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Existence of Crystal Bases for Kirillov-Reshetikhin Modules of Type D

Dedicated to Professor Noriaki Kawanaka on his sixtieth birthday

By

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Abstract

We show that a crystal base exists for any Kirillov-Reshetikhin module of type $D_n^{(1)}$, generalizing the result of Kang et al. for the end nodes of the Dynkin diagram of D_n.

§1. Introduction

Let $\mathfrak{g}$ be an affine algebra and let $U_q'(\mathfrak{g})$ be the corresponding quantum affine algebra without degree operator. Among irreducible finite-dimensional $U_q'(\mathfrak{g})$-modules there exists a distinguished family called Kirillov-Reshetikhin modules (KR modules for short). They were introduced in [17] in connection with a certain conjectural formula of multiplicities of irreducible $U_q(\mathfrak{g}_0)$-modules in a tensor product of those modules. Here $\mathfrak{g}_0$ stands for the finite-dimensional simple Lie algebra whose Dynkin diagram is obtained by removing the 0-vertex, that is prescribed in [10], from that of $\mathfrak{g}$. It is known [5, 4] that irreducible finite-dimensional $U_q'(\mathfrak{g})$-modules are classified by n-tuples of polynomials called Drinfeld polynomials, where n is the rank of $\mathfrak{g}_0$.

Let us define KR modules by the Drinfeld polynomials. Let $k \in \{1, 2, \dots, n\}, l \in \mathbb{Z}_{>0}$ and a an invertible element of $\mathbb{Q}(q)$. A KR module $\tilde{W}_{l,a}^{(k)}$ is defined to be the unique irreducible finite-dimensional $U_q'(\mathfrak{g})$-module that has

$$P_i(u) = \begin{cases} (1 - aq_i^{1-l}u)(1 - aq_i^{3-l}u) \cdots (1 - aq_i^{l-1}u) & \text{if } i = k, \\ 1 & \text{otherwise} \end{cases}$$

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as Drinfeld polynomials. (See Section 2.1 for the definition of q_i.) When $l = 1$ it is also called a fundamental representation. Since a fundamental representation is known to have a crystal base [14], we choose $a^\dagger$ so that $\tilde{W}^{(k)}_{l,a^\dagger}$ has a crystal base and redefine $W^{(k)}_{l,a} = \tilde{W}^{(k)}_{l,a^\dagger a}$. $W^{(k)}_{l,1}$ is also denoted simply by $W^{(k)}_l$.

The above-mentioned conjectural multiplicity formula was shown when $\mathfrak{g}$ is non-twisted by combining two results. The first one is the proof of certain algebraic relations among characters of KR modules, called Q-systems, presented in [20] for simply-laced cases and in [8] for all non-twisted cases. The second one is a derivation of the multiplicity formula, called fermionic formula, in [16] for type A and in [7] for all non-twisted cases, by using the Q-systems. However, it deserves to emphasize that there is also a q-analogue of the conjecture, called $X = M$ conjecture [7, 6]. The definition of X requires the existence of the crystal base of a KR module. Despite many efforts as in [12, 11, 25, 18, 9, 14, 19, 23, 2], this existence problem is yet to be settled. For type D for instance, the crystal base has been shown to exist for $W^{(k)}_l$ where $k = 1, n-1, n; l \in \mathbb{Z}_{>0}$ in [12] and for $W^{(k)}_1$ for arbitrary k in [18, 14].

In this paper we prove the following theorem, thereby settling the problem for type D.

Theorem 1.1. *For $2 \le k \le n-2$ and $l \ge 1$, the $U'_q(D^{(1)}_n)$-module $W^{(k)}_l$ has a crystal pseudobase.*

Here (L, B) is said to be a crystal pseudobase if $(L, B/\{\pm 1\})$ is a crystal base. (See Definition 2.1 for the definition of a crystal base.) Let us give a short sketch of our proof. We follow the technique already developed in [12], namely, from a fundamental representation $W^{(k)}_1$ we construct $W^{(k)}_l$ for any l by fusion construction (Section 2.3), and we apply a criterion of the existence of a crystal pseudobase (Proposition 2.6) to the constructed module $W^{(k)}_l$. Practically, we need to check two conditions ((2.27) and (2.28)). Checking the second one is not difficult, if once we find out the vectors $\{u_j\}$ correctly, whereas checking the first one requires information on the image W and the kernel N of the R-matrix $R(x, y) : W^{(k)}_{1,x} \otimes W^{(k)}_{1,y} \longrightarrow W^{(k)}_{1,y} \otimes W^{(k)}_{1,x}$ at $x/y = q^2$. Up to now such information was obtained by calculating the spectral decomposition of the R-matrix when dealing with $W^{(k)}_l$ for higher l. It seems to be the reason why showing the existence of crystal bases of $W^{(k)}_l$ for higher k has not been succeeded so far, since the calculation of the corresponding R-matrix is too much complicated. However, thanks to the result by Nakajima [20], we are now able to identify W and N with tensor products of KR modules (Lemma 3.4). Using the crystal base of W and a property of a bilinear form between

$W^{(k)}_{1,q} \otimes W^{(k)}_{1,q^{-1}}$ and $W^{(k)}_{1,q^{-1}} \otimes W^{(k)}_{1,q}$, we can check the first condition of the criterion. It is known that a $U_q(\mathfrak{g})$-module with a connected crystal base is irreducible. Therefore, once $W^{(k)}_l$ is shown to have a crystal pseudobase, it follows that $W^{(k)}_l$ is an irreducible finite-dimensional module with the desired Drinfeld polynomials, since it is a simple quotient of $W^{(k)}_{1,q^{1-l}} \otimes W^{(k)}_{1,q^{3-l}} \otimes \cdots \otimes W^{(k)}_{1,q^{l-1}}$.

After the author finished the manuscript, he learned from Kashiwara that the module $W^{(k)}_l$ can be shown to be irreducible by Theorem 9.2 of [14]. Once it is established, the character is known by [3]. Hence it turns out that there is no need to prove the inequality of the character in (i) just after Proposition 3.7. However, this does not seem to prove that $W^{(k)}_l$ is isomorphic to a module of the form of $V^{\otimes l}/\sum_{i=0}^{l-2} V^{\otimes i} \otimes N \otimes V^{\otimes(l-2-i)}$. The author was also informed from Nakajima that the existence of a polarization on the fundamental representation $W^{(k)}_1$ was shown in [24] (see also [21, 1] for more general results). Hence similar calculations of the prepolarization as in Section 5 will give a proof of the existence of crystal bases for KR modules of other quantum affine algebras.

§2. Crystal Base and Fusion Construction

§2.1. Crystal base

In this subsection we briefly recall the definition of crystal bases. For more details along with the definition of $U_q(\mathfrak{g})$, refer to [13].

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra and let M be a $U_q(\mathfrak{g})$-module. M is said to be integrable if $M = \oplus_{\lambda \in P} M_\lambda$, $\dim M_\lambda < \infty$ for any λ, and for any i, M is a union of finite-dimensional $U_q(\mathfrak{g}_i)$-modules. Here P is the weight lattice of $\mathfrak{g}$, M_λ is the weight space of M of weight λ and $U_q(\mathfrak{g}_i)$ is the subalgebra generated by Chevalley generators e_i and f_i. If M is integrable, we have

$$M = \bigoplus_{0 \le n \le \langle h_i, \lambda \rangle} f_i^{(n)}(\mathrm{Ker}\, e_i \cap M_\lambda). \tag{2.1}$$

Note that we use the following notations: $[m]_i = (q_i^m - q_i^{-m})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{m=1}^n [m]_i$, $f_i^{(n)} = f_i^n/[n]_i!$ with $q_i = q^{(\alpha_i,\alpha_i)}$, where $(\ ,\ )$ is an invariant bilinear form on P. We define the endomorphisms $\tilde{e}_i, \tilde{f}_i$ of M by

$$\tilde{f}_i(f_i^{(n)}u) = f_i^{(n+1)}u \quad \text{and} \quad \tilde{e}_i(f_i^{(n)}u) = f_i^{(n-1)}u \tag{2.2}$$

for $u \in \text{Ker } e_i \cap M_\lambda$ with $0 \le n \le \langle h_i, \lambda\rangle$. Similarly, we have

$$M = \bigoplus_{0 \le n \le -\langle h_i, \mu\rangle} e_i^{(n)}(\text{Ker } f_i \cap M_\mu). \tag{2.3}$$

These two decompositions are related as follows:

if $0 \le n \le \langle h_i, \lambda\rangle$ and $u \in \text{Ker } e_i \cap M_\lambda$,
then $v = f_i^{(\langle h_i, \lambda\rangle)} u$ belongs to $\text{Ker } f_i \cap M_{s_i(\lambda)}$ and $f_i^{(n)} u = e_i^{(\langle h_i, \lambda\rangle - n)} v$.

Here $s_i(\lambda) = \lambda - \langle h_i, \lambda\rangle \alpha_i$. Hence we obtain

$$\tilde{f}_i(e_i^{(n)} v) = e_i^{(n-1)} v \quad \text{and} \quad \tilde{e}_i(e_i^{(n)} v) = e_i^{(n+1)} v \tag{2.4}$$

for $v \in \text{Ker } f_i \cap M_\mu$ with $0 \le n \le -\langle h_i, \mu\rangle$.

Let us now look at the definition of a crystal base. Let A be the subring of $\mathbb{Q}(q)$ consisting of rational functions without poles at $q = 0$. Let M be an integrable $U_q(\mathfrak{g})$-module.

Definition 2.1. A pair (L, B) is called a crystal base of M if it satisfies the following 6 conditions:

(2.5) L is a free sub-A-module of M such that $M \simeq \mathbb{Q}(q) \otimes_A L$,

(2.6) B is a base of the $\mathbb{Q}$-vector space L/qL,

(2.7) $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for any i.

By (2.7) $\tilde{e}_i$ and $\tilde{f}_i$ act on L/qL.

$$\tilde{e}_i B \subset B \cup \{0\} \text{ and } \tilde{f}_i B \subset B \cup \{0\}. \tag{2.8}$$

$$L = \oplus_{\lambda \in P} L_\lambda \text{ and } B = \sqcup_{\lambda \in P} B_\lambda \tag{2.9}$$

where $L_\lambda = L \cap M_\lambda$ and $B_\lambda = B \cap (L_\lambda / qL_\lambda)$.

$$\text{For } b, b' \in B, b' = \tilde{f}_i b \text{ if and only if } \tilde{e}_i b' = b. \tag{2.10}$$

Standard notations are in order. For $b \in B$ we set

(2.11) $\varepsilon_i(b) = \max\{m \in \mathbb{Z}_{\ge 0} \mid \tilde{e}_i^m b \ne 0\}, \quad \varphi_i(b) = \max\{m \in \mathbb{Z}_{\ge 0} \mid \tilde{f}_i^m b \ne 0\}$,

(2.12) $\varepsilon(b) = \sum_i \varepsilon_i(b)\Lambda_i, \quad \varphi(b) = \sum_i \varphi_i(b)\Lambda_i$,

(2.13) $wt\, b = \varphi(b) - \varepsilon(b)$.

Here $\{\Lambda_i\}$ stands for the set of fundamental weights of $\mathfrak{g}$.

The crystal base behaves nicely under the tensor product. Let (L_j, B_j) be the crystal base of an integrable $U_q(\mathfrak{g})$-module M_j $(j = 1, 2)$. Set $L = L_1 \otimes_A L_2$ and $B = \{b_1 \otimes b_2 \mid b_j \in B_j (j = 1, 2)\}$. Then (L, B) is a crystal base of $M_1 \otimes M_2$. Moreover, the action of $\tilde{e}_i$ and $\tilde{f}_i$ becomes very simple as

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \tag{2.14}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \tag{2.15}$$

Here $0 \otimes b$ and $b \otimes 0$ are understood to be 0. We denote this B by $B_1 \otimes B_2$. ε_i, φ_i and wt are given by

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_1) + \varepsilon_i(b_2) - \varphi_i(b_1)), \tag{2.16}$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \varphi_i(b_2) - \varepsilon_i(b_2)), \tag{2.17}$$

$$wt\,(b_1 \otimes b_2) = wt\, b_1 + wt\, b_2. \tag{2.18}$$

Next lemma is used later in Section 4.

Lemma 2.2. *Let (L, B) be a crystal base. Assume that $\tilde{e}_i^3 b = \tilde{f}_i^3 b = 0$ for any $b \in B$. Let $v \in L$ be such that $v \equiv b$ mod qL. Then we have*

$$e_i v \equiv q_i^{-\varphi_i(b)} \tilde{e}_i v \ \textit{mod}\ q q_i^{-\varphi_i(b)} L,$$
$$f_i v \equiv q_i^{-\varepsilon_i(b)} \tilde{f}_i v \ \textit{mod}\ q q_i^{-\varepsilon_i(b)} L.$$

In particular, $e_i v \equiv 0$ (resp. $f_i v \equiv 0$) if $\varepsilon_i(b) = 0$ (resp. $\varphi_i(b) = 0$).

Proof. We prove the second relation. Let λ be the weight of v. From the assumption it suffices to check the relation for the following cases, since the other cases are trivial.

(i) $\varepsilon_i(b) = 0, \langle h_i, \lambda\rangle = 1$ or 2,

(ii) $\varepsilon_i(b) = 0$ or $1, \langle h_i, \lambda\rangle = 0$.

In case (i) we have $f_i v = \tilde{f}_i v$ by (2.1) and (2.2). In case (ii) let us write $v = f_i v_1 + v_2$ with $v_j \in \operatorname{Ker} e_i \cap L$ $(j = 1, 2)$ such that $\langle h_i, wt\, v_1\rangle = 2, \langle h_i, wt\, v_2\rangle = 0$. Then we have $f_i v = f_i^2 v_1 = [2]_i \tilde{f}_i v \equiv q_i^{-1} \tilde{f}_i v$ mod $q q_i^{-1} L$.

The first relation can be checked similarly by using (2.3) and (2.4). □

§2.2. Polarization

We define a total order on $\mathbb{Q}(q)$ by

$$f > g \text{ if and only if } f - g \in \bigsqcup_{n\in\mathbb{Z}} \{q^n(c + qA) \mid c > 0\}$$

and $f \geq g$ if $f > g$ or $f = g$.

Let M and N be $U_q(\mathfrak{g})$-modules. A bilinear form $(\ ,\ ) : M \otimes_{\mathbb{Q}(q)} N \to \mathbb{Q}(q)$ is called an admissible pairing if it satisfies

$$\begin{aligned} (q^h u, v) &= (u, q^h v), \\ (e_i u, v) &= (u, q_i^{-1} t_i^{-1} f_i v), \\ (f_i u, v) &= (u, q_i^{-1} t_i e_i v), \end{aligned} \tag{2.19}$$

for all $u \in M$ and $v \in N$. (2.19) implies

$$(e_i^{(n)} u, v) = (u, q_i^{-n^2} t_i^{-n} f_i^{(n)} v), \quad (f_i^{(n)} u, v) = (u, q_i^{-n^2} t_i^{n} e_i^{(n)} v). \tag{2.20}$$

A symmetric bilinear form $(\ ,\ )$ on M is called a preporlarization of M if it satisfies (2.19) for $u, v \in M$. A preporlarization is called a porlarization if it is positive definite with respective to the order on $\mathbb{Q}(q)$.

§2.3. Fusion construction

In what follows we assume that $\mathfrak{g}$ is of affine type. Let P be the weight lattice, $\{\Lambda_i\}$ the set of fundamental weights and δ the generator of null roots of $\mathfrak{g}$. Then we have $P = \bigoplus_i \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$. We set

$$P_{cl} = P/\mathbb{Z}\delta.$$

Similar to the quantum algebra $U_q(\mathfrak{g})$ which is associated with P, we can also consider $U_q'(\mathfrak{g})$, which is associated with P_{cl}, namely, the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, q^h $(h \in (P_{cl})^*)$.

Let K be a commutative ring containing $\mathbb{Q}(q)$ and let x be an invertible element of K. We introduce a $K \otimes_{\mathbb{Q}(q)} U_q'(\mathfrak{g})$-module V_x by replacing the actions of e_i, f_i with $x^{\delta_{i0}} e_i, x^{-\delta_{i0}} f_i$. The action of q^h is not changed. Let y also be an invertible element of K. A $K \otimes_{\mathbb{Q}(q)} U_q'(\mathfrak{g})$-linear map

$$R(x, y) : \ V_x \otimes V_y \longrightarrow V_y \otimes V_x$$

is called a R-matrix. Here we need to specify the coproduct Δ of $U_q(\mathfrak{g})$ we use in this paper. Our choice is the "lower" one (see [13]) given by

$$\Delta(q^h) = q^h \otimes q^h \quad \text{for } h \in (P_{cl})^*,$$
$$\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i,$$
$$\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i.$$

For a finite-dimensional $U'_q(\mathfrak{g})$-module V we assume the following.

(2.21) V is irreducible.

(2.22) There exists $\lambda_0 \in P_{cl}$ such that $wt\, V \subset \lambda_0 + \sum_{i\neq 0} \mathbb{Z}_{\leq 0}\alpha_i$ and $\dim V_{\lambda_0} = 1$.

Here $\{\alpha_i\}$ is the set of simple roots. Under these assumptions it is known that there exists a unique R-matrix up to a scalar multiple. Moreover, $R(x,y)$ depends only on x/y. Take a non zero vector u_0 from V_{λ_0}. We normalize $R(x,y)$ in such a way that $R(x,y)(u_0 \otimes u_0) = u_0 \otimes u_0$. It is known in [14] that if V is a "good" module then the normalized R-matrix does not have a pole at $x/y = a \in A$.

Next we review the fusion construction following section 3 of [12]. Let l be a positive integer and $\mathfrak{S}_l$ the l-th symmetric group. Let s_i be the simple reflection which interchanges i and $i+1$, and let $\ell(w)$ be the length of $w \in \mathfrak{S}_l$. Let V be a finite-dimensional $U'_q(\mathfrak{g})$-module. Let $R(x,y)$ denote the R-matrix for $V \otimes V$. For any $w \in \mathfrak{S}_l$ we construct a map $R_w(x_1,\dots,x_l) : V_{x_1} \otimes \cdots \otimes V_{x_l} \to V_{x_{w(1)}} \otimes \cdots \otimes V_{x_{w(l)}}$ by

$$R_1(x_1,\dots,x_l) = 1,$$
$$R_{s_i}(x_1,\dots,x_l) = \left(\bigotimes_{j<i} \mathrm{id}_{V_{x_j}}\right) \otimes R(x_i,x_{i+1}) \otimes \left(\bigotimes_{j>i+1} \mathrm{id}_{V_{x_j}}\right),$$
$$R_{ww'}(x_1,\dots,x_l) = R_{w'}(x_{w(1)},\dots,x_{w(l)}) \circ R_w(x_1,\dots,x_l)$$
$$\text{for } w, w' \text{ such that } \ell(ww') = \ell(w) + \ell(w').$$

Fix $r \in \mathbb{Z}_{>0}$. For each $l \in \mathbb{Z}_{>0}$, we put

$$R_l = R_{w_0}(q^{r(l-1)}, q^{r(l-3)}, \dots, q^{-r(l-1)}) :$$
$$V_{q^{r(l-1)}} \otimes V_{q^{r(l-3)}} \otimes \cdots \otimes V_{q^{-r(l-1)}} \to V_{q^{-r(l-1)}} \otimes V_{q^{-r(l-3)}} \otimes \cdots \otimes V_{q^{r(l-1)}},$$

where w_0 is the longest element of $\mathfrak{S}_l$. Then R_l is a $U'_q(\mathfrak{g})$-linear homomorphism. Define

$$V_l = \mathrm{Im}\, R_l.$$

Let us denote by W the image of

$$R(q^r,q^{-r}) : V_{q^r} \otimes V_{q^{-r}} \longrightarrow V_{q^{-r}} \otimes V_{q^r}$$

and by N its kernel. Then we have

(2.23) V_l considered as a submodule of $V^{\otimes l} = V_{q^{-r(l-1)}} \otimes \cdots \otimes V_{q^{r(l-1)}}$ is contained in $\displaystyle\bigcap_{i=0}^{l-2} V^{\otimes i} \otimes W \otimes V^{\otimes(l-2-i)}$.

Similarly, we have

(2.24) V_l is a quotient of $\displaystyle V^{\otimes l} / \sum_{i=0}^{l-2} V^{\otimes i} \otimes N \otimes V^{\otimes(l-2-i)}$.

§2.4. Preliminary propositions

In this subsection, following [12] we define a prepolarization on V_l and prepare a necessary proposition to show the main theorem. First we recall

Lemma 2.3. *Let M_j and N_j be $U'_q(\mathfrak{g})$-modules and let $(\ ,\ )_j$ be an admissible pairing between M_j and N_j $(j=1,2)$. Then the pairing $(\ ,\ )$ between $M_1 \otimes M_2$ and $N_1 \otimes N_2$ defined by $(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)_1 (u_2, v_2)_2$ for all $u_j \in M_j$ and $v_j \in N_j$ is admissible.*

Let V be a finite-dimensional $U'_q(\mathfrak{g})$-module satisfying (2.21),(2.22). Suppose V has a polarization. The polarization on V gives an admissible pairing between V_x and $V_{x^{-1}}$. Hence it induces an admissible pairing between $V_{x_1} \otimes \cdots \otimes V_{x_l}$ and $V_{x_1^{-1}} \otimes \cdots \otimes V_{x_l^{-1}}$.

Lemma 2.4. *If $x_j = x_{l+1-j}^{-1}$ for $j = 1, \ldots, l$, then for any $u, u' \in V_{x_1} \otimes \cdots \otimes V_{x_l}$, we have*

$$(u, R_{w_0}(x_1, \ldots, x_l)u') = (u', R_{w_0}(x_1, \ldots, x_l)u).$$

By taking $x_1 = q^{r(l-1)}, x_2 = q^{r(l-3)}$, etc., we obtain the admissible pairing $(\ ,\ )$ between $W = V_{q^{r(l-1)}} \otimes V_{q^{r(l-3)}} \otimes \cdots \otimes V_{q^{-r(l-1)}}$ and $W' = V_{q^{-r(l-1)}} \otimes V_{q^{-r(l-3)}} \otimes \cdots \otimes V_{q^{r(l-1)}}$ that satisfies

$$(w, R_l w') = (w', R_l w) \quad \text{for any } w, w' \in W. \tag{2.25}$$

This allows us to define a preporlarization $(\ ,\ )_l$ on V_l by

$$(R_l u, R_l u')_l = (u, R_l u')$$

for $u, u' \in V_{q^{r(l-1)}} \otimes V_{q^{r(l-3)}} \otimes \cdots \otimes V_{q^{-r(l-1)}}$.

Next we introduce a $\mathbb{Z}$-form of $U'_q(\mathfrak{g})$. Recall that A is the subring of $\mathbb{Q}(q)$ consisting of rational functions without poles at $q = 0$. We introduce the subalgebras $A_{\mathbb{Z}}$ and $K_{\mathbb{Z}}$ of $\mathbb{Q}(q)$ by

$$A_{\mathbb{Z}} = \{f(q)/g(q) \mid f(q), g(q) \in \mathbb{Z}[q], g(0) = 1\},$$
$$K_{\mathbb{Z}} = A_{\mathbb{Z}}[q^{-1}].$$

Then we have

$$K_{\mathbb{Z}} \cap A = A_{\mathbb{Z}}, \quad A_{\mathbb{Z}}/qA_{\mathbb{Z}} \simeq \mathbb{Z}.$$

We then define $U'_q(\mathfrak{g})_{K_{\mathbb{Z}}}$ as the $K_{\mathbb{Z}}$-subalgebra of $U'_q(\mathfrak{g})$ generated by e_i, f_i, q^h ($h \in (P_{cl})^*$). Set $V_{K_{\mathbb{Z}}} = U'_q(\mathfrak{g})_{K_{\mathbb{Z}}} u_0$ and assume

$$(V_{K_{\mathbb{Z}}})_{\lambda_0} = K_{\mathbb{Z}} u_0. \tag{2.26}$$

Let us further set

$$(V_l)_{K_{\mathbb{Z}}} = R_l((V_{K_{\mathbb{Z}}})^{\otimes l}) \cap (V_{K_{\mathbb{Z}}})^{\otimes l}.$$

Then one can show

Proposition 2.5. (i) *$(\ ,\ )_l$ is a nondegenerate prepolarization on V_l.*
(ii) $(R_l(u_0^{\otimes l}), R_l(u_0^{\otimes l}))_l = 1$.
(iii) $((V_l)_{K_{\mathbb{Z}}}, (V_l)_{K_{\mathbb{Z}}})_l \subset K_{\mathbb{Z}}$.

Let I be the index set of the vertices of the Dynkin diagram of $\mathfrak{g}$ with the vertex 0 as in [10]. Let $\mathfrak{g}_0$ be the finite-dimensional simple Lie algebra whose Dynkin diagram is obtained by removing the 0-vertex from that of $\mathfrak{g}$. Let $\overline{P}_+$ be the set of dominant integral weights of $\mathfrak{g}_0$ and $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g}_0)$-module of highest weight λ for $\lambda \in \overline{P}_+$. The following proposition, which is essentially stated in Proposition 2.6.1 and 2.6.2 of [12], is a key to prove the main theorem.

Proposition 2.6. *Let M be a finite-dimensional integrable $U'_q(\mathfrak{g})$-module. Let $(\ ,\ )$ be a prepolarization on M, and $M_{K_{\mathbb{Z}}}$ a $U'_q(\mathfrak{g})_{K_{\mathbb{Z}}}$-submodule of M such that $(M_{K_{\mathbb{Z}}}, M_{K_{\mathbb{Z}}}) \subset K_{\mathbb{Z}}$. Let $\lambda_1, \ldots, \lambda_m \in \overline{P}_+$, and we assume the following conditions.*

$$\dim M_{\lambda_k} \le \sum_{j=1}^{m} \dim V(\lambda_j)_{\lambda_k} \text{ for } k = 1, \ldots, m. \tag{2.27}$$

(2.28) *There exist* $u_j \in (M_{K_\mathbb{Z}})_{\lambda_j}$ $(j = 1, \dots, m)$ *such that* $(u_j, u_k) \in \delta_{jk} + qA$, *and* $(e_i u_j, e_i u_j) \in q q_i^{-2(1+\langle h_i, \lambda_j \rangle)} A$ *for any* $i \in I \setminus \{0\}$.

Set $L = \{u \in M \mid (u, u) \in A\}$ *and set* $B = \{b \in M_{K_\mathbb{Z}} \cap L / M_{K_\mathbb{Z}} \cap qL \mid (b, b)_0 = 1\}$. *Here* $(\ ,\ )_0$ *is the* $\mathbb{Q}$-*valued symmetric bilinear form on* L/qL *induced by* $(\ ,\ )$. *Then we have the following.*

(i) $(\ ,\ )$ *is a polarization on* M.

(ii) $M \simeq \bigoplus_j V(\lambda_j)$ *as* $U_q(\mathfrak{g}_0)$-*modules.*

(iii) (L, B) *is a crystal pseudobase of* M.

§3. KR Module of Type D

§3.1. KR module $W_1^{(k)}$

First we review the Dynkin datum of type $D_n^{(1)}$. Let $I = \{0, 1, \dots, n\}$ be the index set of the Dynkin diagram, $\{\alpha_i\}_{i \in I}$ the set of simple roots, $\{\Lambda_i\}_{i \in I}$ the set of fundamental weights. The standard null root δ is given by

$$\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n. \tag{3.1}$$

We denote the weight lattice by P, that is, $P = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$. The sublattice $\overline{P} = \bigoplus_{i \in I_0} \mathbb{Z}\overline{\Lambda}_i$ can be viewed as the weight lattice for D_n. Here $I_0 = I \setminus \{0\}$ and $\overline{\Lambda}_i = \Lambda_i - a_i \Lambda_0$ with a_i being the coefficient of α_i in (3.1). It is sometimes useful to introduce an orthonormal basis $\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\}$ of $\mathbb{Q} \otimes_\mathbb{Z} \overline{P}$ in such a way that we have

$$\alpha_i = \begin{cases} \epsilon_i - \epsilon_{i+1} & (i = 1, \dots, n-1) \\ \epsilon_{n-1} + \epsilon_n & (i = n), \end{cases}$$

$$\overline{\Lambda}_i = \begin{cases} \epsilon_1 + \cdots + \epsilon_i & (i = 1, \dots, n-2) \\ (\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n)/2 & (i = n-1) \\ (\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)/2 & (i = n). \end{cases}$$

Then we have $\alpha_0 = \delta - \epsilon_1 - \epsilon_2$. Since the lengths of the simple roots are all equal, we have $q_i = q$ for any $i \in I$. Hence we shall abbreviate i from $[m]_i$ or $[m]_i!$.

Let $W_1^{(k)}$ be the k-th fundamental representation of $U_q'(D_n^{(1)})$. It is known that it has the following decomposition into $U_q(D_n)$-modules.

$$W_1^{(k)} \simeq \begin{cases} V(\overline{\Lambda}_k) \oplus V(\overline{\Lambda}_{k-2}) \oplus \cdots \oplus V(\overline{\Lambda}_1 \text{ or } 0) & \text{if } 1 \le k \le n-2, \\ V(\overline{\Lambda}_k) & \text{if } k = n-1, n. \end{cases} \tag{3.2}$$

On $W_1^{(k)}$ the following results are known.

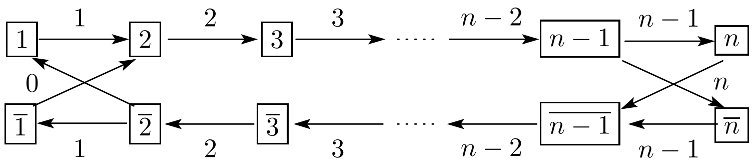
Proposition 3.1. (1) $W_1^{(k)}$ is "good" in the sense of Kashiwara. In particular, it has a crystal base.

(2) $W_1^{(k)}$ has a polarization.

The first claim is due to Kashiwara [14] and the second to Koga [18], who got the result by exploiting the fusion construction among the spin representations.

§3.2. Crystal of $W_1^{(k)}$

We denote the crystal of $W_1^{(k)}$ by $B^{k,1}$. We review in this subsection Schilling's variation of Koga's result on the crystal structure of $B^{k,1}$. First we treat the case of $k = 1$. The crystal graph of $B^{1,1}$ is depicted as follows.



Here for $b, b' \in B^{1,1}$ $b \xrightarrow{i}$ means $\tilde{f}_i b = b'$ ($\Leftrightarrow b = \tilde{e}_i b'$).

Next in view of (3.2) we recall the $U_q(D_n)$-crystal structure of $B(\overline{\Lambda}_l)$, the crystal of $U_q(D_n)$-module $V(\overline{\Lambda}_l)$, by [15]. Consider the alphabet $\mathcal{A} = \{1, 2, \ldots, n, \overline{n}, \overline{n-1}, \ldots, \overline{1}\}$ consisting of the crystal elements of $B^{1,1}$. It is given the following (partial) order.

$$1 \prec 2 \prec \cdots \prec n-1 \prec \begin{matrix} n \\ \overline{n} \end{matrix} \prec \overline{n-1} \prec \cdots \prec \overline{1}.$$

Then, for $1 \le l \le n-2$ $B(\overline{\Lambda}_l)$ is identified with the set of columns

$$\begin{array}{|c|} \hline m_1 \\ \hline m_2 \\ \hline \vdots \\ \hline m_l \\ \hline \end{array}$$

of height l satisfying

(3.3) $$m_j \not\succ m_{j+1} \quad \text{for } j = 1, \ldots, l-1,$$

(3.4) $$\text{if } m_a = p \text{ and } m_b = \overline{p}, \text{ then } \mathrm{dist}(p, \overline{p}) \le p.$$

Here $\mathrm{dist}(p,\overline{p}) = a+l+1-b$ if $m_a = p$ and $m_b = \overline{p}$. The column tableau as above is also written as $m_1 m_2 \cdots m_l$. Note that we allow $(m_j, m_{j+1}) = (n, \overline{n})$ and $(\overline{n}, n)$. The actions of $\tilde{e}_i, \tilde{f}_i$ $(i \in I_0)$ is given by considering the embedding

$$\begin{array}{ccc} B(\overline{\Lambda}_l) & \hookrightarrow & (B^{1,1})^{\otimes l} \\ m_1 m_2 \cdots m_l & \mapsto & m_1 \otimes m_2 \otimes \cdots \otimes m_l \end{array}$$

and apply the tensor product rule of crystals on the r.h.s. $B(0)$ is realized as $\{\phi\}$ with the trivial actions of $\tilde{e}_i, \tilde{f}_i$ $(i \in I_0)$, that is, $\tilde{e}_i \phi = \tilde{f}_i \phi = 0$.

For $1 \le k \le n-2$, we are to represent $B^{k,1}$ as the set of column tableaux of height k satisfying (3.3). By (3.2) $B^{k,1}$ is the union of the sets corresponding to $B(\overline{\Lambda}_l)$ with $0 \le l \le k$ and $l \equiv k \pmod 2$. In [22] maps from $B(\overline{\Lambda}_l)$ to column tableaux of height k were defined. If $b \in B(\overline{\Lambda}_l)$, then fill the column of height l of b succesively by a pair $(i_j, \overline{i_j})$ for $1 \le j \le (k-l)/2$ in the following way to obtain a column of height k. Set $i_0 = 0$. Let $i_{j-1} < i_j$ be minimal such that

(1) neither i_j or $\overline{i_j}$ is in the column;
(2) adding i_j and $\overline{i_j}$ to the column we have $\mathrm{dist}(i_j, \overline{i_j}) \ge i_j + j$;
(3) adding i_j and $\overline{i_j}$ to the column, all other pairs $(a, \overline{a})$ in the new column with $a > i_j$ satisfy $\mathrm{dist}(a, \overline{a}) \le a + j$.

The filling map and $\tilde{f}_i$ for $i \in I_0$ commute. Denote the filling map to height k by F_k or simply F. Let D_k or D, the dropping map, be the inverse of F_k. Explicitly, given a one-column tableau of b of height k, let $i_0 = 0$ and successively find $i_j > i_{j-1}$ minimal such that the pair $(i_j, \overline{i_j})$ is in b and $\mathrm{dist}(i_j, \overline{i_j}) \ge i_j + j$. Drop all such pairs $(i_j, \overline{i_j})$ from b. Thus we have

$$B^{k,1} \simeq \bigoplus_{0 \le l \le k,\ l \equiv k\ (2)} F_k(B(\overline{\Lambda}_l)) \quad \text{as } U_q(D_n)\text{-crystals.}$$

It is the set of all column tableaux of height k satisfying (3.3) only.

We are left to give the rule of the actions of $\tilde{e}_0$ and $\tilde{f}_0$. For this purpose we need slight variants of F_k and D_k, denoted by $\tilde{F}_k$ and $\tilde{D}_k$, respectively, which act on columns that do not contain $1, 2, \overline{2}, \overline{1}$. On these columns $\tilde{F}_k$ and $\tilde{D}_k$ are defined by replacing $i \mapsto i-2$ and $\overline{i} \mapsto \overline{i-2}$, then applying F_k and D_k, and finally replacing $i \mapsto i+2$ and $\overline{i} \mapsto \overline{i+2}$. The following proposition is given in [22].

Proposition 3.2. *For $b \in B^{k,1}$,*

$$\tilde{e}_0 b = \begin{cases} F_k(\tilde{D}_{k-2}(x)) & \text{if } b = 12x \\ \tilde{F}_{k-1}(x)\overline{2} & \text{if } b = 12x\overline{2} \\ \tilde{F}_{k-1}(x)\overline{1} & \text{if } b = 12x\overline{1} \\ \tilde{F}_{k-2}(x)\overline{21} & \text{if } b = 12x\overline{21} \\ F_k(\tilde{D}_{k-1}(x)\overline{2}) & \text{if } b = 1x \\ F_k(\tilde{D}_{k-1}(x)\overline{1}) & \text{if } b = 2x \\ x\overline{21} & \text{if } b = 1x\overline{1} \text{ and } \tilde{D}_{k-2}(x) = x \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{f}_0 b = \begin{cases} F_k(\tilde{D}_{k-2}(x)) & \text{if } b = x\overline{21} \\ 2\tilde{F}_{k-1}(x) & \text{if } b = 2x\overline{21} \\ 1\tilde{F}_{k-1}(x) & \text{if } b = 1x\overline{21} \\ 12\tilde{F}_{k-2}(x) & \text{if } b = 12x\overline{21} \\ F_k(2\tilde{D}_{k-1}(x)) & \text{if } b = x\overline{1} \\ F_k(1\tilde{D}_{k-1}(x)) & \text{if } b = x\overline{2} \\ 12x & \text{if } b = 1x\overline{1} \text{ and } \tilde{D}_{k-2}(x) = x \\ 0 & \text{otherwise} \end{cases}$$

where x does not contain $1, 2, \overline{2}, \overline{1}$.

§3.3. Existence of crystal pseudobase for $W_l^{(k)}$

In this subsection we prove our main theorem by using Proposition 2.6. We prepare several lemmas and propositions.

Lemma 3.3. *Let $R(x/y)$ be the R-matrix from $W_{1,x}^{(k)} \otimes W_{1,y}^{(k)}$ to $W_{1,y}^{(k)} \otimes W_{1,x}^{(k)}$. Then it has the following form.*

$$R(z) = P_{2\varpi_k} + \frac{z - q^2}{1 - q^2 z} P_{\varpi_{k+1}+\varpi_{k-1}} + \cdots .$$

Here $z = x/y, \varpi_j = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_j \in \overline{P}_+$ for $0 \le j \le n-1$, and P_λ stands for the projector onto the irreducible $U_q(D_n)$-module $V(\lambda)$ in $(W_1^{(k)})^{\otimes 2}$.

Proof. Let u_0 be a $U_q(D_n)$-highest weight vector of $W_1^{(k)}$ of highest weight $\overline{\Lambda}_k (= \varpi_k)$. Since $V(\varpi_{k+1} + \varpi_{k-1})$ is multiplicity free, a unique highest weight vector up to a scalar is given by

$$v = u_0 \otimes f_k u_0 - q f_k u_0 \otimes u_0.$$

Since $f_iu_0=0$ for $i\in I\setminus\{k\}$, we have

$$F^{(1)}v=u_0\otimes F^{(1)}u_0-qF^{(1)}u_0\otimes u_0$$

where $F^{(1)}=f_{k+1}\cdots f_{n-2}f_nf_{n-1}\cdots f_{k+1}f_1\cdots f_{k-1}$. Hence we have

$$F^{(2)}v=qu_0\otimes F^{(2)}u_0-qF^{(2)}u_0\otimes u_0+(\text{unwanted terms})$$

where $F^{(2)}=f_2\cdots f_kF^{(1)}$ and we know $f_0(\text{unwanted terms})=0$ by weight consideration. Hence we have

$$F^{(3)}v=q^{-1}y^{-1}u_0\otimes F^{(3)}u_0-qx^{-1}F^{(3)}u_0\otimes u_0 \text{ on } V_x\otimes V_y$$

where $F^{(3)}=f_0F^{(2)}$ and $V=W_1^{(k)}$. Note that $F^{(3)}u_0=\alpha u_0$ with some $\alpha\neq 0$, since the corresponding crystal element is not killed from Proposition 3.2. Thus we have

$$F^{(3)}v=\alpha(q^{-1}y^{-1}-qx^{-1})u_0\otimes u_0.$$

Now let

$$R(z)\propto\varphi(z)P_{2\varpi_k}+\varphi'(z)P_{\varpi_{k+1}+\varpi_{k-1}}+\cdots.$$

Then we have

$$\begin{aligned}&R(z)F^{(3)}v=\alpha(q^{-1}y^{-1}-qx^{-1})\varphi(z)(u_0\otimes u_0)\\ =&F^{(3)}R(z)v=\varphi'(z)F^{(3)}v'=\alpha(q^{-1}x^{-1}-qy^{-1})\varphi'(z)(u_0\otimes u_0).\end{aligned}$$

Here by v' we mean that it is considered to be in $V_y\otimes V_x$. Thus we have

$$\varphi'(z)/\varphi(z)=\frac{z-q^2}{1-q^2z}.$$

□

Set $W=\text{Im } R(q^2), N=\text{Ker } R(q^2)$. They are $U_q'(D_n^{(1)})$-modules. Using the main result of [20] one can show the following.

Lemma 3.4. *We have*

$$W\simeq W_2^{(k)},\quad N\simeq\bigotimes_{j\sim k}W_1^{(j)}$$

as $U_q'(D_n^{(1)})$-modules. Here $j\sim k$ means that the corresponding vertices are tied by an edge in the Dynkin diagram. Moreover, both W and N are irreducible.

Proof. In [20] it is shown that there exists an exact sequence of $U'_q(D_n^{(1)})$-modules

$$0 \longrightarrow \bigotimes_{j\sim k} W_1^{(j)} \longrightarrow W_{1,q}^{(k)} \otimes W_{1,q^{-1}}^{(k)} \longrightarrow W_2^{(k)} \longrightarrow 0.$$

(An acute reader should have noticed that the exact sequence is different from [20]. It is because the definition of the KR modules and the choice of the coproduct are different.) Moreover, it is also known that $\bigotimes_{j\sim k} W_1^{(j)}$ and $W_2^{(k)}$ are irreducible. Set $W' = \bigotimes_{j\sim k} W_1^{(j)}$ and consider $N \cap W'$. Since W' is irreducible, we have $N \cap W' = \{0\}$ or W'. Recall that $W_{1,q}^{(k)} \otimes W_{1,q^{-1}}^{(k)}$ contains a unique irreducible $U_q(D_n)$-module $V(\varpi_{k+1}+\varpi_{k-1})$. From the previous lemma and (3.2) we know it is contained both in N and in W'. Hence we have $N \supset W'$. Now suppose $N \supsetneq W'$. Then we have a surjective $U'_q(D_n^{(1)})$-linear map

$$W_{1,q}^{(k)} \otimes W_{1,q^{-1}}^{(k)}/W' \longrightarrow W_{1,q}^{(k)} \otimes W_{1,q^{-1}}^{(k)}/N.$$

Since the l.h.s. is irreducible, $N = W'$ or $W_{1,q}^{(k)} \otimes W_{1,q^{-1}}^{(k)}$. Since N cannot be the second choice by the previous lemma. One obtains $N = W'$ and $W \simeq W_2^{(k)}$. □

Since W is known to be a KR module by the previous lemma, we have

Lemma 3.5. *As a $U_q(D_n)$-module W has the following decomposition.*

$$W \simeq \bigoplus_{0\le m_1\le m_2\le [k/2]} V(\overline{\Lambda}_{k-2m_1} + \overline{\Lambda}_{k-2m_2})$$

We set $B = B^{k,1}$. We fix a basis $\{v_I\}_{I\in B}$ of $W_1^{(k)}$ in such a way that $v_I \bmod qL = I$ as an element of B.

Proposition 3.6. *N contains a vector of the form*

$$v_{I_1} \otimes v_{I_2} - \sum_{J_1\otimes J_2\in B_1} a_{J_1J_2} v_{J_1} \otimes v_{J_2} \quad (a_{J_1J_2} \in A)$$

for any I_1, I_2 such that $I_1 \otimes I_2 \in B^{\otimes 2} \setminus B_1$.

See (4.2),(4.4) for the definition of B_1.

We now apply the fusion construction in Section 2.3 to $V = W_1^{(k)}$ with $r = 1$. The assumptions (2.21),(2.22) are valid with $\lambda_0 = \overline{\Lambda}_k$. (2.26) can also be checked. Other necessary properties are guaranteed by Proposition 3.1. For $l \in \mathbb{Z}_{>0}$ we define $W_l^{(k)} = \operatorname{Im} R_l$. Let $k' = [k/2]$. Let $\mathbf{c} = (c_1, c_2, \ldots, c_{k'})$ be a sequence of integers such that $l \geq c_1 \geq c_2 \geq \cdots \geq c_{k'} \geq 0$. For such $\mathbf{c}$ we define a vector u_m $(0 \leq m \leq k')$ in $W_l^{(k)}$ inductively by

$$u_m = (e_{k-2m}^{(c_m)} \cdots e_2^{(c_m)} e_1^{(c_m)})(e_{k-2m+1}^{(c_m)} \cdots e_3^{(c_m)} e_2^{(c_m)}) e_0^{(c_m)} u_{m-1}, \tag{3.5}$$

where u_0 here is $u_0^{\otimes l}$ in $(W_1^{(k)})^{\otimes l}$. Set $u(\mathbf{c}) = u_{k'}$. The weight of $u(\mathbf{c})$ is given by

$$\lambda(\mathbf{c}) = \sum_{j=0}^{k'} (c_j - c_{j+1}) \overline{\Lambda}_{k-2j},$$

where we have set $c_0 = l, c_{k'+1} = 0$. For $l, m \in \mathbb{Z}_{\geq 0}$ such that $m \leq l$ we define the q-binomial coefficient by

$$\begin{bmatrix} l \\ m \end{bmatrix} = \frac{[l]!}{[m]![l-m]!}.$$

The following proposition calculates values of prepolarizations necessary to prove the main theorem.

Proposition 3.7. (1) $(u(\mathbf{c}), u(\mathbf{c}))_l = \prod_{j=1}^{k'} q^{c_j(2l-c_j)} \begin{bmatrix} 2l \\ c_j \end{bmatrix}$,

(2) $(e_j u(\mathbf{c}), e_j u(\mathbf{c}))_l = 0$ *unless* $k - j \in 2\mathbb{Z}_{\geq 0}$. *If* $k - j \in 2\mathbb{Z}_{\geq 0}$, *then setting* $p = (k-j)/2 + 1$ $(e_j u(\mathbf{c}), e_j u(\mathbf{c}))_l$ *is given by*

$$q^{2l-c_{p-1}-1}[2l - c_{p-1}] \prod_{j=1}^{k'} q^{(c_j - \delta_{j,p})(2l-c_j)} \begin{bmatrix} 2l - \delta_{j,p} \\ c_j - \delta_{j,p} \end{bmatrix}.$$

Proofs of these propositions are given in subsequent sections.

The rest of this section is devoted to the proof of Theorem 1.1. From Proposition 2.6 it suffices to show

(i) $ch\, W_l^{(k)} \leq \sum_{l \geq c_1 \geq \cdots \geq c_{k'} \geq 0} ch\, V(\lambda(\mathbf{c}))$, where $V(\lambda)$ is the irreducible $U_q(D_n)$-module with highest weight λ and chV stands for the formal character of V.

(ii) $(u(\mathbf{c}), u(\mathbf{c}'))_l \in \delta_{\mathbf{c}\mathbf{c}'} + qA$ and $(e_j u(\mathbf{c}), e_j u(\mathbf{c}))_l \in q^{-1-2\langle h_j, \lambda(\mathbf{c}) \rangle} A$ for $j \neq 0$.

Let us show (i). First notice that $\sum_{l \geq c_1 \geq \cdots \geq c_{k'} \geq 0} ch\ V(\lambda(\mathbf{c})) = \sum_{0 \leq m_1 \leq \cdots \leq m_l \leq k'} ch\, V(\overline{\Lambda}_{k-2m_1} + \cdots + \overline{\Lambda}_{k-2m_l})$. In view of (2.24) and Proposition 3.6 $W_l^{(k)}$ is a quotient of a module generated by the set of vectors

$$\{v_{I_1} \otimes v_{I_2} \otimes \cdots \otimes v_{I_l} \mid I_j \otimes I_{j+1} \in B_1 \text{ for } j = 1, \ldots, l-1\}.$$

Assume $I_j \in B(\overline{\Lambda}_{k-2m_{l+1-j}}) \subset B$. Then $I_1 \otimes I_2 \otimes \cdots \otimes I_l$ belongs to

$$\bigcap_{j=1}^{l-1} B(\overline{\Lambda}_{k-2m_l}) \otimes \cdots \otimes B(\overline{\Lambda}_{k-2m_{j+2}}) \otimes B(\overline{\Lambda}_{k-2m_{j+1}} + \overline{\Lambda}_{k-2m_j}) \otimes B(\overline{\Lambda}_{k-2m_{j-1}}) \otimes \cdots \otimes B(\overline{\Lambda}_{k-2m_1}).$$

However, the above crystal is known to be identified with $B(\overline{\Lambda}_{k-2m_1} + \cdots + \overline{\Lambda}_{k-2m_l})$ (see Proposition 2.2.1 of [15]). This fact verifies (i).

For the proof of (ii) note that

$$[m] \in q^{1-m}A, \quad \begin{bmatrix} m \\ n \end{bmatrix} \in q^{-n(m-n)}A.$$

If $\mathbf{c} \neq \mathbf{c}'$, $(u(\mathbf{c}), u(\mathbf{c}'))_l = 0$ since the weights of $u(\mathbf{c})$ and $u(\mathbf{c}')$ are different. $(u(\mathbf{c}), u(\mathbf{c}'))_l \in 1 + qA$ by Proposition 3.7 (1). For the second part it suffices to notice that $\langle h_i, \lambda(\mathbf{c})\rangle \geq 0$. The proof is completed.

§4. Proof of Proposition 3.6

We prepare several lemmas. The next one is a direct consequence of Proposition 3.2.

Lemma 4.1. *Suppose $k \geq 2$. Set $k' = [k/2]$. For elements b, b' in $B^{k,1}$ let $b \xrightarrow{\tilde{e}_0} b'$ mean $\tilde{e}_0 b = b'$. Then we have the following rules of 0-actions. In (2)-(4) $\bullet$ stands for a crystal element whose explicit form is not used later.*
(1)

$$\begin{matrix} 1 \\ 2 \\ \vdots \\ k \end{matrix} \xrightarrow{\tilde{e}_0} \begin{matrix} 1 \\ 3 \\ \vdots \\ k \\ \overline{1} \end{matrix} \xrightarrow{\tilde{e}_0} \begin{matrix} 3 \\ 4 \\ \vdots \\ k \\ \overline{2} \\ \overline{1} \end{matrix}$$

(2) *Let* $1 \leq m \leq k' - 1$.

$$
\begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m \\ \overline{k-m} \\ \vdots \\ \overline{k-2m+1} \end{array}
\xrightarrow{\tilde{e}_0}
\begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m-1 \\ \overline{k-m-1} \\ \vdots \\ \overline{k-2m+1} \\ \overline{2} \\ \overline{1} \end{array}
\xrightarrow{\tilde{e}_0} \quad \bullet
$$

(3) *Let* $0 \leq m_1 \leq m_2 \leq k' - 1, m_1 \leq p \leq \min(m_{21}, m_s - 1)$, *where* $m_{21} = m_2 - m_1, m_s = m_1 + m_2$.

$$
\begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m_{21}-p \\ k-2p+1 \\ \vdots \\ k-2m_1 \\ \overline{k-m_{21}-p} \\ \vdots \\ \overline{k-2m_2+1} \end{array}
\xrightarrow{\tilde{e}_0}
\begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m_{21}-p-1 \\ k-2p+1 \\ \vdots \\ k-2m_1 \\ \overline{k-m_{21}-p-1} \\ \vdots \\ \overline{k-2m_2+1} \\ \overline{2} \\ \overline{1} \end{array}
\xrightarrow{\tilde{e}_0} \quad \bullet
$$

(4) *Let* $1 \leq m_1 \leq m_2 \leq k' - 1, m_{21} + 1 \leq p \leq m_2$, *where* $m_{21} = m_2 - m_1$. *Set* $m_s = m_1 + m_2$.

$$\begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m_s+p \\ \overline{k-m_s+p} \\ \vdots \\ \overline{k-2m_1+1} \\ \overline{k-2p} \\ \vdots \\ \overline{k-2m_2+1} \end{array} \xrightarrow{\tilde{e}_0} \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m_s+p-1 \\ \overline{k-m_s+p-1} \\ \vdots \\ \overline{k-2m_1+1} \\ \overline{k-2p} \\ \vdots \\ \overline{k-2m_2+1} \\ \overline{2} \\ \overline{1} \end{array} \xrightarrow{\tilde{e}_0} \bullet$$

Lemma 4.2. *Let V be a $U_q(\mathfrak{g}_0)$-module with a crystal base (L, B). Let W a submodule of V. Let $\{b_j \mid j \in I\} \subset B$. Suppose v is a vector in W such that $v \equiv \sum_j b_j \bmod qL$. Decompose I as $I = I_1 \sqcup I_2$ by*

$$I_1 = \{j \in I \mid \tilde{e}_i b_j = 0 \text{ for any } i\}, \quad I_2 = \{j \in I \mid \tilde{e}_i b_j \neq 0 \text{ for some } i\}.$$

Then there exits a highest weight vector w in W such that $w \equiv \sum_{j \in I_1} b_j \bmod qL$.

Proof. By applying $\tilde{f}_i \tilde{e}_i$ we know that there exists a vector v' in W such that $v' \equiv \sum_{j \in I'} b_j \bmod qL$, where $I' = \{j \in I_2 \mid \tilde{e}_i b_j \neq 0\}$. Hence there also exists a vector v'' in W such that $v'' \equiv \sum_{j \in (I')^c} b_j \bmod qL$. Continuing this with different i's, we obtain a vector v''' in W such that $v''' \equiv \sum_{j \in I_1} b_j \bmod qL$. Hence we can write v''' as $v''' = w + w'$ in such a way that $w \equiv \sum_{j \in I_1} b_j \bmod qL$ is a highest weight vector and $w' \in qL$ is not, but we can remove w' from v'''. $\square$

In what follows in this section, by abuse of notation we represent a basis vector v_I in $W_1^{(k)}$ also as I.

Lemma 4.3. *Let $0 \leq m \leq k'$. A highest weight vector of $V(2\overline{\Lambda}_{k-2m})$ in W is given by*

$$\begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m \\ \overline{k-m} \\ \vdots \\ \overline{k-2m+1} \end{array} \otimes \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m \\ \overline{k-m} \\ \vdots \\ \overline{k-2m+1} \end{array} \quad \textit{mod } qL.$$

Proof. A highest weight vector of $V(2\overline{\Lambda}_k)$ is given by

$$\begin{array}{c}1\\2\\\vdots\\k\end{array}\otimes\begin{array}{c}1\\2\\\vdots\\k\end{array}.$$

Noting that W is a submodule of $W^{(k)}_{1,q^{-1}}\otimes W^{(k)}_{1,q}$, apply e_0^2 and use Lemma 2.2 and 4.1 (1). We obtain

$$q\begin{array}{c}3\\4\\\vdots\\k\\\overline{2}\\\overline{1}\end{array}\otimes\begin{array}{c}1\\2\\\vdots\\k\end{array}+(1+q^2)\begin{array}{c}1\\3\\\vdots\\k\\\overline{1}\end{array}\otimes\begin{array}{c}1\\3\\\vdots\\k\\\overline{1}\end{array}+q\begin{array}{c}1\\2\\\vdots\\k\end{array}\otimes\begin{array}{c}3\\4\\\vdots\\k\\\overline{2}\\\overline{1}\end{array}\equiv\begin{array}{c}1\\3\\\vdots\\k\\\overline{1}\end{array}\otimes\begin{array}{c}1\\3\\\vdots\\k\\\overline{1}\end{array}\mod qL$$

as a vector in W. Apply further $e^{(2)}_{k-2}\cdots e^{(2)}_2e^{(2)}_1e^{(2)}_{k-1}\cdots e^{(2)}_3e^{(2)}_2$, one obtains

$$\begin{array}{c}1\\2\\\vdots\\k-1\\\overline{k-1}\end{array}\otimes\begin{array}{c}1\\2\\\vdots\\k-1\\\overline{k-1}\end{array}\mod qL$$

as a vector in W.

For $m>1$, we prove by induction on m. By Lemma 2.2 and 4.1 (2), one has

$$e^{(2)}_{k-2m-2}\cdots e^{(2)}_2 e^{(2)}_1 e^{(2)}_{k-2m-1}\cdots e^{(2)}_3 e^{(2)}_2 e^2_0 \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m \\ \overline{k-m} \\ \vdots \\ \overline{k-2m+1} \end{array} \otimes \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m \\ \overline{k-m} \\ \vdots \\ \overline{k-2m+1} \end{array}$$

$$\equiv \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m-1 \\ \overline{k-m-1} \\ \vdots \\ \overline{k-2m-1} \end{array} \otimes \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m-1 \\ \overline{k-m-1} \\ \vdots \\ \overline{k-2m-1} \end{array} \mod qL,$$

as a vector in W. Lemma 4.2 completes the proof. ☐

Lemma 4.4. *Let $0 \leq m_1 \leq m_2 \leq k'$. Set $m_{21} = m_2 - m_1, m_s = m_1 + m_2, M = \max(m_1, m_{21})$. A highest weight vector of $V(\overline{\Lambda}_{k-2m_1} + \overline{\Lambda}_{k-2m_2})$ in W is given by*

$$(4.1)\qquad \sum_{p=m_1}^{M} \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-p \\ \overline{k-p} \\ \vdots \\ \overline{k-2p+1} \end{array} \otimes \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m_{21}-p \\ k-2p+1 \\ \vdots \\ k-2m_1 \\ \overline{k-m_{21}-p} \\ \vdots \\ \overline{k-2m_2+1} \end{array} + \sum_{p=M+1}^{m_2} \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-p \\ \overline{k-p} \\ \vdots \\ \overline{k-2p+1} \end{array} \otimes \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m_s+p \\ \overline{k-m_s+p} \\ \vdots \\ \overline{k-2m_1+1} \\ \overline{k-2p} \\ \vdots \\ \overline{k-2m_2+1} \end{array}$$

mod qL.

Proof. We prove by induction on m_2. The case of $m_2 = m_1$ is proved in the previous lemma. Assume $m_1 > 0$. Apply $e_{k-2m_2-2}\cdots e_1 e_{k-2m_2-1}\cdots e_2 e_0$ to (4.1) and use Lemma 2.2 and 4.1 (2),(3),(4). Since one can always neglect terms corresponding to the crystal elements that are not killed by $\tilde{e}_i$ for some $i \neq 0$ by Lemma 4.2, we obtain

$$\sum_{p=m_1}^{M} \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-p \\ \overline{k-p} \\ \vdots \\ \overline{k-2p+1} \end{array} \otimes \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m_{21}-p-1 \\ k-2p+1 \\ \vdots \\ k-2m_1 \\ \overline{k-m_{21}-p-1} \\ \vdots \\ \overline{k-2m_2-1} \end{array} + \sum_{p=M+1}^{m_2} \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-p \\ \overline{k-p} \\ \vdots \\ \overline{k-2p+1} \end{array} \otimes \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m_s+p-1 \\ \overline{k-m_s+p-1} \\ \vdots \\ \overline{k-2m_1+1} \\ \overline{k-2p} \\ \vdots \\ \overline{k-2m_2-1} \end{array}$$

$$+ \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m_2-1 \\ \overline{k-m_2-1} \\ \vdots \\ \overline{k-2m_2-1} \end{array} \otimes \begin{array}{c} 1 \\ 2 \\ \vdots \\ k-m_1 \\ \overline{k-m_1} \\ \vdots \\ \overline{k-2m_1+1} \end{array}$$

mod qL as a vector in W. Note that the last term can be regarded as the term in the summand of the middle term for $p = m_2+1$. If $m_1 > m_{21}$, the induction proceeds. If $m_1 \leq m_{21}$, note also that the term in the summand of the middle term for $p = M+1$ can be regarded as the one of the first term for $p = M+1$.

The proof for $m_1 = 0$ is similar, but needs some attention. Note that only the first summation survives in (4.1). Divide the cases into three: $p = 0, 1 \leq p \leq m_2 - 1, p = m_2$ for the calculation of the action of e_0. □

The following lemma is an easy consequence of (2.25) with $l = 2$ and the nondegeneracy of the admissible pairing.

Lemma 4.5. *Let* $(\ ,\ )$ *be the admissible pairing between* $W^{(k)}_{1,q} \otimes W^{(k)}_{1,q^{-1}}$ *and* $W^{(k)}_{1,q^{-1}} \otimes W^{(k)}_{1,q}$ *induced from the poralization of* $W^{(k)}_1$*. Then* $u \in N$ *if and only if* $(u, v) = 0$ *for any* $v \in W$*.*

Now we are to prove Proposition 3.6. Set

$$b(\overline{\Lambda}_{k-2m}) = 12\cdots(k-m)(\overline{k-m})\cdots(\overline{k-2m+1}) \in B.$$

$b(\overline{\Lambda}_{k-2m})$ is a highest weight vector of $V(\overline{\Lambda}_{k-2m})$ in $W_1^{(k)}$. We define the following subsets of $B^{\otimes 2}$.

$$B_1^h = \{b(\overline{\Lambda}_{k-2m_2}) \otimes b(\overline{\Lambda}_{k-2m_1}) \mid 0 \le m_1 \le m_2 \le k'\}, \tag{4.2}$$

$$B_2^h = \{\text{elements appearing in the summand of (4.1)}\} \setminus B_1^h, \tag{4.3}$$

$$B_a = \left(\bigcup_{i_1,\dots,i_m \in I_0} \tilde{f}_{i_m} \cdots \tilde{f}_{i_1} B_a^h \right) \setminus \{0\} \quad \text{for } a = 1,2, \tag{4.4}$$

$$B_3 = B^{\otimes 2} \setminus (B_1 \sqcup B_2), \quad B_3^h = \{b \in B_3 \mid \tilde{e}_i b = 0 \text{ for any } i \neq 0\}. \tag{4.5}$$

Note that (4.1) always contains $b(\overline{\Lambda}_{k-2m_2}) \otimes b(\overline{\Lambda}_{k-2m_1})$. For $J = I_1 \otimes I_2 \in B^{\otimes 2}$ set $v_J = v_{I_1} \otimes v_{I_2}$.

Lemma 4.6. *For any $J \in B_2^h \sqcup B_3^h$ there exists a highest weight vector w_J in N such that*

$$w_J \equiv \begin{cases} v_J - v_{J_1} & \text{if } J \in B_2^h \\ v_J & \text{if } J \in B_3^h \end{cases}$$

mod $qL^{\otimes 2}$. Here J_1 is the unique element of B_1^h that has the same weight as J.

Proof. If the weight of J is not of the form of $\overline{\Lambda}_{k-2m_1} + \overline{\Lambda}_{k-2m_2}$, the assertion is clear. Suppose $wt\, J = \overline{\Lambda}_{k-2m_1} + \overline{\Lambda}_{k-2m_2}$. Consider a highest weight vector of the form

$$v = c_{J_1} v_{J_1} + \sum_{J_2} c_{J_2} v_{J_2} + \sum_{J_3} c_{J_3} v_{J_3} + v'.$$

Here J_1 is the unique element in B_1^h of the fixed weight, $c_{J_a} \in A$ for $a = 1,2,3$, the summation $\sum_{J_a}$ ranges over $J_a \in B_a^h$ that has the fixed weight for $a = 2,3$, and v' is some vector in $qL^{\otimes 2}$. Let w be the highest weight vector (4.1) in W. From Lemma 4.5 $v \in N$ if and only if $(v,w) = 0$. On the other hand, we have $(v,w) \equiv c_{J_1} + \sum_{J_2} c_{J_2}$ mod qA since $(L^{\otimes 2}, L^{\otimes 2}) \subset A$. Thus we have

$$c_{J_1} + \sum_{J_2} c_{J_2} \equiv 0 \text{ mod } qA.$$

Hence for any $c_{J_2}, c_{J_3} \in A$, there exists a highest weight vector v in N such that $v \equiv \sum_{J_2} c_{J_2}(v_{J_2} - v_{J_1}) + \sum_{J_3} c_{J_3} v_{J_3}$ mod $qL^{\otimes 2}$. The assertion follows from this by setting $c_J = 1, c_{J'} = 0$ for $J' \neq J$. □

Applying $\tilde{f}_i$'s to w_J for all J in $B_2^h \sqcup B_3^h$ we obtain all weight vectors of N. Any such weight vector should have the following form

$$av_J + \sum_{J' \neq J, J' \in B_2 \sqcup B_3} c_{J'} v_{J'} + \sum_{J'' \in B_1} d_{J''} v_{J''},$$

where $J \in B_2 \sqcup B_3, a \in 1 + qA, c_{J'} \in qA, d_{J''} \in A$. By Gaussian elimination, we obtain the desired result.

§5. Proof of Proposition 3.7

We begin this section with an easy lemma.

Lemma 5.1. *Let v be a weight vector in $W_l^{(k)}$. If $(wt\, v, \epsilon_i) > l$ for some $i \in \{1, \ldots, n\}$, then $v = 0$.*

Proof. The claim follows from the fact that $(wt\, u, \epsilon_i) \leq 1$ for a nonzero weight vector u in $W_1^{(k)}$ and $W_l^{(k)}$ is a subspace of $W_1^{(k)}$. □

The following formula will be used frequently.

$$f_i^{(a)} e_i^{(b)} = \sum_{j=0}^{\min(a,b)} e_i^{(b-j)} f_i^{(a-j)} \left\{ \begin{matrix} q^{a-b} t_i^{-1} \\ j \end{matrix} \right\}, \tag{5.1}$$

where $\left\{ \begin{smallmatrix} t \\ j \end{smallmatrix} \right\} = \prod_{k=1}^{j} (q^{1-k} t - q^{k-1} t^{-1}) / (q^k - q^{-k})$.

In this section we abbreviate l of the preporlarization $(\ ,\ )_l$ on $W_l^{(k)}$. We also write $|u|^2$ for (u, u). Recall the definition of u_m (3.5). $wt\, u_m$ is given by

$$wt\, u_m = (l - c_1)\overline{\Lambda}_k + (c_1 - c_2)\overline{\Lambda}_{k-2} + \cdots + (c_{m-1} - c_m)\overline{\Lambda}_{k-2m+2} + c_m \overline{\Lambda}_{k-2m}.$$

Lemma 5.2.

$$|u_m|^2 = q^{c_m(2l - c_m)} \begin{bmatrix} 2l \\ c_m \end{bmatrix} |u_{m-1}|^2.$$

Proof. Since the other case is similar, we prove when k is even. Using (2.20), we have

$$|u_m|^2 = ((e_{k-2m-1}^{(c_m)} \cdots e_1^{(c_m)})(e_{k-2m+1}^{(c_m)} \cdots e_2^{(c_m)}) e_0^{(c_m)} u_{m-1}, f_{k-2m}^{(c_m)} u_m).$$

By (5.1) we obtain

$$\begin{aligned} &f_{k-2m}^{(c_m)} u_m \\ &= \sum_j e_{k-2m}^{(c_m - j)} f_{k-2m}^{(c_m - j)} \begin{bmatrix} c_m \\ j \end{bmatrix} (e_{k-2m-1}^{(c_m)} \cdots e_1^{(c_m)})(e_{k-2m+1}^{(c_m)} \cdots e_2^{(c_m)}) e_0^{(c_m)} u_{m-1}. \end{aligned} \tag{5.2}$$

Note that $wt\, f_{k-2m}^{(c_m-j)}(e_{k-2m-1}^{(c_m)}\cdots e_1^{(c_m)})(e_{k-2m+1}^{(c_m)}\cdots e_2^{(c_m)})e_0^{(c_m)}u_{m-1} = wt\, u_m - (2c_m - j)\alpha_{k-2m}$. From Lemma 5.1 the summand in the r.h.s. of (5.2) becomes 0 unless $j = c_m$. Hence we have

$$|u_m|^2 = |(e_{k-2m-1}^{(c_m)}\cdots e_1^{(c_m)})(e_{k-2m+1}^{(c_m)}\cdots e_2^{(c_m)})e_0^{(c_m)}u_{m-1}|^2.$$

Similar calculations continue until we arrive at $|u_m|^2 = |e_0^{(c_m)}u_{m-1}|^2$. Using (2.20),(5.1) and Lemma 5.1 again, we this time have $|u_m|^2 = q^{c_m(2l-c_m)}\begin{bmatrix}2l\\c_m\end{bmatrix}|u_{m-1}|^2$. □

Lemma 5.3. (1) $e_j u_{k'} = 0$ *when* k *is even, if* $j \geq k+1$ *or* $j = 1$ *when* k is *odd.*

(2) $|f_j u_p|^2 = q^{c_p(2l-1-c_p)}\begin{bmatrix}2l-1\\c_p\end{bmatrix}q^{c_{p-1}-1}[c_{p-1}]|u_p|^2$ *if* $j = k-2p+2, p = 1, \ldots, k'$.

(3) $|f_j u_p|^2 = 0$ *if* $j = k-2p+1, p = 1, \ldots, k'$.

Proof. (1) Write $u_{k'} = Eu_0$. If $j \geq k+1$, e_j commutes with E. The claim follows from $e_j u_0 = 0$. When k is odd, $e_1 u_{k'} = 0$ follows from Lemma 5.1.

(2) When $c_p = 0$, the equality is shown as follows.

$$\begin{aligned}|f_j u_p|^2 &= |f_{k-2p+2}u_{p-1}|^2\\ &= (u_{p-1}, q^{-1}t_{k-2p+2}e_{k-2p+2}f_{k-2p+2}u_{p-1})\\ &= q^{c_{p-1}-1}[c_{p-1}]|u_{p-1}|^2.\end{aligned}$$

Here we have used the relation $e_{k-2p+2}u_{p-1} = 0$, that can be confirmed by Lemma 5.1.

Now assume $c_p > 0$. Imitating the proof of Lemma 5.2, one obtains

$$|f_j u_p|^2 = |(e_{k-2p+1}^{(c_p)}\cdots e_2^{(c_p)})e_0^{(c_p)}\cdot f_{f-2p+2}u_{p-1}|^2.$$

Next we calculate

$$\begin{aligned}&q^{-c_p^2}t_{k-2p+1}^{-c_p}f_{k-2p+1}^{(c_p)}(e_{k-2p+1}^{(c_p)}\cdots e_2^{(c_p)})e_0^{(c_p)}\cdot f_{k-2p+2}u_{p-1}\\ &= q^{-c_p}\sum_j e_{k-2p+1}^{(c_p-j)}f_{k-2p+1}^{(c_p-j)}\begin{bmatrix}c_p-1\\j\end{bmatrix}e_{k-2p}^{(c_p)}\cdots e_2^{(c_p)}e_0^{(c_p)}\cdot f_{k-2p+2}u_{p-1}.\end{aligned}$$

Since $(wt\, f_{k-2p+1}^{(c_p-j)}e_{k-2p}^{(c_p)}\cdots e_2^{(c_p)}e_0^{(c_p)}f_{k-2p+2}u_{p-1}, \epsilon_{k-2p+2}) = l-1+c_p-j$, the summand of the above expression survives only when $j = c_p, c_p-1$. Noting $\begin{bmatrix}c_p-1\\c_p\end{bmatrix} = 0$, we obtain

$$|f_j u_p|^2 = |(e_{k-2p}^{(c_p)}\cdots e_2^{(c_p)})e_0^{(c_p)}\cdot f_{k-2p+1}f_{k-2p+2}u_{p-1}|^2.$$

Calculating similarly, one gets $|f_j u_{p-1}|^2 = |e_0^{(c_p)} \cdot f_2 \cdots f_{k-2p+2} u_{p-1}|^2$. After removing $e_0^{(c_p)}$, we arrive at

$$|f_j u_p|^2 = q^{c_p(2l-1-c_p)} \begin{bmatrix} 2l-1 \\ c_p \end{bmatrix} |f_2 \cdots f_{k-2p+2} u_{p-1}|^2.$$

Calculating similarly, we obtain

$$|f_2 \cdots f_{k-2p+2} u_{p-1}|^2 = |f_{k-2p+2} u_{p-1}|^2 = q^{c_{p-1}-1}[c_{p-1}]|u_{p-1}|^2.$$

(3) The proof goes parallel to that of (2). When $c_p = 0$,

$$|f_j u_p|^2 = |f_{k-2p+1} u_{p-1}|^2 = 0$$

from Lemma 5.1. Assume $c_p > 0$. One obtains

$$|f_j u_p|^2 = |f_1 f_2 \cdots f_{k-2p+1} u_{p-1/2}|^2.$$

Noting that $f_i u_{p-1} = 0$ for $i = 1, \ldots, k-2p+1$, we have

$$\begin{aligned} f_1 f_2 \cdots f_{k-2p+1} u_{p-1/2} &= f_1 f_2 \cdots f_{k-2p+1} (e_{k-2p+1}^{(c_p)} \cdots e_2^{(c_p)}) e_0^{(c_p)} u_{p-1} \\ &= \alpha \cdot f_1 (e_{k-2p+1}^{(c_p-1)} \cdots e_2^{(c_p-1)}) \cdot e_0^{(c_p)} u_{p-1} = 0. \end{aligned}$$

Here α is a product of q-integers.

The proof is complete. □

Now we are in a position to prove Proposition 3.7. (1) is a simple consequence of Lemma 5.2. (2) when $j \geq k+1$ is settled by Lemma 5.3 (1). To show when $j \leq k$ note that

$$\begin{aligned} |e_j u_{k'}|^2 &= (u_{k'}, q^{-1} t_j^{-1} f_j e_j u_{k'}) \\ &= q^{2\beta_j} |f_j u_{k'}|^2 + q^{\beta_j - 1}[\beta_j]|u_{k'}|^2, \end{aligned}$$

where

$$\beta_j = -\langle h_j, \mathrm{wt}\, u_{k'} \rangle = \begin{cases} c_{\frac{k-j}{2}+1} - c_{\frac{k-j}{2}} & \text{if } j \equiv k \ (2), \\ 0 & \text{if } j \not\equiv k \ (2). \end{cases}$$

Thus we are left to evaluate $|f_j u_{k'}|^2$. Examining the proof of Lemma 5.2 carefully, one notices that the same recursion formula is valid when $m > p$, namely, one has

$$|f_j u_m|^2 = q^{c_m(2l-c_m)} \begin{bmatrix} 2l \\ c_m \end{bmatrix} |f_j u_{m-1}|^2 \quad \text{for } m > p.$$

The formula for $|f_j u_{k'}|^2$ is obtained from this, Lemma 5.3 (2) or (3) and Lemma 5.2. Calculating explicitly we obtain (2).

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