Publ. RIMS, Kyoto Univ.
43 (2007), 625–659

An Existence Theorem for Tempered Solutions of $\mathcal{D}$-Modules on Complex Curves

By

Giovanni Morando*

Contents

Communicated by M. Kashiwara. Received June 20, 2006. Revised November 8, 2006.
2000 Mathematics Subject Classification(s): Primary 32S40; Secondary 34M35; 32B20; 58J15.
Key words: $\mathcal{D}$-modules, Ordinary differential equations, Tempered holomorphic, Subanalytic.
Research supported in part by grant CPDA061823 of Padova University.
*Dipartimento di Matematica Pura e Applicata, Università di Padova, Via Trieste, 63, Padova 35121, Italy.
Institut de Mathématiques de Jussieu, Université Paris 6, 175 rue du Chevaleret, 75013 Paris, France.

Abstract

Let X be a complex curve, X_{sa} the subanalytic site associated to X, $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module. Let $\mathcal{O}^t_{X_{sa}}$ be the sheaf on X_{sa} of tempered holomorphic functions and $\mathscr{S}ol(\mathcal{M})$ (resp. $\mathscr{S}ol^t(\mathcal{M})$) the complex of holomorphic (resp. tempered holomorphic) solutions of $\mathcal{M}$. We prove that the natural morphism

$$H^1(\mathscr{S}ol^t(\mathcal{M})) \longrightarrow H^1(\mathscr{S}ol(\mathcal{M}))$$

is an isomorphism. As a consequence, we prove that $\mathscr{S}ol^t(\mathcal{M})$ is $\mathbb{R}$-constructible in the sense of sheaves on X_{sa}. Such a result is conjectured by M. Kashiwara and P. Schapira in [15] in any dimension.

Introduction

The problem of existence for ordinary linear differential equations (and even non-linear) is classical and the litterature presents many results on this subject. In particular, existence theorems for solutions with growth conditions have been obtained by many authors such as J.-P. Ramis and Y. Sibuya ([21]), B. Malgrange ([20]) and N. Honda ([7]). In [21] and [20], the authors proved existence for functions with Gevrey-type growth conditions at the origin on sectors of sufficiently small amplitude. Using similar techniques, in [7], the author proved existence for ultra-distributions with support on $\mathbb{R}_{\geq 0}$.

The functional spaces considered in [21] and [20] correspond to sheaves on the real blow-up at the origin of $\mathbb{C}$. Essentially they are sheaves on the unit circle. Indeed, growth conditions did not allow a global sheaf theoretical approach.

Nonetheless, tempered distributions were a basic tool in M. Kashiwara's functorial proof of the Riemann-Hilbert correspondence in [9] and [10]. In order to use tempered distributions functorially, M. Kashiwara introduced the new functor $T\mathcal{H}om$ of tempered cohomology. Such a functor represented the first step in a different approach to sheaves which, through [13], led to the full use of sheaves on sites in [14]. Indeed in [14], M. Kashiwara and P. Schapira combined classical analytical results of S. Łojasiewicz ([16], see also [19]) with sheaves on sites. They realized tempered distributions, tempered $\mathcal{C}^\infty$ functions and Whitney $\mathcal{C}^\infty$ functions as sheaves on the subanalytic site. They also defined tempered holomorphic functions $\mathcal{O}^t_{X_{sa}}$ as the complex of the solutions of the Cauchy-Riemann system in the space of tempered distributions.

In a subsequent paper, [15], M. Kashiwara and P. Schapira extended the notion of microsupport of sheaves to the subanalytic site. In this way they established the framework for the study of tempered holomorphic solutions of $\mathcal{D}$-modules. They also gave an example which is the starting point of the study of tempered holomorphic solutions of an irregular ordinary differential equation.

Given a complex analytic manifold X, we denote by $\mathrm{Op}^c_{X_{sa}}$ the category of relatively compact subanalytic open subsets of X and by X_{sa} the subanalytic site, that is the site whose underlying category is $\mathrm{Op}^c_{X_{sa}}$ and whose coverings are the finite coverings. We denote by $\mathrm{Mod}(k_X)$ (resp. $\mathrm{Mod}(k_{X_{sa}})$) the category of sheaves of k-modules on the site X (resp. X_{sa}). Let $\varrho : X \longrightarrow X_{sa}$ be the natural morphism of sites.

Given a $\mathcal{D}_X$-module $\mathcal{M}$, it is natural to compare

$$\mathscr{S}ol\mathcal{M} := R\varrho_* \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}, \mathcal{O}_X\big)$$

and

$$\mathscr{S}ol^t\mathcal{M} := \mathrm{R}\mathcal{H}om_{\varrho_!\mathcal{D}_X}\big(\varrho_!\mathcal{M}, \mathcal{O}^t_{X_{sa}}\big)$$

(for the definition of $\varrho_!$, see Section 2).

Along his proof of the Riemann-Hilbert correspondence, M. Kashiwara proved that, if $\mathcal{M}$ is a regular $\mathcal{D}_X$-module, then $\mathscr{S}ol^t\mathcal{M} \xrightarrow{\sim} \mathscr{S}ol\mathcal{M}$.

In [15], the authors studied $\mathscr{S}ol^t\mathcal{M}$ comparing it to $\mathscr{S}ol\mathcal{M}$, for a particular example on a complex curve X.

In the present paper, we go into the study of $\mathscr{S}ol^t(\mathcal{M})$ for $\mathcal{M}$ a holonomic $\mathcal{D}$-module on a complex curve X. In particular we prove an existence theorem for tempered solutions of ordinary differential equations in the subanalytic topology, thus refining the classical results on small open sectors. Such a result has two consequences.

First, we obtain that the natural morphism

$$H^1(\mathscr{S}ol^t\mathcal{M}) \longrightarrow H^1(\mathscr{S}ol\mathcal{M}) \tag{0.0.1}$$

is an isomorphism.

Second, we prove that the complex $\mathscr{S}ol^t(\mathcal{M})$ is $\mathbb{R}$-constructible in the sense of [15]. In that paper the authors conjectured such a result in any dimension.

Our results being on a complex curve, it is natural to look for extensions of them in higher dimensions. In [22], C. Sabbah conjectured and widely developed the higher dimensional version of Hukuhara-Turrittin's Theorem. Recently Y. André announced the proof of Sabbah's conjecture. Such results would be at the base of a possible extensions of our results.

The contents of the present paper are subdivided as follows.

In **Section 1**, we briefly review subanalytic sets recalling the classical results that we will need. We study in detail relatively compact subanalytic open subsets of $\mathbb{R}^2$. We give a decomposition of $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$ using sets biholomorphic to open sectors. Such a result will be essential in Section 3.

In **Section 2**, we recall definitions and basic results of sheaves on the subanalytic site and tempered holomorphic functions on a complex curve. In Subsection 2.2 we prove a result concerning the composition of a tempered holomorphic function and a biholomorphism.

In **Section 3**, we consider an open disc $X \subset \mathbb{C}$ centered at 0 and

$$P := z^N \frac{d}{dz} I_m + A(z) \ , \tag{0.0.2}$$

where $m \in \mathbb{Z}_{>0}$, $N \in \mathbb{N}$, $A \in gl(m, \mathcal{O}_{\mathbb{C}}(X))$ and I_m is the identity matrix of order m. The aim of this section is to study the solvability of P in the space of tempered holomorphic functions on $U \in \mathrm{Op}^c_{X_{sa}}$ with $0 \in \partial U$. We prove that there exist an open neighborhood $W \subset \mathbb{C}$ of 0 and a finite subanalytic covering $\{U_j\}_{j \in J}$ of $U \cap W$ such that for any $g_j \in \mathcal{O}^t_{X_{sa}}(U_j)^m$ there exists $u_j \in \mathcal{O}^t_{X_{sa}}(U_j)^m$ such that $Pu_j = g_j$ $(j \in J)$. We start the section by recalling Hukuhara-Turrittin's Theorem which is a basic tool in the study of ordinary differential equations.

In **Section 4**, we deal with $\mathcal{D}_X$-modules on a complex analytic curve X. We begin by recalling some classical results on $\mathcal{D}_X$-modules. In Subsection 4.2, we prove a first consequence of the results of Section 3, that is, (0.0.1) is an isomorphism. In Subsection 4.4 we prove a second consequence of the results of Section 3, that is, $\mathscr{S}ol^t(\mathcal{M})$ is $\mathbb{R}$-constructible in the sense of sheaves on X_{sa}.

We thank P. Schapira for proposing this problem to our attention and for many fruitful discussions and A. D'Agnolo for many useful remarks.

§1. Subanalytic Sets

In the first subsection, we recall the definition and some classical results on subanalytic sets. In the second subsection we focus on relatively compact subanalytic open subsets of $\mathbb{R}^2$. We prove some results mixing the complex and the real analytic structure on $\mathbb{R}^2$. Indeed, we describe the local structure of relatively compact subanalytic open subsets of $\mathbb{R}^2$ via biholomorphic images of open sectors (Theorem 1.4).

§1.1. Review on subanalytic sets

Let M be a real analytic manifold, $\mathcal{A}$ the sheaf of real-valued real analytic functions on M.

Definition 1.1. Let $X \subset M$.

(i) X is said to be *semi-analytic at* $x \in M$ if there exists an open neighborhood W of x such that $X \cap W = \cup_{i \in I} \cap_{j \in J} X_{i,j}$ where I and J are finite sets and either $X_{i,j} = \{y \in W;\ f_{i,j}(y) > 0\}$ or $X_{i,j} = \{y \in W;\ f_{i,j}(y) = 0\}$ for some $f_{i,j} \in \mathcal{A}(W)$. X is said *semi-analytic* if it is semi-analytic at each $x \in M$.

(ii) X is said *subanalytic* if for any $x \in M$ there exist an open neighborhood W of x, a real analytic manifold N and a relatively compact semi-analytic set $A \subset M \times N$ such that $\pi(A) = X \cap W$, where $\pi : M \times N \to M$ is the projection.

(iii) Let N be a real analytic manifold. A map $f : X \to N$ is said *subanalytic* if its graph,
$$\Gamma_f := \left\{(x, y) \in X \times N; y = f(x)\right\} ,$$
is subanalytic in $M \times N$.

Given $X \subset M$, denote by $\overset{\circ}{X}$ (resp. $\overline{X}$, ∂X) the interior (resp. the closure, the boundary) of X.

Proposition 1.1 (See [2]). *Let X and Y be subanalytic subsets of M. Then $X \cup Y$, $X \cap Y$, $\overline{X}$, $\overset{\circ}{X}$ and $X \setminus Y$ are subanalytic. Moreover the connected components of X are subanalytic, the family of connected components of X is locally finite and X is locally connected.*

Definition 1.2, Theorem 1.1 and Proposition 1.2 below are stated and proved in [4] for the more general case of o-minimal structures.

Definition 1.2 (Cylindrical Cell Decomposition). Let $n \in \mathbb{Z}_{>0}$. A *cylindrical cell decomposition* (*ccd* for short) $\left\{C_k\right\}_{k \in K}$ of $\mathbb{R}^n$ is a finite partition of $\mathbb{R}^n$ into subanalytic sets C_k obtained inductively on n in the following way. The sets C_k are called *cells*.

$n = 1$: The cells defining a ccd of $\mathbb{R}$ are open intervals $]a, b[$ or points $\{c\}$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$, $a < b$, and $c \in \mathbb{R}$.

$n > 1$: A ccd $\{D_h\}_{h\in H}$ of $\mathbb{R}^n$ is given by a ccd $\{C_k\}_{k\in K}$ of $\mathbb{R}^{n-1}$, $l_k \in \mathbb{N}$ and subanalytic continuous functions

$$\zeta_{k,1}, \dots, \zeta_{k,l_k} : C_k \to \mathbb{R}$$

such that, for any $x \in C_k$, $\zeta_{k,j}(x) < \zeta_{k,j+1}(x)$, $j = 1, \dots, l_k - 1$ $(k \in K)$.

The cells D_h are the graphs of $\zeta_{k,j}$,

$$\Gamma_{\zeta_{k,j}} := \{(x, \zeta_{k,j}(x)) \in C_k \times \mathbb{R}\} \qquad (1 \le j \le l_k)\ ,$$

and the sets

$$\{(x,y) \in C_k \times \mathbb{R}; \zeta_{k,j}(x) < y < \zeta_{k,j+1}(x)\} \tag{1.1.1}$$

for $0 \le j \le l_k$, where $\zeta_{k,0} = -\infty$ and $\zeta_{k,l_k+1} = +\infty$.

Theorem 1.1 (See [4], Theorem 2.10). *Let $A_1, \dots, A_d$ be relatively compact subanalytic subsets of $\mathbb{R}^n$. There exists a cylindrical cell decomposition of $\mathbb{R}^n$ adapted to each A_j. That is, each A_j is a union of cells.*

Proposition 1.2 (See [4], Theorem 3.4). *Let Z be a subanalytic subset of $\mathbb{R}^n$. The following properties are equivalent.*

(i) *Z is closed and bounded.*

(ii) *Every subanalytic continuous map $\zeta :]0,1[\to Z$ extends by continuity to a map $[0,1[\to Z$.*

(iii) *For any subanalytic continuous function $\zeta : Z \to \mathbb{R}$, $\zeta(Z)$ is closed and bounded.*

For Theorem 1.2 below, see [2, Theorem 6.4].

Theorem 1.2 (Łojasiewicz's Inequality). *Let M be a real analytic manifold, $K \subset M$. Let $f, g : K \to \mathbb{R}$ be subanalytic functions with compact graphs. If $f^{-1}(\{0\}) \subset g^{-1}(\{0\})$, then there exist $c, r \in \mathbb{R}_{>0}$ such that, for any $x \in K$,*

$$|f(x)| \ge c|g(x)|^r\ .$$

For Theorem 1.3 below, see [12, Proposition 8.2.3].

Theorem 1.3 (Curve Selection Lemma). *Let Z be a subanalytic subset of M and let $z_0 \in \overline{Z}$. Then there exists an analytic map*

$$\gamma :]-1, 1[\longrightarrow M\ ,$$

such that $\gamma(0) = z_0$ and $\gamma(t) \in Z$ for $t \neq 0$.

§1.2. Subanalytic subsets of $\mathbb{R}^2$

Notation 1.1. Given a real analytic manifold M, we denote by $\mathrm{Op}_{M_{sa}}$ (resp. $\mathrm{Op}^c_{M_{sa}}$) the category of subanalytic open (resp. relatively compact subanalytic open) subsets of M.

Let

$$\begin{aligned} \tilde{\pi} : \mathbb{R}_{\geq 0} \times]-\pi, 3\pi[&\longrightarrow \mathbb{R}^2 \\ (\varrho, \vartheta) &\longmapsto \varrho e^{i\vartheta} \ . \end{aligned}$$

One has that, given $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$ with $0 \notin U$, $\tilde{\pi}^{-1}(U)$ is a subanalytic open subset of $\mathbb{R}_{>0} \times]-\pi, 3\pi[$, relatively compact in $\mathbb{R}^2$.

For $R \in \mathbb{R}_{>0}$, $\eta, \xi : [0, R] \longrightarrow]-\pi, 3\pi[$ subanalytic continuous functions such that $\eta(\varrho) < \xi(\varrho)$, for any $\varrho \in]0, R[$, denote

$$B^{\xi}_{\eta} := \Big\{(\varrho, \vartheta) \in]0, R[\times]-\pi, 3\pi[\, ; \, \eta(\varrho) < \vartheta < \xi(\varrho)\Big\} \ .$$

Remark that $\overline{B^{\xi}_{\eta}} \subset [0, R] \times]-\pi, 3\pi[$.

Proposition 1.3. *Let $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$, $0 \in \partial U$. There exists an open neighborhood $W \subset \mathbb{R}^2$ of 0, such that $U \cap W$ is a finite union of sets of the form $\tilde{\pi}(B^{\xi}_{\eta}) \cap W$.*

Proof. The set $\tilde{\pi}^{-1}(U)$ is a subanalytic open subset of $\mathbb{R}_{>0} \times]-\pi, 3\pi[$, relatively compact in $\mathbb{R}^2$. Let $\epsilon \in \mathbb{R}_{>0}$, $\epsilon < \pi$. Take a cylindrical cell decomposition of $\mathbb{R}^2$ adapted to

$$\tilde{\pi}^{-1}(U) \cap \Big(\mathbb{R}_{>0} \times \,]-\epsilon, 2\pi + \epsilon[\Big) \ .$$

The conclusion follows. □

For $z \in \mathbb{C}$ and $\epsilon \in \mathbb{R}_{>0}$, denote by $B(z, \epsilon)$ the open ball of center z and radius ϵ.

Let us introduce semi-analytic arcs and prove an easy result which states the local equivalence between semi-analytic arcs and graphs of subanalytic functions.

Definition 1.3. Let $\gamma :]-1, 1[\longrightarrow \mathbb{R}^2$ be an analytic map, $\delta \in \mathbb{R}_{>0}$ such that $\gamma|_{[0,\delta]}$ is injective. We call

$$\Gamma := \gamma\big(]0, \delta[\big)$$

a *semi-analytic arc with an endpoint at* $\gamma(0)$.

Recall that, given a function η, we denote by Γ_η the graph of η.

Lemma 1.1. *Let $R \in \mathbb{R}_{>0}$, $\eta : [0, R[\to \mathbb{R}$ a subanalytic continuous map. There exist $\delta \in \mathbb{R}_{>0}$ and an analytic map $\gamma :]-1, 1[\longrightarrow \mathbb{R}^2$ such that $\gamma(0) = \big(0, \eta(0)\big)$ and*

$$\gamma\big(]-1,1[\setminus\{0\}\big) = \Gamma_\eta \cap \big(]0,\delta[\times\mathbb{R}\big) \ . \tag{1.2.1}$$

In particular, there exist a semi-analytic arc Γ with an endpoint at $\big(0, \eta(0)\big)$ and an open neighborhood $W \subset \mathbb{R}^2$ of $\big(0, \eta(0)\big)$, such that

$$\Gamma_\eta \cap W = \overline{\Gamma} \cap W \ .$$

Proof. Let $p_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first coordinate.

By Theorem 1.3 there exists an analytic map $\gamma :]-1, 1[\longrightarrow \mathbb{R}^2$ such that $\gamma(0) = \big(0, \eta(0)\big)$ and

$$\gamma\Big(]-1,1[\setminus\{0\}\Big) \ \subset \ \Gamma_\eta\setminus\big\{\big(0,\eta(0)\big)\big\} \ . \tag{1.2.2}$$

Remark that we can suppose that $\gamma|_{[0,1[}$ and $\gamma|_{]-1,0]}$ are injective. Since $\gamma\big(]-1,1[\big)$ is arcwise-connected, $p_1\Big(\gamma\big(]-1,1[\big)\Big)$ is arcwise-connected as well. Hence, since $\{0\} \subsetneqq p_1\Big(\gamma\big(]-1,1[\big)\Big) \subset \mathbb{R}_{\geq 0}$, there exists $\delta \in \mathbb{R}_{>0}$ such that $p_1\big(\gamma\big(]-1,1[\big)\big) = [0,\delta[$.

Further, by (1.2.2),

$$p_1\Big(\gamma\big(]-1,1[\setminus\{0\}\big)\Big) =]0,\delta[\ . \tag{1.2.3}$$

Let us prove that if $0 < x < \delta$, then $\big(x, \eta(x)\big) \in \gamma\big(]-1,1[\setminus\{0\}\big)$, this will conclude the proof. Let $x \in]0, \delta[$. By (1.2.3), there exists $y \in \mathbb{R}$ such that $(x, y) \in \gamma\big(]-1,1[\setminus\{0\}\big)$. By (1.2.2), it follows that $y = \eta(x)$. Hence $\big(x, \eta(x)\big) \in \gamma\big(]-1,1[\setminus\{0\}\big)$. □

Roughly speaking, from Lemma 1.1 and Proposition 1.3, it follows that $(\partial U) \cap W$ is a finite collection of semi-analytic arcs with an endpoint at 0.

Let us now introduce biholomorphic images of open sectors. We start with a well known result on the local nature of holomorphic functions on $\mathbb{C}$. For Proposition 1.4 below, see [6, Theorem 2.1].

Proposition 1.4. *Let $U \subset \mathbb{C}$ be an open neighborhood of 0, $\varphi : U \to \mathbb{C}$ a non constant holomorphic map such that 0 is a zero of order n for φ.*

There exist an open neighborhood $U' \subset U$ of 0, $\epsilon \in \mathbb{R}_{>0}$, and a holomorphic isomorphism $\psi : U' \to B(0,\epsilon)$ such that, for any $z \in U'$,

$$\varphi|_{U'}(z) = \big(\psi(z)\big)^n \ .$$

Definition 1.4. Let $\alpha, \beta \in \mathbb{R}$, $r \in \mathbb{R}_{>0}$, $\alpha < \beta$. The set

$$S_{\alpha,\beta,r} := \left\{\varrho e^{i\vartheta} \in \mathbb{C}^{\times};\ 0 < \varrho < r,\ \vartheta \in]\alpha, \beta[\right\}$$

is called an *open sector of amplitude $\beta - \alpha$ and radius r* or simply *an open sector.*

We will need to stress on the amplitude and the direction of a sector so we will also use the following slightly different notation

$$S_{\tau\pm\eta,r} := S_{\tau-\eta,\tau+\eta,r}$$

for $\tau \in \mathbb{R}$ and $\eta, r \in \mathbb{R}_{>0}$.

Corollary 1.1. *Let $U \subset \mathbb{C}$ be an open neighborhood of 0, $\varphi : U \to \mathbb{C}$ a non constant holomorphic map such that $\varphi(0) = 0$.*

(i) *There exist $r, \tau \in \mathbb{R}_{>0}$ such that $\overline{B(0,r)} \subset U$ and, for any $\vartheta \in \mathbb{R}$, $\varphi|_{\overline{S_{\vartheta\pm\tau,r}}}$ is an injective map.*

(ii) *Suppose that, given $\alpha, \beta \in \mathbb{R}$, there exist $\mu, \delta, R \in \mathbb{R}_{>0}$ such that*

$$\varphi\big(]0,\delta[\times\{0\}\big) \subset S_{\alpha+\mu,\beta-\mu,R} \ .$$

Then, there exist $\eta, r' \in \mathbb{R}_{>0}$ such that

$$\varphi\big(S_{0\pm\eta,r'}\big) \subset S_{\alpha,\beta,R} \ .$$

Proof. It is based on Proposition 1.4 and the fact that holomorphic isomorphisms are conformal maps. □

We are now ready to state and prove the main result of this section. Denote by $\mathcal{O}_{\mathbb{C}}$ the sheaf of holomorphic functions on $\mathbb{C}$.

Theorem 1.4. *Let $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$, $0 \in \partial U$. There exist an open neighborhood $W \subset \mathbb{C}$ of 0, a finite set J, open sectors $S_{j,k}$, $\varphi_{j,k} \in \mathcal{O}_{\mathbb{C}}\big(\overline{S_{j,k}}\big)$ $(j \in J, k = 1,2)$ such that*

(i) *$\varphi_{j,k}(0) = 0$ and $\varphi_{j,k}|_{\overline{S_{j,k}}}$ is injective $(j \in J, k = 1,2)$,*

(ii)

$$U \cap W = \bigcup_{j \in J} \Big(\varphi_{j,1}(S_{j,1}) \cap \varphi_{j,2}(S_{j,2}) \Big) \ .$$

Proof of Theorem 1.4. By Proposition 1.3, it is sufficient to prove the statement for $U = \tilde{\pi}(B_\eta^\xi) \cap W$, for $W \subset \mathbb{C}$ an open neighborhood of 0.

First we need two technical lemmas.

Lemma 1.2. *Let S be an open sector, $\varphi \in \mathcal{O}_\mathbb{C}(\overline{S})$ such that $\varphi(0) = 0$ and $\varphi|_{\overline{S}}$ is injective. Suppose that there exists $\epsilon \in \mathbb{R}_{>0}$ such that $\varphi(S) \cap B(0,\epsilon)$ is contained in an open sector of amplitude strictly smaller than 2π.*

Then there exist $r \in \mathbb{R}_{>0}$, an open neighborhood $V \subset \mathbb{C}$ of 0 and $\zeta_1, \zeta_2 : [0,\epsilon] \to]-\pi, 3\pi[$ subanalytic continuous functions such that, for any $\varrho \in [0,\epsilon]$, $\zeta_1(\varrho) < \zeta_2(\varrho)$ and

$$\varphi\big(S \cap B(0,r)\big) = \tilde{\pi}\big(B_{\zeta_1}^{\zeta_2}\big) \cap V \ .$$

Proof. We limit to give a sketch of the proof which is essentially of topological nature.

There exist $\eta \in [0, 2\pi]$, $\mu \in \mathbb{R}_{>0}$, $\mu < \pi$, such that $\varphi(S) \cap B(0,\epsilon) \subset S_{\eta \pm \mu, \epsilon}$.

Remark that $[\eta - \mu, \eta + \mu] \subset]\eta - \pi, \eta + \pi[\subset]-\pi, 3\pi[$. Take a ccd of $\mathbb{R}^2$ adapted to

$$\tilde{\pi}^{-1}\Big(\varphi(S) \cap B(0,\epsilon)\Big) \cap \Big(\mathbb{R}_{>0} \times]\eta - \mu, \eta + \mu[\Big) \ .$$

Since, for any $\delta \in \mathbb{R}_{>0}$, $\varphi(S) \cap B(0,\delta)$ has just one connected component having 0 in its boundary, the conclusion follows. □

Lemma 1.3. *Let $R \in \mathbb{R}_{>0}$, $\eta : [0,R] \longrightarrow]-\pi, 3\pi[$ a subanalytic continuous map. There exist an open neighborhood $V \subset \mathbb{C}$ of 0, $\tau, r, \epsilon \in \mathbb{R}_{>0}$, $\varphi \in \mathcal{O}_\mathbb{C}\big(\overline{B(0,r)}\big)$ and $\zeta_1, \zeta_2 : [0,\epsilon] \to]-\pi, 3\pi[$ subanalytic continuous functions satisfying the following conditions.*

(i) *$\varphi|_{\overline{S_{-\tau,\tau,r}}}$ is injective.*

(ii) *For any $\varrho \in [0,\epsilon]$, $-\pi < \zeta_1(\varrho) < \eta(\varrho) < \zeta_2(\varrho) < 3\pi$ and*

(1.2.4) $$\varphi\big(S_{0,\tau,r}\big) = \tilde{\pi}\Big(B_\eta^{\zeta_2}\Big) \cap V \ ,$$

(1.2.5) $$\varphi\big(S_{-\tau,0,r}\big) = \tilde{\pi}\Big(B_{\zeta_1}^{\eta}\Big) \cap V \ .$$

Proof. Remark that it is sufficient to prove the statement for $\eta|_{[0,\epsilon']}$ for some $\epsilon' \in \mathbb{R}_{>0}$, $\epsilon' < R$. We set for short $\eta_{\epsilon'} := \eta|_{[0,\epsilon']}$.

Since $\eta(0) \in]-\pi, 3\pi[$, there exist $\epsilon, \mu_1 \in \mathbb{R}_{>0}$, $\mu_1 < \pi$, such that $[\eta_\epsilon(0) - \mu_1, \eta_\epsilon(0) + \mu_1] \subset]-\pi, 3\pi[$ and

$$\Gamma_{\eta_\epsilon} \setminus \big(0, \eta_\epsilon(0)\big) \subset]0, R[\times]\eta_\epsilon(0) - \mu_1, \eta_\epsilon(0) + \mu_1[\ . \tag{1.2.6}$$

First, let us show that there exist an open neighborhood $W \subset \mathbb{C}$ of $]-1,1[$, $\varphi \in \mathcal{O}_{\mathbb{C}}(W)$ and $\delta \in \mathbb{R}_{>0}$ such that $\varphi(0) = 0$ and

$$\varphi\Big(\big(]-1,1[\setminus\{0\}\big) \times \{0\}\Big) = \tilde{\pi}\Big(\Gamma_{\eta_\epsilon} \cap \big(]0,\delta[\times\mathbb{R}\big)\Big)\ . \tag{1.2.7}$$

By Lemma 1.1, there exist $\delta \in \mathbb{R}_{>0}$ and an analytic map $\gamma :]-1,1[\to \mathbb{R}^2$ such that $\gamma(0) = \big(0, \eta(0)\big)$ and

$$\gamma\big(]-1,1[\setminus\{0\}\big) = \Gamma_{\eta_\epsilon} \cap \big(]0,\delta[\times\mathbb{R}\big)\ .$$

Since $\tilde{\pi} \circ \gamma$ is an analytic map, there exist a complex neighborhood $W$of $]-1,1[$ and $\varphi \in \mathcal{O}_{\mathbb{C}}(W)$ such that $\varphi|_{]-1,1[\times\{0\}} = \tilde{\pi} \circ \gamma|_{]-1,1[}$. In particular, $\varphi(0) = 0$ and

$$\varphi\Big(\big(]-1,1[\setminus\{0\}\big) \times \{0\}\Big) = \tilde{\pi}\Big(\Gamma_{\eta_\epsilon} \cap \big(]0,\delta[\times\mathbb{R}\big)\Big)\ . \tag{1.2.8}$$

Hence, (1.2.7) follows.

Now, remark that (1.2.6) implies that

$$\tilde{\pi}\big(\Gamma_{\eta_\epsilon} \setminus \big(0, \eta_\epsilon(0)\big)\big) \subset S_{\eta_\epsilon(0)\pm\mu_1, R}\ . \tag{1.2.9}$$

Combining (1.2.8) and (1.2.9), we have that

$$\begin{aligned}\varphi\big(]0,1[\times\{0\}\big) &\subset \tilde{\pi}\Big(\Gamma_{\eta_\epsilon} \cap \big(]0,\delta[\times\mathbb{R}\big)\Big)\\ &\subset \tilde{\pi}\big(\Gamma_{\eta_\epsilon} \setminus \big(0, \eta_\epsilon(0)\big)\big)\\ &\subset S_{\eta_\epsilon(0)\pm\mu_1, R}\ .\end{aligned} \tag{1.2.10}$$

Since $[\eta_\epsilon(0) - \mu_1, \eta_\epsilon(0) + \mu_1] \subset]-\pi, 3\pi[$ and $\mu_1 < \pi$, there exists $\mu_2 \in \mathbb{R}_{>0}$ such that $\mu_1 < \mu_2 < \pi$ and

$$[\eta_\epsilon(0) - \mu_2, \eta_\epsilon(0) + \mu_2] \subset]-\pi, 3\pi[\ . \tag{1.2.11}$$

Let $r \in \mathbb{R}_{>0}$ be such that $\overline{B(0,r)} \subset W$. Then, Corollary 1.1 (ii) applies and there exist $\tau \in \mathbb{R}_{>0}$ such that, up to shrinking r,

$$\varphi\big(S_{0,\tau,r}\big) \subset S_{\eta_\epsilon(0)\pm\mu_2, R}\ . \tag{1.2.12}$$

Further, by Corollary 1.1 (i), up to shrinking τ and r, we have that $\varphi|_{\overline{S_{0,\tau,r}}}$ is injective.

Then, Lemma 1.2 applies and there exist $r' \in \mathbb{R}_{>0}$, an open neighborhood $V \subset \mathbb{C}$ of 0 and $\zeta_1, \zeta_2 : [0,\epsilon] \to]-\pi, 3\pi[$ subanalytic continuous functions such that, for any $\varrho \in [0,\epsilon]$, $\zeta_1(\varrho) < \zeta_2(\varrho)$ and

$$\varphi(S_{0,\tau,r'}) = \tilde{\pi}(B_{\zeta_1}^{\zeta_2}) \cap V \ .$$

Then, (1.2.11) and (1.2.12) imply that one can chose $\zeta_1 = \eta_\epsilon$. Hence (1.2.4) follows.

Clearly, (1.2.5) can be proved using the same arguments. □

End of the Proof of Theorem 1.4. As said above, by Proposition 1.3, it is sufficient to prove the statement for $U = \tilde{\pi}(B_\eta^\xi) \cap W$, for $W \subset \mathbb{C}$ an open neighborhood of 0.

Consider B_η^ξ, by Lemma 1.3, there exist $\zeta_1, \zeta_2 : [0,\epsilon] \to]-\pi, 3\pi[$, $r, \tau \in \mathbb{R}_{>0}$, $\varphi_1, \varphi_2 \in \mathcal{O}_\mathbb{C}\big(\overline{B(0,r)}\big)$, $V_1, V_2 \subset \mathbb{C}$ open neighborhoods of 0 such that, for any $\varrho \in [0,\epsilon]$, $\eta(\varrho) < \zeta_2(\varrho) < 3\pi$, $-\pi < \zeta_1(\varrho) < \xi(\varrho)$, $\varphi_1|_{\overline{S_{0,\tau,r}}}$ $\varphi_2|_{\overline{S_{-\tau,0,r}}}$ are injective and

$$\tilde{\pi}\big(B_\eta^{\zeta_2}\big) \cap V_1 = \varphi_1(S_{0,\tau,r}) \ ,$$
$$\tilde{\pi}\big(B_{\zeta_1}^{\xi}\big) \cap V_2 = \varphi_2(S_{-\tau,0,r}) \ .$$

We distinguish two cases: $\xi(0) = \eta(0)$ and $\eta(0) < \xi(0)$.

(i) Suppose $\xi(0) = \eta(0)$.

We have that

$$-\pi < \zeta_1(0) < \eta(0) = \xi(0) < \zeta_2(0) < 3\pi \ .$$

It follows that there exists $\epsilon' \in \mathbb{R}_{>0}$ such that, for any $\varrho \in [0,\epsilon']$,

$$\zeta_1(\varrho) < \eta(\varrho) \le \xi(\varrho) < \zeta_2(\varrho) \ .$$

Hence, considering $\eta, \xi, \zeta_1, \zeta_2$ as restricted to $[0,\epsilon']$, we have that

$$B_\eta^\xi = B_\eta^{\zeta_2} \cap B_{\zeta_1}^\xi \ .$$

Now, up to take smaller τ, ϵ', we can suppose that $\tilde{\pi}(B_{\zeta_1}^\xi)$ and $\tilde{\pi}(B_\eta^{\zeta_2})$ are contained in an open sector of amplitude strictly smaller than 2π. In particular, $\tilde{\pi}$ is a bijection on $B_\eta^{\zeta_2} \cup B_{\zeta_1}^\xi$. It follows that

$$\tilde{\pi}\big(B_\eta^\xi\big) = \tilde{\pi}\big(B_\eta^{\zeta_2}\big) \cap \tilde{\pi}\big(B_{\zeta_1}^\xi\big) \ .$$

Taking $V := V_1 \cap V_2$, the conclusion follows.

(ii) Suppose $\eta(0) < \xi(0)$.

Up to take smaller τ, there exist $\epsilon' \in \mathbb{R}_{>0}$ and $\alpha, \beta : [0, \epsilon'] \to \mathbb{R}$ constant functions such that, for any $\varrho \in [0, \epsilon']$,

$$\eta(\varrho) < \alpha(\varrho) < \zeta_2(\varrho) < \zeta_1(\varrho) < \beta(\varrho) < \xi(\varrho) \ .$$

It follows that, considering $\eta, \xi, \zeta_1, \zeta_2$ as restricted to $[0, \epsilon']$,

$$B_\eta^\xi = B_\eta^{\zeta_2} \cup B_\alpha^\beta \cup B_{\zeta_1}^\xi \ .$$

The conclusion follows. □

Detailing the proof of Theorem 1.4, one can give a more precise statement in the following way.

Remark. Let U, W, $\varphi_{j,k}$ and $S_{j,k}$ as given in Theorem 1.4. Given $r, \eta \in \mathbb{R}_{>0}$, there exist an open neighborhood $W' \subset W$ of the origin, a finite set J' and open sectors $S'_{j',k} \subset S_{j,k}$ $(j' \in J')$ such that the amplitude (resp. the radius) of $S'_{j',k}$ is smaller than η (resp. r) and

$$U \cap W' = \bigcup_{j \in J'} \Big(\varphi_{j,1}(S'_{j,1}) \cap \varphi_{j,2}(S'_{j,2}) \Big) \ .$$

§2. Tempered Holomorphic Functions

In the first subsection we recall the definition and some classical results on the subanalytic site X_{sa} underlying a complex curve X and sheaves on X_{sa}. In the second subsection we recall the definition of the subanalytic sheaf of tempered holomorphic functions on a complex curve. In the third subsection we prove a result on the pull back of tempered holomorphic functions through biholomorphisms. We refer to [15] and [14] for the first and the second subsections.

§2.1. The subanalytic site

Let X be a complex analytic manifold, denote by $\overline{X}$ the complex conjugate manifold and by $X_\mathbb{R}$ the underlying real analytic manifold. For k a commutative ring, we denote by $\mathrm{Mod}(k_X)$ the category of sheaves of k-modules on X.

We endow $\mathrm{Op}^c_{X_{sa}} := \mathrm{Op}^c_{X_{\mathbb{R}sa}}$ with a Grothendieck topology, called the subanalytic topology, by deciding that an usual open covering $U = \cup_{i \in I} U_i$ in

$\mathrm{Op}^c_{X_{sa}}$ is a covering for the subanalytic topology if there exists a finite subset $J \subset I$ such that $U = \cup_{j\in J} U_j$. Denote by X_{sa} this site and call it the *subanalytic site*. Further, denote by $\mathrm{Cov}_{sa}(U)$ the family of coverings of $U \in \mathrm{Op}^c_{X_{sa}}$ for the subanalytic topology and by $\mathrm{Mod}(k_{X_{sa}})$ the category of sheaves of k-modules on the subanalytic site.

One can show (see [14, Remark 6.3.6]) that $\mathrm{Mod}(k_{X_{sa}})$ is equivalent to the category of sheaves on the site $X_{sa,lf}$, where the class of open sets of $X_{sa,lf}$ is $\mathrm{Op}_{X_{sa}}$ and, for $U \in \mathrm{Op}_{X_{sa}}$, the family of coverings of U for $X_{sa,lf}$ consists of subanalytic open coverings $\{U_\sigma\}_{\sigma\in\Sigma}$ of U such that for any compact K of X, there exists a finite subset $J \subset \Sigma$ such that $K \cap \big(\cup_{j\in J} U_j\big) = K \cap U$.

Let $\mathrm{PSh}(k_{X_{sa}})$ be the category of presheaves of k-modules on X_{sa}. Denote by $for : \mathrm{Mod}(k_{X_{sa}}) \to \mathrm{PSh}(k_{X_{sa}})$ the forgetful functor which associates to a sheaf F on X_{sa} its underlying presheaf. It is well known that for admits a left adjoint $\cdot^a : \mathrm{PSh}(k_{X_{sa}}) \to \mathrm{Mod}(k_{X_{sa}})$.

For $F \in \mathrm{PSh}(k_{X_{sa}})$, let us briefly recall the construction of F^a.

For $U \in \mathrm{Op}^c_{X_{sa}}$ and $S = \{U_1, \dots, U_n\} \in \mathrm{Cov}_{sa}(U)$, set

(2.1.1)
$$F(S) := \left\{(s_1, \dots, s_n) \in \prod_{j=1}^n F(U_j);\ s_j|_{U_j\cap U_k} = s_k|_{U_j\cap U_k}, j,k = 1,\dots,n\right\} .$$

If S is a covering of U and S' is a refinement of S, then there exists a natural restriction morphism $F(S) \underset{\varrho_{SS'}}{\longrightarrow} F(S')$.

Then, for $U \in \mathrm{Op}^c_{X_{sa}}$, set

$$F^+(U) := \varinjlim_{S\in\mathrm{Cov}_{sa}(U)} F(S) . \tag{2.1.2}$$

It turns out that $F^a \simeq F^{++}$.

Now, let $s \in F^a(U)$. Since the inductive limit considered in (2.1.2) is filtrant, s can be identified to an n-uple $(s_1, \dots, s_n) \in F(S)$, for $S = \{U_j\}_{j=1,\dots,n} \in \mathrm{Cov}_{sa}(U)$, $s_j \in F(U_j)$ $(j = 1, \dots, n)$.

Further, if $s \in F^a(U)$ can be identified to $s_1 \in F(S_1)$ and to $s_2 \in F(S_2)$, for $S_1, S_2 \in \mathrm{Cov}_{sa}(U)$, then there exists a refinement $S \in \mathrm{Cov}_{sa}(U)$ of S_1 and S_2 and $\bar{s} \in F(S)$ such that s can be identified to $\bar{s}$.

For Proposition 2.1 below, see [14, Proposition 2.1.12].

Proposition 2.1. *Consider the complex in* $\mathrm{Mod}(k_{X_{sa}})$

$$F' \xrightarrow{\varphi} F \xrightarrow{\psi} F'' . \tag{2.1.3}$$

The following conditions are equivalent.

(i) (2.1.3) *is exact.*

(ii) *For any* $U \in \mathrm{Op}^c_{X_{sa}}$ *and any* $t \in F(U)$ *such that* $\psi(t) = 0$*, there exist* $\{U_j\}_{j\in J} \in \mathrm{Cov}_{sa}(U)$ *and* $s_j \in F(U_j)$ *such that* $\varphi(s_j) = t|_{U_j}$ $(j \in J)$*.*

We shall denote by

$$\varrho : X \longrightarrow X_{sa} \ ,$$

the natural morphism of sites associated to $\mathrm{Op}^c_{X_{sa}} \longrightarrow \mathrm{Op}_X = \{U \subset X; U \text{ open}\}$. We refer to [14] for the definitions of the functors $\varrho_* : \mathrm{Mod}(k_X) \longrightarrow \mathrm{Mod}(k_{X_{sa}})$ and $\varrho^{-1} : \mathrm{Mod}(k_{X_{sa}}) \longrightarrow \mathrm{Mod}(k_X)$ and for Proposition 2.2 below.

Proposition 2.2.

(i) ϱ_* *is right adjoint to* ϱ^{-1}*.*

(ii) ϱ^{-1} *has a left adjoint denoted by* $\varrho_! : \mathrm{Mod}(k_X) \longrightarrow \mathrm{Mod}(k_{X_{sa}})$.

(iii) ϱ^{-1} *and* $\varrho_!$ *are exact,* ϱ_* *is exact on constructible sheaves.*

(iv) ϱ_* *and* $\varrho_!$ *are fully faithful.*

Through ϱ_*, we will consider $\mathrm{Mod}(k_X)$ as a subcategory of $\mathrm{Mod}(k_{X_{sa}})$.

The functor $\varrho_!$ is described as follows. If $U \in \mathrm{Op}^c_{X_{sa}}$ and F is a sheaf on X, then $\varrho_!(F)$ is the sheaf on X_{sa} associated to the presheaf $U \mapsto F(\overline{U})$.

§2.2. Definition and main properties of $\mathcal{O}^t_{X_{sa}}$

Througout this subsection, X will be a complex analytic curve with structure sheaf $\mathcal{O}_X$. For higher dimensions we refer to [14].

Denote by $\mathcal{D}_X$ the sheaf of differential operators with holomorphic coefficients on X. Denote by $\mathcal{D}b_{X_\mathbb{R}}$ the sheaf of distributions on $X_\mathbb{R}$ and, for a closed subset Z of X, by $\Gamma_Z(\mathcal{D}b_{X_\mathbb{R}})$ the subsheaf of sections supported by Z. One denotes by $\mathcal{D}b^t_{X_{sa}}$ the presheaf of *tempered distributions* on $X_\mathbb{R}$ defined as follows

$$\mathrm{Op}_{X_{sa}} \ni U \longmapsto \mathcal{D}b^t_{X_{sa}}(U) := \Gamma(X; \mathcal{D}b_{X_\mathbb{R}}) / \Gamma_{X\setminus U}(X; \mathcal{D}b_{X_\mathbb{R}}) \ .$$

In [14] it is proved that $\mathcal{D}b^t_{X_{sa}}$ is a sheaf on X_{sa}. This sheaf is well defined in the category $\mathrm{Mod}(\varrho_!\mathcal{D}_X)$. Moreover, for any $U \in \mathrm{Op}^c_{X_{sa}}$, $\mathcal{D}b^t_{X_{sa}}$ is $\Gamma(U, \cdot)$-acyclic.

One defines the sheaf $\mathcal{O}^t_{X_{sa}} \in D^b(\varrho_!\mathcal{D}_X)$ of tempered holomorphic functions as

$$\mathcal{O}^t_{X_{sa}} := R\mathcal{H}om_{\varrho_!\mathcal{D}_{\overline{X}}}\big(\varrho_!\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X_\mathbb{R}}\big) \ .$$

In [14] it is proved that, since $\dim X = 1$, $R\varrho_* \mathcal{O}_X$ and $\mathcal{O}^t_{X_{sa}}$ are concentrated in degree 0 . Hence we can write the following exact sequence of sheaves on X_{sa}

$$0 \longrightarrow \mathcal{O}^t_{X_{sa}} \longrightarrow \mathcal{D}b^t_{X_{sa}} \xrightarrow{\bar{\partial}} \mathcal{D}b^t_{X_{sa}} \longrightarrow 0 \ .$$

Lemma 2.1. *Let $X = \mathbb{C}$, $X_{\mathbb{R}} = \mathbb{R}^2$, $U, V \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$.*

(i) *$H^j(U, \mathcal{O}^t_{X_{sa}}) = 0$, for $j > 0$.*

(ii) *The following sequence is exact*

$$0 \to \mathcal{O}^t_{X_{sa}}(U \cup V) \to \mathcal{O}^t_{X_{sa}}(U) \oplus \mathcal{O}^t_{X_{sa}}(V) \to \mathcal{O}^t_{X_{sa}}(U \cap V) \to 0 \ . \tag{2.2.1}$$

Proof. (i) By the definition of $\mathcal{D}b^t_{X_{sa}}$, given $h \in \mathcal{D}b^t_{X_{sa}}(U)$, there exists $\widetilde{h} \in \mathcal{D}b_{X_{\mathbb{R}}}(\mathbb{R}^2)$ such that $\widetilde{h}\big|_U = h$. It is well known that there exists $g \in \mathcal{D}b_{X_{\mathbb{R}}}(\mathbb{R}^2)$ such that $\bar{\partial} g = \widetilde{h}$. This implies that $\bar{\partial}\big(g|_U\big) = h$. So we have the exact sequence

$$0 \longrightarrow \mathcal{O}^t_{X_{sa}}(U) \longrightarrow \mathcal{D}b^t_{X_{sa}}(U) \xrightarrow{\bar{\partial}} \mathcal{D}b^t_{X_{sa}}(U) \longrightarrow 0 \ .$$

Since $\mathcal{D}b^t_{X_{sa}}$ is acyclic with respect to the functor $\Gamma(U; \cdot)$, for $U \in \mathrm{Op}^c_{X_{sa}}$, it follows that, for all $j \in \mathbb{Z}_{>0}$, $H^j(U, \mathcal{O}^t_{X_{sa}}) = 0$.

(ii) Obvious from (i). □

Now we recall the definition of polynomial growth for $\mathcal{C}^\infty$ functions on $X_{\mathbb{R}}$ and in (2.2.5) we give an alternative expression for tempered holomorphic functions on $U \in \mathrm{Op}^c_{X_{sa}}$.

Definition 2.1. Let U be an open subset of $X_{\mathbb{R}}$, $f \in \mathcal{C}^\infty_{X_{\mathbb{R}}}(U)$. One says that f has *polynomial growth at* $p \in X$ if it satisfies the following condition. For a local coordinate system $x = (x_1, x_2)$ around p, there exist a sufficiently small compact neighborhood K of p and $M \in \mathbb{Z}_{>0}$ such that

$$\sup_{x \in K \cap U} \mathrm{dist}(x, K \setminus U)^M \big|f(x)\big| < +\infty \ . \tag{2.2.2}$$

We say that $f \in \mathcal{C}^\infty_{X_{\mathbb{R}}}(U)$ has *polynomial growth on* U if it has polynomial growth at any $p \in X$. We say that f is *tempered at* p if all its derivatives have polynomial growth at $p \in X$. We say that f is *tempered on* U if it is tempered at any $p \in X$. Denote by $\mathcal{C}^{\infty,t}_X$ the presheaf on $X_{\mathbb{R}}$ of tempered $\mathcal{C}^\infty$-functions.

It is obvious that f has polynomial growth at any point of U. If no confusion is possible we will write "f is tempered" instead of "f is tempered on U".

In [14] it is proved that $\mathcal{C}_X^{\infty,t}$ is a sheaf on X_{sa}.

For $U \subset \mathbb{R}^2$ a relatively compact open set, we can characterize functions with polynomial growth on U by means of a family of norms.

For $x \in \mathbb{R}^2$, $f \in \mathcal{C}^\infty_{\mathbb{R}^2}(U)$, $g = (g_1, \dots, g_m) \in \big(\mathcal{C}^\infty_{\mathbb{R}^2}(U)\big)^m$ and $M \in \mathbb{Z}_{>0}$, denote

$$\begin{aligned} \delta_{\partial U}(x) &:= \operatorname{dist}(x, \partial U)\ , \\ \|f\|_U^M &:= \sup_{x\in U}\ \delta_{\partial U}(x)^M |f(x)|\ , \\ \|g\|_U^M &:= \max\left\{\|g_j\|_U^M;\ j = 1,\dots,m\right\}\ . \end{aligned} \tag{2.2.3}$$

Proposition 2.3. *Let $U \subset \mathbb{R}^2$ be a relatively compact open set and let $f \in \mathcal{C}^\infty_{\mathbb{R}^2}(U)$. Then f has polynomial growth if and only if there exists $M \in \mathbb{R}_{>0}$ such that*

$$\|f\|_U^M < +\infty\ , \tag{2.2.4}$$

or equivalently: there exist $C, M \in \mathbb{R}_{>0}$ such that for any $x \in U$,

$$\big|f(x)\big| \le C\delta_{\partial U}(x)^{-M}\ .$$

Proof. Suppose that f satisfies (2.2.4), that is,

$$\sup_{x\in U}\ \delta_{\partial U}(x)^M |f(x)| < +\infty\ .$$

Let K be a compact neighborhood of $\overline{U}$. For any $p \in \overline{U}$, K is a compact neighborhood of p such that

$$\begin{aligned} \sup_{x\in K\cap U} \operatorname{dist}(x, K\setminus U)^M |f(x)| &\le \sup_{x\in U}\ \delta_{\partial U}(x)^M |f(x)| \\ &< +\infty\ . \end{aligned}$$

Hence, f has polynomial growth.

Conversely, suppose that h has polynomial growth. That is, for $p \in \partial U$, there exists a compact neighborhood K_p of p verifying (2.2.2).

Set

$$V := \Big\{x \in K_p;\ \delta_{\partial U\setminus K_p}(x) > \delta_{\partial U}(x)\Big\}\ .$$

Then for any $x \in V$, $\delta_{\partial U}(x) = \delta_{\partial U\cap K_p}(x)$.

Since $p \in V$, there exists $\epsilon \in \mathbb{R}_{>0}$ such that $\overline{B(p,\epsilon)} \subset V$. Set

$$Z_p := \overline{B(p,\epsilon)} \cup (K_p \cap \partial U) \ .$$

Then $Z_p \cap \partial U = K_p \cap \partial U$ and, for any $x \in Z_p \cap U$, $\delta_{\partial U}(x) = \delta_{\partial U \cap K_p}(x) = \delta_{\partial U \cap Z_p}(x)$.

Hence,

$$\begin{aligned} \sup_{x \in Z_p \cap U} \delta_{\partial U}(x)^M |f(x)| &= \sup_{x \in Z_p \cap U} \delta_{\partial U \cap Z_p}(x)^M |f(x)| \\ &\leq \sup_{x \in K_p \cap U} \delta_{\partial U \cap K_p}(x)^M |f(x)| \\ &< +\infty \ . \end{aligned}$$

Since ∂U is compact, the conclusion follows. □

Lemma 2.2 below is an easy consequence of Cauchy's Formula. See [23, Lemma 3].

Lemma 2.2. *Let U be a relatively compact open subset of X, $f \in \mathcal{O}_X(U)$ with polynomial growth on U. Then $f \in \mathcal{O}^t_{X_{sa}}(U)$.*

For Proposition 2.4 below, see [14].

Proposition 2.4. *One has the following isomorphism*

$$\mathcal{O}^t_{X_{sa}} \simeq R\mathcal{H}om_{\varrho_! \mathcal{D}_{\overline{X}}}\big(\varrho_! \mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,t}_{X_{\mathbb{R}}}\big) \ .$$

Hence, for $U \in \mathrm{Op}^c_{X_{sa}}$, we deduce the short exact sequence

$$0 \longrightarrow \mathcal{O}^t_{X_{sa}}(U) \longrightarrow \mathcal{C}^{\infty,t}_{X_{\mathbb{R}}}(U) \xrightarrow{\bar\partial} \mathcal{C}^{\infty,t}_{X_{\mathbb{R}}}(U) \longrightarrow 0 \ . \tag{2.2.5}$$

§2.3. Pull-back of tempered holomorphic functions

Recall that, for U a relatively compact open subset of $\mathbb{R}^2$ and $z \in \mathbb{R}^2$, we set $\delta_{\partial U}(z) := \mathrm{dist}(z, \partial U)$.

Lemma 2.3. *Let X be an open subset of $\mathbb{R}^2$, $f : X \to \mathbb{R}^2$ be a $\mathcal{C}^\infty$-subanalytic map. Let $U \in \mathrm{Op}^c_{X_{sa}}$, $V \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$ satisfying $f(U) = V$ and $f\big(\partial U\big) = \partial V$. Let $h \in \mathcal{C}^\infty_{\mathbb{R}^2}(V)$.*

Then h has polynomial growth on V if and only if $h \circ f$ has polynomial growth on U.

Proof. Consider the subanalytic continuous functions $\delta_{\partial U}, \delta_{\partial V} \circ f|_{\overline{U}} : \overline{U} \to \mathbb{R}_{\geq 0}$. Since $f\bigl(\partial U\bigr) = \partial V$ and $f(U) = V$,

$$\bigl(\delta_{\partial V} \circ f|_{\overline{U}}\bigr)^{-1}\bigl(\{0\}\bigr) = \partial U \ .$$

In particular,

$$\bigl(\delta_{\partial V} \circ f|_{\overline{U}}\bigr)^{-1}\bigl(\{0\}\bigr) = \delta_{\partial U}^{-1}\bigl(\{0\}\bigr) \ .$$

By Theorem 1.2, there exist $a, b, \alpha, \beta \in \mathbb{R}_{>0}$ such that, for any $x \in \overline{U}$,

$$a\Bigl(\delta_{\partial V} \circ f|_{\overline{U}}(x)\Bigr)^{\alpha} \leq \delta_{\partial U}(x) \ , \tag{2.3.1}$$

and

$$b\bigl(\delta_{\partial U}(x)\bigr)^{\beta} \leq \delta_{\partial V} \circ f|_{\overline{U}}(x) \ . \tag{2.3.2}$$

(i) Suppose that $h \circ f$ has polynomial growth on U, that is, there exist $C, M \in \mathbb{R}_{>0}$ such that, for any $x \in U$,

$$\bigl|h\bigl(f(x)\bigr)\bigr| \leq C\bigl(\delta_{\partial U}(x)\bigr)^{-M} \ .$$

By (2.3.1), we obtain

$$\bigl|h\bigl(f(x)\bigr)\bigr| \leq Ca^{-M}\bigl(\delta_{\partial V} \circ f|_{\overline{U}}(x)\bigr)^{-M\alpha} \ .$$

Since $f(U) = V$, it follows that, for any $y \in V$,

$$\bigl|h(y)\bigr| \leq Ca^{-M}\bigl(\delta_{\partial V}(y)\bigr)^{-M\alpha} \ ,$$

that is, h has polynomial growth on V.

(ii) Suppose that h has polynomial growth on V, that is, there exist $C', M' \in \mathbb{R}_{>0}$ such that, for any $y \in V$,

$$|h(y)| \leq C'\bigl(\delta_{\partial V}(y)\bigr)^{-M'} \ .$$

Since $f(U) = V$, we have, for any $x \in U$,

$$\bigl|h(f(x))\bigr| \leq C'\bigl(\delta_{\partial V} \circ f(x)\bigr)^{-M'} \ .$$

By (2.3.2), we obtain

$$\bigl|h(f(x))\bigr| \leq C'b^{-M'}\bigl(\delta_{\partial U}(x)\bigr)^{-M'\beta} \ ,$$

that is, $h \circ f$ has polynomial growth on U. □

Theorem 2.1. *Let X be an open subset of $\mathbb{C}$, $f \in \mathcal{O}_{\mathbb{C}}(X)$. Let $U \in \mathrm{Op}^c_{X_{sa}}$ such that $f|_{\overline{U}}$ is an injective map. Let $h \in \mathcal{O}_X\big(f(U)\big)$. Then, $h \in \mathcal{O}^t_{\mathbb{C}_{sa}}\big(f(U)\big)$ if and only if $h \circ f \in \mathcal{O}^t_{X_{sa}}(U)$.*

Proof. Since f is an open mapping, $f|_U : U \to f(U)$ is a holomorphic isomorphism.

It is sufficient to prove that $f(\partial U) = \partial\big(f(U)\big)$ in order to apply Lemma 2.3.

(i) $f(\partial U) \subset \partial\big(f(U)\big)$. For $x \in \partial U$, there exists $\{x_n\}_{n\in\mathbb{N}} \subset U$ such that $x_n \underset{n\to+\infty}{\longrightarrow} x$. It follows that $f(x_n) \underset{n\to+\infty}{\longrightarrow} f(x)$, hence $f(x) \in \overline{f(U)}$. Suppose that $f(x) \in f(U)$. Since $f|_U$ is an isomorphism onto $f(U)$, there exists $\overline{x} \in U$ such that $f(\overline{x}) = f(x)$, this contradicts the hypothesis that $f|_{\overline{U}}$ is injective. It follows that $f(x) \in \partial\big(f(U)\big)$.

(ii) $f(\partial U) \supset \partial\big(f(U)\big)$. For $y \in \partial(f(U))$, there exists $\{y_n\}_{n\in\mathbb{N}} \subset f(U)$ such that $y_n \underset{n\to+\infty}{\longrightarrow} y$. Set $x_n := \big(f|_U\big)^{-1}(y_n)$. Then $\{x_n\}_{n\in\mathbb{N}} \subset U$ is a bounded sequence. Hence there exists a subsequence converging to $x \in \overline{U}$. Since $f(x) = y$ and $f|_U$ is an isomorphism onto $f(U)$, $x \in \partial U$. □

§3. Existence Theorem

Let $X \subset \mathbb{C}$ be an open neighborhood of 0, P a differential operator defined on X, whose only possible singular point is 0. In this section we study the non-homogeneous ordinary differential system relative to P.

In the first subsection we recall some classical results on the holomorphic solutions of P.

In the second subsection we start by recalling an existence theorem for tempered holomorphic functions on small open sectors. As said in the introduction such a result is classical and it has been treated in more general cases by many authors. We recall the version obtained by N. Honda in [7]. Then we state and prove the main result of this section which states that given $U \in \mathrm{Op}^c_{X_{sa}}$, with $0 \in \partial U$, there exist an open neighborhood W of 0 and $\big\{U_j\big\}_{j\in J} \in \mathrm{Cov}_{sa}(U \cap W)$ such that P is a surjective endomorphism on $\mathcal{O}^t_{X_{sa}}(U_j)$ $(j \in J)$. The proof is based on the decomposition of the germ of U at 0 in sets biholomorphic to open sectors (Theorem 1.4) and on an existence theorem for sets biholomorphic to open sectors. The proof of this latter result uses a result on the composition of a biholomorphism and a tempered holomorphic function (Theorem 2.1) in order to reduce the problem to open sectors of small amplitude.

As a corollary we obtain that P is a surjective endomorphism of the sheaf $\mathcal{O}^t_{X_{sa}}$.

§3.1. Some classical results

Denote by $\mathcal{C}^0_{\mathbb{C}}$ the sheaf of continuous functions on $\mathbb{C}$. For R a ring, we denote by $gl(m,R)$ (resp. $GL(m,R)$) the ring of (resp. multiplicative group of invertible) $m \times m$ matrices. In this chapter we are going to consider $z^{1/l}$, $l \in \mathbb{Z}_{>0}$, as a holomorphic function on open sets contained in open sectors of amplitude smaller than 2π, by chosing the branch of $z^{1/l}$ which has positive real values on $\mathbb{R}_{>0} \times \{0\}$.

Let $X \subset \mathbb{C}$ be an open disc centered at 0. Let

$$P := z^N \frac{d}{dz} I_m + A(z) \ , \tag{3.1.1}$$

where $m \in \mathbb{Z}_{>0}$, $N \in \mathbb{N}$, $A \in gl\big(m, \mathcal{O}_{\mathbb{C}}(X)\big)$ and I_m is the identity matrix of order m.

Theorem 3.1 below is a fundamental result on ordinary differential systems. A complete proof of Theorem 3.1 below, originally due to Hukuhara and Turrittin, is given in [24].

Theorem 3.1 (See [24]). *Let P be the differential operator* (3.1.1). *There exist $l \in \mathbb{Z}_{>0}$, a diagonal matrix $\Lambda \in gl\big(m, z^{-1/l} \cdot \mathbb{C}[z^{-1/l}]\big)$ and for any $\vartheta_0 \in \mathbb{R}$, there exist $\vartheta_1, \vartheta_2 \in \mathbb{R}$, $\vartheta_1 < \vartheta_0 < \vartheta_2$, $R, K, M \in \mathbb{R}_{>0}$ and $F_{S_{\vartheta_1,\vartheta_2,R}} \in GL\Big(m, \mathcal{O}_{\mathbb{C}}(S_{\vartheta_1,\vartheta_2,R}) \cap \mathcal{C}^0_{\mathbb{C}}\big(\overline{S_{\vartheta_1,\vartheta_2,R}} \setminus \{0\}\big)\Big)$, satisfying the following conditions*

(i) *for any $z \in S_{\vartheta_1,\vartheta_2,R}$,*

$$K^{-1}|z|^M \leq \big|F_{S_{\vartheta_1,\vartheta_2,R}}(z)\big| \leq K|z|^{-M} \ , \tag{3.1.2}$$

(ii) *the m columns of the matrix $F_{S_{\vartheta_1,\vartheta_2,R}}(z) \exp\big(\Lambda(z)\big)$ are $\mathbb{C}$-linearly independent holomorphic solutions of the system $Pu = 0$.*

If no confusion is possible we will write $F(z)$ instead of $F_{S_{\vartheta_1,\vartheta_2,R}}(z)$.

Note that (3.1.2) implies that $F, F^{-1} \in GL\big(m, \mathcal{O}^t_{X_{sa}}(S_{\vartheta_1,\vartheta_2,R})\big)$.

Definition 3.1. We call the matrix $F(z) \exp\big(\Lambda(z)\big)$, given in Theorem 3.1, a *fundamental holomorphic solution of P on $S_{\vartheta_1,\vartheta_2,R}$*. If U is an open subset of $S_{\vartheta_1,\vartheta_2,R}$, we say that P a *fundamental holomorphic solution on U*.

Lemma 3.1. *Let $U \in \mathrm{Op}^c_{X_{sa}}$, connected and simply connected. Suppose that P has a fundamental holomorphic solution $F(z)\exp(\Lambda(z))$ on U. Let $g \in \mathcal{O}(U)^m$, $z_1 \in U$.*

Then, for $\Gamma_{z_1,z} \subset U$ a path from z_1 to $z \in U$,

$$F(z)\exp\big(\Lambda(z)\big)\int_{\Gamma_{z_1,z}} \exp\big(-\Lambda(\zeta)\big)F(\zeta)^{-1}\frac{g(\zeta)}{\zeta^N}d\zeta$$

is a holomorphic solution of $Pu = g$.

Proof. Obvious. □

§3.2. Existence theorem for tempered holomorphic functions

Let $l \in \mathbb{Z}_{>0}$, $p(z) \in z^{-1/l} \cdot \mathbb{C}[z^{-1/l}]$, S an open sector of amplitude smaller than 2π, $g \in \mathcal{O}_X(S)$. Set

$$I_{p,z_0}(g)(z) := \exp\big(p(z)\big)\int_{\Gamma_{z_0,z}} \exp\big(-p(\zeta)\big)g(\zeta)d\zeta\ , \tag{3.2.1}$$

where $z_0 \in \overline{S}$ and $\Gamma_{z_0,z} \subset \overline{S}$ is a path from z_0 to $z \in S$.

Proposition 3.1 (See [7], Proposition 2.3)**.** *Let $l \in \mathbb{Z}_{>0}$ and $p(z) \in z^{-1/l} \cdot \mathbb{C}[z^{-1/l}]$. There exists $\alpha \in \mathbb{R}_{>0}$ such that for any open sector S of amplitude $\eta \leq \alpha$, there exist $z_0 \in \overline{S}$ and a path $\Gamma_{z_0,z} \subset \overline{S}$ from z_0 to $z \in S$ such that if $g \in \mathcal{O}_X(S)$ satisfies $\|g\|_S^M < +\infty$, for some $M \in \mathbb{R}_{>0}$, then $I_{p,z_0}(g) \in \mathcal{O}_X(S)$ and*

$$\|I_{p,z_0}(g)\|_S^M < +\infty\ .$$

Now we prove an analogue of Proposition 3.1 for sets biholomorphic to an open sector of sufficiently small amplitude. Then we will use such a result to prove an existence theorem for P on $U \in \mathrm{Op}^c_{X_{sa}}$, $0 \in \partial U$.

Proposition 3.2. *Let $W \subset \mathbb{C}$ be an open neighborhood of 0, $\varphi \in \mathcal{O}_{\mathbb{C}}(W)$ non constant, $\varphi(0) = 0$, $l \in \mathbb{Z}_{>0}$, $p \in z^{-1/l}\mathbb{C}[z^{-1/l}]$.*

There exist $r, \eta \in \mathbb{R}_{>0}$ such that for any open sector $S \subset\subset B(0,r) \subset W$ of amplitude smaller than η, there exist $z_0 \in \varphi\big(\overline{S}\big)$ and a path $\Gamma_{z_0,z} \subset \varphi\big(\overline{S}\big)$ from z_0 to $z \in \varphi(S)$ such that, for any $g \in \mathcal{O}^t_{X_{sa}}\big(\varphi(S)\big)$,

$$I_{p,z_0}(g)(z) = \exp\big(p(z)\big)\int_{\Gamma_{z_0,z}} \exp\big(-p(\zeta)\big)g(\zeta)d\zeta\ \in\ \mathcal{O}^t_{X_{sa}}\big(\varphi(S)\big)\ .$$

Proof. The proof is based on the following sequence of equivalences which will be made rigorous along the proof.

$$(3.2.2)\qquad \begin{gathered} I_{p,z_0}(g)(z) \in \mathcal{O}^t_{X_{sa}}(\varphi(S)) \\ \Updownarrow \\ I_{p,z_0}(g)\circ\varphi(w) \in \mathcal{O}^t_{W_{sa}}(S) \\ \Updownarrow \\ I_{\tilde{p},w_0}(\tilde{g})(w) \in \mathcal{O}^t_{W_{sa}}(S) \end{gathered}$$

for some $\tilde{p}(w) \in w^{-1/l'} \cdot \mathbb{C}[w^{-1/l'}]$, $l' \in \mathbb{Z}_{>0}$, $w_0 \in \overline{S}$ and $\tilde{g} \in \mathcal{O}^t_{W_{sa}}(S)$. We will obtain (3.2.2) from Proposition 3.1 by taking the amplitude of S small enough.

There exists $c \in \mathbb{Z}_{>0}$ such that, $\varphi(w) = w^c\varphi_1(w)$ and $\varphi_1(0) \neq 0$, for any $w \in W$. There exist $r, \eta_0 \in \mathbb{R}_{>0}$, $\eta_0 < 2\pi$, such that, for any open sector $S \subset\subset B(0,r) \subset W$ of amplitude smaller than η_0, $\varphi|_{\overline{S}}$ is injective. For the rest of the proof, a sector S will be supposed to have amplitude (resp. of radius) smaller than η_0 (resp. r).

Let

$$p(z) := \sum_{j=1}^{n} \frac{a_j}{z^{j/l}}\ ,$$

for $q \in \mathbb{Z}_{>0}$ and $a_j \in \mathbb{C}$ $(j = 1, \ldots, n)$.

We have

$$\begin{aligned} p\big(\varphi(w)\big) &= \sum_{j=1}^{n} \frac{a_j}{\big(w^c\varphi_1(w)\big)^{j/l}} \\ &= \sum_{j=1}^{n} a_j \frac{\varphi_{2,j}(w)}{w^{cj/l}} \\ &= \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{q_j} \frac{\beta_{j,k}}{w^{k/\lambda_j}} + \varphi_{3,j}(w) \right) \\ &= \sum_{j=1}^{q'} \frac{a'_j}{w^{j/l'}} + \psi_j(w)\ , \end{aligned}$$

for some $l', \lambda_j, q_j, q' \in \mathbb{Z}_{>0}$, $\beta_{j,k}, a'_j \in \mathbb{C}$ and $\varphi_{2,j}, \varphi_{3,j}, \psi_j$ power series in $z^{1/l''}$, for some $l'' \in \mathbb{Z}_{>0}$, converging on S and defined on $\overline{S}$.

Set

$$\tilde{p}(w) := \sum_{j=1}^{q'} \frac{a'_j}{w^{j/l'}}\ \in\ w^{-1/l'}\mathbb{C}[w^{-1/l'}]$$

and

$$h(w) := \exp\Big(\sum_{j=1}^{q'} \psi_j(w)\Big) \ \in \ \mathcal{O}_{\mathbb{C}}(S) \cap \mathcal{C}^0_{\mathbb{C}}(\overline{S}) \ .$$

It follows that

$$\exp\big(p(\varphi(w))\big) = \exp\big(\tilde{p}(w)\big)h(w) \ .$$

Consider $\tilde{p} \in w^{-1/l'}\mathbb{C}[w^{-1/l'}]$. By Proposition 3.1, there exists $\eta \in \mathbb{R}_{>0}$, such that for S an open sector of amplitude smaller that η, there exist $w_0 \in \overline{S}$, a path $\Gamma_{w_0,w} \subset \overline{S}$ from w_0 to w such that, for any $\tilde{g} \in \mathcal{O}^t_{W_{sa}}(S)$,

$$\exp\big(\tilde{p}(w)\big)\int_{\Gamma_{w_0,w}} \exp\big(-\tilde{p}(\zeta)\big)\tilde{g}(\zeta)d\zeta \ \in \ \mathcal{O}^t_{W_{sa}}(S) \ . \tag{3.2.3}$$

Since the multiplication by h and h^{-1} is a bijection on $\mathcal{O}^t_{W_{sa}}(S)$, (3.2.3) implies that, for any $\tilde{g} \in \mathcal{O}^t_{W_{sa}}(S)$,

$$\begin{aligned} &h(w)I_{\tilde{p},w_0}\big(h^{-1}\cdot \tilde{g}\big)(w) \\ &\qquad = h(w)\exp\big(\tilde{p}(w)\big)\int_{\Gamma_{w_0,w}} \exp\big(-\tilde{p}(\zeta)\big)h(\zeta)^{-1}\tilde{g}(\zeta)d\zeta \ \in \ \mathcal{O}^t_{W_{sa}}(S) \ . \end{aligned} \tag{3.2.4}$$

Set $z_0 := \varphi(w_0) \in \varphi\big(\overline{S}\big)$ and let $\Gamma_{z_0,z} := \varphi\big(\Gamma_{w_0,w}\big)$. Then, for any $g \in \mathcal{O}^t_{X_{sa}}\big(\varphi(S)\big)$,

$$I_{p,z_0}(g) \circ \varphi(w) = h(w)I_{\tilde{p},w_0}\big(h^{-1}\cdot(g\circ\varphi)\cdot\varphi'\big)(w) \ . \tag{3.2.5}$$

Up to shrinking η, we can suppose that $\eta < \eta_0$. In particular $\varphi|_{\overline{S}}$ is injective for any open sector S of amplitude smaller than η.

Since $(g\circ\varphi)\cdot\varphi' \in \mathcal{O}^t_{W_{sa}}(S)$, (3.2.4) and (3.2.5) imply that

$$I_{p,z_0}(g) \circ \varphi(w) \in \mathcal{O}^t_{W_{sa}}(S) \ .$$

Since $\varphi|_{\overline{S}}$ is injective, the conclusion follows by Theorem 2.1. □

Let us now consider the differential operator P given in (3.1.1).

Proposition 3.3. *Let J be a finite set, $W_j \subset \mathbb{C}$ open neighborhoods of 0, $\varphi_j \in \mathcal{O}_{\mathbb{C}}(W_j)$ non constant, $\varphi_j(0) = 0$ ($j \in J$). There exist $r, \eta \in \mathbb{R}_{>0}$ such that for any sector $S \subset\subset B(0,r) \subset \cap_{j\in J} W_j$ of amplitude smaller than η,*

$$P : \mathcal{O}^t_{X_{sa}}(\varphi_j(S))^m \longrightarrow \mathcal{O}^t_{X_{sa}}(\varphi_j(S))^m$$

is an epimorphism ($j \in J$).

Proof. There exists $\eta_0 \in \mathbb{R}_{>0}$ such that for any sector $S \subset\subset \cap_{j\in J} W_j$ of amplitude smaller than η_0, P has fundamental holomorphic solutions $F(z)\exp(\Lambda(z))$ on $\varphi_j(S)$, for any $j \in J$.

For $k = 1, \ldots, m$, let $p_k \in z^{-1/l}\mathbb{C}[z^{-1/l}]$ be the (k,k)-entry of Λ, for some $l \in \mathbb{Z}_{>0}$.

By Proposition 3.2, for any $j \in J, k = 1, \ldots, m$, there exist $r_{j,k}, \eta_{j,k}$ such that for any open sector $S \subset\subset B(0, r_{j,k}) \subset \cap_{j\in J} W_j$ of amplitude smaller than $\eta_{j,k}$, there exist $z_{0,j,k} \in \varphi_j\big(\overline{S}\big)$ and paths $\Gamma_{z_{0,j,k},z}$ from $z_{0,j,k}$ to $z \in \varphi_j(S)$ such that for any $g_j \in \mathcal{O}^t_{X_{sa}}(\varphi_j(S))$

$$\exp\big(p_k(z)\big)\int_{\Gamma_{z_{0,j,k},z}} \exp\big(-p_k(\zeta)\big)g_j(\zeta)d\zeta \;\in\; \mathcal{O}^t_{X_{sa}}\big(\varphi_j(S)\big)\ .$$

Set

$$r := \min\{r_{j,k};\ j \in J, k = 1, \ldots, m\}\ ,$$
$$\eta := \min\{\eta_0, \eta_{j,k};\ j \in J, k = 1, \ldots, m\}\ .$$

Let $S \subset\subset B(0,r)$ be an open sector of amplitude smaller than η. Let Γ_j be the collection of m paths $\Gamma_{z_{0,j,k},z}$, then for any $g \in \mathcal{O}^t_{X_{sa}}(\varphi_j(S))^m$

$$\exp\big(\Lambda(z)\big)\int_{\Gamma_j} \exp\big(-\Lambda(\zeta)\big)g(\zeta)d\zeta \;\in\; \mathcal{O}^t_{X_{sa}}\big(\varphi_j(S)\big)^m$$

$(j \in J)$.

Since multiplication by $F(z)$ and $\frac{F(z)^{-1}}{z^N}$ are bijections on $\mathcal{O}^t_{X_{sa}}\big(\varphi(V_j)\big)^m$, we obtain that, for any $g \in \mathcal{O}^t_{X_{sa}}(\varphi_j(S))^m$,

$$F(z)\exp\big(\Lambda(z)\big)\int_{\Gamma_j} \exp\big(-\Lambda(\zeta)\big)\frac{F(\zeta)^{-1}}{\zeta^N}g(\zeta)d\zeta \;\in\; \mathcal{O}^t_{X_{sa}}\big(\varphi_j(S)\big)^m$$

$(j \in J)$.

The conclusion follows by Lemma 3.1. □

Lemma 3.2. *Let $U, V \in \mathrm{Op}^c_{X_{sa}}$. If P is a surjective endomorphism both on $\mathcal{O}^t_{X_{sa}}(U)^m$ and $\mathcal{O}^t_{X_{sa}}(V)^m$. Then P is a surjective endomorphism on $\mathcal{O}^t_{X_{sa}}(U \cap V)^m$.*

Proof. The result follows from the exact sequence (2.2.1). □

Theorem 3.2. *Let $U \in \mathrm{Op}^c_{X_{sa}}$ with $0 \in \partial U$. There exist an open neighborhood $W \subset X$ of 0 and $\{U_j\}_{j\in J} \in \mathrm{Cov}_{sa}(U \cap W)$ such that, for any $j \in J$,*

$$P : \mathcal{O}^t_{X_{sa}}(U_j)^m \longrightarrow \mathcal{O}^t_{X_{sa}}(U_j)^m$$

is an epimorphism.

Proof. By Theorem 1.4, there exist an open neighborhood W of 0, a finite set J, open sectors $S_{j,k}$, $\varphi_{j,k} \in \mathcal{O}_{\mathbb{C}}\big(\overline{S_{j,k}}\big)$, such that $\varphi_{j,k}(0) = 0$, $\varphi_{j,k}|_{\overline{S_{j,k}}}$ is injective ($j \in J, k = 1, 2$) and

$$U \cap W = \bigcup_{j \in J} \Big(\varphi_{j,1}\big(S_{j,1}\big) \cap \varphi_{j,2}\big(S_{j,2}\big)\Big) \ . \tag{3.2.6}$$

Further, by Remark 1.2, we can suppose that the amplitude and the radius of $S_{j,k}$ are arbitrarily small. In particular, Proposition 3.3 applies and we have that P is an epimorphism on $\varphi_{j,k}\big(S_{j,k}\big)$, for any $j \in J$, $k = 1, 2$.

The conclusion follows from (3.2.6) and Lemma 3.2. □

The following Corollary is an obvious consequence of Theorem 3.2. In view of Proposition 2.1, it states that P is an epimorphism of sheaves on X_{sa}.

Corollary 3.1. *Let $U \in \mathrm{Op}^c_{X_{sa}}$ with $0 \in \partial U$. There exist an open neighborhood $W \subset X$ of 0 such that for any $g \in \mathcal{O}^t_{X_{sa}}(U)^m$, there exist $\{U_j\}_{j \in J} \in \mathrm{Cov}_{sa}(U \cap W)$ and $u_j \in \mathcal{O}^t_{X_{sa}}(U_j)^m$ satisfying*

$$Pu_j = g|_{U_j} \qquad\qquad (j \in J) \ .$$

Proof. Obvious. □

§4. Tempered Holomorphic Solutions

In this section we deal with solutions of $\mathcal{D}_X$-modules, for X a complex analytic curve.

In the first subsection we recall some classical results about $\mathcal{D}_X$-modules. First, for a coherent $\mathcal{D}_X$-module $\mathcal{M}$, we define the complex of holomorphic (resp. tempered holomorphic) solutions of $\mathcal{M}$, $\mathscr{S}ol\,\mathcal{M}$ (resp. $\mathscr{S}ol^t\mathcal{M}$). Then, we recall that, if $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module, then $\mathscr{S}ol^t\mathcal{M} \simeq \mathscr{S}ol\,\mathcal{M}$. Moreover we recall that a holonomic $\mathcal{D}_X$-module is locally an extension of a $\mathcal{D}_X$-module supported on a point (hence regular) and a $\mathcal{D}_X$-module locally isomorphic to a differential operator.

In the second subsection we state the existence theorem in the framework of $\mathcal{D}$-modules. It asserts that, for a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, $H^1\big(\mathscr{S}ol^t\mathcal{M}\big)$ is isomorphic to $H^1\big(\mathscr{S}ol\mathcal{M}\big)$. Using the results recalled in the first subsection, we reduce to the case of a differential operator. Such case is the object of the third subsection.

In the third subsection we treat the case of a differential operator. Making use of the language of sheaves on X_{sa}, we give a more natural setting and statement to the results obtained in Section 3.

In the fourth subsection we prove that $\mathscr{S}ol^t(\mathcal{M})$ is $\mathbb{R}$-constructible in the sense of sheaves on X_{sa}.

Throughout this section, X will be a complex analytic curve.

§4.1. Classical results on $\mathcal{D}$-modules

For a detailed and comprehensive exposition of $\mathcal{D}_X$-modules we refer to [3] and [11]. For an introduction to derived categories and cohomology of sheaves, we refer to [12].

We denote by $\mathcal{D}_X$ the sheaf of differential operators with holomorphic coefficients on X, $\mathrm{Mod}(\mathcal{D}_X)$ the category of $\mathcal{D}_X$-modules, $\mathrm{Mod}_{coh}(\mathcal{D}_X)$ the full subcategory of $\mathrm{Mod}(\mathcal{D}_X)$ consisting of coherent $\mathcal{D}_X$-modules.

For $\mathcal{M} \in \mathrm{Mod}_{coh}(\mathcal{D}_X)$ we denote by $\mathrm{char}\mathcal{M}$ the characteristic variety of $\mathcal{M}$. Recall that $\mathcal{M} \in \mathrm{Mod}_{coh}(\mathcal{D}_X)$ is said *holonomic* if $\dim \mathrm{char}\mathcal{M} = 1$. We denote by $\mathrm{Mod}_h(\mathcal{D}_X) \subset \mathrm{Mod}_{coh}(\mathcal{D}_X)$ the abelian category of holonomic $\mathcal{D}_X$-modules.

We denote by $D^b(X_{sa})$ (resp. $D^b(X)$, $D^b(\mathcal{D}_X)$) the bounded derived category of sheaves of $\mathbb{C}$-vector spaces on X_{sa} (resp. sheaves of $\mathbb{C}$-vector spaces on X, $\mathcal{D}_X$-modules). We denote by $D^b_{coh}(\mathcal{D}_X)$ (resp. $D^b_h(\mathcal{D}_X)$) the full subcategory of $D^b(\mathcal{D}_X)$ consisting of bounded complexes whose cohomology groups are coherent (resp. holonomic). For $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$, set $\mathrm{char}\mathcal{M} := \cup_{j\in\mathbb{Z}} \mathrm{char} H^j(\mathcal{M})$.

Let T^*X be the cotanget bundle on X, $\pi_X : T^*X \to X$ the canonical projection, $T^*_X X$ the zero section of T^*X and $\dot{T}^*X := T^*X \setminus T^*_X X$.

For $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$, set

$$S(\mathcal{M}) := \pi_X \left(\mathrm{char}\mathcal{M} \cap \dot{T}^*X \right) .$$

It is well known that, if $\mathcal{M} \in D^b_h(\mathcal{D}_X)$, then $S(\mathcal{M})$ is a discrete subset of X.

Definition 4.1. An object $\mathcal{M} \in D^b_h(\mathcal{D}_X)$ is said *regular holonomic* if, for any $x \in X$,

$$\mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X,x}) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, \widehat{\mathcal{O}}_{X,x}) ,$$

where $\widehat{\mathcal{O}}_{X,x}$ is the $\mathcal{D}_{X,x}$-module of formal power series at x. We denote by $D^b_{rh}(\mathcal{D}_X)$ the full subcategory of $D^b_h(\mathcal{D}_X)$ of regular holonomic $\mathcal{D}_X$-modules.

Recall that $\varrho : X \to X_{sa}$ is the natural morphism of sites. For a coherent $\mathcal{D}_X$-module $\mathcal{M}$, we set for short

$$\mathscr{S}ol\mathcal{M} = R\varrho_* \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathcal{M}, \mathcal{O}_X\big) \in D^b\big(X_{sa}\big)$$
$$\mathscr{S}ol^t\mathcal{M} = \mathrm{R}\mathcal{H}om_{\varrho_!\mathcal{D}_X}\big(\varrho_!\mathcal{M}, \mathcal{O}^t_{X_{sa}}\big) \in D^b\big(X_{sa}\big)\ .$$

For Theorem 4.1 below, see [10]. We recall it here with the notation of [15].

Theorem 4.1. *Let X be a complex analytic manifold, $\mathcal{M} \in D^b_{rh}(\mathcal{D}_X)$. The natural morphism in $D^b(X_{sa})$*

$$\mathscr{S}ol^t\mathcal{M} \longrightarrow \mathscr{S}ol\mathcal{M}$$

is an isomorphism.

For $a \in X$, let $\Gamma_{[\{a\}]}(\cdot)$ be the tempered support functor on a. For $\mathcal{M} \in D^b(\mathcal{D}_X)$, denote

$$\mathcal{M}(*a) := \mathcal{O}_X(*a) \otimes_{\mathcal{O}_X} \mathcal{M}\ ,$$

where $\mathcal{O}_X(*a)$ is the $\mathcal{D}_X$-module of meromorphic functions at a.

Proposition 4.1 below follows from Kashiwara's Lemma (see [11, Theorem 4.30]) and Kashiwara's thesis [8].

Proposition 4.1. *Let $a \in X$.*

(i) *For $\mathcal{M} \in D^b_h(\mathcal{D}_X)$, there exists a distinguished triangle*

$$R\Gamma_{[\{a\}]}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}(*a) \xrightarrow{+1}\ .$$

(ii) *If $\mathcal{M} \in D^b_h(\mathcal{D}_X)$, then $R\Gamma_{[\{a\}]}\mathcal{M} \in D^b_{rh}(\mathcal{D}_X)$.*

(iii) *If $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_X)$, then there exist an open neighborhood W of a and $P \in \mathcal{D}_X(W)$ such that*

$$\mathcal{M}(*a)|_W \simeq \frac{\mathcal{D}_X|_W}{\mathcal{D}_X|_W \cdot P}\ .$$

§4.2. Existence theorem for holonomic $\mathcal{D}_X$-modules

Theorem 4.2. *Let $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_X)$. The natural morphism of sheaves on X_{sa}*

$$H^1\Big(\mathscr{S}ol^t(\mathcal{M})\Big) \longrightarrow H^1\Big(\mathscr{S}ol(\mathcal{M})\Big) \tag{4.2.1}$$

is an isomorphism.

Proof. The problem is local on X_{sa}. Since $S(\mathcal{M})$ is a discrete set, it is sufficient to prove the statement in the case $S(\mathcal{M}) \subset \{a\}$, for $a \in X$.

Now, using Theorem 4.1 and Proposition 4.1, it is sufficient to prove the statement for $\mathcal{M} = \frac{\mathcal{D}_W}{\mathcal{D}_W \cdot P}$, for $W \subset X$ an open neighborhood of a and $P \in \mathcal{D}_X(W)$.

That is, up to shrinking X, we are reduced to prove that the natural morphism of sheaves on X_{sa}

$$H^1\Big(\mathscr{S}ol^t\Big(\frac{\mathcal{D}_X}{\mathcal{D}_X \cdot P}\Big)\Big) \longrightarrow H^1\Big(\mathscr{S}ol\Big(\frac{\mathcal{D}_X}{\mathcal{D}_X \cdot P}\Big)\Big)$$

is an isomorphism.

This is the object of Subsection 4.3 below. □

§4.3. The case of a single operator

For this subsection, let $X \subset \mathbb{C}$ be an open disc centered at the origin and

$$P = \sum_{j=0}^{m} a_j(z) \frac{d^j}{dz^j} \ , \tag{4.3.1}$$

for $a_j(z) \in \mathcal{O}_X(X)$ $(j = 1, \ldots, m)$, a_m not identically zero.

Set $S(P) := S\big(\mathcal{D}_X/\mathcal{D}_X \cdot P\big)$, then we have

$$S(P) = \big\{z \in X;\ a_m(z) = 0\big\} \ .$$

Remark that, since ϱ_* is exact on constructible sheaves and $\mathcal{O}_X$ is ϱ_*-acyclic,

$$\frac{\varrho_* \mathcal{O}_X}{P \varrho_* \mathcal{O}_X} \simeq \varrho_* \frac{\mathcal{O}_X}{P \mathcal{O}_X} \ .$$

Proposition 4.2. *The natural morphism of sheaves on* X_{sa}

$$\frac{\mathcal{O}^t_{X_{sa}}}{P\mathcal{O}^t_{X_{sa}}} \longrightarrow \varrho_* \frac{\mathcal{O}_X}{P\mathcal{O}_X} \ , \tag{4.3.2}$$

is an isomorphism.

We need two preliminary lemmas.

Lemma 4.1. *Let* $U \in \mathrm{Op}^c_{X_{sa}}$, $S(P) \cap U = \emptyset$. *For any* $g \in \mathcal{O}^t(U)$, *there exist* $\{U_j\}_{j \in J} \in \mathrm{Cov}_{sa}(U)$ *and* $u_j \in \mathcal{O}^t(U_j)$ *such that* $Pu_j = g|_{U_j}$.

Proof. Since the problem is local on X_{sa} and $S(P)$ is a discrete set, we can suppose that $S(P) = \{0\}$.

First case: $0 \notin \partial U$. It follows that P is a regular operator on a neighborhood of $\overline{U}$. The result follows immediately by Theorem 4.1.

Second case: $0 \in \partial U$. The result follows from Theorem 3.1 and the first case. $\square$

Lemma 4.2. *Let $U \subset X$ be an open ball and assume $\partial U \cap S(P) = \emptyset$. Then, the natural morphism*

$$\frac{\mathcal{O}^t_{X_{sa}}(U)}{P\mathcal{O}^t_{X_{sa}}(U)} \xrightarrow{\varphi_t} \frac{\mathcal{O}_X(U)}{P\mathcal{O}_X(U)}$$

is an isomorphism.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \dfrac{\mathcal{O}_X(\overline{U})}{P\mathcal{O}_X(\overline{U})} & \xrightarrow{\varphi_c} & \dfrac{\mathcal{O}_X(U)}{P\mathcal{O}_X(U)} \\ & \searrow & \uparrow{\scriptstyle \varphi_t} \\ & & \dfrac{\mathcal{O}^t_{X_{sa}}(U)}{P\mathcal{O}^t_{X_{sa}}(U)} \end{array} .$$

The proof consists of two steps:

(i) φ_c is an isomorphism,

(ii) φ_t is injective.

(i) Consider the complex

$$\mathscr{F} := \quad 0 \longrightarrow \mathcal{O}_X \xrightarrow{P} \mathcal{O}_X \longrightarrow 0 .$$

Since $\partial U \cap S(P) = \emptyset$, then $R\Gamma_{\partial U}\big(\mathscr{F}|_{\overline{U}}\big) \simeq 0$. It follows that $R\Gamma\big(\overline{U}, \mathscr{F}\big) \xrightarrow{\sim} R\Gamma\big(U, \mathscr{F}\big)$. In particular, since $\mathcal{O}_X$ is $\Gamma\big(U, \cdot\big)$ and $\Gamma\big(\overline{U}, \cdot\big)$-acyclic, it follows that φ_c is an isomorphism.

(ii) Let $h \in \ker(\varphi_t)$, that is, $h \in \mathcal{O}^t_{X_{sa}}(U)$ and there exists $u \in \mathcal{O}_X(U)$ satisfying $Pu = h$. Let us prove that $u \in \mathcal{O}^t_{X_{sa}}(U)$.

The problem is local on X_{sa}. Clearly, $u|_{U_0} \in \mathcal{O}^t_{X_{sa}}(U_0)$ for any $U_0 \in \mathrm{Op}^c_{U_{sa}}$. So, let $x \in \partial U$, there exists an open neighborhood W of x such that $S(P) \cap \overline{U \cap W} = \emptyset$. In particular, P is a regular operator on $\overline{U \cap W}$.

By Theorem 4.1, the complex

$$0 \longrightarrow \mathcal{O}^t_{X_{sa}}|_{U \cap W} \xrightarrow{P} \mathcal{O}^t_{X_{sa}}|_{U \cap W} \longrightarrow 0$$

is concentrated in degree 0. In particular, there exists $\{V_j\}_{j\in J} \in \mathrm{Cov}_{sa}(U\cap W)$ and $v_j \in \mathcal{O}^t_{X_{sa}}(V_j)$ such that $Pv_j = h|_{V_j}$, that is, $P(v_j - u|_{V_j}) = 0$. Since $S(P)\cap \overline{V}_j = \emptyset$, then $v_j - u|_{V_j}$ extends holomorphically up to the boundary of V_j. That is $v_j - u|_{V_j} = w$, for some $w \in \mathcal{O}_X(\overline{V}_j)$. In particular $u|_{V_j} \in \mathcal{O}^t_{X_{sa}}(V_j)$. The conclusion follows. □

Now we can prove Proposition 4.2.

Proof of Proposition 4.2. Since $S(P)$ is a discrete set and the statement is local on X_{sa}, we can suppose that $S(P) \subset \{0\}$.

We are going to prove that, for any $U \in \mathrm{Op}^c_{X_{sa}}$, the natural morphism

$$\frac{\mathcal{O}^t_{X_{sa}}}{P\mathcal{O}^t_{X_{sa}}}(U) \xrightarrow{\varphi} \frac{\mathcal{O}_X}{P\mathcal{O}_X}(U) \tag{4.3.3}$$

is an isomorphism.

Consider the presheaves on $\mathrm{Op}^c_{X_{sa}}$ defined by

$$\mathrm{Op}^c_{X_{sa}} \ni U \longmapsto F^t(U) := \frac{\mathcal{O}^t_{X_{sa}}(U)}{P\mathcal{O}^t_{X_{sa}}(U)}\ ,$$
$$\mathrm{Op}^c_{X_{sa}} \ni U \longmapsto F(U) := \frac{\mathcal{O}_X(U)}{P\mathcal{O}_X(U)}\ .$$

Recall that, for a presheaf G on X_{sa}, we denote by G^a the associated sheaf on X_{sa}. We have that $\frac{\mathcal{O}^t_{X_{sa}}}{P\mathcal{O}^t_{X_{sa}}} := F^{t,a}$ and $\varrho_* \frac{\mathcal{O}_X}{P\mathcal{O}_X} \simeq F^a$.

Suppose that $0 \notin U$. Then $F^a(U) \simeq 0$ and Lemma 4.1 implies that $F^{t,a}(U) \simeq 0$.

Suppose now that $0 \in U$. First, let us prove that φ is surjective.

Recall (2.1.1). Let $s \in F^a(U)$. Since the inductive limit considered in (2.1.2) is filtrant, s can be identified to $(s_0, \dots, s_n) \in F(S)$, for $S = \{U_0, \dots, U_n\} \in \mathrm{Cov}_{sa}(U)$ and $s_j \in F(U_j)$ $(j = 0, \dots, n)$. Up to take a refinement, we can suppose that $0 \in U_0$ is an open ball, $s_0 \in F^t(U_0)$, $0 \notin U_k$, $s_k = 0$ and $s_0|_{U_0\cap U_k} = 0$ as an element of $F^t(U_0 \cap U_k)$ $(k \neq 0)$.

It follows that $(s_0, 0, \dots, 0)$ defines an element of $F^t(S)$. In particular, it defines a section $s^t \in F^{t,a}(U)$ such that $\varphi(s^t) = s$. Hence φ is surjective.

Now, let us show that φ is injective.

Let $s^t \in F^{t,a}(U)$ such that $\varphi(s^t) = 0$. As before, s^t can be identified with $(s^t_0, \dots, s^t_n) \in F^t(S)$, for $S = \{U_0, \dots, U_n\} \in \mathrm{Cov}_{sa}(U)$ and $s^t_j \in F^t(U_j)$ $(j = 0, \dots, n)$. Up to take a refinement of S, we can suppose that $0 \in U_0$ is an open ball, $0 \notin U_k$ and $s^t_k = 0$ for $k \neq 0$. That is, s^t can be identified to $(s^t_0, 0, \dots, 0) \in F^t(S)$.

Now, let $\varphi_t : F^t(U_0) \to F(U_0)$. By Lemma 4.2, φ_t is injective. Clearly, $\varphi(s^t) = 0$ implies that $\varphi_t(s^t_0) = 0$. Hence $s^t_0 = 0$ and φ is injective. □

§4.4. $\mathbb{R}$-constructibility for tempered holomorphic solutions

In the study of classical solution sheaves of $\mathcal{D}$-modules, the notions of micro-support and $\mathbb{R}$-constructibility play a central role. We refer to [12] for definitions and classical results. In [15], M. Kashiwara and P. Schapira defined such notions in the context of sheaves on X_{sa}. Further, they conjectured some results on tempered holomorphic solutions of holonomic $\mathcal{D}$-modules involving $\mathbb{R}$-constructibility corresponding to classical results on holomorphic solutions.

Proposition 4.3 below follows from the results obtained in Section 3. It proves, in dimension 1, a conjecture from [15] stating that, for $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module, $\mathscr{S}ol^t(\mathcal{M})$ is $\mathbb{R}$-constructible in the sense of sheaves on X_{sa}.

Denote by $D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ the full triangulated subcategory of the bounded derived category of $\mathrm{Mod}(\mathbb{C}_X)$ consisting of complexes whose cohomology groups are $\mathbb{R}$-constructible. In what follows, for $F \in D^b(\mathbb{C}_{X_{sa}})$ and $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, we set for short

$$R\mathcal{H}om_{\mathbb{C}_X}(G, F) := \varrho^{-1} R\mathcal{H}om_{\mathbb{C}_{X_{sa}}}(G, F) \in D^b(\mathbb{C}_X)$$

and

$$RHom_{\mathbb{C}_X}(G, F) := R\Gamma(X, R\mathcal{H}om_{\mathbb{C}_X}(G, F)) \; .$$

Proposition 4.3. *Let X be a complex curve and let $\mathcal{M} \in D^b_h(\mathcal{D}_X)$. Then, for any $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, $R\mathcal{H}om_{\mathbb{C}_X}(G, \mathscr{S}ol^t(\mathcal{M})) \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$.*

Proof. We may suppose that $X \subset \mathbb{C}$ is an open ball centered at the origin. By dévissage we may suppose that $\mathcal{M} \simeq \frac{\mathcal{D}_X}{\mathcal{D}_X P}$, for P a differential operator as in (4.3.1) such that $S(P) \subset \{0\}$. Since the triangulated category $D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ is generated by the objects $\mathbb{C}_U$, for $U \in \mathrm{Op}^c_{X_{sa}}$, we may assume that $G = \mathbb{C}_U$ for such an U.

Let $V \in \mathrm{Op}^c_{X_{sa}}$ such that $0 \notin \overline{V}$, then Theorem 4.1 implies that $\mathscr{S}ol^t(\mathcal{M})|_V \simeq \mathscr{S}ol(\mathcal{M})|_V$. In particular,

$$R\mathcal{H}om_{\mathbb{C}_X}(\mathbb{C}_U, \mathscr{S}ol^t(\mathcal{M}))|_{X\setminus\{0\}} \simeq R\mathcal{H}om_{\mathbb{C}_X}(\mathbb{C}_U, \mathscr{S}ol(\mathcal{M}))|_{X\setminus\{0\}} \; .$$

It follows that $R\mathcal{H}om_{\mathbb{C}_X}(\mathbb{C}_U, \mathscr{S}ol^t(\mathcal{M}))$ is weakly-$\mathbb{R}$-constructible on X and $\mathbb{R}$-constructible on $X \setminus \{0\}$.

Since $\mathcal{H}\mathrm{om}_{\mathbb{C}_X}(\mathbb{C}_U, \mathscr{S}ol^t(\mathcal{M}))$ is a subsheaf of $\mathcal{H}\mathrm{om}_{\mathbb{C}_X}(\mathbb{C}_U, \mathscr{S}ol(\mathcal{M}))$ and $R\mathcal{H}om_{\mathbb{C}_X}(\mathbb{C}_U, \mathscr{S}ol^t(\mathcal{M}))$ is concentrated in degrees 0 and 1, it remains to prove that the stalk at 0 of $\mathcal{E}xt^1_{\mathbb{C}_X}(\mathbb{C}_U, \mathscr{S}ol^t(\mathcal{M}))$ has finite dimension.

Since $R\mathcal{H}om_{\mathbb{C}_X}(\mathbb{C}_U, \mathscr{S}ol^t(\mathcal{M}))$ is weakly-$\mathbb{R}$-constructible, there exists an open ball B such that

$$\mathcal{E}xt^1_{\mathbb{C}_X}\big(\mathbb{C}_U, \mathscr{S}ol^t(\mathcal{M})\big)_0 \simeq R^1 Hom_{\mathbb{C}_X}\big(\mathbb{C}_{U\cap B}, \mathscr{S}ol^t(\mathcal{M})\big) \ . \tag{4.4.1}$$

Recall that $\mathscr{S}ol^t(\mathcal{M})$ is represented by the complex

$$0 \longrightarrow \mathcal{O}^t_{X_{sa}} \xrightarrow{P} \mathcal{O}^t_{X_{sa}} \longrightarrow 0$$

and that $\mathcal{O}^t_{X_{sa}}$ is $\Gamma(V,\cdot)$-acyclic for $V \in \mathrm{Op}^c_{X_{sa}}$. It follows that the object $RHom_{\mathbb{C}_X}(\mathbb{C}_U, \mathscr{S}ol^t(\mathcal{M}))$ is represented by the complex

$$\Gamma\big(U, \mathscr{S}ol^t(\mathcal{M})\big) := 0 \longrightarrow \mathcal{O}^t_{X_{sa}}(U) \xrightarrow{P} \mathcal{O}^t_{X_{sa}}(U) \longrightarrow 0 \ .$$

In particular

$$R^1 Hom_{\mathbb{C}_X}\big(\mathbb{C}_U, \mathscr{S}ol^t(\mathcal{M})\big) \simeq H^1\big(\Gamma\big(U, \mathscr{S}ol^t(\mathcal{M})\big)\big) \ .$$

We conclude the proof by showing that $H^1\big(\Gamma\big(U, \mathscr{S}ol^t(\mathcal{M})\big)\big)$ has finite dimension.

First consider the case $0 \in U$. By (4.4.1), we can suppose that U is an open ball. Then, by Lemma 4.2, we have

$$H^1\big(\Gamma\big(U, \mathscr{S}ol^t(\mathcal{M})\big)\big) \simeq H^1\big(\Gamma\big(U, \mathscr{S}ol(\mathcal{M})\big)\big)$$

and the conclusion follows.

Suppose now that $0 \in \partial U$.

By Theorem 3.2 and Lemma 3.2, there exists a finite covering $\{U_j\}_{j\in J} \in \mathrm{Cov}_{sa}(U)$ such that, for any $K \subset J$

$$H^1\big(\Gamma\big(U_K, \mathscr{S}ol^t(\mathcal{M})\big)\big) \simeq 0 \ , \tag{4.4.2}$$

where we have set for short $U_K := \cap_{k\in K} U_k$.

Arguing by induction on $n \geq 1$, we are going to prove that, for any $n \geq 1$ and $K_1, \dots, K_n \subset J$,

$$H^1\big(\Gamma\big(\cup_{h=1}^n U_{K_h}, \mathscr{S}ol^t(\mathcal{M})\big)\big)$$

has finite dimension. This will conclude the proof.

If $n = 1$, the result follows at once from (4.4.2).

Suppose now that, for any $K'_1, \dots, K'_{n-1} \subset J$,

$$\dim\ H^1\big(\Gamma\big(\cup_{h=1}^{n-1} U_{K'_h}, \mathscr{S}ol^t(\mathcal{M})\big)\big) < +\infty\ . \tag{4.4.3}$$

Consider $K_1, \dots, K_n \subset J$ and the following distinguished triangle

$$\begin{aligned}\Gamma\big(\cup_{h=1}^{n} U_{K_h}, \mathscr{S}ol^t(\mathcal{M})\big) &\longrightarrow \\ \Gamma\big(\cup_{h=1}^{n-1} U_{K_h}, \mathscr{S}ol^t(\mathcal{M})\big) \oplus \Gamma\big(U_{K_n}, \mathscr{S}ol^t(\mathcal{M})\big) &\longrightarrow \\ \Gamma\big(\cup_{h=1}^{n-1} U_{K_h} \cap U_{K_n}, \mathscr{S}ol^t(\mathcal{M})\big) &\xrightarrow{+1}\ .\end{aligned} \tag{4.4.4}$$

Clearly $U_{K_h} \cap U_{K_n} = U_{K_h \cup K_n}$. Then (4.4.3) implies that the second and the third term of the distinguished triangle (4.4.4) have finite dimensional cohomology groups.

The conclusion follows. □

References

[1] E. Bierstone and P. D. Milman, *The local geometry of analytic mappings*, ETS, Pisa, 1988.

[2] ———, Semianalytic and subanalytic sets, Inst. Hautes Études Sci. Publ. Math. No. 67 (1988), 5–42. `http://www.numdam.org`

[3] J.-E. Björk, *Analytic $\mathcal{D}$-modules and applications*, Kluwer Acad. Publ., Dordrecht, 1993.

[4] M. Coste, *An introduction to o-minimal geometry.* Pisa. Istituti Ed. e Poligrafici Intern., Università di Pisa, Dipartimento di Matematica, 2000.

[5] L. van den Dries, *Tame topology and o-minimal structures*, Cambridge Univ. Press, Cambridge, 1998.

[6] O. Forster, *Lectures on Riemann surfaces*, Translated from the German by Bruce Gilligan, Springer, New York, 1981.

[7] N. Honda, On the solvability of ordinary differential equations in the space of distributions, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **39** (1992), no. 2, 207–232.

[8] M. Kashiwara, Algebraic study of systems of partial differential equations, Mém. Soc. Math. France (N.S.) No. 63 (1995), xiv+72 pp.

[9] ———, Faisceaux constructibles et systèmes holonômes d'équations aux dérivées partielles linéaires à points singuliers réguliers, in *Séminaire Goulaouic-Schwartz, 1979–1980 (French)*, Exp. 19, 7 pp, École Polytech., Palaiseau.

[10] ———, The Riemann-Hilbert problem for holonomic systems, Publ. Res. Inst. Math. Sci. **20** (1984), no. 2, 319–365.

[11] ———, *$\mathcal{D}$-modules and microlocal calculus*, Translated from the 2000 Japanese original by Mutsumi Saito, Amer. Math. Soc., Providence, RI, 2003.

[12] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Springer, Berlin, 1990.

[13] ———, Moderate and formal cohomology associated with constructible sheaves, Mém. Soc. Math. France (N.S.) No. 64 (1996), iv+76 pp.

[14] ———, Ind-sheaves, Astérisque No. 271 (2001), 136 pp.

[15] ———, Microlocal study of ind-sheaves. I. Micro-support and regularity, Astérisque No. 284 (2003), 143–164.

[16] S. Łojasiewicz, Sur le problème de la division, Studia Math. **18** (1959), 87–136.

[17] ———, *Ensembles semi-analytiques*, IHES 1965.

[18] S. Łojasiewicz, Sur la géométrie semi- et sous-analytique, Ann. Inst. Fourier (Grenoble) **43** (1993), no. 5, 1575–1595. `http://www.numdam.org`

[19] B. Malgrange, *Ideals of differentiable functions*, Tata Inst. Fund. Res., Bombay, 1967.

[20] ———, *Équations différentielles à coefficients polynomiaux*, Progr. Math., 96, Birkhäuser Boston, Boston, MA, 1991.

[21] J.-P. Ramis and Y. Sibuya, Hukuhara domains and fundamental existence and uniqueness theorems for asymptotic solutions of Gevrey type, Asymptotic Anal. **2** (1989), no. 1, 39–94.

[22] C. Sabbah, Équations différentielles à points singuliers irréguliers et phénomène de Stokes en dimension 2, Astérisque, No. 263 (2000), viii+190 pp.

[23] Y. Siu, Holomorphic functions of polynomial growth on bounded domains, Duke Math. J. **37** (1970), 77–84.

[24] W. Wasow, *Asymptotic expansions for ordinary differential equations*, Interscience Publishers John Wiley & Sons, Inc., New York, 1965.