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Asymptotic Behavior of the Semigroup Associated with the Linearized Compressible Navier-Stokes Equation in an Infinite Layer

By

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Abstract

Asymptotic behavior of solutions to the linearized compressible Navier-Stokes equation around a given constant state is considered in an infinite layer $\mathbf{R}^{n-1} \times (0, a)$, $n \geq 2$, under the no slip boundary condition for the momentum. The L^p decay estimates of the associated semigroup are established for all $1 \leq p \leq \infty$. It is also shown that the time-asymptotic leading part of the semigroup is given by an $n - 1$ dimensional heat semigroup.

§1. Introduction

This paper is concerned with the large time behavior of solutions to the following system of equations:

$$\partial_t u + Lu = 0, \tag{1.1}$$

where $u = \begin{pmatrix} \phi \\ m \end{pmatrix}$ with $\phi = \phi(x, t) \in \mathbf{R}$ and $m = {}^T(m^1(x, t), \ldots, m^n(x, t)) \in \mathbf{R}^n$, $n \geq 2$, and L is an operator defined by

$$L = \begin{pmatrix} 0 & \gamma \mathrm{div} \\ \gamma \nabla & -\nu \Delta I_n - \widetilde{\nu} \nabla \mathrm{div} \end{pmatrix}$$

with positive constants ν and γ and a nonnegative constant $\widetilde{\nu}$. Here $t > 0$ denotes the time variable and $x \in \mathbf{R}^n$ denotes the space variable; the superscript$^T \cdot$

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stands for the transposition; I_n is the $n \times n$ identity matrix; and div, ∇ and Δ are the usual divergence, gradient and Laplacian with respect to x. We consider (1.1) in an infinite layer

$$\Omega = \mathbf{R}^{n-1} \times (0, a) = \left\{ x = \begin{pmatrix} x' \\ x_n \end{pmatrix} ; \, x' \in \mathbf{R}^{n-1}, \, 0 < x_n < a \right\}$$

under the boundary condition

$$m|_{\partial\Omega} = 0, \tag{1.2}$$

together with the initial condition

$$u|_{t=0} = u_0 = \begin{pmatrix} \phi_0 \\ m_0 \end{pmatrix}. \tag{1.3}$$

Problem (1.1)–(1.3) is obtained by the linearization of the compressible Navier-Stokes equation around a motionless state with a positive constant density, where ϕ is the perturbation of the density and m is the momentum.

In [7] we showed that $-L$ generates an analytic semigroup $\mathscr{U}(t)$ in $W^{1,p} \times L^p$ for $1 < p < \infty$. In this paper we establish an L^p decay estimate of $\mathscr{U}(t)$ for all $1 \le p \le \infty$ and give a more detailed description of the behavior of $\mathscr{U}(t)$ as $t \to \infty$.

One of the primary factors affecting the large time behavior of solutions to (1.1)–(1.3) is that (1.1) is a symmetric hyperbolic-parabolic system. Due to this structure, solutions of (1.1) exhibit characters of solutions of both wave and heat equations. In the case of the Cauchy problem on the whole space $\mathbf{R}^n$, detailed descriptions of large time behavior of solutions have been obtained ([5, 6, 11, 13, 14]). Hoff and Zumbrun [5, 6] showed that there appears some interesting interaction of hyperbolic and parabolic aspects of (1.1) in the decay properties of L^p norms with $1 \le p \le \infty$. It was shown in [5, 6] that the solution is asymptotically written in the sum of two terms, one is the solution of the heat equation and the other is given by the convolution of the heat kernel and the fundamental solution of the wave equation. The latter one is called the diffusion wave and it decays faster than the heat kernel in L^p norm for $p > 2$ while slower for $p < 2$. This decay property of the diffusion wave also appears in the exterior domain problem ([12]). In the case of the half space problem, it was shown in [8, 9] that not only the above mentioned behavior of the diffusion wave appears but also some difference to the Cauchy problem appears in the decay property of the spatial derivatives due to the presence of the unbounded boundary.

There is one more factor that affects the large time behavior of solutions to (1.1)–(1.3). In contrast to the domains mentioned above, the infinite layer Ω has a finite thickness in the x_n direction. This implies that the Poincaré inequality holds. If one considers, for example, the incompressible Navier-Stokes equation under the no slip boundary condition (1.2), then it is easy to see that, by the Poincaré inequality, the L^2 norm of the solution tends to zero exponentially as $t \to \infty$. In the case of problem (1.1)–(1.3), the Poincaré inequality holds for m but not for ϕ. This leads to that the spectrum reaches the origin but it is like the one such as the $n-1$ dimensional Laplace operator. As a result, no hyperbolic feature appears in the leading part of the solution.

In fact, we will show that the solution $u = \mathscr{U}(t)u_0$ of (1.1)–(1.3) satisfies

$$(1.4) \qquad \|u(t)\|_{L^p} = O(t^{-\frac{n-1}{2}(1-\frac{1}{p})}), \quad \|u(t) - u^{(0)}(t)\|_{L^p} = O(t^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}})$$

for any $1 \le p \le \infty$ as $t \to \infty$. Here $u^{(0)} = (\phi^{(0)}(x', t), 0)$ and $\phi^{(0)}(x', t)$ is a function satisfying

$$\partial_t \phi^{(0)} - \kappa \Delta' \phi^{(0)} = 0, \quad \phi^{(0)}\Big|_{t=0} = \frac{1}{a}\int_0^a \phi_0(x', x_n)\, dx_n,$$

where $\kappa = \frac{a^2\gamma^2}{12\nu}$ and $\Delta' = \partial_{x_1}^2 + \cdots + \partial_{x_{n-1}}^2$.

The proof of (1.4) is based on a detailed analysis of the resolvent $(\lambda + L)^{-1}$ associated with (1.1)–(1.3). We will consider the Fourier transform $(\lambda + \widehat{L}_{\xi'})^{-1}$ of the resolvent in $x' \in \mathbf{R}^{n-1}$, where $\xi' \in \mathbf{R}^{n-1}$ denotes the dual variable. The semigroup $\mathscr{U}(t)$ generated by $-L$ is then written as $\mathscr{U}(t) = \mathscr{F}_{\xi'}^{-1}\left[\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda + \widehat{L}_{\xi'})^{-1}\, d\lambda\right]$. Since $(\lambda + \widehat{L}_{\xi'})^{-1}$ has different characters between the cases $|\xi'| >> 1$ and $|\xi'| << 1$, we decompose the semigroup $\mathscr{U}(t)$ into the two parts according to the partition: $|\xi'| \ge r_0$ and $|\xi'| \le r_0$ for some $r_0 > 0$.

In [7] we established the estimates of $(\lambda + \widehat{L}_{\xi'})^{-1}$ with $|\xi'| \ge r_0$, which will lead to the exponential decay of the corresponding part of $\mathscr{U}(t)$. We derived an integral representation for $(\lambda + \widehat{L}_{\xi'})^{-1}$ and applied the Fourier multiplier theorem as in [1, 2, 3], where L^p estimates for the incompressible Stokes equation were established.

In this paper we study $(\lambda + \widehat{L}_{\xi'})^{-1}$ with $|\xi'| << 1$. We regard $\widehat{L}_{\xi'}$ as a perturbation from $\widehat{L}_0$ to investigate the spectrum of $-L$ near $\lambda = 0$. We will find that the spectrum near the origin is given by $-\kappa|\xi'|^2 + O(|\xi'|^4)$ with $|\xi'| << 1$. It should be noted that the structure of the spectrum near the origin is quite similar to that of the linearized operator appearing in the free surface problem of viscous incompressible fluid studied in [4]. As in [4] we will appeal the analytic perturbation theory to compute the eigenvalue and the associated eigenprojection of $\lambda + \widehat{L}_{\xi'}$ for $|\xi'| << 1$. We will then derive the estimates for the integral kernel of the eigenprojection which are used to obtain the L^p estimates of the semigroup.

This paper is organized as follows. In Section 2 we introduce some notation and state the main result of this paper. In Section 3 we investigate $(\lambda + \widehat{L}_{\xi'})^{-1}$ with $|\xi'| << 1$. Section 4 is devoted to the proof of the main result. In the Appendix we will give the integral representation for $(\lambda + \widehat{L}_{\xi'})^{-1}$ obtained in [7] to estimate some part of the Dunford integral for the semigroup.

§2. Main Result

We first introduce some notation which will be used throughout the paper. For a domain D and $1 \le p \le \infty$ we denote by $L^p(D)$ the usual Lebesgue space on D and its norm is denoted by $\|\cdot\|_{L^p(D)}$. Let ℓ be a nonnegative integer. The symbol $W^{\ell,p}(D)$ denotes the ℓ-th order L^p Sobolev space on D with norm $\|\cdot\|_{W^{\ell,p}(D)}$. When $p = 2$, the space $W^{\ell,2}(D)$ is denoted by $H^\ell(D)$ and its norm is denoted by $\|\cdot\|_{H^\ell(D)}$. $C_0^\ell(D)$ stands for the set of all C^ℓ functions which have compact support in D. We denote by $W_0^{1,p}(D)$ the completion of $C_0^1(D)$ in $W^{1,p}(D)$. In particular, $W_0^{1,2}(D)$ is denoted by $H_0^1(D)$.

We simply denote by $L^p(D)$ (resp., $W^{\ell,p}(D)$, $H^\ell(D)$) the set of all vector fields $m = {}^T(m^1, \dots, m^n)$ on D with $m^j \in L^p(D)$ (resp., $W^{\ell,p}(D)$, $H^\ell(D)$), $j = 1, \dots, n$, and its norm is also denoted by $\|\cdot\|_{L^p(D)}$ (resp., $\|\cdot\|_{W^{\ell,p}(D)}$, $\|\cdot\|_{H^\ell(D)}$). For $u = \begin{pmatrix} \phi \\ m \end{pmatrix}$ with $\phi \in W^{k,p}(D)$ and $m = {}^T(m^1, \dots, m^n) \in W^{\ell,q}(D)$, we define $\|u\|_{W^{k,p}(D)\times W^{\ell,q}(D)}$ by $\|u\|_{W^{k,p}(D)\times W^{\ell,q}(D)} = \|\phi\|_{W^{k,p}(D)} + \|m\|_{W^{\ell,q}(D)}$. When $k = \ell$ and $p = q$, we simply write $\|u\|_{W^{k,p}(D)\times W^{k,p}(D)} = \|u\|_{W^{k,p}(D)}$.

In case $D = \Omega$ we abbreviate $L^p(\Omega)$ (resp., $W^{\ell,p}(\Omega)$, $H^\ell(\Omega)$) as L^p (resp., $W^{\ell,p}$, H^ℓ). In particular, the norm $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{L^p}$ is denoted by $\|\cdot\|_p$.

In case $D = (0, a)$ we denote the norm of $L^p(0, a)$ by $|\cdot|_p$. The inner product of $L^2(0, a)$ is denoted by

$$(f, g) = \int_0^a f(x_n)\overline{g(x_n)}\, dx_n, \quad f, g \in L^2(0, a).$$

Here $\overline{g}$ denotes the complex conjugate of g. Furthermore, we define $\langle\cdot,\cdot\rangle$ and $\langle\cdot\rangle$ by

$$\langle f, g\rangle = \frac{1}{a}(f, g) \quad \text{and} \quad \langle f\rangle = \langle f, 1\rangle = \frac{1}{a}\int_0^a f(x_n)\, dx_n$$

for $f, g \in L^2(0, a)$, respectively.

The norms of $W^{\ell,p}(0,a)$ and $H^\ell(0,a)$ are denoted by $|\cdot|_{W^{\ell,p}}$ and $|\cdot|_{H^\ell}$, respectively.

We often write $x \in \Omega$ as $x = \begin{pmatrix} x' \\ x_n \end{pmatrix}$, $x' = {}^T(x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$. Partial derivatives of a function u in x, x', x_n and t are denoted by $\partial_x u$, $\partial_{x'} u$, $\partial_{x_n} u$ and $\partial_t u$, respectively. We also write higher order partial derivatives of u in x as $\partial_x^k u = (\partial_x^\alpha u;\ |\alpha| = k)$.

We denote the $k \times k$ identity matrix by I_k. In particular, when $k = n+1$, we simply write I for I_{n+1}. We also define $(n+1)\times(n+1)$ diagonal matrices Q_0 and $\widetilde{Q}$ by

$$Q_0 = \mathrm{diag}\,(1, 0, \dots, 0), \quad \widetilde{Q} = \mathrm{diag}\,(0, 1, \dots, 1).$$

We then have, for $u = \begin{pmatrix} \phi \\ m \end{pmatrix} \in \mathbf{R}^{n+1}$,

$$Q_0 u = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad \widetilde{Q} u = \begin{pmatrix} 0 \\ m \end{pmatrix}.$$

We next introduce some notation about integral operators. For a function $f = f(x')$ $(x' \in \mathbf{R}^{n-1})$, we denote its Fourier transform by $\widehat{f}$ or $\mathscr{F} f$:

$$\widehat{f}(\xi') = (\mathscr{F} f)(\xi') = \int_{\mathbf{R}^{n-1}} f(x') e^{-i\xi'\cdot x'}\, dx'.$$

The inverse Fourier transform is denoted by $\mathscr{F}^{-1}$:

$$(\mathscr{F}^{-1} f)(x) = (2\pi)^{-(n-1)} \int_{\mathbf{R}^{n-1}} f(\xi') e^{i\xi'\cdot x'}\, d\xi'.$$

For a function $K(x_n, y_n)$ on $(0,a)\times(0,a)$ we will denote by Kf the integral operator $\int_0^a K(x_n, y_n) f(y_n)\, dy_n$.

We denote the resolvent set of a closed operator A by $\rho(A)$ and the spectrum of A by $\sigma(A)$. For $\Lambda \in \mathbf{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$ we will denote

$$\Sigma(\Lambda, \theta) = \{\lambda \in \mathbf{C};\ |\arg\,(\lambda - \Lambda)| \le \theta\}.$$

We now state the main result of this paper. In [7] we showed that $-L$ generates an analytic semigroup $\mathscr{U}(t)$ on $W^{1,r}(\Omega) \times L^r(\Omega)$ $(1 < r < \infty)$ and established the estimates of $\mathscr{U}(t)$ for $0 < t \le 1$. As for the large time behavior of $\mathscr{U}(t)$, we have the following result.

Theorem 2.1. *Let $1 < r < \infty$ and let $\mathscr{U}(t)$ be the semigroup generated by $-L$. Suppose that $u_0 \in L^1(\Omega) \cap \big[W^{1,r}(\Omega) \times L^r(\Omega)\big]$. Then the solution $u = \mathscr{U}(t)u_0$ of problem* (1.1)–(1.3) *is decomposed as*

$$\mathscr{U}(t)u_0 = \mathscr{U}^{(0)}(t)u_0 + \mathscr{U}^{(\infty)}(t)u_0,$$

where each term on the right-hand side has the following properties.
(i) *$\mathscr{U}^{(0)}(t)u_0$ is written in the form*

$$\mathscr{U}^{(0)}(t)u_0 = \mathscr{W}^{(0)}(t)u_0 + \mathscr{R}^{(0)}(t)u_0.$$

Here $\mathscr{W}^{(0)}(t)u_0 = \begin{pmatrix} \phi^{(0)}(x',t) \\ 0 \end{pmatrix}$ and $\phi^{(0)}(x',t)$ is a function independent of x_n and satisfies the following heat equation on $\mathbf{R}^{n-1}$:

$$\partial_t\phi^{(0)} - \kappa\Delta'\phi^{(0)} = 0, \quad \phi^{(0)}|_{t=0} = \langle\phi_0(x',\cdot)\rangle,$$

where $\kappa = \frac{a^2\gamma^2}{12\nu}$ and $\Delta' = \partial_{x_1}^2 + \cdots + \partial_{x_{n-1}}^2$. The function $\mathscr{R}^{(0)}(t)u_0$ satisfies the following estimate. For any $1 \le p \le \infty$ and $\ell = 0, 1$, there exists a positive constant C such that

$$\|\partial_x^\ell \mathscr{R}^{(0)}(t)u_0\|_p \le Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}}\|u_0\|_1$$

holds for $t \ge 1$. Furthermore, it holds that

$$\|\partial_x \mathscr{R}^{(0)}(t)\widetilde{Q}u_0\|_p \le Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-1}\|\widetilde{Q}u_0\|_1$$

and

$$\|\mathscr{R}^{(0)}(t)[\partial_x\widetilde{Q}u_0]\|_p \le Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}}\|\widetilde{Q}u_0\|_1.$$

(ii) *There exists a positive constant c such that $\mathscr{U}^{(\infty)}(t)u_0$ satisfies*

$$\|\partial_x^\ell \mathscr{U}^{(\infty)}(t)u_0\|_r \le Ce^{-ct}\|u_0\|_{W^{\ell,r}\times L^r}, \quad \ell = 0, 1,$$

for all $t \ge 1$. Furthermore, the following estimates

$$\|\partial_x^\ell \mathscr{U}^{(\infty)}(t)u_0\|_\infty \le Ce^{-ct}\|u_0\|_{H^{[\frac{n}{2}]+1+\ell}\times H^{[\frac{n}{2}]+\ell}},$$

$$\|\partial_x^\ell \mathscr{U}^{(\infty)}(t)u_0\|_p \le Ce^{-ct}\|u_0\|_{W^{\ell+1,p}\times W^{\ell,p}}, \quad p = 1, \infty,$$

hold for all $t \ge 1$, provided that u_0 belongs to the Sobolev spaces on the right of the above inequalities. Here $[q]$ denotes the greatest integer less than or equal to q.

Remark 2.1. Young's inequality for convolution integral, together with a direct computation of the L^p-norm of the heat kernel, shows that $\|\mathscr{W}^{(0)}(t)u_0\|_p$ decays exactly in the order $t^{-\frac{n-1}{2}(1-\frac{1}{p})}$. We thus have the optimal decay estimate

$$\|\mathscr{U}^{(0)}(t)u_0\|_p \leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})}\|u_0\|_1.$$

Furthermore, noting that $\mathscr{W}^{(0)}(t)\widetilde{Q}u_0 = 0$, we have the estimate

$$\|\partial_x \mathscr{U}^{(0)}(t)\widetilde{Q}u_0\|_p \leq Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-1}\|\widetilde{Q}u_0\|_1$$

for $t \geq 1$.

We will prove Theorem 2.1 in Section 4.

§3. Spectral Analysis for $-L$

The proof of Theorem 2.1 is based on the analysis of the resolvent problem associated with (1.1)–(1.3), which takes the form

$$\lambda u + Lu = f, \tag{3.1}$$

where L is the operator on $H^1 \times L^2$ defined in (1.1) with domain of definition $D(L) = H^1 \times (H^2 \cap H^1_0)$. To investigate (3.1) we take the Fourier transform in $x' \in \mathbf{R}^{n-1}$. We then have the following boundary value problem for functions $\phi(x_n)$ and $m(x_n)$ on the interval $(0, a)$:

$$\lambda u + \widehat{L}_{\xi'} u = f, \tag{3.2}$$

where $u = \begin{pmatrix} \phi(x_n) \\ m'(x_n) \\ m^n(x_n) \end{pmatrix}$, $f = \begin{pmatrix} f^0(x_n) \\ f'(x_n) \\ f^n(x_n) \end{pmatrix}$, and $\widehat{L}_{\xi'}$ is the operator of the form

$$\widehat{L}_{\xi'} = \begin{pmatrix} 0 & i\gamma^T\xi' & \gamma\partial_{x_n} \\ i\gamma\xi' & \nu(|\xi'|^2 - \partial_{x_n}^2)I_{n-1} + \widetilde{\nu}\xi'^T\xi' & -i\widetilde{\nu}\xi'\partial_{x_n} \\ \gamma\partial_{x_n} & -i\widetilde{\nu}^T\xi'\partial_{x_n} & \nu(|\xi'|^2 - \partial_{x_n}^2) - \widetilde{\nu}\partial_{x_n}^2 \end{pmatrix},$$

which is a closed operator on $H^1(0, a) \times L^2(0, a)$ with domain of definition $D(\widehat{L}_{\xi'}) = H^1(0, a) \times (H^2(0, a) \cap H^1_0(0, a))$.

In [7] we studied $(\lambda + \widehat{L}_{\xi'})^{-1}$ with $|\xi'| \geq r$ for any $r > 0$. In this section we investigate the spectrum of $-\widehat{L}_{\xi'}$ for $|\xi'| << 1$. We analyze it regarding the problem as a perturbation from the one with $\xi' = 0$.

We write $\widehat{L}_{\xi'}$ in the following form:

$$\widehat{L}_{\xi'} = \widehat{L}_0 + \sum_{j=1}^{n-1} \xi_j \widehat{L}_j^{(1)} + \sum_{j,k=1}^{n-1} \xi_j \xi_k \widehat{L}_{jk}^{(2)},$$

where $\xi' = {}^T(\xi_1, \ldots, \xi_{n-1})$,

$$\widehat{L}_0 = \begin{pmatrix} 0 & 0 & \gamma\partial_{x_n} \\ 0 & -\nu\partial_{x_n}^2 I_{n-1} & 0 \\ \gamma\partial_{x_n} & 0 & -\nu_1\partial_{x_n}^2 \end{pmatrix}, \quad \nu_1 = \nu + \widetilde{\nu},$$

$$\widehat{L}_j^{(1)} = \begin{pmatrix} 0 & i\gamma^T \boldsymbol{e}'_j & 0 \\ i\gamma \boldsymbol{e}'_j & 0 & -i\widetilde{\nu}\boldsymbol{e}'_j\partial_{x_n} \\ 0 & -i\widetilde{\nu}^T\boldsymbol{e}'_j\partial_{x_n} & 0 \end{pmatrix},$$

$$\widehat{L}_{jk}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu\delta_{jk} I_{n-1} + \widetilde{\nu}\boldsymbol{e}'_j {}^T\boldsymbol{e}'_k & 0 \\ 0 & 0 & \nu\delta_{jk} \end{pmatrix}.$$

We will treat $\widehat{L}_{\xi'}$ as a perturbation from $\widehat{L}_0$. We begin with the analysis of (3.2) with $\xi' = 0$:

$$(\lambda + \widehat{L}_0)u = f.$$

We introduce some quantities. For $k = 1, 2, \ldots$, we set $a_k = k\pi/a$. We define $\lambda_{1,k}$ and $\lambda_{\pm,k}$ by

$$\lambda_{1,k} = -\nu a_k^2$$

and

$$\lambda_{\pm,k} = -\frac{\nu_1}{2}a_k^2 \pm \frac{1}{2}\sqrt{\nu_1^2 a_k^4 - 4\gamma^2 a_k^2}$$

for $k = 1, 2, \ldots$. An elementary observation shows that $\lambda_{\pm,k}$ are the two roots of $\lambda^2 + \nu_1 a_k^2 \lambda + \gamma^2 a_k^2 = 0$; $\lambda_{-,k} = \overline{\lambda_{+,k}}$ with $\operatorname{Im}\lambda_{+,k} = \gamma a_k \sqrt{1 - \frac{\nu_1^2}{4\gamma^2}a_k^2}$ when $a_k < 2\gamma/\nu_1$ and $\lambda_{\pm,k} \in \mathbf{R}$ when $a_k > 2\gamma/\nu_1$; and it holds that

$$\lambda_{+,k} = -\frac{\gamma^2}{\nu_1} + O(k^{-2}), \quad \lambda_{-,k} = -\nu_1 a_k^2 + O(1) \tag{3.3}$$

as $k \to \infty$. (See [7, Remarks 3.2 and 3.5].)

Lemma 3.1. (i) *The spectrum* $\sigma(-\widehat{L}_0)$ *is given by*

$$\sigma(-\widehat{L}_0)=\{0\}\cup\{\lambda_{1,k}\}_{k=1}^{\infty}\cup\{\lambda_{+,k},\lambda_{-,k}\}_{k=1}^{\infty}\cup\{-\tfrac{\gamma^2}{\nu_1}\}.$$

Here 0 *is an eigenvalue with eigenspace spanned by* ${}^T(1,0,\dots,0)$.

(ii) *There exist positive numbers* η_0 *and* θ_0 *with* $\theta_0\in(\frac{\pi}{2},\pi)$ *such that the following estimates hold uniformly for* $\lambda\in\rho(-\widehat{L}_0)\cap\Sigma(-\eta_0,\theta_0)$:

$$\left|(\lambda+\widehat{L}_0)^{-1}f\right|_{H^\ell\times L^2}\le\frac{C}{|\lambda|}|f|_{H^\ell\times L^2},\quad \ell=0,1,$$

$$\left|\partial_{x_n}^{\ell}\widetilde{Q}(\lambda+\widehat{L}_0)^{-1}f\right|_2\le\frac{C}{(|\lambda|+1)^{1-\frac{\ell}{2}}}|f|_{H^{\ell-1}\times L^2},\quad \ell=1,2,$$

$$\left|\partial_{x_n}^{2}Q_0(\lambda+\widehat{L}_0)^{-1}f\right|_2\le\frac{C}{(|\lambda|+1)^{\frac{1}{2}}}|f|_{H^2\times H^1}.$$

Proof. We write (3.2) with $\xi'=0$ as

$$\lambda m'-\nu\partial_{x_n}^2 m'=f',\quad m'|_{x_n=0,a}=0, \tag{3.4}$$

and

$$\begin{cases}\lambda\phi+\gamma\partial_{x_n}m^n=f^0,\\ \lambda m^n-\nu_1\partial_{x_n}^2m^n+\gamma\partial_{x_n}\phi=f^n,\quad m^n|_{x_n=0,a}=0.\end{cases} \tag{3.5}$$

By using the Fourier series expansion, it is easy to see that (3.4) has a unique solution $m'\in H^2(0,a)\cap H_0^1(0,a)$ for any $f'\in L^2(0,a)$ if and only if $\lambda\ne\lambda_{1,k}$ for any $k=1,2,\dots$. Furthermore, it is also possible to deduce the estimates

$$\left|\partial_{x_n}^{\ell}m'\right|_2\le\frac{C}{(|\lambda|+1)^{1-\frac{\ell}{2}}}|f'|_2,\quad \ell=0,1,2,$$

uniformly in $\lambda=-\frac{\nu\pi^2}{2a^2}+\eta e^{\pm i\theta}$ with $\eta\ge0$ and $\theta\in[0,\theta_0)$. Here θ_0 is any fixed constant in $(\frac{\pi}{2},\pi)$ and C is a positive constant depending only on θ_0.

We next consider (3.5). Let $\lambda=0$ and $f^0=f^n=0$ in (3.5). We see from the first equation of (3.5) that $\partial_{x_n}m^n=0$. Then the boundary condition $m^n|_{x_n=0,a}=0$ implies that $m^n=0$. It follows from the second equation of (3.5) that ϕ is a constant. Therefore, 0 is an eigenvalue and the geometric eigenspace is spanned by $\psi^{(0)}={}^T(1,0,\dots,0)$.

Let $\lambda\ne0$ in (3.5). We then see that problem (3.5) is equivalent to

$$\phi=\frac{1}{\lambda}\left\{f^0-\gamma\partial_{x_n}m^n\right\}, \tag{3.6}$$

$$\lambda^2 m^n - (\nu_1\lambda + \gamma^2)\partial_{x_n}^2 m^n = \lambda f^n - \gamma\partial_{x_n} f^0, \quad m^n|_{x_n=0,a} = 0. \tag{3.7}$$

In case $\nu_1\lambda + \gamma^2 = 0$, it is easy to see that problem (3.6)–(3.7) has only the trivial solution $\phi = m^n = 0$ for $f^0 = f^n = 0$. For general $f^0 \in H^1(0,a)$ and $f^n \in L^2(0,a)$, (3.7) implies that $m^n = \lambda^{-2}\left\{\lambda f^n - \gamma\partial_{x_n} f^0\right\}$ which is not necessarily in $H^1(0,a)$. This implies that $-\frac{\gamma^2}{\nu_1} \in \sigma(-\widehat{L}_0)$.

Let us consider the case $\lambda \neq 0$ and $\nu_1\lambda + \gamma^2 \neq 0$. In this case, (3.7) is equivalent to

$$\sigma m^n - \partial_{x_n}^2 m^n = \frac{1}{\nu_1\lambda + \gamma^2}\left\{\lambda f^n - \gamma\partial_{x_n} f^0\right\}, \quad m^n|_{x_n=0,a} = 0, \tag{3.8}$$

where $\sigma = \frac{\lambda^2}{\nu_1\lambda+\gamma^2}$. Since $\lambda f^n - \gamma\partial_{x_n} f^0 \in L^2(0,a)$, problem (3.8) has a unique solution $m^n \in H^2(0,a) \cap H_0^1(0,a)$ if and only if $\sigma \neq -a_k^2$ for any $k = 1, 2, \ldots$, namely, $(\lambda - \lambda_{+,k})(\lambda - \lambda_{-,k}) \neq 0$ for any $k = 1, 2, \ldots$. If (3.8) has a solution $m^n \in H^2(0,a) \cap H_0^1(0,a)$, then (3.6) determines ϕ which is in $H^1(0,a)$. Consequently we see that $\sigma(-\widehat{L}_0) = \{0\} \cup \{\lambda_{1,k}\}_{k=1}^\infty \cup \{\lambda_{+,k}, \lambda_{-,k}\}_{k=1}^\infty \cup \{-\frac{\gamma^2}{\nu_1}\}$.

We next derive estimates for ϕ and m^n uniformly in $\lambda \in \rho(-\widehat{L}_0) \cap \Sigma(-\eta_0, \theta_0)$ with suitable η_0 and θ_0. To do so, we expand the solution m^n of (3.8) into the Fourier sine series $m^n = \sum_{k=1}^\infty m_k^n \sin a_k x_n$. It is easy to see that the Fourier coefficients m_k^n are given by

$$m_k^n = \frac{1}{\sigma + a_k^2}\frac{1}{\nu_1\lambda + \gamma^2}\left\{\lambda f_k^n + \gamma a_k f_k^0\right\}$$

for $k = 1, 2, \ldots$, where f_k^0 and f_k^n are the coefficients of the Fourier cosine and sine series expansion of f^0 and f^n, respectively.

Since $(\sigma + a_k^2)(\nu_1\lambda + \gamma^2) = (\lambda - \lambda_{+,k})(\lambda - \lambda_{-,k})$, we have

$$|m^n|_2^2 \leq C\sum_{k=1}^\infty \frac{1}{|(\lambda - \lambda_{+,k})(\lambda - \lambda_{-,k})|^2}\left\{|\lambda|^2\left|f_k^n\right|^2 + a_k^2\left|f_k^0\right|^2\right\}.$$

It then follows from (3.3) that there are positive numbers η_0 and $\theta_0 \in (\frac{\pi}{2}, \pi)$ such that, for λ with $|\arg(\lambda + \eta_0)| \leq \theta_0$,

$$\begin{aligned}|m^n|_2^2 &\leq C\sum_{k=1}^\infty \frac{1}{(|\lambda|+1)^2(|\lambda|+k^2)^2}\left\{|\lambda|^2\left|f_k^n\right|^2 + a_k^2\left|f_k^0\right|^2\right\}\\ &\leq \frac{C|f|_2^2}{(|\lambda|+1)^2}.\end{aligned}$$

This, together with (3.8), then implies that

$$\begin{aligned}\left|\partial_{x_n}^2 m^n\right|_2 &\leq |\sigma|\,|m^n|_2 + \frac{|\lambda|}{|\nu_1\lambda + \gamma^2|}\left|f^n\right|_2 + \frac{\gamma}{|\nu_1\lambda + \gamma^2|}\left|\partial_{x_n} f^0\right|_2\\ &\leq C|f|_{H^1\times L^2}\end{aligned}$$

uniformly in λ with $|\arg(\lambda+\eta_0)| \leq \theta_0$. Taking the L^2 inner product of (3.8) with m^n and integrating by parts, we have

$$\begin{aligned}|\partial_{x_n} m^n|_2^2 &\leq C\left\{|\sigma|\,|m^n|_2^2 + |f^n|_2\,|m^n|_2 + \frac{1}{|\lambda|+1}\left|f^0\right|_2 |\partial_{x_n} m^n|_2\right\}\\ &\leq \frac{C|f|_2^2}{|\lambda|+1} + \frac{1}{2}|\partial_{x_n} m^n|_2^2\end{aligned}$$

uniformly in λ with $|\arg(\lambda+\eta_0)| \leq \theta_0$, and hence, $|\partial_{x_n} m^n|_2 \leq \frac{C|f|_2}{(|\lambda|+1)^{\frac{1}{2}}}$. Consequently, we have

$$\left|\partial_{x_n}^{\ell} m^n\right|_2 \leq \frac{C|f|_{H^{(\ell-1)_+}\times L^2}}{(|\lambda|+1)^{1-\frac{\ell}{2}}} \tag{3.9}$$

for $\ell = 0,1,2$ uniformly in λ with $|\arg(\lambda+\eta_0)| \leq \theta_0$. It then follows from (3.6) and (3.9) that

$$|\phi|_2 \leq \frac{1}{|\lambda|}\left\{\left|f^0\right|_2 + \gamma\,|\partial_{x_n} m^n|_2\right\} \leq \frac{C}{|\lambda|}|f|_2.$$

We next estimate the derivatives of ϕ. Differentiating the first equation of (3.5) we have

$$\lambda\partial_{x_n}\phi + \gamma\partial_{x_n}^2 m^n = \partial_{x_n} f^0. \tag{3.10}$$

We see from the second equation of (3.5) that

$$-\nu_1\partial_{x_n}^2 m^n + \gamma\partial_{x_n}\phi = f^n - \lambda m^n. \tag{3.11}$$

By adding (3.11) $\times \frac{\gamma}{\nu_1}$ to (3.10) we obtain

$$\left(\lambda + \frac{\gamma^2}{\nu_1}\right)\partial_{x_n}^{\ell+1}\phi = \partial_{x_n}^{\ell+1} f^0 + \frac{\gamma}{\nu_1}\left\{\partial_{x_n}^{\ell} f^n - \lambda\partial_{x_n}^{\ell} m^n\right\},\quad \ell = 0,1.$$

This, together with (3.9), implies that

$$\begin{aligned}\left|\partial_{x_n}^{\ell+1}\phi\right|_2 &\leq \frac{C}{|\lambda|+1}\left\{\left|\partial_{x_n}^{\ell+1} f^0\right|_2 + \left|\partial_{x_n}^{\ell} f^n\right|_2 + |\lambda|\left|\partial_{x_n}^{\ell} m^n\right|_2\right\}\\ &\leq \frac{C}{(|\lambda|+1)^{1-\frac{\ell}{2}}}|f|_{H^{\ell+1}\times H^{\ell}},\quad \ell = 0,1,\end{aligned}$$

for λ with $|\arg(\lambda+\eta_0)| \leq \theta_0$, by changing $\eta_0 > 0$ and $\theta_0 \in (\frac{\pi}{2}, \pi)$ suitably if necessary. This completes the proof. $\square$

We next investigate the eigenvalue 0 of $-\widehat{L}_0$.

Lemma 3.2. *The eigenvalue* 0 *of* $-\widehat{L}_0$ *is simple and the associated eigenprojection is given by*

$$\widehat{\Pi}^{(0)}u = \begin{pmatrix} \langle\phi\rangle \\ 0 \end{pmatrix} \quad \textit{for} \quad u = \begin{pmatrix} \phi \\ m \end{pmatrix}.$$

Proof. To show the simplicity of the eigenvalue 0, let us first consider the problem $\widehat{L}_0 u = \psi^{(0)}$, where $\psi^{(0)} = {}^T(1, 0, \ldots, 0)$ is an eigenfunction for the eigenvalue 0. This problem is equivalent to (3.4)–(3.5) with $\lambda = 0$, $f' = 0$, $f^0 = 1$, $f^n = 0$. By (3.4), we have $m' = 0$, and by the first equation of (3.5), we have $m^n = \frac{1}{\gamma}x_n + c$ for some constant c. There is no such m^n satisfying the boundary condition $m^n|_{x_n=0,a} = 0$. Therefore, 0 is a simple eigenvalue.

Let us prove that the eigenprojection $\widehat{\Pi}^{(0)}$ has the desired form. Since $\dim \mathrm{Range}\, \widehat{\Pi}^{(0)} = 1$, we have $\widehat{\Pi}^{(0)}u = c_u \psi^{(0)}$ for some $c_u \in \mathbf{C}$. It then follows that

$$\langle \widehat{\Pi}^{(0)}u, \psi^{(0)} \rangle = c_u. \tag{3.12}$$

Consider now the formal adjoint problem

$$\lambda u + \widehat{L}_0^* u = 0,$$

where

$$\widehat{L}_0^* = \begin{pmatrix} 0 & 0 & -\gamma\partial_{x_n} \\ 0 & -\nu\partial_{x_n}^2 I_{n-1} & 0 \\ -\gamma\partial_{x_n} & 0 & -\nu_1\partial_{x_n}^2 \end{pmatrix}$$

with domain of definition $D(\widehat{L}_0^*) = D(\widehat{L}_0)$. Similarly to above, we can see that $\sigma(-\widehat{L}_0^*) = \sigma(-\widehat{L}_0)$, and, in particular, 0 is a simple eigenvalue and $\widehat{L}_0^*\psi^{(0)} = 0$. Furthermore, let $\widehat{\Pi}^{(0)*}$ be the eigenprojection for the eigenvalue 0 of $-\widehat{L}_0^*$. Then we have

$$\widehat{\Pi}^{(0)}u = \frac{1}{2\pi i}\int_\Gamma (\lambda + \widehat{L}_0)^{-1}u\, d\lambda, \quad \widehat{\Pi}^{(0)*}u = \frac{1}{2\pi i}\int_\Gamma (\lambda + \widehat{L}_0^*)^{-1}u\, d\lambda,$$

where Γ is a circle with center 0 and sufficiently small radius. By integration by parts, we have

$$\langle (\lambda + \widehat{L}_0)\widetilde{u}, \widetilde{v} \rangle = \langle \widetilde{u}, (\overline{\lambda} + \widehat{L}_0^*)\widetilde{v} \rangle$$

for $\widetilde{u}, \widetilde{v} \in D(\widehat{L}_0)$. Taking $\widetilde{u} = (\lambda + \widehat{L}_0)^{-1}u$ and $\widetilde{v} = (\overline{\lambda} + \widehat{L}_0^*)^{-1}v$, we have

$$\langle (\lambda + \widehat{L}_0)^{-1}u, v \rangle = \langle u, (\overline{\lambda} + \widehat{L}_0^*)^{-1}v \rangle$$

for $u, v \in H^1(0,a) \times L^2(0,a)$. We then obtain

$$\begin{aligned}\left\langle \widehat{\Pi}^{(0)} u, \psi^{(0)} \right\rangle &= \left\langle \frac{1}{2\pi i} \int_\Gamma (\lambda + \widehat{L}_0)^{-1} u \, d\lambda, v \right\rangle \\ &= \left\langle u, \frac{1}{2\pi i} \int_\Gamma (\lambda + \widehat{L}_0^*)^{-1} v \, d\lambda \right\rangle \\ &= \left\langle u, \widehat{\Pi}^{(0)*} \psi^{(0)} \right\rangle = \left\langle u, \psi^{(0)} \right\rangle = \langle \phi \rangle\end{aligned}$$

for $u = \begin{pmatrix} \phi \\ m \end{pmatrix}$. This, together with (3.12), gives the desired expression of $\widehat{\Pi}^{(0)}$. This completes the proof. $\square$

We next estimate $(\lambda + \widehat{L}_{\xi'})^{-1}$ for small ξ'. Based on Lemma 3.1 we obtain the following estimates.

Theorem 3.1. *Let η_0 and θ_0 be the numbers given in Lemma* 3.1. *Then there exists a positive number $\widetilde{r}_0 = \widetilde{r}_0(\eta_0, \theta_0)$ such that the set $\Sigma(-\eta_0, \theta_0) \cap \left\{\lambda;\, |\lambda| \geq \frac{\eta_0}{2}\right\}$ is in $\rho(-\widehat{L}_{\xi'})$ for $|\xi'| \leq \widetilde{r}_0$. Furthermore, the following estimates hold for any multi-index α' with $|\alpha'| \leq n$ uniformly in $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \left\{\lambda;\, |\lambda| \geq \frac{\eta_0}{2}\right\}$ and ξ' with $|\xi'| \leq \widetilde{r}_0$:*

$$\left|\partial_{\xi'}^{\alpha'} (\lambda + \widehat{L}_{\xi'})^{-1} f\right|_{H^\ell \times L^2} \leq \frac{C}{|\lambda|} |f|_{H^\ell \times L^2}, \quad \ell = 0, 1,$$

$$\left|\partial_{\xi'}^{\alpha'} \partial_{x_n}^\ell \widetilde{Q} (\lambda + \widehat{L}_{\xi'})^{-1} f\right|_2 \leq \frac{C}{(|\lambda| + 1)^{1 - \frac{\ell}{2}}} |f|_{H^{\ell - 1} \times L^2}, \quad \ell = 1, 2,$$

$$\left|\partial_{\xi'}^{\alpha'} \partial_{x_n}^2 Q_0 (\lambda + \widehat{L}_{\xi'})^{-1} f\right|_2 \leq \frac{C}{(|\lambda| + 1)^{\frac{1}{2}}} |f|_{H^2 \times H^1}.$$

Proof. In the following we will write

$$\widehat{L}^{(1)}(\xi') = \sum_{j=1}^{n-1} \xi_j \widehat{L}_j^{(1)} \quad \text{and} \quad \widehat{L}^{(2)}(\xi') = \sum_{j,k=1}^{n-1} \xi_j \xi_k \widehat{L}_{jk}^{(2)}.$$

We first observe that

$$\left|\widehat{L}_j^{(1)} u\right|_{H^\ell \times H^{(\ell-1)_+}} \leq C \left\{ |Q_0 u|_{H^{(\ell-1)_+}} + |\widetilde{Q} u|_{H^{(\ell-1)_+ + 1}} \right\} \tag{3.13}$$

and

$$\left|\widehat{L}_{jk}^{(2)} u\right|_{H^\ell \times H^{(\ell-1)_+}} \leq C |\widetilde{Q} u|_{H^{(\ell-1)_+}}. \tag{3.14}$$

It then follows from Lemma 3.1 and (3.14) that

$$\left|\widehat{L}^{(2)}_{jk}(\lambda+\widehat{L}_0)^{-1}f\right|_{H^\ell\times L^2}\le C\left|\widetilde{Q}(\lambda+\widehat{L}_0)^{-1}f\right|_2\le C|f|_2 \tag{3.15}$$

for $\ell=0,1$ and $\lambda\in\Sigma(-\eta_0,\theta_0)\cap\{\lambda;|\lambda|\ge\frac{\eta_0}{2}\}$ with $C=C(\eta_0,\theta_0)>0$. Also, by Lemma 3.1 and (3.13), we have

$$\left|\widehat{L}^{(1)}_{j}(\lambda+\widehat{L}_0)^{-1}f\right|_{H^\ell\times L^2}\le C|f|_2 \tag{3.16}$$

for $\ell=0,1$ and $\lambda\in\Sigma(-\eta_0,\theta_0)\cap\{\lambda;|\lambda|\ge\frac{\eta_0}{2}\}$ with $C=C(\eta_0,\theta_0)>0$. It then follows that there exists a positive number $\widetilde{r}_0$ such that if $|\xi'|\le\widetilde{r}_0$, then

$$\left|\left(\widehat{L}^{(1)}(\xi')+\widehat{L}^{(2)}(\xi')\right)(\lambda+\widehat{L}_0)^{-1}f\right|_{H^\ell\times L^2}\le\frac{1}{2}|f|_2$$

for $\ell=0,1$ and $\lambda\in\Sigma(-\eta_0,\theta_0)\cap\{\lambda;|\lambda|\ge\frac{\eta_0}{2}\}$. By the Neumann series expansion, we see that $I+\left(\widehat{L}^{(1)}(\xi')+\widehat{L}^{(2)}(\xi')\right)(\lambda+\widehat{L}_0)^{-1}$ is invertible on $H^\ell(0,a)\times L^2(0,a)$, $\ell=0,1$, for $\lambda\in\Sigma(-\eta_0,\theta_0)\cap\{\lambda;|\lambda|\ge\frac{\eta_0}{2}\}$ and ξ' with $|\xi'|\le\widetilde{r}_0$. In particular, we conclude that $\Sigma(-\eta_0,\theta_0)\cap\{\lambda;|\lambda|\ge\frac{\eta_0}{2}\}\subset\rho(-\widehat{L}_{\xi'})$ and

$$(\lambda+\widehat{L}_{\xi'})^{-1}=(\lambda+\widehat{L}_0)^{-1}\sum_{N=0}^{\infty}(-1)^N\left[\left(\widehat{L}^{(1)}(\xi')+\widehat{L}^{(2)}(\xi')\right)(\lambda+\widehat{L}_0)^{-1}\right]^N \tag{3.17}$$

for $\lambda\in\Sigma(-\eta_0,\theta_0)\cap\{\lambda;|\lambda|\ge\frac{\eta_0}{2}\}$ and ξ' with $|\xi'|\le\widetilde{r}_0$. Furthermore, we see from Lemma 3.1, (3.13) and (3.14) that

$$\begin{aligned}&\left|\partial_{\xi'}^{\alpha'}(\lambda+\widehat{L}_{\xi'})^{-1}f\right|_{H^\ell\times L^2}\\&\le\frac{C}{|\lambda|}\left|\partial_{\xi'}^{\alpha'}\sum_{N=0}^{\infty}(-1)^N\left[\left(\widehat{L}^{(1)}(\xi')+\widehat{L}^{(2)}(\xi')\right)(\lambda+\widehat{L}_0)^{-1}\right]^N f\right|_{H^\ell\times L^2}\\&\le\frac{C}{|\lambda|}|f|_{H^\ell\times L^2},\quad \ell=0,1.\end{aligned}$$

Similarly, we have, for $\ell=1,2$,

$$\begin{aligned}&\left|\partial_{\xi'}^{\alpha'}\partial_{x_n}^{\ell}\widetilde{Q}(\lambda+\widehat{L}_{\xi'})^{-1}f\right|_2\\&\le\frac{C}{(|\lambda|+1)^{1-\frac{\ell}{2}}}\left|\partial_{\xi'}^{\alpha'}\sum_{N=0}^{\infty}(-1)^N\left[\left(\widehat{L}^{(1)}(\xi')+\widehat{L}^{(2)}(\xi')\right)(\lambda+\widehat{L}_0)^{-1}\right]^N f\right|_{H^{\ell-1}\times L^2}\\&\le\frac{C}{(|\lambda|+1)^{1-\frac{\ell}{2}}}|f|_{H^{\ell-1}\times L^2}.\end{aligned}$$

Let us estimate $\partial_{x_n}^2 Q_0(\lambda + \widehat{L}_{\xi'})^{-1} f$. We see from Lemma 3.1, (3.13) and (3.14) that

$$\left| \widehat{L}_j^{(1)} (\lambda + \widehat{L}_0)^{-1} f \right|_{H^2 \times H^1} \leq C |f|_{H^1 \times L^2}$$

and

$$\left| \widehat{L}_{jk}^{(2)} (\lambda + \widehat{L}_0)^{-1} f \right|_{H^2 \times H^1} \leq C |f|_2$$

uniformly for $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \left\{\lambda; |\lambda| \geq \frac{\eta_0}{2}\right\}$. Therefore, taking $\widetilde{r}_0$ smaller if necessary, we have

$$\left| \left(\widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi') \right) (\lambda + \widehat{L}_0)^{-1} f \right|_{H^2 \times H^1} \leq \frac{1}{2} |f|_{H^1 \times L^2}$$

for $\lambda \in \Sigma(-\eta_0, \theta_0) \cap \left\{\lambda; |\lambda| \geq \frac{\eta_0}{2}\right\}$ and ξ' with $|\xi'| \leq \widetilde{r}_0$. It then follows from Lemma 3.1 and (3.17) that

$$\begin{aligned}
&\left| \partial_{\xi'}^{\alpha'} \partial_{x_n}^2 Q_0 (\lambda + \widehat{L}_{\xi'})^{-1} f \right|_2 \\
&\leq \frac{C}{(|\lambda|+1)^{\frac{1}{2}}} \left| \partial_{\xi'}^{\alpha'} \sum_{N=0}^{\infty} (-1)^N \left[\left(\widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi') \right) (\lambda + \widehat{L}_0)^{-1} \right]^N f \right|_{H^2 \times H^1} \\
&\leq \frac{C}{(|\lambda|+1)^{\frac{1}{2}}} |f|_{H^2 \times H^1}.
\end{aligned}$$

This completes the proof. □

We next investigate the spectrum of $-\widehat{L}_{\xi'}$ near $\lambda = 0$.

Theorem 3.2. *Let η_0 and $\widetilde{r}_0$ be the numbers given in Theorem 3.1. Then there exists a positive number r_0 with $r_0 \leq \widetilde{r}_0$ such that for each ξ' with $|\xi'| \leq r_0$ it holds that*

$$\sigma(-\widehat{L}_{\xi'}) \cap \{\lambda; |\lambda| \leq \eta_0\} = \{\lambda_0(\xi')\},$$

where $\lambda_0(\xi') \in \mathbf{R}$ and $\lambda_0(\xi')$ is a simple eigenvalue of $-\widehat{L}_{\xi'}$ that has the form

$$\lambda_0(\xi') = -\frac{a^2 \gamma^2}{12\nu} |\xi'|^2 + O(|\xi'|^4)$$

as $|\xi'| \to 0$.

Proof. By Theorem 3.1, (3.13) and (3.14), we see that if $|\lambda| = \eta_0$, then $\lambda \in \rho(-\widehat{L}_{\xi'})$ for $|\xi'| \leq \widetilde{r}_0$. In particular,

$$\widehat{\Pi}(\xi') = \frac{1}{2\pi i} \int_{|\lambda| = \eta_0} (\lambda + \widehat{L}_{\xi'})^{-1} \, d\lambda$$

is the eigenprojection for the eigenvalues of $-\widehat{L}_{\xi'}$ lying inside the circle $|\lambda| = \eta_0$. The continuity of $(\lambda + \widehat{L}_{\xi'})^{-1}$ in (λ, ξ') then implies that $\dim \operatorname{Range} \widehat{\Pi}(\xi') = \dim \operatorname{Range} \widehat{\Pi}^{(0)} = 1$. (See [10, Chap. 1, Lemma 4.10 and Chap. 4, Theorem 3.16].) Therefore, we see from Lemma 3.2 that $\sigma(-\widehat{L}_{\xi'}) \cap \{\lambda;\, |\lambda| \le \eta_0\}$ consists of only one simple eigenvalue, say $\lambda_0(\xi')$.

To show that $\lambda_0(\xi')$ has the desired asymptotic form, we first observe that λ is an eigenvalue of $-\widehat{L}_{\xi'}$ if and only if it is an eigenvalue of $-\widehat{L}_{T'\xi'}$ for any $(n-1) \times (n-1)$ orthogonal matrix T', since $\widehat{L}_{\xi'} = T^{-1}\widehat{L}_{T'\xi'}T$, where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & T' & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It then follows that $\lambda_0(\xi')$ is a function of $|\xi'|$, and hence, it suffices to consider $\widehat{L}_{\xi'}$ with $\xi' = \eta \boldsymbol{e}'_1$, where $\eta \in \mathbf{R}$ and $|\eta| = |\xi'|$.

We write $\widehat{L}_{\xi'}$ with $\xi' = \eta \boldsymbol{e}'_1$ as $\widehat{L}_\eta$, and $\widehat{L}_\eta = \widehat{L}_0 + \eta \widehat{L}_1^{(1)} + \eta^2 \widehat{L}_{1,1}^{(2)}$. We also denote the corresponding eigenvalue by $\lambda_0(\eta)$. With this $\widehat{L}_\eta$, taking $T' = -I_{n-1}$, we see that $\lambda_0(\eta) = \lambda_0(-\eta)$ since $\lambda_0(\xi')$ is simple. Furthermore, we have a relation $\overline{\widehat{L}_\eta u} = \widehat{L}_{-\eta}\overline{u}$, which implies that $\overline{\lambda_0(\eta)} = \lambda_0(-\eta) = \lambda_0(\eta)$. This means that $\lambda_0(\eta) \in \mathbf{R}$.

In view of (3.13) and (3.14) we can apply the analytic perturbation theory [10, Chap. 2, Sect. 2.2 and Chap. 7, Remark 2.10] to see that

$$\lambda_0(\eta) = \lambda^{(0)} + \eta\lambda^{(1)} + \eta^2\lambda^{(2)} + \eta^3\lambda^{(3)} + O(\eta^4)$$

with $\lambda^{(0)} = 0$. Since $\lambda_0(\eta) = \lambda_0(-\eta)$, we have $\lambda^{(1)} = \lambda^{(3)} = 0$. The coefficient $\lambda^{(2)}$ of η^2 is given by

$$\lambda^{(2)} = -\langle \widehat{L}_{1,1}^{(2)}\psi^{(0)}, \psi^{(0)} \rangle + \langle \widehat{L}_1^{(1)} S \widehat{L}_1^{(1)} \psi^{(0)}, \psi^{(0)} \rangle,$$

where $S = \left[(I - \widehat{\Pi}^{(0)})\widehat{L}_0(I - \widehat{\Pi}^{(0)})\right]^{-1}$. It is easy to see that $\widehat{L}_{1,1}^{(2)}\psi^{(0)} = 0$. Let us compute $\langle \widehat{L}_1^{(1)} S \widehat{L}_1^{(1)} \psi^{(0)}, \psi^{(0)} \rangle$. Since $\widehat{L}_1^{(1)}\psi^{(0)} = i\gamma \begin{pmatrix} 0 \\ \boldsymbol{e}'_1 \\ 0 \end{pmatrix}$, we have

$$S\widehat{L}_1^{(1)}\psi^{(0)} = \frac{i\gamma}{\nu} \begin{pmatrix} 0 \\ \boldsymbol{e}'_1 \\ 0 \end{pmatrix} (-\partial_{x_n}^2)^{-1} \cdot 1,$$

and hence,

$$\widehat{L}_1^{(1)} S \widehat{L}_1^{(1)} \psi^{(0)} = - \begin{pmatrix} \frac{\gamma^2}{\nu}(-\partial_{x_n}^2)^{-1} \cdot 1 \\ 0 \\ -\frac{\widetilde{\nu}\gamma}{\nu} \partial_{x_n} (-\partial_{x_n}^2)^{-1} \cdot 1 \end{pmatrix}.$$

Here $(-\partial_{x_n}^2)^{-1}$ denotes the inverse of $-\partial_{x_n}^2$ under the 0-Dirichlet boundary condition at $x_n = 0, a$. We thus obtain

$$\langle \widehat{L}_1^{(1)} S \widehat{L}_1^{(1)} \psi^{(0)}, \psi^{(0)} \rangle = -\frac{\gamma^2}{\nu} \langle (-\partial_{x_n}^2)^{-1} \cdot 1 \rangle = -\frac{a^2 \gamma^2}{12\nu}.$$

Consequently, we obtain

$$\lambda_0(\eta) = -\frac{a^2\gamma^2}{12\nu}\eta^2 + O(\eta^4).$$

This completes the proof. □

We next investigate the eigenprojection $\widehat{\Pi}(\xi')$ associated with $\lambda_0(\xi')$. To do so, we will consider the formal adjoint problem

$$\lambda u + \widehat{L}_{\xi'}^* u = f,$$

where $\widehat{L}_{\xi'}^*$ is the operator of the form

$$\widehat{L}_{\xi'}^* = \widehat{L}_0^* + \sum_{j=1}^{n-1} \xi_j \widehat{L}_j^{(1)*} + \sum_{j,k=1}^{n-1} \xi_j \xi_k L_{jk}^{(2)*}$$

with domain of definition $D(\widehat{L}_{\xi'}^*) = D(\widehat{L}_{\xi'})$. Here $\xi' = {}^T(\xi_1, \dots, \xi_{n-1})$,

$$\widehat{L}_0^* = \begin{pmatrix} 0 & 0 & -\gamma\partial_{x_n} \\ 0 & -\nu\partial_{x_n}^2 I_{n-1} & 0 \\ -\gamma\partial_{x_n} & 0 & -\nu_1\partial_{x_n}^2 \end{pmatrix},$$

$$\widehat{L}_j^{(1)*} = \begin{pmatrix} 0 & -i\gamma^T\boldsymbol{e}_j' & 0 \\ -i\gamma\boldsymbol{e}_j' & 0 & -i\widetilde{\nu}\boldsymbol{e}_j'\partial_{x_n} \\ 0 & -i\widetilde{\nu}^T\boldsymbol{e}_j'\partial_{x_n} & 0 \end{pmatrix},$$

$$\widehat{L}_{jk}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu\delta_{jk} I_{n-1} + \widetilde{\nu}\boldsymbol{e}_k'{}^T\boldsymbol{e}_j' & 0 \\ 0 & 0 & \nu\delta_{jk} \end{pmatrix}.$$

Theorem 3.3. *Let $\widehat{\Pi}(\xi')$ be the eigenprojection associated with $\lambda_0(\xi')$. Then there exists a positive number r_0 such that for any ξ' with $|\xi'| \le r_0$ the projection $\widehat{\Pi}(\xi')$ is written in the form*

$$\widehat{\Pi}(\xi')u = \int_0^a \widehat{\Pi}(\xi', x_n, y_n)u(y_n)\, dy_n$$

with

$$\widehat{\Pi}(\xi', x_n, y_n) = \widehat{\Pi}^{(0)} + \sum_{j=1}^{n-1} \xi_j \widehat{\Pi}_j^{(1)}(x_n, y_n) + \widehat{\Pi}^{(2)}(\xi', x_n, y_n).$$

Here $\widehat{\Pi}^{(0)} = \frac{1}{a}Q_0$; $\widehat{\Pi}_j^{(1)} \in W^{1,\infty}((0,a)\times(0,a))$, $j = 1, \ldots, n-1$; and $\widehat{\Pi}^{(2)}\times(\xi', x_n, y_n)$ satisfies

$$\left|\partial_{\xi'}^{\alpha'} \widehat{\Pi}^{(2)}(\xi', \cdot, \cdot)\right|_{W^{1,\infty}((0,a)\times(0,a))} \le C|\xi'|^{2-|\alpha'|}$$

for any multi-index α' with $|\alpha'| \le n$ uniformly in ξ' with $|\xi'| \le r_0$. Furthermore, $\widehat{\Pi}(\xi')$ has the properties

$$\widehat{\Pi}(\xi')\left[\partial_{x_n}\widetilde{Q}u\right] = -\sum_{j=1}^{n-1} \xi_j \left(\partial_{y_n}\widehat{\Pi}_j^{(1)}(\xi')\right)\left[\widetilde{Q}u\right] - \left(\partial_{y_n}\widehat{\Pi}^{(2)}(\xi')\right)\left[\widetilde{Q}u\right]$$

and

$$\partial_{x_n}\widehat{\Pi}(\xi')\widetilde{Q}u = \partial_{x_n}\widehat{\Pi}^{(2)}(\xi')\widetilde{Q}u.$$

Proof. By (3.13)–(3.17) we see that $(\lambda + \widehat{L}_{\xi'})^{-1}$ has the form

$$(\lambda + \widehat{L}_{\xi'})^{-1} = (\lambda + \widehat{L}_0)^{-1} - \sum_{j=1}^{n-1} \xi_j(\lambda + \widehat{L}_0)^{-1}L_j^{(1)}(\lambda + \widehat{L}_0)^{-1} + \widetilde{R}(\lambda, \xi'),$$

where

$$\begin{aligned} (3.18)\qquad & \widetilde{R}(\lambda, \xi') \\ &= -(\lambda + \widehat{L}_0)^{-1}\widehat{L}^{(2)}(\xi')(\lambda + \widehat{L}_0)^{-1} \\ &\quad + (\lambda + \widehat{L}_0)^{-1}\sum_{N=2}^{\infty}(-1)^N\left[(\widehat{L}^{(1)}(\xi') + \widehat{L}^{(2)}(\xi'))(\lambda + \widehat{L}_0)^{-1}\right]^N \end{aligned}$$

and $\widetilde{R}(\lambda, \xi')$ satisfies

$$\left|\partial_{\xi'}^{\alpha'}\widetilde{R}(\lambda, \xi')f\right|_{H^2} \le C|\xi'|^{2-|\alpha'|}|f|_{H^2\times H^1}. \tag{3.19}$$

Similarly, one can prove that

$$(\lambda + \widehat{L}^*_{\xi'})^{-1} = (\lambda + \widehat{L}^*_0)^{-1} - \sum_{j=1}^{n-1} \xi_j (\lambda + \widehat{L}^*_0)^{-1} L_j^{(1)*} (\lambda + \widehat{L}^*_0)^{-1} + \widetilde{R}^*(\lambda, \xi'),$$

where

$$\begin{aligned} \widetilde{R}^*(\lambda, \xi') & \\ &= -(\lambda + \widehat{L}^*_0)^{-1} \widehat{L}^{(2)*}(\xi')(\lambda + \widehat{L}^*_0)^{-1} \\ &\quad + (\lambda + \widehat{L}^*_0)^{-1} \sum_{N=2}^{\infty} (-1)^N \left[(\widehat{L}^{(1)*}(\xi') + \widehat{L}^{(2)*}(\xi'))(\lambda + \widehat{L}^*_0)^{-1} \right]^N \end{aligned} \tag{3.20}$$

and $\widetilde{R}^*(\lambda, \xi')$ satisfies

$$\left| \partial_{\xi'}^{\alpha'} \widetilde{R}^*(\lambda, \xi') f \right|_{H^2} \leq C |\xi'|^{2-|\alpha'|} |f|_{H^2 \times H^1}. \tag{3.21}$$

We now define $\psi(\xi', x_n)$ and $\widetilde{\psi}^*(\xi', x_n)$ by

$$\psi(\xi', x_n) = \frac{1}{2\pi i} \int_{|\lambda| = \eta_0} (\lambda + \widehat{L}_{\xi'})^{-1} \psi^{(0)} \, d\lambda$$

and

$$\widetilde{\psi}^*(\xi', x_n) = \frac{1}{2\pi i} \int_{|\lambda| = \eta_0} (\lambda + \widehat{L}^*_{\xi'})^{-1} \psi^{(0)} \, d\lambda,$$

where $\psi^{(0)} = {}^T(1, 0, \ldots, 0)$. It then follows from (3.18)–(3.21) that ψ and $\widetilde{\psi}^*$ have the form

$$\begin{aligned} \psi(\xi', x_n) &= \psi^{(0)} + \sum_{j=1}^{n-1} \xi_j \psi_j^{(1)}(x_n) + \psi^{(2)}(\xi', x_n), \\ \widetilde{\psi}^*(\xi', x_n) &= \psi^{(0)} + \sum_{j=1}^{n-1} \xi_j \widetilde{\psi}_j^{(1)*}(x_n) + \widetilde{\psi}^{(2)*}(\xi', x_n), \end{aligned} \tag{3.22}$$

where $\psi_j^{(1)}$, $\widetilde{\psi}_j^{(1)*}$, $\psi^{(2)}$ and $\widetilde{\psi}^{(2)*}$ satisfy

$$\left| \psi_j^{(1)} \right|_{H^2} + \left| \widetilde{\psi}_j^{(1)*} \right|_{H^2} \leq C, \quad j = 1, \ldots, n-1,$$

$$\left| \partial_{\xi'}^{\alpha'} \psi^{(2)}(\xi') \right|_{H^2} + \left| \partial_{\xi'}^{\alpha'} \widetilde{\psi}^{(2)*}(\xi') \right|_{H^2} \leq C |\xi'|^{2-|\alpha'|}.$$

Therefore, we have

$$\left| \psi_j^{(1)} \right|_{W^{1,\infty}} + \left| \widetilde{\psi}_j^{(1)*} \right|_{W^{1,\infty}} \leq C, \quad j = 1, \ldots, n-1,$$

and

$$\left|\partial_{\xi'}^{\alpha'}\psi^{(2)}(\xi')\right|_{W^{1,\infty}}+\left|\partial_{\xi'}^{\alpha'}\widetilde{\psi}^{(2)*}(\xi')\right|_{W^{1,\infty}}\le C|\xi'|^{2-|\alpha'|}. \tag{3.23}$$

We note that $\langle\psi(\xi'),\widetilde{\psi}^*(\xi')\rangle$ is analytic in ξ' and

$$\langle\psi(\xi'),\widetilde{\psi}^*(\xi')\rangle=1+\sum_{j=1}^{n-1}\xi_j\left\{\langle\psi^{(0)},\widetilde{\psi}_j^{(1)*}\rangle+\langle\psi_j^{(1)},\psi^{(0)}\rangle\right\}+\widetilde{\Psi}^{(2)}(\xi'),$$

where $\widetilde{\Psi}^{(2)}(\xi')$ satisfies $\left|\partial_{\xi'}^{\alpha'}\widetilde{\Psi}^{(2)}(\xi')\right|\le C|\xi'|^{2-|\alpha'|}$. In particular, taking r_0 smaller if necessary, we see that

$$\left|\langle\psi(\xi'),\widetilde{\psi}^*(\xi')\rangle\right|\ge\frac{1}{2}$$

for $|\xi'|\le r_0$.

We set

$$\psi^*(\xi',x_n)=\frac{1}{\langle\psi(\xi'),\widetilde{\psi}^*(\xi')\rangle}\widetilde{\psi}^*(\xi',x_n).$$

Then we have

$$\langle\psi(\xi'),\psi^*(\xi')\rangle=1$$

and

$$\psi^*(\xi',x_n)=\psi^{(0)}+\sum_{j=1}^{n-1}\xi_j\psi_j^{(1)*}(x_n)+\psi^{(2)*}(\xi',x_n), \tag{3.24}$$

where $\psi_j^{(1)*}$ and $\psi^{(2)*}$ satisfy

$$\begin{aligned}&\left|\psi_j^{(1)*}\right|_{W^{1,\infty}}\le C,\quad j=1,\dots,n-1,\\&\left|\partial_{\xi'}^{\alpha'}\psi_j^{(2)*}(\xi')\right|_{W^{1,\infty}}\le C|\xi'|^{2-|\alpha'|}.\end{aligned} \tag{3.25}$$

It is not difficult to see that $\langle u,\psi^*(\xi')\rangle\psi(\xi')$ is the eigenprojection $\widehat{\Pi}(\xi')$ associated with $\lambda_0(\xi')$.

Setting

$$\begin{aligned}\widehat{\Pi}^{(0)}&=\tfrac{1}{a}Q_0\\\widehat{\Pi}_j^{(1)}(x_n,y_n)&=\psi^{(0)}(x_n)^T\psi_j^{(1)*}(y_n)+\psi_j^{(1)}(x_n)^T\psi^{(0)}(y_n),\\\widehat{\Pi}^{(2)}(\xi',x_n,y_n)&=\psi(\xi',x_n)^T\psi^{(2)*}(\xi',y_n)+\psi^{(2)}(\xi',x_n)^T\psi^*(\xi',y_n),\end{aligned}$$

we see from (3.22)–(3.25) that the integral kernel $\widehat{\Pi}(\xi', x_n, y_n)$ of $\widehat{\Pi}(\xi')$ is written as

$$\widehat{\Pi}(\xi', x_n, y_n) = \psi(\xi', x_n)^T \psi^*(\xi', y_n) = \widehat{\Pi}^{(0)} + \sum_{j=1}^{n-1} \xi_j \widehat{\Pi}_j^{(1)}(x_n, y_n) + \widehat{\Pi}^{(2)}(\xi', x_n, y_n)$$

with $\widehat{\Pi}_j^{(1)} \in W^{1,\infty}((0,a)\times(0,a))$, $j = 1, \dots, n-1$, and

$$\left|\partial_{\xi'}^{\alpha'} \widehat{\Pi}^{(2)}(\xi', \cdot, \cdot)\right|_{W^{1,\infty}((0,a)\times(0,a))} \le C|\xi'|^{2-|\alpha'|}.$$

We thus conclude that $\widehat{\Pi}(\xi')$ is written in the desired form.

We finally show that $\widehat{\Pi}(\xi')\left[\partial_{x_n}\widetilde{Q}u\right](x_n)$ and $\partial_{x_n}\widehat{\Pi}(\xi')\widetilde{Q}u$ have the desired forms. Since $\psi^*(\xi', y_n)$ is an eigenfunction of $\widehat{L}_{\xi'}^*$, we have $\widetilde{Q}\psi^*\Big|_{y_n=0,a} = 0$, which implies that $\widehat{\Pi}(\xi', x_n, y_n)\widetilde{Q}\Big|_{y_n=0,a} = 0$. An integration by parts then yields

$$\begin{aligned}\widehat{\Pi}(\xi')\left[\partial_{x_n}\widetilde{Q}u\right](x_n) &= \int_0^a \widehat{\Pi}(\xi', x_n, y_n)\partial_{y_n}\widetilde{Q}u(y_n)\,dy_n \\ &= -\int_0^a \partial_{y_n}\widehat{\Pi}(\xi', x_n, y_n)\widetilde{Q}u(y_n)\,dy_n \\ &= -\left(\partial_{y_n}\widehat{\Pi}(\xi')\right)\left[\widetilde{Q}u\right](x_n).\end{aligned}$$

Since $\partial_{y_n}\widehat{\Pi}^{(0)} = 0$, we have the desired form of $\widehat{\Pi}(\xi')\left[\partial_{x_n}\widetilde{Q}u\right]$. Furthermore, since $\partial_{x_n}\psi^{(0)} = 0$ and $\widetilde{Q}\psi^{(0)} = 0$, we have $\partial_{x_n}\widehat{\Pi}_j^{(1)}(x_n, y_n)\widetilde{Q} = 0$, and hence, $\partial_{x_n}\widehat{\Pi}(\xi')\widetilde{Q}u = \partial_{x_n}\widehat{\Pi}^{(2)}(\xi')\widetilde{Q}u$. This completes the proof. □

We next consider $(\lambda + \widehat{L}_{\xi'})^{-1}$ with $|\xi'| \ge r_0$. The analysis of $(\lambda + \widehat{L}_{\xi'})^{-1}$ with $|\xi'| \ge r$ for any $r > 0$ is given in [7]. Applying [7, Theorems 2.5–2.7], we obtain the following estimates.

Let r_0 be the number given in Theorem 3.3. We take a cut-off function $\chi(\xi') \in C^\infty(\mathbf{R}^{n-1})$ satisfying $0 \le \chi \le 1$ on $\mathbf{R}^{n-1}$, $\chi(\xi') = 1$ for $|\xi'| \le \frac{r_0}{2}$ and $\chi(\xi') = 0$ for $|\xi'| \ge r_0$. We set

$$\chi^{(0)}(\xi') = \chi(\xi'), \quad \chi^{(1)}(\xi') = 1 - \chi(\xi'). \tag{3.26}$$

We define the operators $R^{(j)}(\lambda)$, $j = 0, 1$, by

$$R^{(j)}(\lambda)f = \mathscr{F}_{\xi'}^{-1}\left[\chi^{(j)}(\xi')(\lambda + \widehat{L}_{\xi'})^{-1}\hat{f}\right], \quad j = 0, 1. \tag{3.27}$$

By [7, Theorems 2.5–2.7] we have the following estimates.

Theorem 3.4. *Let r_0 be the positive number given in Theorem 3.3.*

(i) *There exist positive numbers $\widetilde{\eta}$ and $\widetilde{\theta}$ with $\widetilde{\theta} \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(-\widetilde{\eta}, \widetilde{\theta}) \subset \rho(-\widehat{L}_{\xi'})$ for $|\xi'| \geq \frac{r_0}{2}$.*

(ii) *Let $1 < p < \infty$ and define $R^{(1)}(\lambda)$ as above. Then the following estimates hold uniformly in $\lambda \in \Sigma(-\widetilde{\eta}, \widetilde{\theta})$:*

$$\|\partial_x^k R^{(1)}(\lambda) f\|_p \leq \left\{ \frac{\|Q_0 f\|_{W^{k,p}}}{|\lambda| + 1} + \frac{\|\widetilde{Q} f\|_p}{(|\lambda| + 1)^{1 - \frac{k}{2}}} \right\}, \quad k = 0, 1.$$

Theorem 3.5. *Let $\widetilde{\eta}$ and $\widetilde{\theta}$ be the numbers as in Theorem 3.4. Then the following estimates hold uniformly in $\lambda \in \Sigma(-\widetilde{\eta}, \widetilde{\theta})$:*

$$\|\partial_x^k Q_0 R^{(1)}(\lambda) f\|_\infty \leq C \left\{ \frac{\|Q_0 f\|_{H^{[\frac{n}{2}]+1+k}}}{|\lambda| + 1} + \frac{\|\widetilde{Q} f\|_{H^{[\frac{n}{2}]+k}}}{(|\lambda| + 1)^{\frac{3}{4}}} \right\}, \quad k = 0, 1,$$

and

$$\|\partial_x^k \widetilde{Q} R^{(1)}(\lambda) f\|_\infty \leq C \left\{ \frac{\|Q_0 f\|_{H^{[\frac{n}{2}]+k}}}{(|\lambda| + 1)^{\frac{3}{4}}} + \frac{\|\widetilde{Q} f\|_{H^{[\frac{n}{2}]-1+k}}}{(|\lambda| + 1)^{\frac{\varepsilon}{4}}} \right\}, \quad k = 0, 1.$$

Here ε is some number satisfying $0 < \varepsilon < \frac{1}{3}$.

Theorem 3.6. *Let $p = 1, \infty$ and let $\widetilde{\eta}$ and $\widetilde{\theta}$ be the numbers as in Theorem 3.4. Then the following estimates hold uniformly in $\lambda \in \Sigma(-\widetilde{\eta}, \widetilde{\theta})$:*

$$\|\partial_x^k Q_0 R^{(1)}(\lambda) f\|_p \leq \frac{C}{|\lambda| + 1} \|f\|_{W^{k+1,p} \times W^{k,p}}, \quad k = 0, 1,$$

and

$$\|\partial_x^k \widetilde{Q} R^{(1)}(\lambda) f\|_p \leq C \left\{ \frac{\|Q_0 f\|_{W^{k,p}}}{|\lambda| + 1} + \frac{\|\widetilde{Q} f\|_p}{(|\lambda| + 1)^{1 - \frac{k}{2}}} \right\}, \quad k = 0, 1.$$

§4. Proof of Theorem 2.1

In this section we prove Theorem 2.1 by applying Theorems 3.1–3.6.

Proof of Theorem 2.1. Let $\eta > 0$ be a positive number. By Theorem 2.1 in [7] there exists a number $\theta \in (\frac{\pi}{2}, \pi)$ such that $\mathscr{U}(t) u_0$ is written as

$$\mathscr{U}(t) u_0 = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda + L)^{-1} u_0 \, d\lambda,$$

where $\Gamma = \{\lambda = \eta + se^{\pm i\theta};\ s \geq 0\}$.

We decompose $\mathscr{U}(t)u_0$ into the following form:

$$\mathscr{U}(t)u_0 = U^{(0)}(t)u_0 + U^{(1)}(t)u_0,$$

where $U^{(j)}(t)u_0$, $j = 0, 1$, are defined by

$$U^{(j)}(t)u_0 = \mathscr{F}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}\chi^{(j)}(\xi')(\lambda + \widehat{L}_{\xi'})^{-1}\, d\lambda\right], \quad j = 0, 1,$$

with $\chi^{(j)}(\xi')$ defined in (3.26).

We first consider $U^{(1)}(t)u_0$. In view of Theorem 3.4, we can deform the contour Γ into $\Gamma_\infty = \{\lambda = -\widetilde{\eta} + se^{\pm i\widetilde{\theta}};\ s \geq 0\}$, where $\widetilde{\eta}$ and $\widetilde{\theta}$ are the numbers given in Theorem 3.4. We then obtain

$$U^{(1)}(t)u_0 = \frac{1}{2\pi i}\int_{\Gamma_\infty} e^{\lambda t}R^{(1)}(\lambda)u_0\, d\lambda,$$

where $R^{(1)}(\lambda)$ is the operator defined in (3.27). It follows from Theorems 3.4–3.6 that

$$\begin{aligned}
&\left\|\partial_x^\ell U^{(1)}(t)u_0\right\|_p \leq Ce^{-ct}\|u_0\|_{W^{\ell,p}\times L^p}, \quad 1 < p < \infty, \quad \ell = 0, 1,\\
&\left\|\partial_x^\ell U^{(1)}(t)u_0\right\|_\infty \leq Ce^{-ct}\|u_0\|_{H^{[\frac{n}{2}]+1+\ell}\times H^{[\frac{n}{2}]+\ell}}, \quad \ell = 0, 1,\\
&\left\|\partial_x^\ell U^{(1)}(t)u_0\right\|_p \leq Ce^{-ct}\|u_0\|_{W^{\ell+1,p}\times W^{\ell,p}}, \quad p = 1, \infty, \quad \ell = 0, 1,
\end{aligned}$$

for $t \geq 1$.

We next consider $U^{(0)}(t)u_0$. By Theorem 3.1, we can deform the contour Γ into $\Gamma_0 \cup \widetilde{\Gamma}$ and a suitable circle around 0, where

$$\Gamma_0 = \{\lambda = -\eta_0 + is;\ |s| \leq s_0\}, \quad \widetilde{\Gamma} = \{\lambda = \eta + se^{\pm i\theta};\ s \geq \widetilde{s}_0\}.$$

Here we choose positive numbers s_0 and $\widetilde{s}_0$ so that Γ_0 connects with $\widetilde{\Gamma}$ at the end points of Γ_0. It then follows from Theorems 3.2, 3.3 and the residue theorem that $U^{(0)}(t)u_0$ is written as

$$U^{(0)}(t)u_0 = W^{(0)}(t)u_0 + W^{(1)}(t)u_0,$$

where

$$W^{(0)}(t)u_0 = \mathscr{F}^{-1}\left[\chi^{(0)}(\xi')e^{\lambda_0(\xi')t}\widehat{\Pi}(\xi')\widehat{u}_0\right]$$

and

$$W^{(1)}(t)u_0 = \mathscr{F}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_0\cup\widetilde{\Gamma}} e^{\lambda t}\chi^{(0)}(\xi')(\lambda + \widehat{L}_{\xi'})^{-1}\widehat{u}_0\, d\lambda\right].$$

By using the integral representation of $(\lambda + \widehat{L}_{\xi'})^{-1}$ given in [7, Theorem 3.8], one can see that $W^{(1)}(t)u_0$ has the same estimates as those for $U^{(1)}(t)u_0$. We will give an outline of the estimates for $W^{(1)}(t)u_0$ in the Appendix.

Let us consider $W^{(0)}(t)u_0$. We write it as

$$W^{(0)}(t)u_0 = \mathscr{W}^{(0)}(t)u_0 + \mathscr{R}^{(0)}(t)u_0,$$

where

$$\mathscr{W}^{(0)}(t)u_0 = \mathscr{F}^{-1}\left[e^{-\kappa|\xi'|^2 t}\widehat{\Pi}^{(0)}\widehat{u}_0\right], \quad \kappa = -\frac{a^2\gamma^2}{12\nu},$$

and

$$\mathscr{R}^{(0)}(t)u_0 = \mathscr{W}^{(1)}(t)u_0 + \mathscr{R}_1^{(0)}(t)u_0 + \mathscr{R}_2^{(0)}(t)u_0 + \mathscr{R}_3^{(0)}(t)u_0.$$

Here

$$\begin{aligned}
\mathscr{W}^{(1)}(t)u_0 &= \mathscr{F}^{-1}\left[(\chi^{(0)}(\xi') - 1)e^{-\kappa|\xi'|^2 t}\widehat{\Pi}^{(0)}\widehat{u}_0\right], \\
\mathscr{R}_1^{(0)}(t)u_0 &= \mathscr{F}^{-1}\left[\chi^{(0)}(\xi')e^{-\kappa|\xi'|^2 t}\widehat{\Pi}^{(1)}(\xi')\widehat{u}_0\right], \\
\mathscr{R}_2^{(0)}(t)u_0 &= \mathscr{F}^{-1}\left[\chi^{(0)}(\xi')e^{-\kappa|\xi'|^2 t}\widehat{\Pi}^{(2)}(\xi')\widehat{u}_0\right]
\end{aligned}$$

and

$$\mathscr{R}_3^{(0)}(t)u_0 = \mathscr{F}^{-1}\left[\chi^{(0)}(\xi')(e^{\lambda_0(\xi')t} - e^{-\kappa|\xi'|^2 t})\widehat{\Pi}(\xi')\widehat{u}_0\right]$$

with $\kappa = \frac{a^2\gamma^2}{12\nu}$ and

$$\widehat{\Pi}^{(1)}(\xi') = \sum_{j=1}^{n-1} \xi_j \widehat{\Pi}_j^{(1)}.$$

Clearly, $\mathscr{W}^{(0)}(t)u_0 = \begin{pmatrix} \phi^{(0)}(t) \\ 0 \end{pmatrix}$ and $\phi^{(0)}$ satisfies

$$\partial_t \phi^{(0)} - \kappa\Delta'\phi^{(0)} = 0, \quad \phi^{(0)}\Big|_{t=0} = \langle \phi_0 \rangle.$$

It is easy to see that

$$\left\|\partial_x^\ell \mathscr{W}^{(1)}(t)u_0\right\|_p \le Ce^{-ct}\|u_0\|_1, \quad \ell = 0, 1.$$

By Theorem 3.3, we easily deduce that

$$\left\|\partial_x^\ell \mathscr{R}_1^{(0)}(t)u_0\right\|_p \le Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}}\|u_0\|_1, \quad \ell = 0, 1.$$

Let us consider $\mathscr{R}_2^{(0)}(t)u_0$. We will estimate it based on the Riemann-Lebesgue lemma as in the estimates for solutions of the Cauchy problem given in [13]. Since

$$\mathscr{R}_2^{(0)}(t)u_0 = \int_{\mathbf{R}^{n-1}} \int_0^a \mathscr{R}_2^{(0)}(t, x'-y', x_n, y_n)u(y', y_n)\, dy'dy_n$$

with

$$\begin{aligned} (4.1) \qquad \mathscr{R}_2^{(0)}(t, x', x_n, y_n) &= \mathscr{F}^{-1}\left[\chi^{(0)}(\xi')e^{-\kappa|\xi'|^2 t}\widehat{\Pi}^{(2)}(\xi', x_n, y_n)\right](x') \\ &= \int_{\mathbf{R}^{n-1}} \chi^{(0)}(\xi')e^{-\kappa|\xi'|^2 t}\widehat{\Pi}^{(2)}(\xi', x_n, y_n)e^{i\xi'\cdot x}\, d\xi', \end{aligned}$$

we have

$$\left\|\partial_x^\ell \mathscr{R}_2^{(0)}(t)u_0\right\|_1 \le \sup_{0\le y_n\le a}\left\|\partial_x^\ell \mathscr{R}_2^{(0)}(t,\cdot,\cdot,y_n)\right\|_1 \|u_0\|_1.$$

By Theorem 3.3, we see that

$$\sup_{0\le y_n\le a}\left|\partial_{\xi'}^{\alpha'}\left(\xi'^{\beta'}\partial_{x_n}^j \chi^{(0)}(\xi')e^{-\kappa|\xi'|^2 t}\widehat{\Pi}^{(2)}(\xi', x_n, y_n)\right)\right|_{L^1_{x_n}} \le C|\xi'|^{2-|\alpha'|}e^{-\frac{\kappa}{2}|\xi'|^2 t}$$

for $|\beta'| + j \le 1$. Therefore, since $e^{i\xi'\cdot x'} = \displaystyle\sum_{j=1}^{n-1}\frac{x_j}{i|x'|^2}\partial_{\xi_j}e^{i\xi'\cdot x'}$, we perform the integration by parts in (4.1) to obtain, for any $k = 0, 1, 2, \ldots,$

$$\begin{aligned} \sup\nolimits_{0\le y_n\le a}\left|\partial_{x'}^{\beta'}\partial_{x_n}^j \mathscr{R}_2^{(0)}(t, x', \cdot, y_n)\right|_1 &\le C|x'|^{-k}\int_{\mathbf{R}^{n-1}}|\xi'|^{2-k}e^{-\frac{\kappa}{2}|\xi'|^2 t}\, d\xi' \\ &\le C|x'|^{-k}t^{\frac{k}{2}}t^{-\frac{n-1}{2}-1}, \quad |\beta'| + j \le 1. \end{aligned}$$

This implies that

$$\begin{aligned} &\sup_{0\le y_n\le a}\left\|\partial_x^\ell \mathscr{R}_2^{(0)}(t,\cdot,\cdot,y_n)\right\|_1 \\ &\quad\le C\int_{|x'|\le t^{\frac12}} t^{-\frac{n-1}{2}-1}\, dx' + \int_{|x'|\ge t^{\frac12}}|x'|^{-n}t^{\frac{n}{2}}t^{-\frac{n-1}{2}-1}\, dx' \\ &\quad\le Ct^{-1} \end{aligned}$$

for $\ell = 0, 1$. Similarly, one can estimate $\mathscr{R}_3^{(0)}(t)u_0$. In fact, by Theorem 3.2, we have $\lambda_0(\xi') = -\kappa|\xi'|^2 + \lambda^{(4)}(\xi')$, where $\lambda^{(4)}(\xi')$ is analytic in ξ' and $|\lambda^{(4)}(\xi')| \le C|\xi'|^4$. Since

$$e^{\lambda_0(\xi')t} - e^{-\kappa|\xi'|^2 t} = \lambda^{(4)}(\xi')te^{-\kappa|\xi'|^2 t}\int_0^1 e^{\theta\lambda^{(4)}(\xi')t}\, d\theta,$$

we see from Theorem 3.3 that

$$\sup_{0\le y_n\le a}\left|\partial_{\xi'}^{\alpha'}\left[\xi'^{\beta'}\partial_{x_n}^{j}\chi^{(0)}(\xi')(e^{\lambda_0(\xi')t}-e^{-\kappa|\xi'|^2t})\widehat{\Pi}(\xi',\cdot,y_n)\right]\right|_1 \\ \le C|\xi'|^{2-|\alpha'|}e^{-\frac{\kappa}{2}|\xi'|^2t}$$

for $|\beta'|+j\le 1$. Similarly to above, one can obtain $\sup_{0\le y_n\le a}\left\|\partial_x^\ell \mathscr{R}_3^{(0)}(t,\cdot,\cdot,y_n)\right\|_1 \le Ct^{-1}$ for $\ell=0,1$. Consequently, we have

$$\left\|\partial_x^\ell \mathscr{R}^{(0)}(t)u_0\right\|_1 \le Ct^{-\frac{1}{2}}\|u_0\|_1, \quad \ell=0,1.$$

On the other hand, it is easy to see that

$$\left\|\partial_x^\ell \mathscr{R}^{(0)}(t)u_0\right\|_\infty \le Ct^{-\frac{n-1}{2}-\frac{1}{2}}\|u_0\|_1, \quad \ell=0,1.$$

Therefore, by interpolation, we have

$$\left\|\partial_x^\ell \mathscr{R}^{(0)}(t)u_0\right\|_p \le Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}}\|u_0\|_1, \quad \ell=0,1.$$

By Theorem 3.3, we have $\widehat{\Pi}^{(0)}\widetilde{Q}=0$, $\partial_{x_n}\widehat{\Pi}^{(0)}=0$ and $\partial_{x_n}\widehat{\Pi}^{(1)}(\xi')\widetilde{Q}=0$. It then follows that

$$\begin{aligned}\left\|\partial_{x_n}\mathscr{R}^{(0)}(t)\widetilde{Q}u_0\right\|_p &= \left\|\partial_{x_n}\left(\mathscr{R}_2^{(0)}(t)+\mathscr{R}_3^{(0)}(t)\right)\widetilde{Q}u_0\right\|_p \\ &\le Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-1}\left\|\widetilde{Q}u_0\right\|_1.\end{aligned}$$

Since $\widehat{\Pi}^{(j)}(\xi')\left[\partial_{x_n}\widetilde{Q}\widehat{u}_0\right]=-\left(\partial_{y_n}\widehat{\Pi}^{(j)}(\xi')\right)\left[\widetilde{Q}\widehat{u}_0\right]$, $j=1,2$, we see that

$$\begin{aligned}\left\|\mathscr{R}^{(0)}(t)\left[\partial_{x_n}\widetilde{Q}u_0\right]\right\|_p &= \left\|\left(\partial_{y_n}\mathscr{R}^{(0)}(t)\right)\widetilde{Q}u_0\right\|_p \\ &\le Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-\frac{1}{2}}\left\|\widetilde{Q}u_0\right\|_1.\end{aligned}$$

Clearly, $\partial_{x'}\mathscr{R}^{(0)}(t)\widetilde{Q}u_0=\mathscr{R}^{(0)}(t)\left[\partial_{x'}\widetilde{Q}u_0\right]$ and

$$\left\|\partial_{x'}\mathscr{R}^{(0)}(t)\widetilde{Q}u_0\right\|_p \le Ct^{-\frac{n-1}{2}(1-\frac{1}{p})-1}\left\|\widetilde{Q}u_0\right\|_1.$$

The desired results of Theorem 2.1 are thus obtained by setting

$$\mathscr{U}^{(0)}(t)=\mathscr{W}^{(0)}(t)+\mathscr{R}^{(0)}(t)$$

and

$$\mathscr{U}^{(\infty)}(t)=U^{(1)}(t)+W^{(1)}(t).$$

This completes the proof. □

§5. Appendix

We here give an outline of the estimate for $W^{(1)}(t)u_0$. The proof is based on the integral representation for $(\lambda+\widehat{L}_{\xi'})^{-1}\widehat{u}_0$ given in [7, Theorem 3.8]. (See Proposition A.2 below.)

We write

$$W^{(1)}(t)u_0 = W^{(1,0)}(t)u_0 + W^{(1,1)}(t)u_0,$$

where

$$W^{(1,0)}(t)u_0 = \mathscr{F}^{-1}\left[\frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}\chi^{(0)}(\xi')(\lambda+\widehat{L}_{\xi'})^{-1}\widehat{u}_0\,d\lambda\right]$$

and

$$W^{(1,1)}(t)u_0 = \mathscr{F}^{-1}\left[\frac{1}{2\pi i}\int_{\widetilde{\Gamma}} e^{\lambda t}\chi^{(0)}(\xi')(\lambda+\widehat{L}_{\xi'})^{-1}\widehat{u}_0\,d\lambda\right].$$

The desired estimates for $W^{(1,1)}(t)u_0$ can be obtained exactly in a similar manner to the proof of [7, Theorems 2.1–2.3].

Let us consider $W^{(1,0)}(t)u_0$. We first recall the integral representation given in [7, Theorem 3.8].

We introduce the characteristic roots of the ordinary differential system $(\lambda+\widehat{L}_{\xi'})u=0$, which are given by $\pm\mu_j(\lambda,\xi')$, $j=1,2$, where

$$\mu_1 = \mu_1(\lambda,|\xi'|^2) = \sqrt{\frac{\lambda+\nu|\xi'|^2}{\nu}}$$

and

$$\mu_2 = \mu_2(\lambda,|\xi'|^2) = \sqrt{\frac{\lambda^2+\nu_1|\xi'|^2\lambda+\gamma^2|\xi'|^2}{\nu_1\lambda+\gamma^2}}.$$

Observe that $\mu_2 = \sqrt{\frac{(\lambda-\lambda_{+,0})(\lambda-\lambda_{-,0})}{(\nu_1\lambda+\gamma^2)}}$ with $\lambda_{\pm,0} = -\frac{\nu_1}{2}|\xi'|^2 \pm \frac{1}{2}\sqrt{\nu_1^2|\xi'|^4-4\gamma^2|\xi'|^2}$, where $\lambda_{\pm,0}$ satisfies $\lambda_{-,0}=\overline{\lambda_{+,0}}$ with $\operatorname{Im}\lambda_{+,0} = \gamma|\xi'|\sqrt{1-\frac{\nu_1^2}{4\gamma^2}|\xi'|^2}$ when $|\xi'| < 2\gamma/\nu_1$ and

$$\lambda_{\pm,0} = -\frac{\nu_1}{2}|\xi'|^2 \pm i\gamma|\xi'| + O(|\xi'|^3)$$

as $|\xi'|\to 0$.

We next define complex valued functions b_j $(j=1,2,3)$ by

$$b_1(\lambda,\xi',x_n) = b_1(\lambda,|\xi'|^2,x_n) = \cosh\mu_1 x_n - \cosh\mu_2 x_n,$$

$$b_2(\lambda,\xi',x_n) = b_2(\lambda,|\xi'|^2,x_n) = \mu_1\sinh\mu_1 x_n - \tfrac{|\xi'|^2}{\mu_2}\sinh\mu_2 x_n,$$

$$b_3(\lambda,\xi',x_n) = b_3(\lambda,|\xi'|^2,x_n) = \mu_2\sinh\mu_2 x_n - \tfrac{|\xi'|^2}{\mu_1}\sinh\mu_1 x_n,$$

with $\mu_j = \mu_j(\lambda, \xi')$, $j = 1, 2$. We set

$$D(\lambda, \xi') = D(\lambda, |\xi|^2) = b_3(\lambda, \xi', a) b_2(\lambda, \xi', a) + |\xi'|^2 b_1(\lambda, \xi', a)^2.$$

In the following we will frequently abbreviate $b_j(\lambda, \xi', x_n)$ to $b_j(x_n)$.

Roughly speaking, $\lambda \in \rho(-\widehat{L}_{\xi'})$ if and only if $D(\lambda, \xi') \neq 0$. In fact, by [7, Lemma 3.4] we have the following result.

Proposition A.1. $D(\lambda, \xi') \neq 0$ *for* $(\lambda, \xi') \in \Gamma_0 \times \{|\xi'| \leq r_0\}$.

To give an integral representation of $(\lambda + \widehat{L}_{\xi'})^{-1} f$, we prepare several functions. We define $\beta_0(\lambda, \xi', y_n)$ by

$$\beta_0(\lambda, \xi', y_n) = \frac{\gamma\lambda}{\nu_1 \lambda + \gamma^2} \frac{1}{D(\lambda, \xi')} \left\{ b_3(a) \frac{1}{\mu_2} \sinh \mu_2 y_n + b_1(a) \cosh \mu_2 y_n \right\}$$

and set

$$\boldsymbol{b}_0(\lambda, \xi', y_n) = i\xi' \beta_0(\lambda, \xi', y_n).$$

We define $\boldsymbol{b}_n(\lambda, \xi', y_n)$ by

$$\boldsymbol{b}_n(\lambda, \xi', y_n) = -\frac{i\xi'}{D(\lambda, \xi')} \{b_3(a) b_1(y_n) - b_1(a) b_3(y_n)\}$$

and $B'(\lambda, \xi', y_n)$ by

$$B'(\lambda, \xi', y_n) = -\frac{\sinh \mu_1 y_n}{\sinh \mu_1 a} I_{n-1} + \beta(\lambda, \xi', y_n) \frac{\xi'^T \xi'}{|\xi'|^2}$$

with

$$\beta(\lambda, \xi', y_n) = \frac{\sinh \mu_1 y_n}{\sinh \mu_1 a} - \beta_1(\lambda, \xi', y_n)$$

and

$$\beta_1(\lambda, \xi', y_n) = \frac{1}{D(\lambda, \xi')} \{b_3(a) b_2(y_n) + |\xi'|^2 b_1(a) b_1(y_n)\}.$$

We finally introduce the Green functions of the equation $\mu_j^2 v - \partial_{x_n}^2 v = 0$ under the Dirichlet and Neumann boundary conditions at $\{x_n = 0, a\}$. We define $g^D_{\mu_j}(x_n, y_n)$ $(j = 1, 2)$ by

$$g^D_{\mu_j}(x_n, y_n) = \begin{cases} \dfrac{1}{\mu_j \sinh \mu_j a} \sinh \mu_j (a - x_n) \sinh \mu_j y_n, \, y_n \leq x_n, \\ \dfrac{1}{\mu_j \sinh \mu_j a} \sinh \mu_j (a - y_n) \sinh \mu_j x_n, \, x_n \leq y_n. \end{cases}$$

Similarly, we define $g^N_{\mu_j}(x_n, y_n)$ by

$$g^N_{\mu_j}(x_n, y_n) = \begin{cases} \dfrac{1}{\mu_j \sinh \mu_j a} \cosh \mu_j (a - x_n) \cosh \mu_j y_n, & y_n \le x_n, \\ \dfrac{1}{\mu_j \sinh \mu_j a} \cosh \mu_j (a - y_n) \cosh \mu_j x_n, & x_n \le y_n. \end{cases}$$

We set

$$g^M_{\mu_1,\mu_2}(x_n, y_n) = g^M_{\mu_1}(x_n, y_n) - g^M_{\mu_2}(x_n, y_n), \quad M = D, N.$$

Note that $g^D_{\mu_j}$ (resp. $g^N_{\mu_j}$) are the Green functions of the equation $\mu_j^2 v - \partial^2_{x_n} v = 0$ under the Dirichlet (resp. Neumann) boundary condition at $\{x_n = 0, a\}$. We also define $h_{\mu_j}(x_n)$ and $h_{\mu_1,\mu_2}(x_n)$ by

$$h_{\mu_j}(x_n) = \frac{1}{\mu_j \sinh \mu_j a} \cosh \mu_j x_n$$

and

$$h_{\mu_1,\mu_2}(x_n) = h_{\mu_1}(x_n) - h_{\mu_2}(x_n).$$

We denote the Dirac measure by δ.

Using the functions described above, we can obtain an integral representation for $(\lambda + \widehat{L}_{\xi'})^{-1} f$ ([7, Theorem 3.8]). In particular, we have the following result.

Proposition A.2. *If $(\lambda, \xi') \in \Gamma_0 \times \{0 < |\xi'| \le r_0\}$, then $(\lambda + \widehat{L}_{\xi'})^{-1} f$ is written as*

$$(\lambda + \widehat{L}_{\xi'})^{-1} f(x_n) = \int_0^a \widehat{R}(\lambda, \xi', x_n, y_n) f(y_n)\, dy_n,$$

where

$$\widehat{R}(\lambda, \xi', x_n, y_n) = \widehat{G}(\lambda, \xi', x_n, y_n) + \widehat{K}(\lambda, \xi', x_n, y_n).$$

Here $\widehat{G}(\lambda, \xi', x_n, y_n)$ is an $(n+1) \times (n+1)$ matrix of the form

$$\widehat{G}(\lambda, \xi', x_n, y_n)$$

$$= \tfrac{\nu_1}{d(\lambda)} \delta(x_n - y_n) Q_0$$

$$+ \tfrac{\gamma}{d(\lambda)} \begin{pmatrix} \tfrac{\gamma\lambda}{d(\lambda)} g^N_{\mu_2}(x_n, y_n) & -i^T \xi' g^N_{\mu_2}(x_n, y_n) & -\partial_{x_n} g^D_{\mu_2}(x_n, y_n) \\ -i\xi' g^N_{\mu_2}(x_n, y_n) & 0 & 0 \\ -\partial_{x_n} g^N_{\mu_2}(x_n, y_n) & 0 & 0 \end{pmatrix}$$

$$+\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\nu} g^N_{\mu_1}(x_n, y_n) I_{n-1} & 0 \\ 0 & 0 & \frac{1}{\nu} g^D_{\mu_1}(x_n, y_n) \end{pmatrix}$$

$$+\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\xi'^T\xi'}{\lambda} g^N_{\mu_1,\mu_2}(x_n, y_n) & -\frac{i\xi'}{\lambda}\partial_{x_n} g^D_{\mu_1,\mu_2}(x_n, y_n) \\ 0 & -\frac{i\xi'}{\lambda}\partial_{x_n} g^N_{\mu_1,\mu_2}(x_n, y_n) & -\frac{1}{\lambda}\partial^2_{x_n} g^D_{\mu_1,\mu_2}(x_n, y_n) \end{pmatrix},$$

where $d(\lambda) = \nu_1\lambda + \gamma^2$ *and* $\mu_j = \mu_j(\lambda, \xi')$, $j = 1, 2$; *and*

$$\widehat{K}(\lambda, \xi', x_n, y_n) = \widehat{H}(\lambda, \xi', x_n, y_n) + \check{H}(\lambda, \xi', a - x_n, a - y_n),$$

where

$$\check{H}(\lambda, \xi', a - x_n, a - y_n) = \widehat{H}(\lambda, \xi', a - x_n, a - y_n)\mathrm{diag}\,(I_n, -1)$$

and

$$\widehat{H}(\lambda, \xi', x_n, y_n)$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ \frac{i\xi'}{\nu} h_{\mu_1}(x_n)\beta_0(y_n) & \frac{1}{\nu} h_{\mu_1}(x_n) B'(y_n) & \frac{1}{\nu} h_{\mu_1}(x_n)\boldsymbol{b}_n(y_n) \\ 0 & 0 & 0 \end{pmatrix}$$

$$+\begin{pmatrix} \frac{\gamma|\xi'|^2}{d(\lambda)} h_{\mu_2}(x_n)\beta_0(y_n) & \frac{i\gamma^T\xi'}{d(\lambda)} h_{\mu_2}(x_n)\beta_1(y_n) & -\frac{i\gamma^T\xi'}{d(\lambda)} h_{\mu_2}(x_n)\boldsymbol{b}_n(y_n) \\ \frac{i|\xi'|^2\xi'}{\lambda} h_{\mu_1,\mu_2}(x_n)\beta_0(y_n) & 0 & 0 \\ -\frac{|\xi'|^2}{\lambda}\partial_{x_n} h_{\mu_1,\mu_2}(x_n)\beta_0(y_n) & 0 & 0 \end{pmatrix}$$

$$+\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\xi'^T\xi'}{\lambda} h_{\mu_1,\mu_2}(x_n)\beta_1(y_n) & \frac{|\xi'|^2}{\lambda} h_{\mu_1,\mu_2}(x_n)\boldsymbol{b}_n(y_n) \\ 0 & -\frac{i'^T\xi'}{\lambda}\partial_{x_n} h_{\mu_1,\mu_2}(x_n)\beta_1(y_n) & \frac{i^T\xi'}{\lambda}\partial_{x_n} h_{\mu_1,\mu_2}(x_n)\boldsymbol{b}_n(y_n) \end{pmatrix}$$

with $d(\lambda) = \nu_1\lambda + \gamma^2$, $\mu_j = \mu_j(\lambda, \xi')$, $j = 1, 2$, $\beta_j(y_n) = \beta_j(\lambda, \xi', y_n)$, $j = 0, 1$, $B'(y_n) = B'(\lambda, \xi', y_n)$ *and* $\boldsymbol{b}_n(y_n) = \boldsymbol{b}_n(\lambda, \xi', y_n)$.

We are now in a position to give an outline of the proof of the estimate

$$\|W^{(1,0)}(t)u_0\|_p \le Ce^{-ct}\|u_0\|_p, \quad 1 \le p \le \infty. \tag{A.1}$$

Let $\widehat{\Psi}(\lambda, \xi', x_n, y_n)$ be defined by

$$\widehat{\Psi}(\lambda, \xi', x_n, y_n) = \chi^{(0)}(\xi') \left(\widehat{R}(\lambda, \xi', x_n, y_n) - \frac{\nu_1}{d(\lambda)} \delta(x_n - y_n) Q_0 \right).$$

Note that $\operatorname{supp} \widehat{\Psi}(\lambda, \cdot, x_n, y_n) \subset \{|\xi'| \le r_0\}$.

An elementary observation shows that

$$0 < c_1 \le |\mu_j(\lambda, \xi')| \le c_2, \quad j = 1, 2,$$

uniformly in $(\lambda, \xi') \in \Gamma_0 \times \{|\xi'| \le r_0\}$ (by changing η_0 and r_0 suitably if necessary). We also observe that $\sinh \mu_j a = 0$ $(j = 1, 2)$ if and only if

$$\lambda = -\nu |\xi^{(k)}|^2 \quad \text{for } j = 1,$$

$$\lambda = -\tfrac{\nu_1}{2} |\xi^{(k)}|^2 \pm \tfrac{1}{2} \sqrt{\nu_1^2 |\xi^{(k)}|^4 - 4\gamma^2 |\xi^{(k)}|^2} \quad \text{for } j = 2,$$

where $|\xi^{(k)}|^2 = |\xi'|^2 + a_k^2$, $k = 0, 1, 2, \dots$. Using these facts, together with Proposition A.1, we can see

$$\left| \partial_{\xi'}^{\alpha'} \left[\widehat{\Psi}(\lambda, \xi', x_n, y_n) \right] \right| \le C_{\alpha'} \tag{A.2}$$

for any multi-index α' uniformly in $(\lambda, \xi') \in \Gamma_0 \times \{|\xi'| \le r_0\}$ and $0 \le x_n, y_n \le a$. In fact, it is not difficult to show (A.2) for each term in $\widehat{\Psi}(\lambda, \xi', x_n, y_n)$ except the one including $B'(\lambda, \xi', y_n) = -\frac{\sinh \mu_1 y_n}{\sinh \mu_1 a} I_{n-1} + \beta(\lambda, \xi', y_n) \frac{\xi'^T \xi'}{|\xi'|^2}$ which seems to be singular as $\xi' \to 0$. But by a direct computation one can show $\beta(\lambda, \xi', y_n) = |\xi'|^2 \widetilde{\beta}(\lambda, \xi', y_n)$ with $\widetilde{\beta}(\lambda, \xi', y_n)$ being smooth in $(\lambda, \xi', y_n) \in \Gamma_0 \times \{|\xi'| \le r_0\} \times [0, a]$. (Cf. Proof of [7, Lemma 4.8].) Therefore, we can obtain the estimate (A.2).

We set

$$\begin{aligned} \Psi(\lambda, x', x_n, y_n) &= \mathscr{F}^{-1} \left[\widehat{\Psi}(\lambda, \cdot, x_n, y_n) \right] \\ &= (2\pi)^{-(n-1)} \int_{\mathbf{R}^{n-1}} \widehat{\Psi}(\lambda, \xi', x_n, y_n) e^{i \xi' \cdot x'} \, d\xi'. \end{aligned}$$

By integration by parts we see from (A.2) that

$$\sup_{0 \le x_n, y_n \le a} |\Psi(\lambda, x', x_n, y_n)| \le C_k |x'|^{-k}$$

for all $k = 0, 1, 2, \dots$, uniformly in $\lambda \in \Gamma_0$. This implies that

$$\sup_{0 \le x_n, y_n \le a} \|\Psi(\lambda, \cdot, x_n, y_n)\|_{L^1(\mathbf{R}^{n-1})} \le C$$

uniformly in $\lambda \in \Gamma_0$.

It then follows that

$$\|W^{(1,0)}(t)u_0\|_p \leq Ce^{-ct}\|u_0\|_p, \quad 1 \leq p \leq \infty.$$

The estimates for derivatives can be obtained similarly.

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References

[1] T. Abe and Y. Shibata, On a resolvent estimate of the Stokes equation on an infinite layer, J. Math. Soc. Japan **55** (2003), no. 2, 469–497.

[2] T. Abe and Y. Shibata, On a resolvent estimate of the Stokes equation on an infinite layer. II. $\lambda = 0$ case, J. Math. Fluid Mech. **5** (2003), no. 3, 245–274.

[3] H. Abels and M. Wiegner, Resolvent estimates for the Stokes operator on an infinite layer, Differential Integral Equations **18** (2005), no. 10, 1081–1110.

[4] J. T. Beale and T. Nishida, Large-time behavior of viscous surface waves, in *Recent topics in nonlinear PDE, II (Sendai, 1984)*, 1–14, North-Holland, Amsterdam.

[5] D. Hoff and K. Zumbrun, Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow, Indiana Univ. Math. J. **44** (1995), no. 2, 603–676.

[6] ———, Pointwise decay estimates for multidimensional Navier-Stokes diffusion waves, Z. Angew. Math. Phys. **48** (1997), no. 4, 597–614.

[7] Y. Kagei, Resolvent estimates for the linearized compressible Navier-Stokes equation in an infinite layer, preprint, MHF Preprint Series, Kyushu University, MHF 2006–22.

[8] Y. Kagei and T. Kobayashi, On large-time behavior of solutions to the compressible Navier-Stokes equations in the half space in $\mathbf{R}^3$, Arch. Ration. Mech. Anal. **165** (2002), no. 2, 89–159.

[9] ———, Asymptotic behavior of solutions of the compressible Navier-Stokes equations on the half space, Arch. Ration. Mech. Anal. **177** (2005), no. 2, 231–330.

[10] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[11] S. Kawashima, Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics, Ph. D. Thesis, Kyoto University, 1983.

[12] T. Kobayashi, Some estimates of solutions for the equations of motion of compressible viscous fluid in the three-dimensional exterior domain, J. Differential Equations **184** (2002), no. 2, 587–619.

[13] T. Kobayashi and Y. Shibata, Remark on the rate of decay of solutions to linearized compressible Navier-Stokes equations, Pacific J. Math. **207** (2002), no. 1, 199–234.

[14] A. Matsumura and T. Nishida, The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), no. 9, 337–342.