, $\forall u \in \mathcal{S}(\mathcal{X})$. Using Proposition 5.9 we deduce the existence of a symbol $r_0 \in S^{-\rho-2\delta}_{\rho,\delta}(\Xi)$ such that $\mathfrak{Op}(r \circ \nu^A) = \mathfrak{Op}^A(r_0)$, that is a bounded operator in $L^2(\mathcal{X})$ due to the fact that $\rho + 2\delta \geq 0$ and to Remark 3.2. We conclude that there exists a constant $C_1 > 0$ such that $\big(\mathfrak{Op}(r \circ \nu^A)u, u\big)_{L^2} \geq -C_1 \|u\|^2_{L^2}$, $\forall u \in \mathcal{S}(\mathcal{X})$. $\square$

§5.3. Resolvents and fractional powers of elliptic magnetic pseudodifferential operators

Due to the fact that the Hörmander metric g^A is conformal and geodesically temperate we can use a Theorem of Bony ([Bo1]) characterizing pseudodifferential operators by commutators and prove that the resolvent and the powers of an elliptic magnetic self-adjoint pseudodifferential operator are also of this type.

Theorem 5.12 (Bony). *Let $q \in \mathcal{S}^*(\Xi)$ and $Q := \mathfrak{Op}(q)$. Then $q \in S^{A,m}_{\rho,\delta}$ if and only if $Q \in \mathcal{B}(\boldsymbol{H}^m_A, L^2(\mathcal{X}))$ and for any finite family $\{b_j\}_{1\leq j\leq k} \subset S^{A,+}_{\rho,\delta}$ we have $ad(\mathfrak{Op}(b_1)) \cdots ad(\mathfrak{Op}(b_k))Q \in \mathcal{B}(\boldsymbol{H}^m_A, L^2(\mathcal{X}))$.*

Corollary 5.13. *Under the hypothesis of Theorem* 4.1 *let $P := \mathfrak{Op}^A(p)$. We also denote by P the induced self-adjoint operator in $L^2(\mathcal{X})$ (with domain $\boldsymbol{H}^m_A$). Then for any $z \in \mathbb{C} \setminus \sigma(P)$ we have $(P - z)^{-1} = \mathfrak{Op}^A(\tilde{p}_z)$ with $\tilde{p}_z \in S^{-m}_{\rho,\delta}(\Xi)$.*

Proof. Obviously $(P - z)^{-1} \in \mathcal{B}(\boldsymbol{H}^{-m}_A, L^2(\mathcal{X}))$. Using Proposition 5.7, there exists $q \in S^{A,m}_{\rho,\delta}$ such that $\mathfrak{Op}^A(p) = \mathfrak{Op}(q)$. For a finite family $\{b_j\}_{1\leq j\leq k} \subset S^{A,+}_{\rho,\delta}$, the arguments in [Bo1] imply that $ad(\mathfrak{Op}(b_1)) \cdots ad(\mathfrak{Op}(b_k))\mathfrak{Op}(q)$ is a Weyl operator having a symbol of class $S^{A,m}_{\rho,\delta}$. A simple computation shows that the operator $ad(\mathfrak{Op}(b_1)) \cdots ad(\mathfrak{Op}(b_k))(P - z)^{-1}$ is a finite sum of terms of the form $\pm(P - z)^{-1}K_{a_1}(P - z)^{-1} \cdots (P - z)^{-1}K_{a_l}(P - z)^{-1}$, where $l \leq k$ and each factor K_a is of the form

$$K_a \;=\; \underset{j\in J_a}{\Pi}\, ad(\mathfrak{Op}(b_j))P \;=\; \underset{j\in J_a}{\Pi}\, ad(\mathfrak{Op}(b_j))\mathfrak{Op}(q),$$

with J_a finite subset of $\{1,\dots,k\}$. We conclude that $K_a \in \mathcal{B}(L^2(\mathcal{X}), \boldsymbol{H}_A^{-m})$, and thus

$$ad(\mathfrak{Op}(b_1))\cdots ad(\mathfrak{Op}(b_k))(P-z)^{-1} \in \mathcal{B}(\boldsymbol{H}_A^{-m}, L^2(\mathcal{X})).$$

Using Theorem 5.12, we conclude the existence of a symbol $\tilde{q}_z \in S_{\rho,\delta}^{A,-m}$ such that $(P-z)^{-1} = \mathfrak{Op}(\tilde{q}_z)$. By Proposition 5.9, we deduce the existence of a symbol $\tilde{p}_z \in S_{\rho,\delta}^{-m}(\Xi)$ such that $\mathfrak{Op}^A(\tilde{p}_z) = \mathfrak{Op}(\tilde{q}_z) = (P-z)^{-1}$. ☐

Remark 5.14. From Theorem 5.12 it follows directly that an operator $\mathfrak{Op}^A(p)$ (with $p \in \mathcal{S}^*(\Xi)$) is a "smoothing" one, i.e. transforms $\boldsymbol{H}_A^{-\infty}$ into $\boldsymbol{H}_A^{\infty}$, if and only if it belongs to $\boldsymbol{\Psi}^{A,-\infty}(\Xi)$.

In order to study the fractional powers of operators as in Corollary 5.13 we first remark from Corollary 4.4 that (for the case $n \geq 2$ and replacing if necessary p by $-p$) $\mathfrak{Op}^A(p)$ is lower semibounded. Thus in this case (adding if necessary a sufficiently large constant) we may suppose that $p \geq 1$ and $\mathfrak{Op}^A(p) \geq 1$. We can work with the usual Weyl quantization, because (having assumed that the magnetic field B admits a vector potential with bounded derivatives of any strictly positive order) the two quantization are in a one-to-one correspondence that associates to elliptic magnetic operators, operators that are elliptic with respect to the metric g^A.

Given $p \in S_{\rho,\delta}^m(\Xi)$, resp. $p \in S_{\rho,\delta}^{A,m}$, we call a *principal symbol* of $\mathfrak{Op}^A(p)$, resp. $\mathfrak{Op}(p)$, any element $p_0 \in S_{\rho,\delta}^m(\Xi)$, resp. $p_0 \in S_{\rho,\delta}^{A,m}$, satisfying $p - p_0 \in S_{\rho,\delta}^{m-(\rho-\delta)}(\Xi)$, resp. $p - p_0 \in S_{\rho,\delta}^{A,m-(\rho-\delta)}$.

Theorem 5.15. *Let $m > 0$, $p \in S_{\rho,\delta}^{A,m}$ a real elliptic symbol, such that $p \geq 1$ and $P := \mathfrak{Op}(p) \geq 1$. Then for any $s \in \mathbb{R}$ we have $P^s = \mathfrak{Op}(q_s)$ for some $q_s \in S_{\rho,\delta}^{A,sm}$. Moreover P^s admits p^s as principal symbol.*

Proof. We follow the proof of Proposition 29.1.9 from [Ho3], using Corollary 5.13. ☐

Remark 5.16. Using Theorem 5.15 and Proposition 5.7 we see that the operators $\mathfrak{Op}^A(\langle\xi\rangle)$, $\mathfrak{Op}_A(\langle\xi\rangle) = \mathfrak{Op}(\mu^A)$ and $\sqrt{(D-A)^2+1}$, are elliptic Weyl pseudodifferential operators of first order associated to the metric g^A and having the same principal symbol μ^A. Thus, all three define self-adjoint, lower semibounded operators in $L^2(\mathcal{X})$, having the same domain $\boldsymbol{H}_A^1(\mathcal{X})$ and differing only by bounded L^2 operators. Each one may be a candidate for a magnetic relativistic Schrödinger Hamiltonian. Nevertheless, the last one cannot be obtained by a complete 'quantization' procedure applying to a larger class of classical observables, while the second one (although used in [Ic1], [Ic2], [IT1], [IT2], [ITs1], [ITs2], [NU1], [NU2], etc.) is not covariant for the gauge transformations. Thus, we consider that the only adequate one should be the first one.

§6. The Limiting Absorption Principle

This section is devoted to the spectral analysis of operators of the form $\mathfrak{Op}(p)$, $\mathfrak{Op}(p\circ\nu^A)$ and $\mathfrak{Op}^A(p)$, for an elliptic symbol $p\in S^m(\Xi)$, and a limiting absorption principle for this type of operators is obtained. The main tool we shall use is an abstract result belonging to the 'conjugate operator method', (proved in [ABG]). We shall also make use of some known properties of the Weyl calculus ([Ho1], [Ho2]) and of the magnetic pseudodifferential calculus developed above. The following hypothesis will be assumed all over this section:

Hypothesis 6.1. There exists $\epsilon>0$ (that we can always suppose smaller then $n-1$), such that for any $\alpha\in\mathbb{N}^n$ there exists $C_\alpha>0$ for which $|(\partial^\alpha B_{jk})(x)|\le C_\alpha\langle x\rangle^{-1-\epsilon}$, for any $x\in\mathcal{X}$ and $j,k\in\{1,\dots,n\}$.

Concerning the vector potential A defining B, we shall suppose that it has been chosen to satisfy the conditions in Lemma 6.2 below.

Lemma 6.2. *Suppose that Hypothesis* 6.1 *is satisfied. Then*:

a) *there exists a vector potential A such that $B=dA$ and for any multiindex $\alpha\in\mathbb{N}^n$, $|(\partial^\alpha A_j)(x)|\le C_\alpha\langle x\rangle^{-\epsilon}$ for any $x\in\mathcal{X}$ and any j in $\{1,\dots,n\}$,*

b) *if $|\alpha|\ge 1$, then the above vector potential A also satisfies $|(\partial^\alpha A_j)(x)|\le C_\alpha\langle x\rangle^{-1-\epsilon}\ln(1+\langle x\rangle)$.*

Proof. We choose the Coulomb gauge

$$A_j(x):=\sum_{k=1}^n\int_{\mathcal{X}}dy\,(\partial_k E)(y)B_{kj}(x-y),$$

where E is the standard elementary solution of the Laplace operator on $\mathcal{X}$ and then we use the results in Section 8 of Chapter II in [Mi]. □

Before formulating the main result of this section let us make some remarks. Any vector potential verifying the conditions in the above Lemma 6.2 has the property $\partial^\alpha A_j\in L^\infty(\mathcal{X})$ for any $\alpha\in\mathbb{N}^n$ and any $1\le j\le n$. Thus we can apply the results of our previous Section 5. Moreover it is easy to verify that in the present situation, all the magnetic Sobolev spaces $\boldsymbol{H}^s_A(\mathcal{X}$ (defined in Section 3) coincide with the usual Sobolev spaces $H^s(\mathcal{X}) \equiv \boldsymbol{H}^s_0(\mathcal{X})$.

If $p \in S^m(\Xi)$ is a real elliptic symbol and $m > 0$, the operator $P := \mathfrak{Op}(p)$ is self-adjoint in $L^2(\mathcal{X})$, having the domain $H^m(\mathcal{X})$. We shall denote its form domain by $\mathcal{G} := \mathcal{D}(|P|^{1/2}) = H^{m/2}(\mathcal{X})$. Let us still denote by $\mathcal{G}_{s,p}$ and $\mathcal{G}^*_{s,p}$ ($s \in \mathbb{R}$ and $1 \le p \le \infty$) the spaces of the Besov scale associated to $\mathcal{G}$ and $\mathcal{G}^* \equiv H^{-m/2}(\mathcal{X})$ (see [ABG]). Let us finally remark that for any $z \in \mathbb{C}_\pm$ we have $(P - z)^{-1} \in \mathcal{B}(\mathcal{G}^*; \mathcal{G}) \subset \mathcal{B}(\mathcal{G}^*_{1/2,1}; \mathcal{G}_{-1/2,\infty})$.

We shall denote by g the metric $g_X := |dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2$ and by $M_{m,\delta}$ (for m and δ in $\mathbb{R}$) the weight function $M_{m,\delta}(X) := \langle x\rangle^{-\delta}\langle\xi\rangle^m$, for $X = (x, \xi) \in \Xi$.

Theorem 6.3. *Assume that the magnetic field B satisfies Hypothesis* 6.1. *Let $p \in S^m(\Xi)$, with $m > 0$, satisfying the conditions*:

i) *p is real valued and elliptic*;

ii) *there exists $p_0 \in S^m(\Xi)$ a real elliptic symbol depending only on the variable $\xi \in \mathcal{X}^*$, positive for $|\xi|$ large, and there exists $p_S \in S(M_{m,1+\epsilon}, g)$ and $p_L \in S(M_{m-1,\epsilon}, g)$, such that $p = p_0 + p_S + p_L$.*

Let H, H_0, respectively, the self-adjoint operators defined by $\mathfrak{Op}(p)$ and $\mathfrak{Op}(p_0)$ in $L^2(\mathcal{X})$, both having domain $\boldsymbol{H}^m(\mathcal{X})$. They have the following properties:

a) $\sigma_{ess}(H) = \sigma_{ess}(H_0) = \overline{p_0(\mathcal{X}^*)}$.

b) *The singular continuous spectrum of H* (*if it exists*) *is contained in the set of critical values of p_0 defined as $\Lambda(p_0) := \{p_0(\xi) \mid p_0'(\xi) = 0\}$.*

c) *The eigenvalues of H outside $\Lambda(p_0)$ have finite multiplicity and can accumulate only in $\Lambda(p_0)$.*

d) (*Limiting Absorption Principle*) *The holomorphic function $\mathbb{C}_\pm \ni z \mapsto (H - z)^{-1} \in \mathcal{B}(\mathcal{G}^*_{1/2,1}, \mathcal{G}_{-1/2,\infty})$ has a weak*-continuous extension to $\overline{\mathbb{C}_\pm} \setminus [\Lambda(p_0) \cup \sigma_p(H)]$.*

For the proof of this Theorem we shall need some auxiliary results.

Lemma 6.4. *Let Hypothesis* 6.1 *be verified. We consider the symbol $p \in S^m(\Xi)$.*

a) *There exists $q \in S(M_{m-1,\epsilon}, g)$ such that $\mathfrak{Op}^A(p) = \mathfrak{Op}(p) + \mathfrak{Op}(q)$.*

b) *If, moreover, the symbol* p *verifies for all* $\alpha \in \mathbb{N}^n$ *with* $|\alpha| \geq 1$, *and for all* $\beta \in \mathbb{N}^n$

$$\left|(\partial_x^\alpha \partial_\xi^\beta p)(x,\xi)\right| \leq C_{\alpha,\beta} \langle x\rangle^{-1} \ln(1+\langle x\rangle)\langle\xi\rangle^{m-|\beta|}, \tag{6.1}$$

then, for α *and* β *as above, we have* $\partial_x^\alpha \partial_\xi^\beta q \in S(M_\beta, g)$, *with*

$$M_\beta(x,\xi) := \langle x\rangle^{-1-\epsilon} \ln(1+\langle x\rangle)\langle\xi\rangle^{m-1-|\beta|}.$$

Proof. a) The stated equality is obtained by choosing:

$$q(x,\xi) = \iint_{\mathcal{X}\times\mathcal{X}^*} dt\, đ\eta\, e^{i\langle t,\eta\rangle} r\left(x+\frac{t}{2}, x-\frac{t}{2}, \eta+\xi\right)$$

with

$$r(x,y,\xi) := -\left\langle \Gamma^A(x,y), \int_0^1 d\tau\, (\partial_\xi p)\left(\frac{x+y}{2}, \xi - \tau\Gamma^A(x,y)\right)\right\rangle.$$

Using the estimation

$$|\partial^\alpha A(x+sz)| \leq C_\alpha \langle x\rangle^{-\epsilon}\langle z\rangle^{\epsilon}, \quad \forall x,z\in\mathcal{X}, \forall s\in[-1/2,1/2], \forall\alpha\in\mathbb{N}^n,$$

one verifies easily that $q \in S(M_{m-1,\epsilon}, g)$.

b) We can follow once again the proof of point (a), and remark that for $|\alpha| \geq 1$ and $s \in [-1/2, 1/2]$ we have

$$|\partial^\alpha A(x+sz)| \leq C'_\alpha \langle x\rangle^{-1-\epsilon} \ln(1+\langle x\rangle)\langle z\rangle^{1+\epsilon}\ln(1+\langle z\rangle).$$

The supplementary condition (6.1) is needed when we estimate the derivatives $\partial_x^\alpha \tilde{r}$ (in the differentiation of $\partial_\eta p$ with respect to the first argument). ☐

Lemma 6.5. *Under Hypothesis* 6.1, *the operator* $\mathfrak{Op}(q) : \mathbf{H}^m(\mathcal{X}) \to L^2(\mathcal{X})$ *defined in Lemma* 6.4(a) *is compact.*

Proof. For any $s \in \mathbb{R}$ the operator $\langle D\rangle^s \equiv \mathfrak{Op}(p_s) \in \mathbf{\Psi}(p_s, g)$, and the operators $\langle D\rangle^s$ and $\langle D\rangle^{-s}$ are one the inverse of the other. If we denote $\tilde{p}_t(x,\xi) := \langle x\rangle^t$, it follows that $\mathfrak{Op}(q)\langle D\rangle^{-m} \in \mathbf{\Psi}(\tilde{p}_{-\epsilon}p_{-1}, g)$, we also have $\lim_{|x|+|\xi|\to\infty} \{\langle x\rangle^{-\epsilon}\langle\xi\rangle^{-1}\} = 0$ and the Theorem 18.6.6 in [Ho2] implies that $\mathfrak{Op}(q)\langle D\rangle^{-m}$ is compact. Thus $\mathfrak{Op}(q) = \left(\mathfrak{Op}(q)\langle D\rangle^{-m}\right)\langle D\rangle^m$ is compact as an operator from $\mathbf{H}^m(\mathcal{X})$ to $L^2(\mathcal{X})$. ☐

Proposition 6.6. *Under Hypothesis* 6.1*, if* $p \in S^m(\Xi)$ *is real and elliptic with* $m > 0$*, it follows that* $\mathfrak{Op}(q)$ (*defined in Lemma* 6.4(a)) *is a relatively compact perturbation of* $\mathfrak{Op}(p)$*. In particular* $\sigma_{ess}[\mathfrak{Op}^A(p)] = \sigma_{ess}[\mathfrak{Op}(p)]$*. Moreover, if* $p(x,\xi) = p(\xi)$*, then* $\sigma_{ess}[\mathfrak{Op}^A(p)] = \sigma_{ess}[\mathfrak{Op}(p)] = \overline{p(\mathcal{X}^*)}$ *and, if* $\lim\limits_{|\xi|\to\infty} p(\xi) = \infty$*, then* $\sigma_{ess}[\mathfrak{Op}^A(p)] = \sigma_{ess}[\mathfrak{Op}(p)] = [\gamma, \infty)$*, with* $\gamma := \inf\limits_{\xi\in\mathcal{X}^*} p(\xi)$*.*

Proof. Due to Theorem 4.1 the operator $\mathfrak{Op}^A(p)$ is self-adjoint on the domain $\mathbf{H}^m(\mathcal{X})$, and the same is true for $\mathfrak{Op}(p)$. Lemma 6.5 above implies that $\mathfrak{Op}(q)$ is a relatively compact perturbation of $\mathfrak{Op}(p)$. Then, if $p(x,\xi) = p(\xi)$ we see that $\mathfrak{Op}(p)$ is unitarily equivalent to the operator of multiplication with the function p in $L^2(\mathcal{X}^*)$ and thus we have the second equality. For the last one just remark that $|p(\xi) \geq C|\xi|^m$ for $|\xi| \geq R$, so that p will have constant sign for $|\xi| \geq R$. ☐

Lemma 6.7. *Let* $m \in \mathbb{R}$ *and* $p \in S^m(\Xi)$ *be given such that* $p(x,\xi) = p(\xi)$*; let also* θ_r *be a function in* $C^\infty(\mathcal{X})$ *depending on the parameter* $r \geq 1$ *and such that for any* $\alpha \in \mathbb{N}^n$ *with* $|\alpha| \geq 1$ *satisfies* $|(\partial_x^\alpha \theta_r)(x)| \leq C_\alpha r^{-1}$*,* $\forall x \in \mathbb{R}^n$*,* $\forall r \geq 1$*. Then* $r(p \circ \theta_r - \theta_r \circ p) \in S^{m-1}(\mathbb{R}^n)$ *uniformly in* $r \geq 1$*.*

Proof. For any $u \in \mathcal{S}(\mathcal{X})$

$$\begin{aligned}\{[\mathfrak{Op}(p), \mathfrak{Op}(\theta_r)]\, u\}(x) &= \iint_{\mathcal{X}\times\mathcal{X}^*} dy\, đ\xi\; e^{i\langle x-y,\xi\rangle} p(\xi)\, [\theta_r(y) - \theta_r(x)]\, u(y) \\ &= \sum_{j=1}^{n} \iint_{\mathcal{X}\times\mathcal{X}^*} dy\, đ\xi\; e^{i\langle x-y,\xi\rangle} p_j(\xi) \lambda_j(x,y,r) u(y),\end{aligned}$$

where $p_j(\xi) = (D_j p)(\xi) \in S^{m-1}(\Xi)$, $\lambda_j(x,y,r) = \int\limits_0^1 ds \left(\partial_{x_j}\theta_r\right)(sx + (1-s)y)$, and for any $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^n$, we have $r\partial_x^\alpha \partial_y^\beta \lambda_j \in L^\infty(\Xi \times [1,\infty))$. Thus we have the formula $[\mathfrak{Op}(p), \mathfrak{Op}(\theta_r)] = r^{-1} \sum\limits_{j=1}^{n} \mathfrak{Op}(q_j)$ with

$$q_j(x,\xi,r) := \iint_{\mathcal{X}\times\mathcal{X}^*} dy\, đ\eta\; e^{i\langle y,\eta\rangle} p_j(\xi+\eta) \mu_j(x,y,r)$$

and $\mu_j(x,y,r) := r\lambda_j(x+y/2, x-y/2, r)$. For N_1 and N_2 integers large enough

$$\begin{aligned}&\left|\left(\partial_x^\alpha \partial_\xi^\beta q_j\right)(x,\xi,r)\right| \\ &\leq C_{\alpha\beta} \langle\xi\rangle^{m-1-|\beta|} \iint_{\mathcal{X}\times\mathcal{X}^*} dy\; đ\eta\; \langle y\rangle^{-2N_1} \langle\eta\rangle^{-2N_2+|m-1-|\beta||} \\ &\leq C'_{\alpha\beta} \langle\xi\rangle^{m-1-|\beta|}.\end{aligned}$$

□

Lemma 6.8. *Let* $p \in S(M_{m,\delta}, g)$ *and* $\delta > 0$. *Then there exists* $q \in S^m(\Xi)$ *such that* $\mathfrak{Op}^A(p) = \langle Q\rangle^{-\delta}\mathfrak{Op}(q)$.

Proof. The distribution kernel of the operator $\langle Q\rangle^{\delta}\mathfrak{Op}^A(p)$ is given by

$$K(x,y) := \int_{\mathcal{X}^*} \text{đ}\eta \; e^{i\langle x-y,\eta\rangle}\langle x\rangle^{\delta} p\left(\frac{x+y}{2}, \eta - \Gamma^A(x,y)\right).$$

Thus we can write $\langle Q\rangle^{\delta}\mathfrak{Op}^A(p) = \mathfrak{Op}(q)$ with $q(x,\xi) :=$

$$\begin{aligned} &= \iint_{\mathcal{X}\times\mathcal{X}^*} dy\text{đ}\eta \; e^{i\langle y,\eta-\xi\rangle}\langle x+y/2\rangle^{\delta} p(x, \eta - \Gamma^A(x+y/2, x-y/2)) \\ &= \iint_{\mathcal{X}\times\mathcal{X}^*} dy\, \text{đ}\eta \; e^{i\langle y,\eta\rangle}\langle x+y/2\rangle^{\delta} p(x, \xi+\eta - F(x,y)), \end{aligned}$$

with $F \in BC^{\infty}(\mathcal{X}\times\mathcal{X})$. For N_1 and N_2 large enough we get

$$\begin{aligned} &\left|\left(\partial_x^{\alpha}\partial_{\xi}^{\beta} q\right)(x,\xi)\right| \\ &\quad\le C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}\iint_{\mathcal{X}\times\mathcal{X}^*} dy\, \text{đ}\eta\, \langle y\rangle^{-2N_1+\delta}\langle\eta\rangle^{-2N_2+|m-|\beta||} \\ &\quad\le C'_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}. \end{aligned}$$

□

Proof of Theorem 6.3. We are going to verify the hypothesis of Theorem 7.6.8. in [ABG], that directly implies the conclusion of our theorem; for the second equality in (a) we take into account Proposition 6.6.

1. *The difference* $(H+i)^{-1} - (H_0+i)^{-1}$ *is a compact operator in* $L^2(\mathcal{X})$.
We have $\mathfrak{Op}(p) = \mathfrak{Op}(p_0) + \mathfrak{Op}(p_S) + \mathfrak{Op}(p_L)$. Thus we can write

$$(H+i)^{-1} - (H_0+i)^{-1} = -(H+i)^{-1}\{\mathfrak{Op}(p_S) + \mathfrak{Op}(p_L)\}(H_0+i)^{-1}.$$

From the definition we see that $(p_S + p_L)(x,\xi) = \langle x\rangle^{-\epsilon} r$ with $r \in S^m(\Xi)$. We remark that $(H_0+i)^{-1}$ is in $\mathcal{B}\left[L^2(\mathcal{X}), \boldsymbol{H}^m(\mathcal{X})\right]$, $\mathfrak{Op}(r)$ is in $\mathcal{B}\left[\boldsymbol{H}^m(\mathcal{X}), L^2(\mathcal{X})\right]$, $(H+i)^{-1}$ is in $\mathcal{B}\left[\boldsymbol{H}^{-m}(\mathcal{X}), L^2(\mathcal{X})\right]$ and $\langle Q\rangle^{-\epsilon}$ is a compact operator in $\mathcal{B}(L^2 \times(\mathcal{X}), \boldsymbol{H}^{-m}(\mathcal{X}))$, so that the difference of the resolvents is compact.

2. *For any* $\rho \in C_0^{\infty}(\mathcal{X})$ *with* $\rho(x) = 0$ *for* x *in a neighborhood of* $0 \in \mathcal{X}$, *setting* $\rho_r(x) := \rho(x/r)$, *we have* $\int_1^{\infty} dr\ \|\rho_r(Q)\mathfrak{Op}(p_S)\|_{\mathcal{B}(\mathcal{G},\mathcal{G}^*)} < \infty$.
By construction we have $p_S(x,\xi) = \langle x\rangle^{-1-\epsilon} q_0(x,\xi)$ with $q_0 \in S^m(\Xi)$. Let us

denote $\widetilde{\rho_r}(x) := \rho\left(\frac{x}{r}\right)\left(\frac{r}{\langle x\rangle}\right)^{1+\epsilon}$. Thus $\rho_r(Q)\mathfrak{Op}(p_S) = r^{-(1+\epsilon)}\widetilde{\rho_r}(Q)\mathfrak{Op}(q_0)$, and observing that

$$\|\widetilde{\rho_r}(Q)\mathfrak{Op}(q_0)\|_{\mathcal{B}(\mathcal{G},\mathcal{G}^*)} \leq \left\|\langle D\rangle^{-m/2}\widetilde{\rho_r}(Q)\mathfrak{Op}(q_0)\langle D\rangle^{-m/2}\right\|_{\mathcal{B}(L^2(\mathcal{X}))}$$

it will be enough to prove that

$$\sup_{r\geq 1}\left\|\langle D\rangle^{-m/2}\widetilde{\rho_r}(Q)\mathfrak{Op}(q_0)\langle D\rangle^{-m/2}\right\|_{\mathcal{B}(L^2(\mathcal{X}))} < \infty.$$

Remarking that the function $\widetilde{\rho_r}$ satisfies the hypothesis of Lemma 6.7 we conclude that $r\left\{\mathfrak{p}_{-m/2}\circ\widetilde{\rho_r} - \widetilde{\rho_r}\circ\mathfrak{p}_{-m/2}\right\} \in S^{-(1+m/2)}(\Xi)$, uniformly in $r \geq 1$. But

$$\begin{aligned}&\langle D\rangle^{-m/2}\widetilde{\rho_r}(Q)\mathfrak{Op}(q_0)\langle D\rangle^{-m/2}\\&\quad= \widetilde{\rho_r}(Q)\langle D\rangle^{-m/2}\mathfrak{Op}(q_0)\langle D\rangle^{-m/2}\\&\qquad+\left[\langle D\rangle^{-m/2},\widetilde{\rho_r}(Q)\right]\mathfrak{Op}(q_0)\langle D\rangle^{-m/2}\end{aligned}$$

and by the previous remark the second term above is a Weyl operator of order -1, uniformly for $r \geq 1$. The first term of the above sum is a Weyl operator of order 0, thus defining a bounded operator, uniformly in $r \geq 1$.

3. *For any function $\theta \in C^\infty(\mathcal{X})$ with $\theta(x) = 0$ on a neighborhood of $0 \in \mathcal{X}$ and $\theta(x) = 1$ in a neighbourhood of infinity, we have for any $j = 1, \cdots, n$, $\int_1^\infty \frac{dr}{r}\ \|\theta_r(Q)\left[Q_j, \mathfrak{Op}(p_L)\right]\|_{\mathcal{B}(\mathcal{G},\mathcal{G}^*)} < \infty$, for $\theta_r(x) = \theta(x/r)$.*

In order to prove this estimation we notice that $[Q_j, \mathfrak{Op}(p_L)] = -\mathfrak{Op}(D_{\xi_j}p_L)$ and using Lemmas 6.4 and 6.8 we also have $(D_{\xi_j}p_L)(x,\xi) = \langle x\rangle^{-\epsilon}q_1$ with $q_1 \in S^{m-2}(\Xi)$. Thus $-\theta_r(x)\left(D_{\xi_j}p_L\right)(x,\xi) = r^{-\epsilon}\varphi_r(x)q_1(x,\xi)$ with $\varphi_r(x) := \theta(x/r)(r/\langle x\rangle)^\epsilon$. Thus it will be enough to prove that

$$\sup_{r\geq 1}\left\|\langle D\rangle^{-m/2}\varphi_r(Q)\mathfrak{Op}(q_1)\langle D\rangle^{-m/2}\right\|_{\mathcal{B}(L^2(\mathcal{X}))} < \infty,$$

and this follows by the same argument as in step (2) due to the fact that the function φ_r also verifies the hypothesis of Lemma 6.7.

4. *For any test function $\theta \in C^\infty(\mathcal{X})$ with $\theta(x) = 0$ on a neighborhood of $0 \in \mathcal{X}$ and $\theta(x) = 1$ in a neighbourhood of infinity, we have for any $j = 1, \cdots, n$, $\int_1^\infty \frac{dr}{r}\ \|\theta_r(Q)\langle Q\rangle\left[D_j, \mathfrak{Op}(p_L)\right]\|_{\mathcal{B}(\mathcal{G},\mathcal{G}^*)} < \infty$, for $\theta_r(x) = \theta(x/r)$.*

We start from the equality $[D_j, \mathfrak{Op}(p_L)] = \mathfrak{Op}(D_{x_j}p_L)$ and, using Lemmas 6.4 and 6.8, we see that $\forall\epsilon' \in (0,\epsilon)$, $(D_{x_j}p_L)(x,\xi) = \langle x\rangle^{-(1+\epsilon')}q_2$ with $q_2 \in S^{m-1}(\Xi)$. Thus $\theta_r(x)\langle x\rangle\left(D_{x_j}p_L\right)(x,\xi) = r^{-\epsilon'}\varphi_r(x)q_2(x,\xi)$, with $\varphi_r(x) := \theta(x/r)(r/\langle x\rangle)^{\epsilon'}$ and everything goes on as before. ☐

For t and s in $\mathbb{R}$ let us denote by $\mathcal{H}_t^s$ the usual weighted Sobolev spaces, i.e. $\mathcal{H}_t^s = \{u \in \mathcal{S}^*(\mathcal{X}) \mid \langle D\rangle^s \langle Q\rangle^t u \in L^2(\mathcal{X})\}$. We notice that $L_t^2 \equiv \mathcal{H}_t^0$ and $\mathcal{G} = \mathcal{H}_0^{m/2}$. As shown in [BGS], for $\delta > 0$ and $\gamma > \delta + 1/2$, we have the continuous embedings $L_\gamma^2 \subset \mathcal{H}_{\delta+1/2}^{-m/2} \subset \mathcal{G}^*_{1/2,1}$, the first being also compact. By duality we get the continuous embedings $\mathcal{G}_{-1/2,\infty} \subset \mathcal{H}_{-\delta-1/2}^{m/2} \subset L_{-\gamma}^2$, the last one being compact. One easily gets from the point (d) of our Theorem 6.3 that the limiting absorption principle is valid in $\mathcal{B}(L_\gamma^2; L_{-\gamma}^2)$ for the uniform topology, for any $\gamma > 1/2$.

Remark 6.9. The limiting absorption principle is valid in $\mathcal{B}(\mathcal{H}_\gamma^{-m/2}; \mathcal{H}_{-\gamma}^{m/2})$ for the uniform topology, for any $\gamma > 1/2$ (we may evidently suppose $\gamma \le 1$).

To prove this fact we start with the following identity for $z \in \mathbb{C}_\pm$, consequence of the resolvent equation:

$$(H-z)^{-1} = (H-i)^{-1} + (z-i)(H-i)^{-2} + (z-i)^2 (H-i)^{-1}(H-z)^{-1}(H-i)^{-1}.$$

As $(H-i)^{-1} \in \mathcal{B}(\mathcal{H}_0^{-m/2}; \mathcal{H}_0^{m/2})$, the desired result will follow if we prove that $(H-i)^{-1} \in \mathcal{B}(\mathcal{H}_\gamma^{-m/2}; \mathcal{H}_\gamma^{m/2})$ for any $\gamma \in [-1,1]$. In order to verify this relation, one may proceed as in the proof of our Lemma 6.7, and show that for any function $\varphi \in C^\infty(\mathcal{X})$ with $\partial^\alpha \varphi \in L^\infty(\mathcal{X})$ for $|\alpha| \ge 1$, the commutator $[\mathfrak{Op}(p), \varphi(Q)]$ is a Weyl operator with symbol of class $S^{m-1}(\Xi)$. It follows that for any $u \in \mathcal{S}(\mathcal{X})$

$$\varphi(Q)(H-i)^{-1}u = (H-i)^{-1}(\varphi u) + Tu,$$

with $T \in \mathcal{B}(\mathcal{H}_0^{-m/2}; \mathcal{H}_0^{m/2})$. This equality may then be extended to those elements of $\mathcal{H}_0^{-m/2}$ which verify $\varphi u \in \mathcal{H}_0^{-m/2}$. Choosing $\varphi(x) := \langle x\rangle^\gamma$ with $0 \le \gamma \le 1$, we deduce that $(H-i)^{-1} \in \mathcal{B}(\mathcal{H}_\gamma^{-m/2}; \mathcal{H}_\gamma^{m/2})$ for any $\gamma \in [0,1]$. By duality we obtain the same statement for $\gamma \in [-1,0]$.

Remark 6.10. The conclusion of Theorem 6.3 remains true if we replace the operator $H = \mathfrak{Op}(p)$ with $H' = \mathfrak{Op}^A(p)$.

In fact, from point (a) of Lemma 6.4 we deduce the existence of a symbol $q \in S(M_{m-1,\epsilon}, g)$ such that $\mathfrak{Op}^A(p) = \mathfrak{Op}(p) + \mathfrak{Op}(q) = \mathfrak{Op}(p')$, where $p' = p_0 + p_S + p'_L$ and $p'_L = p_L + q \in S(M_{m-1,\epsilon}, g)$. Since $p' - p \in S(M_{m-1,\epsilon}, g)$, we conclude that p' is elliptic. As $\mathfrak{Op}(p') = \mathfrak{Op}^A(p)$ is symmetric, we deduce that p' is real.

Remark 6.11. The conclusion of Theorem 6.3 remains true if we replace the operator $H = \mathfrak{Op}(p)$ with $H'' = \mathfrak{Op}(p \circ \nu^A)$.

In order to prove this statement, we start from the equality

$$\left[\mathfrak{Op}(p\circ\nu^A)-\mathfrak{Op}(p)\right]u(x)=\iint_{\mathcal{X}\times\mathcal{X}^*}dy\,đ\xi\; e^{i\langle x-y,\xi\rangle}r(x,y,\xi)u(y)$$

for any $u\in\mathcal{S}(\mathcal{X})$, where

$$r(x,y,\xi)=-\int_0^1 d\tau\,\langle A((x+y)/2,(\partial_\xi p)((x+y)/2,\xi-\tau A(x+y)/2))\rangle.$$

Repeating the proof of Lemma 6.4(a) with $\Gamma^A(x,y)$ replaced by $A((x+y)/2)$, we conclude that there exists $q\in S(M_{m-1,\epsilon},g)$ such that $\mathfrak{Op}(p\circ\nu^A)=\mathfrak{Op}(p)+\mathfrak{Op}(q)$ and we are once again in the situation of the previous Remark 6.10.

Example 1. The magnetic relativistic Schrödinger Hamiltonian $\mathfrak{Op}^A(\langle\xi\rangle)$. We consider the situation of Remark 6.10 with $p(\xi)=p_0(\xi)=\langle\xi\rangle$, $p_S=p_L=0$. In this case we have $\overline{p_0(\mathcal{X}^*)}=[1,\infty)$ and $\Lambda(p_0)=\{1\}$. Thus this operator has no singular continuous spectrum.

Example 2. The operator $\mathfrak{Op}(\mu^A)$. We recall that $\mu^A(x,\xi)=\langle\xi-A(x)\rangle$. We are now in the situation of Remark 6.11, with $p(\xi)=p_0(\xi)=\langle\xi\rangle$. T. Umeda ([Um]) has applied the Enss method to this operator obtaining properties (a), (b) and (c) from our Theorem 6.3, but not a limiting absorption principle. Besides, the hypothesis in [Um] are less general then ours: he imposes restrictions on the vector potential A of the form $|\partial^\alpha A_j(x)|\leq C_\alpha\langle x\rangle^{-1-\epsilon}$ for any $\alpha\in\mathbb{N}^n$ (for $1\leq j\leq n$ and $x\in\mathcal{X}$) and $\epsilon>0$. We are making hypothesis only on the magnetic field B and the only properties of A that we use in the proof of Theorem 6.3 are those deduced in Lemma 6.2 from our Hypothesis 6.1.

An interesting result has been obtained by T. Ichinose and H. Tamura [IT2], showing that under very general hypothesis we have $\mathfrak{Op}(\mu^A)\geq 1$. Thus under our hypothesis one has $\sigma(\mathfrak{Op}(\mu^A))=[1,\infty)$.

Example 3. The Remarks 6.10 and 6.11 may also be applied to the Schrödinger operator $H=(D-A)^2$, taking $p(\xi)=p_0(\xi)=|\xi|^2$, $p_S=p_L=0$. In this case Theorem 6.3 does not bring anything new (the situation may be understood from the one without magnetic field), this type of results being known for much more general (singular) magnetic filds of the "short-range" type (see [BMP]). This situation is a consquence of the fact that there exist magnetic fields which verify our Hypothesis 6.1 with $\epsilon\leq 0$ and such that $(D-A)^2$ has dense pure spectrum in an interval of $\mathbb{R}$ (see [CFKS]), and thus Theorem 6.3 clearly may not be applied.

Acknowledgements

We thank the University of Geneva, where a large part of this work was done and especially Werner Amrein for his kind hospitality. We also acknowledge partial support from the CERES Programme (contract no. 4-187/2004).

References

[ABG] W. O. Amrein, A. Boutet de Monvel and V. Georgescu, *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians*, Progr. Math., **135**, Birkhäuser, Basel, 1996.

[Be] R. Beals, A general calculus of pseudodifferential operators, Duke Math. J. **42** (1975), 1–42.

[B] J. Bellissard, Lipshitz continuity of gap boundaries for Hofstadter-like spectra, Comm. Math. Phys. **160** (1994), no. 3, 599–613.

[BGS] A. Boutet de Monvel-Berthier, V. Georgescu and A. Soffer, N-body Hamiltonians with hard-core interactions, Rev. Math. Phys. **6** (1994), no. 4, 515–596.

[BMP] A. Boutet de Monvel and R. Purice, Limiting absorption principle for Schrödinger Hamiltonians with magnetic fields, Comm. Partial Differential Equations **19** (1994), no. 1-2, 89–117.

[BM] L. Boutet de Monvel, Hypoelliptic operators with double characteristics and related pseudo-differential operators, Comm. Pure Appl. Math. **27** (1974), 585–639.

[BMGH] L. Boutet de Monvel, A. Grigis and B. Helffer, Parametrixes d'opérateurs pseudo-différentiels à caractéristiques multiples, in *Journées: Équations aux Dérivées Partielles de Rennes (1975)*, 93–121. Astérisque, 34-35, Soc. Math. France, Paris.

[BNPW] M. Bordemann, N. Neumaier, M. J. Pflaum and S. Waldmann, On representations of star product algebras over cotangent spaces on Hermitian line bundles, J. Funct. Anal. **199** (2003), no. 1, 1–47.

[Bo1] J.-M. Bony, Caractérisations des opérateurs pseudo-différentiels, in *Séminaire sur les Équations aux Dérivées Partielles, 1996–1997*, Exp. XXIII, 17 pp, École Polytech., Palaiseau.

[Bo2] ———, Fourier integral operators and Weyl-Hörmander calculus, in *New trends in microlocal analysis (Tokyo, 1995)*, 3–21, Springer, Tokyo.

[Bo3] ———, Unpublished graduate course 1997–1998.

[CFKS] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*, Springer Study Edition, Springer, Berlin, 1987.

[DR] M. Dimassi and G. D. Raikov, Spectral asymptotics for quantum Hamiltonians in strong magnetic fields, Cubo Mat. Educ. **3** (2001), no. 2, 317–391.

[DS] M. Dimassi and J. Sjöstrand, *Spectral asymptotics in the semi-classical limit*, Cambridge Univ. Press, Cambridge, 1999.

[Fo] G. B. Folland, *Harmonic analysis in phase space*, Ann. of Math. Stud., **122**, Princeton Univ. Press, Princeton, NJ, 1989.

[GMS] C. Gérard, A. Martinez and J. Sjöstrand, A mathematical approach to the effective Hamiltonian in perturbed periodic problems, Comm. Math. Phys. **142** (1991), no. 2, 217–244.

[HS1] B. Helffer and J. Sjöstrand, Équation de Schrödinger avec champ magnétique et équation de Harper, in *Schrödinger operators (Sønderborg, 1988)*, 118–197, Lecture Notes in Phys., 345, Springer, Berlin.

[HS2] ———, On diamagnetism and de Haas-van Alphen effect, Ann. Inst. H. Poincaré Phys. Théor. **52** (1990), no. 4, 303–375.

- [HH] R. Hempel and I. Herbst, Strong magnetic fields, Dirichlet boundaries, and spectral gaps, Comm. Math. Phys. **169** (1995), no. 2, 237–259.
- [Ho] L. Hörmander, Fourier integral operators. I, Acta Math. **127** (1971), no. 1-2, 79–183.
- [Ho1] ______, The Weyl calculus of pseudodifferential operators, Comm. Pure Appl. Math. **32** (1979), no. 3, 360–444.
- [Ho2] ______, *The analysis of linear partial differential operators. III*, Springer, Berlin, 1985.
- [Ho3] ______, *The analysis of linear partial differential operators. IV*, Springer, Berlin, 1985.
- [Ic1] T. Ichinose, The nonrelativistic limit problem for a relativistic spinless particle in an electromagnetic field, J. Funct. Anal. **73** (1987), no. 2, 233–257.
- [Ic2] ______, Essential selfadjointness of the Weyl quantized relativistic Hamiltonian, Ann. Inst. H. Poincaré Phys. Théor. **51** (1989), no. 3, 265–297.
- [II] T. Ichinose and W. Ichinose, On the essential self-adjointness of the relativistic Hamiltonian with a negative scalar potential, Rev. Math. Phys. **7** (1995), no. 5, 709–721.
- [IT1] T. Ichinose and H. Tamura, Path integral for the Weyl quantized relativistic Hamiltonian, Proc. Japan Acad. Ser. A Math. Sci. **62** (1986), no. 3, 91–93.
- [IT2] ______, Imaginary-time path integral for a relativistic spinless particle in an electromagnetic field, Comm. Math. Phys. **105** (1986), no. 2, 239–257.
- [ITs1] T. Ichinose and T. Tsuchida, On Kato's inequality for the Weyl quantized relativistic Hamiltonian, Manuscripta Math. **76** (1992), no. 3-4, 269–280.
- [ITs2] ______, On essential selfadjointness of the Weyl quantized relativistic Hamiltonian, Forum Math. **5** (1993), no. 6, 539–559.
- [If] V. Iftimie, Uniqueness and existence of the integrated density of states for Schrödinger operators with magnetic field and electric potential with singular negative part, Publ. Res. Inst. Math. Sci. **41** (2005), no. 2, 307–327.
- [IMP] V. Iftimie, M. Măntoiu and R. Purice, *Magnetic Pseudodifferential Operators*, preprint arXiv math/0510492.
- [KO1] M. V. Karasev and T. A. Osborn, Symplectic areas, quantization, and dynamics in electromagnetic fields, J. Math. Phys. **43** (2002), no. 2, 756–788.
- [KO2] ______, Quantum magnetic algebra and magnetic curvature, J. Phys. A **37** (2004), no. 6, 2345–2363.
- [MaR] J. E. Marsden and T. S. Ratiu, *Introduction to mechanics and symmetry*, Second edition, Springer, New York, 1999.
- [Mi] S. G. Mikhlin, *Multidimensional singular integrals and integral equations*, Pergamon, Oxford, 1965.
- [MP1] M. Măntoiu and R. Purice, The algebra of observables in a magnetic field, in *Mathematical results in quantum mechanics (Taxco, 2001)*, 239–245, Contemp. Math., **307**, Amer. Math. Soc., Providence, RI.
- [MP2] ______, The magnetic Weyl calculus, J. Math. Phys. **45** (2004), no. 4, 1394–1417.
- [MP3] ______, Strict deformation quantization for a particle in a magnetic field, J. Math. Phys. **46** (2005), no. 5, 052105, 15 pp.
- [MP4] ______, The mathematical formalism of a particle in a magnetic field, in *Mathematical physics of quantum mechanics*, 417–434, Lecture Notes in Phys., **690**, Springer, Berlin.
- [MPR1] M. Măntoiu, R. Purice and S. Richard, Twisted crossed products and magnetic pseudodifferential operators, in *Advances in operator algebras and mathematical physics*, 137–172, Theta, Bucharest.
- [MPR2] ______, *Spectral and propagation results for Schrödinger magnetic operators*, to appear in J. Funct. Anal. DOI 10.1016/j.jfa.2007.05.020.
- [NU1] M. Nagase and T. Umeda, Weyl quantized Hamiltonians of relativistic spinless particles in magnetic fields, J. Funct. Anal. **92** (1990), no. 1, 136–154.
- [NU2] ______, Spectra of relativistic Schrödinger operators with magnetic vector potentials, Osaka J. Math. **30** (1993), no. 4, 839–853.
- [N] G. Nenciu, Dynamics of band electrons in electric and magnetic fields: rigorous justification of the effective Hamiltonians, Rev. Mod. Phys. **63** (1991), 91–127.
- [PST] G. Panati, H. Spohn and S. Teufel, Effective dynamics for Bloch electrons: Peierls

substitution and beyond, Comm. Math. Phys. **242** (2003), no. 3, 547–578.

[Pa] M. Pascu, On the essential spectrum of the relativistic magnetic Schrödinger operator, Osaka J. Math. **39** (2002), no. 4, 963–978.

[Sh] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Springer, Berlin, 1987.

[T] S. Teufel, *Adiabatic perturbation theory in quantum dynamics*, Lecture Notes in Math., **1821**, Springer, Berlin, 2003.

[Um] T. Umeda, Absolutely continuous spectra of relativistic Schrödinger operators with magnetic vector potentials, Proc. Japan Acad. Ser. A Math. Sci. **70** (1994), no. 9, 290–291.