On Holomorphic L^2 Functions on Coverings of Strongly Pseudoconvex Manifolds

By

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Abstract

In this paper we answer an important question posed in the paper [GHS] by Gromov, Henkin and Shubin on existence of sufficiently many holomorphic L^2 functions on arbitrary coverings of strongly pseudoconvex manifolds.

§1.   Introduction

1.1. Let $M \subset\subset N$ be a domain with smooth boundary bM in an n-dimensional complex manifold N, specifically,

$$(1.1) \qquad M = \{z \in N \ : \ \rho(z) < 0\}$$

where ρ is a real-valued function of class $C^2(\Omega)$ in a neighbourhood Ω of the compact set $\overline{M} := M \cup bM$ such that

$$(1.2) \qquad d\rho(z) \neq 0 \quad \text{for all} \quad z \in bM \ .$$

Let $z_1, \ldots, z_n$ be complex local coordinates in N near $z \in bM$. Then the tangent space $T_z N$ at z is identified with $\mathbb{C}^n$. By $T_z^c(bM) \subset T_z N$ we denote the complex tangent space to bM at z, i.e.,

$$(1.3) \qquad T_z^c(bM) = \left\{ w = (w_1, \ldots, w_n) \in T_z(N) \ : \ \sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(z) w_j = 0 \right\} \ .$$

The *Levi form* of ρ at $z \in bM$ is a hermitian form on $T_z^c(bM)$ defined in the local coordinates by the formula

$$(1.4) \qquad L_z(w, \overline{w}) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) w_j \overline{w}_k \ .$$

The manifold M is called *pseudoconvex* if $L_z(w, \overline{w}) \geq 0$ for all $z \in bM$ and $w \in T_z^c(bM)$. It is called *strongly pseudoconvex* if $L_z(w, \overline{w}) > 0$ for all $z \in bM$ and all $w \neq 0$, $w \in T_z^c(bM)$.

Equivalently, strongly pseudoconvex manifolds can be described as the ones which locally, in a neighbourhood of any boundary point, can be presented as strictly convex domains in $\mathbb{C}^n$. It is also known (see [C], [R]) that any strongly pseudoconvex manifold admits a proper holomorphic map with connected fibres onto a normal Stein space.

Without loss of generality we may and will assume that $\pi_1(M) = \pi_1(N)$ for M as above. Let $r : N' \to N$ be an unbranched covering of N. By $M' := r^{-1}(M)$ we denote the corresponding covering of M. Also, by $bM' := r^{-1}(bM)$ and $\overline{M'} := M' \cup bM'$ we denote the boundary and the closure of M' in N'.

Let $dV_{M'}$ be the Riemannian volume form on M' obtained by a Riemannian metric pulled back from N. By $H^2(M')$ we denote the Hilbert space of holomorphic functions g on M' with norm

$$(1.5) \qquad \left(\int_{z \in M'} |g(z)|^2 dV_{M'}(z) \right)^{1/2} .$$

Let X be a subspace of the space $\mathcal{O}(M')$ of all holomorphic functions on M'.

A point $z \in bM'$ is called a *peak point* for X if there exists a function $f \in X$ such that f is unbounded on M' but bounded outside $U \cap M'$ for any neighbourhood U of z in N'.

A point $z \in bM'$ is called a *local peak point* for X if there exists a function $f \in X$ such that f is unbounded in $U \cap M'$ for any neighbourhood U of z in N' and there exists a neighbourhood U of z in N' such that for any neighbourhood V of z in N' the function f is bounded on $U \setminus V$.

The Oka-Grauert theorem [G] implies that if M is strongly pseudoconvex and bM is not empty then every $z \in bM$ is a peak point for $H^2(M)$. In general it is not known whether a similar statement is true for boundary points of an infinite covering M' of M.

Assume that $M' := M_G$ is a regular covering of M with a transformation group G. In [GHS] the von Neumann G-dimension $\dim_G$ was used to measure

the space $H^2(M_G)$. In particular, one of the main results [GHS, Theorem 0.2] states:

Theorem. *If M is strongly pseudoconvex, then*

(a) $\dim_G H^2(M_G) = \infty$ *and*

(b) *each point in bM_G is a local peak point for $H^2(M_G)$.*

Also, in [GHS, p. 554] the following important question was asked: "*A natural question arises*: *is the compact group action really relevant for the existence of many holomorphic L^2-functions or is it just an artifact of the chosen methods which require a use of von Neumann algebras?*" And further: "*It is not clear how to formulate conditions assuring that dim $L^2\mathcal{O}(M') = \infty$*[1] *without any group action.*"

In fact, in the above question the authors do not mean coverings of strongly pseudoconvex manifolds only but rather complex manifolds of bounded geometry satisfying certain geometrical conditions.

The main result of the present paper answers the above formulated question in the case of coverings of strongly pseudoconvex manifolds. In particular, we show that the regularity of M' is irrelevant for the existence of many holomorphic L^2 functions on M'. Moreover, we also prove an extension of the above result of [GHS]. Our method of the proof is different from that used in [GHS] and is based on the L^2 cohomology techniques, as well as, on the geometric properties of M.

1.2. To formulate our result, let $C_M \subset M$ be the union of all compact complex subvarieties of M of complex dimension ≥ 1. It is known that if M is strongly pseudoconvex, then C_M is a compact complex subvariety of M. Let z_i, $1 \leq i \leq m$, be distinct points in $M \setminus C_M$. By $l_2(z_i')$ we denote the Hilbert space of l_2 functions on the fibre $z_i' := r^{-1}(z_i)$.

Theorem 1.1. *If M is strongly pseudoconvex, then*

(a) *For any $f_i \in l_2(z_i')$, $1 \leq i \leq m$, there exists $F \in H^2(M')$ such that $F|_{z_i'} = f_i$, $1 \leq i \leq m$;*

(b) *Each point in bM' is a peak point for $H^2(M')$.*

Similar results are valid for certain weighted L^p spaces of holomorphic functions on M'. These and some other results will be published elsewhere. It

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[1]dim $H^2(M')$ in our notation

is worth noting that results much stronger than Theorem 1.1 can be obtained if M is a strongly pseudoconvex Stein manifold, see [Br1], [Br2] for an exposition.

1.3. One of the purposes of [GHS] and of the present paper was to make a contribution toward settling the Shafarevich conjecture. Let us recall that the Shafarevich conjecture states that the universal covering $\widetilde{L}$ of a complex projective manifold L is holomorphically convex meaning that for every infinite sequence without limit points in $\widetilde{L}$ there exists a holomorphic function on $\widetilde{L}$ that is unbounded on this sequence. According to a theorem of Grauert any complex projective manifold L admits a holomorphic embedding into a strongly pseudoconvex manifold M of complex dimension $\dim_{\mathbb{C}} M = \dim_{\mathbb{C}} L + 1$ with the same fundamental group. Thus the basic idea is to try to develop the complex analysis on coverings M' of M and then, taking restrictions of holomorphic functions on M' to the corresponding coverings $L'(\subset M')$, to study holomorphic functions on L'.

The results of this paper and [GHS] don't imply directly any new results in the area of the Shafarevich conjecture. However, one obtains a rich complex function theory on coverings of strongly pseudoconvex manifolds M' (as above). Thus there is the hope that together with some additional ideas and methods it could give some information about holomorphic functions on L'. We are planning to return to this matter in a forthcoming paper.

Concerning the Shafarevich conjecture, for now the strongest results in this area are due to Eyssidieux [E] and, independently, to Campana [Ca] and the author [Br3]. The main result of the first of these papers states that the regular covering of a complex projective manifold L corresponding to the intersection of the kernels of all linear reductive representations $\pi_1(L) \to GL_n(\mathbb{C})$, with a fixed n, is holomorphically convex. In the last papers it is proved that the regular covering of a complex projective manifold L corresponding to the intersection of the kernels of all representations of $\pi_1(L)$ to complex solvable Lie groups is holomorphically convex. For other fundamental results in the area of the Shafarevich conjecture we refer the reader to the Featured Review by Katzarkov of the paper [E].

§2. Auxiliary Results

2.1. Let X be a complete Kähler manifold of dimension n with a Kähler form ω and E be a hermitian holomorphic vector bundle on X with curvature Θ. Let $L_2^{p,q}(X,E)$ be the space of L^2 E-valued (p,q)-forms on X with the L^2 norm, and let $W_2^{p,q}(X,E)$ be the subspace of forms such that $\overline{\partial}\eta$ is L^2. (The forms

η may be taken to be either smooth or just measurable, in which case $\overline{\partial}\eta$ is understood in the distributional sense.) The cohomology of the resulting L^2 Dolbeault complex $(W_2^{\cdot,\cdot}, \overline{\partial})$ is the L^2 cohomology

$$H_{(2)}^{p,q}(X, E) = Z_2^{p,q}(X, E)/B_2^{p,q}(X, E) \ ,$$

where $Z_2^{p,q}(X, E)$ and $B_2^{p,q}(X, E)$ are the spaces of $\overline{\partial}$-closed and $\overline{\partial}$-exact forms in $L_2^{p,q}(X, E)$, respectively.

If $\Theta \geq \epsilon\omega$ for some $\epsilon > 0$ in the sense of Nakano, then the L^2 Kodaira-Nakano vanishing theorem, see [D], [O], states that

$$(2.1) \qquad\qquad H_{(2)}^{n,r}(X, E) = 0 \quad \text{for} \quad r > 0 \ .$$

2.2. Let $M \subset\subset N$ be a strongly pseudoconvex manifold. Without loss of generality we will assume that $\pi_1(M) = \pi_1(N)$ and N is strongly pseudoconvex, as well. Then there exist a normal Stein space X_N, a proper holomorphic surjective map $p: N \to X_N$ with connected fibres and points $x_1, \ldots, x_l \in X_N$ such that

$$p: N \setminus \bigcup_{1 \leq i \leq l} p^{-1}(x_i) \to X_N \setminus \bigcup_{1 \leq i \leq l} \{x_i\}$$

is biholomorphic, see [C], [R]. By definition, the domain $X_M := p(M) \subset X_N$ is strongly pseudoconvex, and so it is Stein. Without loss of generality we may assume that $x_1, \ldots, x_l \in X_M$. Thus $\cup_{1 \leq i \leq l} p^{-1}(x_i) = C_M$.

Let $L \subset\subset N$ be a strongly pseudoconvex neighbourhood of $\overline{M}$. Then $X_L := p(L)$ is a strongly pseudoconvex neighbourhood of $\overline{X}_M$ in X_N. We introduce a complete Kähler metric on the complex manifold $L \setminus C_M$ as follows.

According to [N] there is a proper one-to-one holomorphic map $i: X_L \hookrightarrow \mathbb{C}^{2n+1}$, $n = \dim_{\mathbb{C}} X_L$, which is an embedding at regular points of X_L. Thus $i(X_L) \subset \mathbb{C}^{2n+1}$ is a closed complex subvariety. By ω_e we denote the $(1,1)$-form on L obtained as the pullback by $i \circ p$ of the Euclidean Kähler form on $\mathbb{C}^{2n+1}$. Clearly, ω_e is d-closed and positive outside C_M.

Similarly we can embed X_N into $\mathbb{C}^{2n+1}$ as a closed complex subvariety. Let $j: X_N \hookrightarrow \mathbb{C}^{2n+1}$ be an embedding such that $j(X_L)$ belongs to the open Euclidean ball B of radius $1/4$ centered at $0 \in \mathbb{C}^{2n+1}$. Set $z_i := j(x_i)$, $1 \leq i \leq l$. By ω_i we denote the restriction to $L \setminus C_M$ of the pullback with respect to $j \circ p$ of the form $-\sqrt{-1} \cdot \partial\overline{\partial} \log(\log ||z - z_i||^2)^2$ on $\mathbb{C}^{2n+1} \setminus \{z_i\}$. (Here $|| \cdot ||$ stands for the Euclidean norm on $\mathbb{C}^{2n+1}$.) Since $j(X_L) \subset B$, the form ω_i is Kähler. Its positivity follows from the fact that the function $-\log(\log ||z||^2)^2$ is strictly

plurisubharmonic for $||z|| < 1$. Also, ω_i is extended to a smooth form on $L \setminus p^{-1}(x_i)$. Now, let us introduce a Kähler form ω_L on $L \setminus C_M$ by the formula

$$(2.2) \qquad \omega_L := \omega_e + \sum_{1 \le i \le l} \omega_i \ .$$

Proposition 2.1. *The path metric d on $L \setminus C_M$ induced by ω_L is complete.*

Proof. Assume, on the contrary, that there is a sequence $\{w_j\}$ convergent either to C_M or to the boundary bL of L such that the sequence $\{d(o, w_j)\}$ is bounded (for a fixed point $o \in L \setminus C_M$). Then, since $\omega_L \ge \omega_e$, the sequence $\{i(p(w_j))\} \subset \mathbb{C}^{2n+1}$ is bounded. This implies that $\{w_j\}$ converges to C_M. But since $\omega_L \ge \sum \omega_i$, the latter is impossible, see, e.g., [GM] for similar arguments. $\square$

In the same way one obtains complete Kähler metrics on unbranched coverings of $L \setminus C_M$ induced by pullbacks to these coverings of the Kähler form ω_L on $L \setminus C_M$.

2.3. We retain the notation of the previous section. Also, for an n-dimensional complex manifold X by T_X and T_X^* we denote complex tangent and cotangent bundles on X and by $K_X = \wedge^n T_X^*$ the canonical line bundle on X.

Let $r : N' \to N$ be an unbranched covering. Consider the corresponding covering $(L \setminus C_M)' := r^{-1}(L \setminus C_M)$ of $L \setminus C_M$. We equip $(L \setminus C_M)'$ with the complete Kähler metric induced by the form $\omega_L' := r^*\omega_L$.

Next we consider the function $f := \sum_{0 \le i \le l} f_i$ on $(L \setminus C_M)'$ such that f_0 is the pullback by $i \circ p \circ r$ of the function $||z||^2$ on $\mathbb{C}^{2n+1}$ and f_i is the pullback by $j \circ p \circ r$ of the function $-\log(\log ||z - z_i||^2)^2$ on $\mathbb{C}^{2n+1} \setminus \{z_i\}$, $1 \le i \le l$. Clearly we have

$$(2.3) \qquad \omega_L' := \sqrt{-1} \cdot \partial \overline{\partial} f \ .$$

Let $E := (L \setminus C_M)' \times \mathbb{C}$ be the trivial holomorphic line bundle on $(L \setminus C_M)'$. Let g be the pullback to $(L \setminus C_M)'$ of a smooth plurisubharmonic function on L. We equip E with the hermitian metric e^{-f-g} (i.e., for $z \times v \in E$ the square of its norm in this metric equals $e^{-f(z)-g(z)}|v|^2$ where $|v|$ is the modulus of $v \in \mathbb{C}$). Then the curvature Θ_E of E satisfies

$$(2.4) \qquad \Theta_E := -\sqrt{-1} \cdot \partial \overline{\partial} \log(e^{-f-g}) = \omega_L' + \sqrt{-1} \cdot \partial \overline{\partial} g \ge \omega_L'.$$

Thus we can apply the L^2 Kodaira-Nakano vanishing theorem of Section 2.1 to get

$$(2.5) \qquad H^{n,r}_{(2)}((L \setminus C_M)', E) = 0 \quad \text{for} \quad r > 0 .$$

Let $K_{(L \setminus C_M)'}$ be the canonical holomorphic line bundle on $(L \setminus C_M)'$ equipped with the hermitian metric induced by ω'_L. Consider the hermitian line bundle $V_g := E \otimes K_{(L \setminus C_M)'}$ equipped with the tensor product of the corresponding hermitian metrics. Then from (2.5) we have

$$(2.6) \qquad H^{0,r}_{(2)}((L \setminus C_M)', V_g) \cong H^{n,r}_{(2)}((L \setminus C_M)', E) = 0 \quad \text{for} \quad r > 0 .$$

2.4. Let $U \subset L$ be a relatively compact neighbourhood of C_M. Consider a finite open cover $(U_i)_{1 \le i \le k}$ of $\overline{L \setminus U}$ by simply connected coordinate charts $U_i \subset\subset N \setminus C_M$. We introduce complex coordinates on $U'_i := r^{-1}(U_i) \subset N'$ by the pullback of the coordinates on U_i. In these coordinates U'_i is naturally identified with $U_i \times S$ where S is the fibre of $r : N' \to N$.

Let η be a smooth (p,q)-form on $(L \setminus C_M)'$ equals 0 on $r^{-1}(U)$. Then in the above holomorphic coordinates (z,s), $z = (z_1, \ldots, z_n) \in U_i \cap L$, $s \in S$, on $U'_i \cap L'$, $L' := r^{-1}(L)$, the form η is presented as

$$(2.7) \quad \eta(z,s) = \sum_{i_1,\ldots,i_p,j_1,\ldots,j_q} \eta_{i;i_1,\ldots,i_p,j_1,\ldots,j_q}(z,s) dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$$

where $\eta_{i;i_1,\ldots,i_p,j_1,\ldots,j_q}$ are smooth functions on $(U_i \cap L) \times S$.

We say that η belongs to the space $\mathcal{E}^{p,q}_{U;2}((L \setminus C_M)')$ if in (2.7) we have

$$(2.8) \qquad \sup_{z \in U_i \cap L, i, i_1,\ldots,i_p,j_1,\ldots,j_q} \left\{ \sum_{s \in S} |\eta_{i;i_1,\ldots,i_p,j_1,\ldots,j_q}(z,s)|^2 \right\} < \infty .$$

Let e be a holomorphic section of $K|_{L \setminus C_M}$. Then $\eta \cdot r^* e$ can be viewed as a (p,q)-form with values in V_g. (Here $r^* e$ is the pullback of e to $(L \setminus C_M)'$, i.e., $r^* e \in \mathcal{O}((L \setminus C_M)', K|_{(L \setminus C_M)'}).$)

Proposition 2.2. *For every $\eta \in \mathcal{E}^{p,q}_{U;2}((L \setminus C_M)')$ and $e \in \mathcal{O}(L \setminus C_M, K|_{L \setminus C_M})$ there is a plurisubharmonic function g in the definition of the metric on V_g such that $\eta \cdot r^* e \in L^{p,q}_2((L \setminus C_M)', V_g)$.*

Proof. In this proof by $|| \cdot ||$ we denote the hermitian metric on the space of V_g-valued (p,q)-forms induced by the hermitian metrics on V_g and $T_{(L \setminus C_M)'}$. Set

$$h_{i;i_1,\ldots,i_p,j_1,\ldots,j_q}(z) := ||r^* e(z,s) \cdot dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}||^2.$$

Then $h_{i;i_1,\ldots,i_p,j_1,\ldots,j_q}$ is a nonnegative continuous function on $U_i \cap L$. Let $\hat{g}$ be such that $r^*\hat{g} = g$. By the definition of metrics on V_g and $T_{(L \setminus C_M)'}$

$$(2.9) \qquad h_{i;i_1,\ldots,i_p,j_1,\ldots,j_q}(z) := \hat{h}_{i;i_1,\ldots,i_p,j_1,\ldots,j_q}(z) \cdot e^{-\hat{g}(z)},$$

where $\hat{h}_{i;i_1,\ldots,i_p,j_1,\ldots,j_q}$ is a nonnegative continuous function on $U_i \cap L$ independent of $\hat{g}$.

Now for some $A \in \mathbb{N}$ we have

$$(2.10) \qquad ||\eta(z,s) \cdot r^*e(z,s)||^2$$
$$\leq A \times \sum_{i_1,\ldots,i_p,j_1,\ldots,j_q} |\eta_{i;i_1,\ldots,i_p,j_1,\ldots,j_q}(z,s)|^2 \cdot h_{i;i_1,\ldots,i_p,j_1,\ldots,j_q}(z) \ .$$

According to the definition of $L_2^{p,q}((L \setminus C_M)', V_g)$ we have to show that

$$|\eta \cdot r^*e|^2 := \int_{(L \setminus C_M)'} ||\eta \cdot r^*e||^2 \cdot (\omega_L')^n < \infty \ .$$

Since $\omega_L' = r^*\omega_L$, from (2.9) and (2.10) we get

$$|\eta \cdot r^*e|^2 \leq$$

$$A \times \sum_{i=1}^{k} \int_{U_i \cap L} \left(\sum_{i_1,\ldots,i_p,j_1,\ldots,j_q,s \in S} |\eta_{i;i_1,\ldots,i_p,j_1,\ldots,j_q}(\cdot,s)|^2 \right) \hat{h}_{i;i_1,\ldots,i_p,j_1,\ldots,j_q} e^{-\hat{g}} \omega_L^n.$$

Also, by the hypothesis of the proposition, see (2.8),

$$\sup_{z \in U_i \cap L} \left\{ \sum_{i_1,\ldots,i_p,j_1,\ldots,j_q,s \in S} |\eta_{i;i_1,\ldots,i_p,j_1,\ldots,j_q}(z,s)|^2 \right\} < \infty \quad \text{for} \quad 1 \leq i \leq k \ .$$

Thus in order to prove the proposition it suffices to check that there is $\hat{g}$ in the definition of the metric on V_g such that for every i

$$(2.11) \qquad \int_{U_i \cap L} \hat{h}_{i;i_1,\ldots,i_p,j_1,\ldots,j_q} e^{-\hat{g}} \omega_L^n < \infty \ .$$

The required result now follows from

Lemma 2.3. *Let h be a nonnegative piecewise continuous function on L equals 0 in some neighbourhood of C_M and bounded on every compact subset of $L \setminus C_M$. Then there exists a smooth plurisubharmonic function $\hat{g}$ on L such that*

$$\int_L h \cdot e^{-\hat{g}} \omega_L^n < \infty \ .$$

Proof. Without loss of generality we identify $L \setminus C_M$ with $X_L \setminus \cup_{1 \leq j \leq l} \{x_j\}$. Also, we identify X_L with a closed subvariety of $\mathbb{C}^{2n+1}$ as in Section 2.2. Let U be a neighbourhood of $\cup_{1 \leq j \leq l} \{x_j\}$ such that $h|_U \equiv 0$. By $\Delta_r \subset \mathbb{C}^{2n+1}$ we denote the open polydisk of radius r centered at $0 \in \mathbb{C}^{2n+1}$. Assume without loss of generality that $0 \in X_L \setminus U$. Consider the monotonically increasing function

$$(2.12) \qquad v(r) := \int_{\Delta_r \cap (X_L \setminus U)} h \cdot \omega_L^n , \quad r \geq 0 .$$

By v_1 we denote a smooth monotonically increasing function satisfying $v \leq v_1$ (such v_1 can be easily constructed by v). Let us determine

$$v_2(r) := \int_0^{r+1} 2v_1(2t) \, dt , \quad r \geq 0 .$$

By the definition v_2 is smooth, convex and monotonically increasing. Moreover,

$$v_2(r) \geq \int_{\frac{r+1}{2}}^{r+1} 2v_1(2t) \, dt \geq (r+1)v(r+1) .$$

Next we define a smooth plurisubharmonic function v_3 on $\mathbb{C}^{2n+1}$ by the formula

$$v_3(z_1, \ldots, z_{2n+1}) := \sum_{j=1}^{2n+1} v_2(|z_j|) .$$

Then the pullback of v_3 to L is a smooth plurisubharmonic function on L. This is the required function $\hat{g}$. Indeed, under the identification described at the beginning of the proof we have

$$\int_L h \cdot e^{-\hat{g}} \omega_L^n = \sum_{k=1}^{\infty} \int_{(\Delta_k \setminus \Delta_{k-1}) \cap (X_L \setminus U)} h \cdot e^{-\hat{g}} \omega_L^n \leq$$

$$\sum_{k=1}^{\infty} v(k) e^{-v_2(k-1)} \leq \sum_{k=1}^{\infty} v(k) e^{-kv(k)} < \infty .$$

$\square$

To complete the proof of the proposition it remains to put in the above lemma

$$h := \sum_{i, i_1, \ldots, i_p, j_1, \ldots, j_q} \rho_i \cdot \hat{h}_{i; i_1, \ldots, i_p, j_1, \ldots, j_q}$$

where ρ_i is the characteristic function of $U_i \cap L$.

$\square$

2.5. Let $O \subset\subset L$ be a neighbourhood of C_M. We set $O' := r^{-1}(O)$, $C'_M := r^{-1}(C_M)$. Assume that the manifold N, see Section 1.1, is equipped with a hermitian metric ρ. We equip the bundle K_L with the hermitian metric induced by ρ. Also, we equip $K_{L'} := r^* K_L$ with the hermitian metric $\rho' := r^* \rho$.

Proposition 2.4. *Any* $h \in L_2((L \setminus C_M)', V_g)$ *holomorphic on* $O' \setminus C'_M$ *admits an extension to a section* $\hat{h}$ *of* $K_{L'}$ *such that* $\hat{h}|_{M'} \in L_2(M', K_{L'})$.

Proof. Consider a coordinate neighbourhood $U \subset\subset O$ of a point $q \in C_M$ with coordinates $z = (z_1, \ldots, z_n)$. Taking the pullback of these coordinates to $r^{-1}(U)$ we identify $r^{-1}(U)$ with $U \times S$ where S is the fibre of r. Then

$$h(z, s) = h_U(z, s) dz_1 \wedge \cdots \wedge dz_n, \quad z \in U \setminus C_M, \ s \in S.$$

By the definition of the metric $|| \cdot ||$ on V_g we have

$$||h(z, s)||^2 \omega_L^n(z) = |h_U(z, s)|^2 ||dz_1 \wedge \cdots \wedge dz_n||^2 \omega_L^n(z) =$$

$$|h_U(z, s)|^2 e^{-\hat{g}(z)} (\sqrt{-1})^n \wedge_{i=1}^n dz_i \wedge d\overline{z}_i.$$

Now, the hypotheses of the proposition imply that

$$(2.13) \qquad \int_{z \in U \setminus C_M} \left(\sum_{s \in S} |h_U(z, s)|^2 \right) e^{-\hat{g}(z)} (\sqrt{-1})^n \wedge_{i=1}^n dz_i \wedge d\overline{z}_i < \infty .$$

Let ω^n be the volume form induced by the hermitian metric ρ on N with the associated $(1,1)$-form ω. Since by our construction $\hat{g}$ is smooth on L, we have on U:

$$e^{-\hat{g}(z)} (\sqrt{-1})^n \wedge_{i=1}^n dz_i \wedge d\overline{z}_i \geq c\, \omega^n$$

for some $c > 0$. From here and (2.13) we get

$$(2.14) \qquad \int_{z \in U \setminus C_M} \left(\sum_{s \in S} |h_U(z, s)|^2 \right) \omega^n(z) < \infty .$$

In particular, this implies that every $h_U(\cdot, s)$, $s \in S$, belongs to the L^2 space on $U \setminus C_M$ defined by integration with respect to the volume form $(\sqrt{-1})^n \wedge_{i=1}^n dz_i \wedge d\overline{z}_i$. Also, every $h_U(\cdot, s)$ is holomorphic on $U \setminus C_M$. Using these facts and the Cauchy integral formulas for coefficients of the Laurent expansion of $h_U(\cdot, s)$, one obtains easily that every $h_U(\cdot, s)$ can be extended holomorphically to U. In turn, this gives an extension $\hat{h}$ of h to $r^{-1}(U)$. Now from (2.14) we obtain that $\hat{h} \in L_2(U, K_{L'})$.

Next assume that $\widetilde{U} \subset\subset L \setminus C_M$ is a simply connected coordinate neighbourhood of a point $q \in \overline{M} \setminus C_M$. Identifying $r^{-1}(\widetilde{U})$ with $\widetilde{U} \times S$ we have anew inequality of type (2.13) for $h|_{r^{-1}(\widetilde{U})}$. Since $\hat{g}$ is smooth on $L \setminus C_M$, repeating literally the previous arguments we get that $h \in L_2(\widetilde{U}, K_{L'})$. Taking a finite open cover of $\overline{M}$ by the above neighbourhoods U and $\widetilde{U}$ and considering the extension of h to M' defined by the above extended forms $\hat{h}$ on $r^{-1}(U)$ we get the required result. $\qquad\square$

§3. Proofs

3.1. Proof of Theorem 1.1 (a). First, we prove Theorem 1.1 (a) for $m = 1$:

Theorem 3.1. Let $z \in M \setminus C_M$ and $z' := r^{-1}(z) \in M'$. Then for any $f \in l_2(z')$ there exists $F \in H^2(M')$ such that $F|_{z'} = f$.

Proof. In the proof we retain the notation of Section 2.

Let $p : N \to X_N$ be the proper holomorphic map onto the normal Stein space X_N from Section 2.2 such that $p : N \setminus C_M \to X_N \setminus \cup_{1 \le i \le l} \{x_i\}$ is biholomorphic. Since X_N is Stein, there is a holomorphic function h on X_N whose set of zeros $\mathcal{Z}_h$ contains $p(z)$ and does not intersect $\cup_{1 \le j \le l} \{x_j\}$. Let $O \subset\subset X_N$ be a Stein neighbourhood of the compact set $\mathcal{Z}_h \cap \overline{X}_L$, $X_L := p(L)$, such that $\overline{O} \cap \cup_{1 \le j \le l} \{x_j\} = \emptyset$ and $\overline{O}$ is holomorphically convex in X_N. We set $O' := (p \circ r)^{-1}(O) \subset N'$. Then according to [Br1, Theorem 1.10] there is a holomorphic function h_1 on O' satisfying

$$(3.1) \qquad h_1|_{z'} = f \quad \text{and} \quad \sup_{y \in p^{-1}(O)} \left\{ \sum_{x \in r^{-1}(y)} |h_1(x)|^2 \right\} < \infty .$$

Let ρ be a C^∞ function on $X_N \setminus \cup_{1 \le i \le l} \{x_i\}$ equals 1 in some neighbourhood of $\mathcal{Z}_h \cap O$ in O and 0 outside O. By $\rho' := (p \circ r)^* \rho$ we denote its pullback to N'. Then $h_2 := (\rho' \cdot h_1)|_{(L \setminus C_M)'}$ is a C^∞ function on $(L \setminus C_M)'$. Let $h' := (p \circ r)^* h|_{L'}$ be the restriction to L' of the pullback of h. Consider the C^∞ $(0,1)$-from $\eta := \overline{\partial} h_2 / h'$. It follows easily from (3.1) and (2.8) that $\eta \in \mathcal{E}^{0,1}_{U;2}((L \setminus C_M)')$ for some $U \subset L \setminus p^{-1}(O)$.

Next, since X_N can be embedded to $\mathbb{C}^{2n+1}$, and since z is a regular point of X_N, there is a section $e \in \mathcal{O}(N, K_N)$ such that $e(z) \ne 0$. Let $V \subset L$ be the set of zeros of e. Then V is contained in the preimage $p^{-1}(V')$ of a complex analytic subspace $V' \subset X_N$. Since X_N is Stein, the latter implies that there is a bounded holomorphic function f_e on L such that $f_e \cdot e^{-1}$ is a bounded section from $\mathcal{O}(L, K_L^*)$ and $f_e(z) = 1$ (recall that $L \subset\subset N$).

Further, by Proposition 2.2 there is a function g in the definition of the metric on the vector bundle V_g from Section 2.3 such that $\widetilde{\eta} := \eta \cdot r^* e \in L_2^{0,1}((L \setminus C_M)', V_g)$. Since $\overline{\partial}\widetilde{\eta} = 0$, by (2.6) there is a section $h_3 \in L_2((L \setminus C_M)', V_g)$ such that $\overline{\partial} h_3 = \widetilde{\eta}$. Choose g in the definition of the metric on V_g so that also $h_2 \cdot r^* e \in L_2((L \setminus C_M)', V_g)$. Thus $h_4 := h_2 \cdot r^* e - h' h_3 \in L_2((L \setminus C_M)', V_g)$, is holomorphic on $(L \setminus C_M)'$. By Proposition 2.4 h_4 is extended to a holomorphic section h_4' of $K_{L'}$ such that $h_4'|_{M'} \in L_2(M', K_{L'})$. Moreover, by our construction $h_4'|_{z'} = f \cdot r^* e|_{z'}$. Finally, we set

$$F := h_4'|_{M'} \cdot (r^* f_e \cdot r^* e^{-1})|_{M'} .$$

Then $F \in H^2(M')$ and $F|_{z'} = f$. $\square$

Now, let us prove Theorem 1.1 (a).

By Theorem 3.1 there are functions $F_i \in H^2(M')$ such that $F_i|_{z_i'} = f_i$, $1 \le i \le m$. Let t_i be a holomorphic function on N such that $t_i(z_i) = 1$ and $t_i(z_j) = 0$ for $j \ne i$. Then the function $F := \sum_{1 \le i \le m} r^* t_i \cdot F_i$ satisfies the required condition. $\square$

3.2. Proof of Theorem 1.1 (b).

Let $x \in bM$ be a boundary point of a strongly pseudoconvex manifold $M \subset\subset N$. We will prove that each point $y \in r^{-1}(x)$ is a peak point for $H^2(M')$.

Choose a coordinate neighbourhood $U_1 \subset\subset N$ of x with complex coordinates $w = (w_1, \dots, w_n)$ centered at x such that the complex tangent space $T_x^c(bM)$ is given by the equation $w_1 = 0$. Consider the Taylor expansion of the defining function ρ for M, see (1.1), (1.2), at x:

$$(3.2) \qquad \rho(w) = \rho(x) + 2\mathrm{Re}f(x, w) + L_x(w - x, \overline{w} - \overline{x}) + O(||w - x||^3) ,$$

where L_x is the Levi form at x and $f(x, w)$ is a complex quadratic polynomial with respect to w:

$$f(x, w) = \frac{\partial \rho}{\partial w_1}(x) w_1 + \frac{1}{2} \sum_{1 \le \mu, \nu \le n} \frac{\partial^2 \rho}{\partial w_\mu \partial w_\nu}(x) w_\mu w_\nu .$$

(Here $\frac{\partial \rho}{\partial w_1}(x) \ne 0$ and $\frac{\partial \rho}{\partial w_i}(x) = 0$ for $2 \le i \le n$ by the choice of the coordinates on U_1). Next, in a sufficiently small simply connected neigbourhood $U \subset\subset U_1$ of x we introduce new coordinates $z = (z_1, \dots, z_n)$ by the formulas

$$z_1 = f(x, w) \quad \text{and} \quad z_i = w_i \quad \text{for} \quad 2 \le i \le n .$$

Since bM is strongly pseudoconvex at x, diminishing if necessary U we have that in U the intersection of the hyperplane $z_1 = 0$ with $\overline{M}$ consists of one

point x and $\mathrm{Re}\,z_1 < 0$ in U. Let $H := \{z \in U \ : \ \mathrm{Re}\,z_1 < 0\}$. Then we can choose a branch of $\log z_1$ so that $h_x(z) = \log z_1$ is a holomorphic function in H. Clearly, $h_x \in L^2(H)$ for L^2 defined by a Riemannian volume form pulled back from N, and h_x has a peak point at x.

Let $H' := r^{-1}(H) \subset\subset N'$. Take a point $y \in r^{-1}(x) \subset bM'$. Let H_y be the connected component of H' containing y. We set $h_y := r^* h_x|_{H_y}$. Next, consider a smooth function ρ on N equals 1 in a neigbourhood $O \subset\subset U$ of x with support $S \subset\subset U$. Let $O_y \subset\subset U_y$ be connected components of $O' = r^{-1}(O)$ and $U' = r^{-1}(U)$ containing y. By ρ_y we denote the pullback of $\rho|_U$ to U_y and by $S'_y \subset\subset U'_y$ support of ρ_y. Then $\overline{\partial}(h_y \rho_y)$ can be extended by 0 to a C^∞ $(0,1)$-form η on $(N' \setminus S') \cup H'_y \cup O'_y$. Note that $(N \setminus S) \cup H \cup O$ is a neighbourhood of $\overline{M}$. Hence, since $M \subset\subset N$ is strongly pseudoconvex, there is a strongly pseudoconvex manifold $L \subset\subset (N \setminus S) \cup H \cup O$ such that $\overline{M} \subset L$. Observe also that $r^{-1}((N \setminus S) \cup H \cup O) \subset (N' \setminus S') \cup H'_y \cup O'_y$. Thus the form η is well-defined on $L' := r^{-1}(L)$. Moreover, by our definitions $\eta \in \mathcal{E}^{0,1}_{W;2}((L \setminus C_M)')$ for some neighbourhood W of C_M, see Section 2.4.

As in the proof of part (a) we will choose a section $e \in \mathcal{O}(N, K_N)$ such that $e(x) \neq 0$ and a function $f_e \in \mathcal{O}(L)$ such that $f_e(x) = 1$ and $f_e \cdot e^{-1}$ is a bounded section from $\mathcal{O}(L, K_L^*)$.

Now, by Proposition 2.2 there is a function g in the definition of the metric on the vector bundle V_g from Section 2.3 such that $\widetilde{\eta} := \eta \cdot r^* e \in L^{0,1}_2((L \setminus C_M)', V_g)$. Since $\overline{\partial}\widetilde{\eta} = 0$, by (2.6) there is a section $h_1 \in L_2((L \setminus C_M)', V_g)$ such that $\overline{\partial} h_1 = \widetilde{\eta}$. According to Proposition 2.4 $h_1|_{(M \setminus C_M)'}$ is extended to a section h_2 of $K_{L'}$ such that $h_2 \in L_2(M', K_{L'})$. From here, using the fact that $(h_y \rho_y)|_{M' \cap H_y}$ is extended by 0 to a smooth L^2 function on M', we get $h' := (h_y \rho_y) \cdot r^* e - h_2 \in H^2(M', K_{L'})$. Finally, we set

$$h := h' \cdot (r^* f_e \cdot r^* e^{-1})|_{M'}.$$

Clearly, $h \in H^2(M')$ and it has a local peak point at y. Let us show that in fact h has a peak point at y. This will complete the proof of the theorem.

Let $L_1 \subset\subset L$ be a neighbourhood of $\overline{M}$. We set $L'_1 := r^{-1}(L_1)$. Then by Proposition 2.4 $h_1|_{r^{-1}(L_1 \setminus C_M)}$ is extended to a section of $K_{L'_1}$ (denoted as before by h_2). By the definition, $h'_2 := h_2 \cdot (r^* f_e \cdot r^* e^{-1})|_{L'_1}$ is a smooth holomorphic on $L'_1 \setminus S'$ function. From the facts that $L_1 \subset\subset L$, the L^2 norm $||\cdot||$ on L'_1 is defined by a Riemannian volume form pulled back from N using the mean-value property for the plurisubharmonic function $|h'_2|^2$ on $L'_1 \setminus S'$ we get for some $c > 0$,

$$\sup_{z \in M} \left\{ \sum_{w \in r^{-1}(z), w \notin U_y} |h_2'(w)|^2 \right\} \leq c \cdot ||h_2'||^2 < \infty \ .$$

In addition, h_2' is continuous on $\overline{U}_y \cap \overline{M'}$. These imply easily that h is bounded outside $U \cap M'$ for any neighbourhood U of y in N'. $\square$

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