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Scalar Conservation Laws with Vanishing and Highly Nonlinear Diffusive-Dispersive Terms

By

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Abstract

We investigate the initial value problem for a scalar conservation law with highly nonlinear diffusive-dispersive terms: $u_t + f(u)_x = \varepsilon(u_x^{2\ell-1})_x - \delta(u_x^{2\ell-1})_{xx}$ ($\ell \geq 1$). In this paper, for a sequence of solutions to the equation with initial data, we give convergence results that a sequence converges to the unique entropy solution to the hyperbolic conservation law. In particular, our main theorem implies the results of Kondo-LeFloch [15] and Schonbek [26], furthermore makes up for insufficiency of the results in Fujino-Yamazaki [9] and LeFloch-Natalini [22]. Applying the technique of compensated compactness, the Young measure and the entropy measure-valued solutions as main tools, we establish the convergence property of the sequence. The final step of our proof is to show that the measure-valued mapping associated to the sequence of solutions is reduced to an entropy solution and this step is mainly based on the approach of LeFloch-Natalini [22].

§1. Introduction and the Main Result

Consider the sequence $\{u^\varepsilon\}$ of smooth solutions of the Cauchy problem for a scalar conservation law in one space dimension with highly nonlinear diffusive-dispersive terms:

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x (\partial_x u)^{2\ell-1} - \delta \partial_x^2 (\partial_x u)^{2\ell-1}, \quad (x,t) \in \mathbf{R} \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0^\varepsilon(x), \quad x \in \mathbf{R}, \tag{1.2}$$

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where $\ell \geq 1$ and ε, $\delta = \delta(\varepsilon) \to 0+$. In this paper, we will show that the sequence $\{u^\varepsilon\}$ converge to the unique entropy solution to

$$u_t + f(u)_x = 0, \quad (x,t) \in \mathbf{R} \times (0,\infty), \tag{1.3}$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}. \tag{1.4}$$

To obtain the convergence property of $\{u^\varepsilon\}$, we assume that there exist the smooth solutions to Eqs. (1.1) and (1.2) defined on $\mathbf{R}\times(0,T^*)$ for some $T^* > 0$, vanishing at infinity and associated with smooth and compactly supported initial data u_0^ε and

$$\exists u_0 \in L^1(\mathbf{R}) \cap L^q(\mathbf{R}),\ \forall q > 1 \text{ s.t. } \lim_{\varepsilon\to 0} u_0^\varepsilon = u_0 \text{ in } L^1(\mathbf{R}) \cap L^q(\mathbf{R}). \tag{1.5}$$

In addition, we assume the following uniform boundedness concerning the initial data with some constant $C_0 > 0$ independent of ε:

$$||u_0^\varepsilon||_{L^2(\mathbf{R})} + ||u_0^\varepsilon||_{L^q(\mathbf{R})} + \delta^{\frac{1}{2\ell}}||u_{0,x}^\varepsilon||_{L^{2\ell}(\mathbf{R})} \leq C_0, \tag{1.6}$$

for $q \in \left(\frac{3\ell-1}{\ell}, \frac{3\ell^2+2\ell-1}{\ell}\right)$ $(\ell \geq 1)$.

On the other hand, we also assume for the flux function $f(u)$ that $f(u)$ is a given smooth function which satisfies the following growth condition:

$$\exists C_1 > 0,\ m > 1 \text{ s.t. } |f'(u)| \leq C_1(1 + |u|^{m-1}) \text{ for any } u \in \mathbf{R}. \tag{I}$$

When F is defined by $F'(u) = f(u)$, we can replace the condition (I) by an assumption (I'):

$$\exists C_2 > 0,\ m > 1 \text{ s.t. } |F(u)| \leq C_2(|u|^2 + |u|^{m+1}) \text{ for any } u \in \mathbf{R}. \tag{I$'$}$$

Then, under the above assumptions, we show the following main result of the present paper:

Theorem 1.1. *Suppose that a condition* (I) *and there exists a sequence* $\{u^\varepsilon\}$ *of the smooth solutions to Eqs.* (1.1) *and* (1.2) *defined on* $\mathbf{R} \times (0,T^*)$, *vanishing at infinity and associated with the initial data satisfying* (1.5) *and* (1.6) *with* $m < q$ $\left(q \in \left[\frac{3\ell^2+3\ell-1}{2\ell}, 3\ell\right]\right)$. *If* $\delta = O\left(\varepsilon^{\frac{(\ell+1)(6\ell-m-1)}{2(3\ell^2-m\ell-q\ell+3\ell-1)}}\right)$, *then the sequence* $\{u^\varepsilon\}$ *converge to the unique entropy solution* $u \in L^\infty(0,T^*;L^q(\mathbf{R}))$ *to Eqs.* (1.3) *and* (1.4) *in* $L^k(0,T^*;L^p(\mathbf{R}))$ ($\forall k < \infty$ *and* $\forall p < q$).

Moreover if we assume that the sequence $\{u^\varepsilon\}$ of solutions to Eqs. (1.1), (1.2) is bounded in $L^\infty(0,T^*;L^q(\mathbf{R}))$, we obtain the same conclusion for any $q > m$ ($m \in (1, 3\ell)$, $\ell \geq 1$) provided that $\delta = o\left(\varepsilon^{\frac{(\ell+1)(6\ell-m-1)}{2\ell(3\ell-m)}}\right)$. Namely it follows that

Theorem 1.2. *Suppose that a condition* (I) *and there exists a sequence* $\{u^\varepsilon\}$ *of the smooth solutions to Eqs.* (1.1) *and* (1.2) *defined on* $\mathbf{R} \times (0, T^*)$, *vanishing at infinity and associated with the initial data satisfying* (1.5) *and* (1.6). *If the sequence is uniformly bounded in* $L^\infty(0, T^*; L^q(\mathbf{R}))$ *for some* $q > m$ $(m \in (1, 3\ell))$ *and* $\delta = o\left(\varepsilon^{\frac{(\ell+1)(6\ell-m-1)}{2\ell(3\ell-m)}}\right)$, *then the sequence* $\{u^\varepsilon\}$ *converge to the unique entropy solution* $u \in L^\infty(0, T^*; L^q(\mathbf{R}))$ *to Eqs.* (1.3) *and* (1.4) *in* $L^k(0, T^*; L^p(\mathbf{R}))$ $(\forall k < \infty$ *and* $\forall p < q)$.

In the consideration of a convergence, the appropriate balance in the relation between ε and δ is claimed so that the sequence of solutions to the conservation laws with diffusion and dispersion terms converges to the solution to the hyperbolic conservation law. In fact, when $\delta = 0$, Eq. (1.1) is reduced to a parabolic equation. In this case, if $\varepsilon \to 0$ (with $\delta = 0$), it is rather trivial that the sequence $\{u^\varepsilon\}$ of solutions to Eq. (1.1) converges to the solution to (1.3) owing to the classical vanishing viscosity method. On the other hand, when $\varepsilon = 0$, then Eq. (1.1) is reduced to the generalized Korteweg-de Vries (KdV) equation [16]. If $\delta \to 0$ in the KdV equation, the sequence of the solutions to Eq. (1.1) does not converge to the solution to Eq. (1.3) in general (cf. [3, 4, 19, 20]).

We recall several fundamental results for the convergence problem to the scalar conservation laws with diffusion and dispersion terms:

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = R^\varepsilon, \quad u^\varepsilon = u^\varepsilon(x,t)$$

where $\varepsilon > 0$ and $R^\varepsilon = R^\varepsilon(u^\varepsilon, u^\varepsilon_x, u^\varepsilon_{xx}, \cdots) \to 0$ as $\varepsilon \to 0$. In particular, our main Theorem 1.1 includes previous results. For the linear diffusion and the linear dispersion terms as $\ell = 1$ in Eq. (1.1) i.e.

$$R^\varepsilon = \varepsilon u^\varepsilon_{xx} - \delta u^\varepsilon_{xxx},$$

a first convergence result is obtained by Schonbek [26] under the assumption that either $\delta = O(\varepsilon^2)$ for Burgers' type flux $\left(f(u) = \frac{u^2}{2}\right)$ and for the family of flux functions:

$$f(u) = -\frac{u^{2h+1}}{2h+1}, \quad h \geq 1,$$

or the stronger condition $\delta = O(\varepsilon^3)$ for general subquadratic flux functions f. This convergence result has been improved by Kondo-LeFloch [15] for the flux satisfying $|f'(u)| \leq M$ (for $\forall u \in \mathbf{R},\ M > 0$). They give that the subsequence of solutions converges in $L^k(0,\infty; L^p(\mathbf{R}))$ $(1 < k < \infty$ and $1 < p < 2)$ to a weak solution of the Cauchy problem (1.3), (1.4) under the assumption $\delta = O(\varepsilon^2)$.

Moreover they obtain that the limit is the unique entropy solution in the sense of Kružkov under the stronger condition $\delta = o(\varepsilon^2)$. They also give a convergence result for multidimensional conservation laws. Clearly, our result is extension to their works. See also a result for systems in Hayes-LeFloch [11].

As compared with the above results, there are the following results for the nonlinear diffusion and the nonlinear dispersion terms:

$$R^\varepsilon = \varepsilon b(u_x^\varepsilon)_x - \delta \left((u_x^\varepsilon)^{2\ell-1}\right)_{xx}, \quad \ell \geq 1 \tag{1.7}$$

under the assumptions that f satisfies the growth condition (I) and moreover that a nondecreasing function b satisfies $b(0) = 0$, $b(\lambda)\lambda \geq 0$ (for $\forall\lambda \in \mathbf{R}$) and

$$C_3|\lambda|^{(2\ell+1)r} \leq b(\lambda)\lambda \leq C_4|\lambda|^{(2\ell+1)r} \text{ for any } |\lambda| \geq N \tag{II}$$

where $C_3,\ C_4,\ N > 0,\ r \geq 1$. In the case as $\ell = 1$ in Eq. (1.7), LeFloch-Natalini [22] show that the sequence $\{u^\varepsilon\}$ is bounded in $L^\infty(0, T^*; L^q(\mathbf{R}))$ for $m < 5 - \frac{1}{r}(=: q)$ and obtain the convergence result that the sequence converges to the unique entropy solution $u \in L^\infty(0, T^*; L^q(\mathbf{R}))$ in $L^k(0, T^*; L^p(\mathbf{R}))$ ($k < \infty,\ p < q$) for $\delta = O(\varepsilon^{\frac{5-m}{r(5-m)-1}})$ ($r \geq 1$). In the case that $\ell \geq 1$ for Eq. (1.7), it is investigated by Fujino-Yamazaki [9]. In [9], we prove the same convergence property to [22] for $\delta = O(\varepsilon^{\frac{6\ell-m-1}{r(6\ell-m-1)-1}})$ ($m < q,\ \forall\ell \geq 1$). On the consideration to Eq. (1.7), the assumption (II) of the diffusion term is very important in the proof of their results in [9, 22]. From the assumption (II), the function b can not imply the identity function $b(\lambda) = \lambda$ because, as $\ell,\ r = 1$ in (II), it follows that

$$C_3'|\lambda|^2 \leq b(\lambda) \leq C_4'|\lambda|^2 \text{ for any } |\lambda| \geq N \tag{II$'$}$$

where $C_3',\ C_4',\ N > 0$. Comparing with this assumption for b in Eq. (1.7), the nonlinear power function $u_x^{2\ell-1}$ ($\forall u \in \mathbf{R}$) of the diffusion term in our scalar conservation law (1.1) imply the identity function as $\ell = 1$ clearly. On the other hand, observing the domain of q for the $L^q(\mathbf{R})$, it is that $(m <)q \in [4, 5)$ in [9, 22] and that $(m <)q \in [\frac{3\ell^2+3\ell-1}{2\ell}, 3\ell]$ ($\ell \geq 1$) in Theorem 1.1 of this paper. Therefore, in that sense, we can also consider the different results for $q > m$.

In this paper, we consider the scalar conservation laws with highly nonlinear diffusive-dispersive terms (1.1) without the assumptions (II) nor (II') by using the technique developed in [9, 22]. Especially, we make use of the compensated compactness, the measure-valued (m.-v.) solutions of the Cauchy problem which are investigated by, for example, DiPerna [8] and Szepessy [27]. Moreover the final step of the proof of the main result relies mainly on the approach of LeFloch-Natalini [22]. To give convergence results Theorems 1.1, 1.2, we recall some elementary notions in Section 2 and we establish the uniform boundedness in $L^q(\mathbf{R})$ by a priori estimates of the solutions to Eq. (1.1) in Section 3. In the last section, owing to a priori estimates and boundedness obtained in Section 3, the convergence argument due to [22] is applied to Eqs. (1.1), (1.2).

§2. Preliminaries

Let us remind of the basic theory for Young measure and entropy measure-valued (m.-v.) solutions concisely. Following DiPerna [8], LeFloch-Natalini [22] and Szepessy [27], we state a generalization of the Young measure.

Proposition 2.1 ([8, 22, 27]). *Suppose that the sequence* $\{u^\varepsilon\}$ *is bounded in* $L^\infty(0,\infty;L^q(\mathbf{R}))$ *and that* $f \in C(\mathbf{R})$ *satisfies the growth condition* (I) *for some* $q' \in (0,q)$, $C>0$. *Then there exists a subsequence* $\{u^{\varepsilon'}\}$ *and a probability measure-valued mapping* $\nu = \nu_{(x,t)}$ *defined on* $\mathbf{R}\times(0,\infty)$, *such that*

$$f(u^{\varepsilon'}) \to \langle \nu_{(x,t)}, f\rangle := \int_{\mathbf{R}} f(\lambda)d\nu_{(x,t)}(\lambda) \ \text{as } \varepsilon' \to 0 \tag{2.1}$$

in $L^s(\mathbf{R}\times(0,\infty))$ *for any* $s \in (1, q/q')$.

A probability measure-valued mapping ν in Proposition 2.1 is called a *Young measure* associated with the subsequence $\{u^{\varepsilon'}\}$. For this Young measure ν, an entropy measure-valued (m.-v.) solution is defined as follows:

Definition 2.1 ([8, 27]). Suppose that $f \in C(\mathbf{R})$ satisfies the growth condition (I) and the initial data $u_0 \in L^1(\mathbf{R})\cap L^q(\mathbf{R})$. If it follows that

$$\partial_t\langle \nu_{(x,t)}(\lambda),\ |\lambda-k|\rangle + \partial_x\langle \nu_{(x,t)}(\lambda),\ \mathrm{sgn}(\lambda-k)(f(\lambda)-f(k))\rangle \le 0 \tag{2.2}$$

in $\mathcal{D}'(\mathbf{R}\times(0,\infty))$ for any $k\in\mathbf{R}$ and that

$$\lim_{T\to 0+}\frac{1}{T}\int_0^T\int_K \langle \nu_{(x,t)}(\lambda), |\lambda-u_0(x)|\rangle dxdt = 0 \tag{2.3}$$

for any compact sets $K\subseteq\mathbf{R}$, then a Young measure $\nu:\mathbf{R}\times(0,\infty)\to \mathrm{Prob}(\mathbf{R})$ associated with the subsequence $\{u^{\varepsilon'}\}$ is called an *entropy measure-valued* (m.-v.) *solution* to Eqs. (1.3), (1.4).

Here we remark that it is not necessary to take a subsequence of $\{u^\varepsilon\}$. As a well-known fact, for an entropy m.-v. solution to Eqs. (1.3), (1.4), uniqueness holds by [27]. Namely if ν and $\tilde{\nu}$ are entropy m.-v. solutions to Eqs. (1.3), (1.4), then there exists a function $w \in L^\infty(\mathbf{R}; L^1(\mathbf{R}) \cap L^q(\mathbf{R}))$ such that $\nu_{(x,t)} = \delta_{w(x,t)} = \tilde{\nu}_{(x,t)}$ for a.e. $(x,t) \in \mathbf{R} \times (0,\infty)$. This uniqueness of the entropy m.-v. solution implies $f(u^\varepsilon) \to \langle \nu_{(x,t)}, f \rangle$ in the sense of distributions. We introduce the convergence theorem as our main tool.

Theorem 2.1 ([22]). *Suppose that f satisfies the growth condition* (I) *and the initial data $u_0 \in L^1(\mathbf{R}) \cap L^q(\mathbf{R})$ for $q \geq 1$. Let ν be a Young measure associated with $\{u^\varepsilon\}$ which is an uniformly bounded sequence in $L^\infty(0,\infty; L^q(\mathbf{R}))$. If a Young measure ν is an entropy m.-v. solution to Eqs.* (1.3), (1.4), *then the sequence $\{u^\varepsilon\}$ converge to the unique entropy solution $u \in L^\infty(0,\infty; L^q(\mathbf{R}))$ in $L^\infty(0,\infty; L^{q'}_{loc}(\mathbf{R}))$ (for any $q' \in [1,q)$) to Eqs.* (1.3), (1.4).

To obtain our convergence results by applying Theorem 2.1, we should show that the uniform boundedness of a sequence $\{u^\varepsilon\}$ in $L^q(\mathbf{R})$ holds for $q > m$ and that a Young measure ν is an entropy m.-v. solution to Eqs. (1.3), (1.4) in the following sections.

§3. A Priori Estimates

In this section, to establish the L^q boundedness, we give several a priori estimates of solutions to a scalar conservation law with highly nonlinear diffusive-dispersive terms (1.1) with initial data u_0^ε which are smooth functions with compact support and satisfy the assumptions (1.5) and (1.6). We suppose that there exists a sequence $\{u^\varepsilon\}$ of the smooth solutions to Eqs. (1.1), (1.2) defined on $\mathbf{R} \times (0, T^*)$, vanishing at infinity and associated with initial data u_0^ε for some $T^* \in (0,\infty]$.

Throughout the calculation of this section and for simplicity, we omit the upper-index ε and describe u^ε into u and so on. Referring to [9], as a first estimate, we find

Lemma 3.1. *For every $T \in (0, T^*)$, We have*

$$\int_{\mathbf{R}} u^2(x,T)dx + 2\varepsilon \int_0^T \int_{\mathbf{R}} u_x^{2\ell}(x,t)dxdt \leq C_0, \tag{3.1}$$

and

$$\begin{aligned} \delta \int_{\mathbf{R}} u_x^{2\ell}(x,T)dx &+ 2\ell(2\ell-1)\varepsilon\delta \int_0^T \int_{\mathbf{R}} u_x^{2(2\ell-2)} u_{xx}^2 dxdt \\ &\leq C_0 + 2\ell \int_{\mathbf{R}} F(u(x,T))dx + 2\ell\varepsilon \int_0^T \int_{\mathbf{R}} f'(u) u_x^{2\ell} dxdt. \end{aligned} \tag{3.2}$$

Proof. Multiplying Eq. (1.1) by u and integrating it in space, we find

$$\int_{\mathbf{R}} \left(\frac{u^2}{2}\right)_t dx = -\varepsilon \int_{\mathbf{R}} u_x^{2\ell} dx.$$

Integrating the above equation in time, We obtain

$$\frac{1}{2}\int_{\mathbf{R}} u^2(x,T)dx + \varepsilon \int_0^T \int_{\mathbf{R}} u_x^{2\ell} dxdt = \frac{1}{2}\int_{\mathbf{R}} u_0^2(x)dx.$$

From an assumption (1.6) of the initial data in L^2 norm, We arrive the first estimate (3.1).

In the same way, multiplying Eq. (1.1) by $f(u)+\delta(u_x^{2\ell-1})_x$ and integrating it in space, We have

$$\int_{\mathbf{R}} F(u)_t dx - \delta \int_{\mathbf{R}} \left(\frac{u_x^{2\ell}}{2\ell}\right)_t dx = -\varepsilon \int_{\mathbf{R}} f'(u)u_x^{2\ell} dx + (2\ell-1)\varepsilon\delta \int_{\mathbf{R}} \left(u_x^{2\ell-2}u_{xx}\right)^2 dx.$$

By integrating in time, We obtain

$$\begin{aligned}
&\delta \int_{\mathbf{R}} u_x^{2\ell} dx + 2\ell(2\ell-1)\varepsilon\delta \int_0^T \int_{\mathbf{R}} \left(u_x^{2\ell-2}u_{xx}\right)^2 dxdt \\
&= \delta \int_{\mathbf{R}} u_{0,x}^{2\ell} dx - 2\ell \int_{\mathbf{R}} F(u_0)dx + 2\ell \int_{\mathbf{R}} F(u)dx + 2\ell\varepsilon \int_0^T \int_{\mathbf{R}} f'(u)u_x^{2\ell} dxdt
\end{aligned}$$

Hence We obtain an inequality (3.2) by the uniform bound of $u_{0,x}$ in the $L^{2\ell}$ norm. □

Combining an assumption (I') and the uniform bound of u in $L^\infty(0,T^*;L^2(\mathbf{R}))$ derived by a estimate (3.1), we can replace by a following assumption;

(I") $\exists C_5 > 0,\ m > 1$ s.t. $|F(u)| \leq C_5(1+|u|^{m+1})$ for any $u \in \mathbf{R}$.

To estimate the solution u to Eq. (1.1) in the L^∞ norm, we use the estimates (3.1), (3.2) and an assumption (I").

Lemma 3.2. *Suppose* $m \in (1, 6\ell-1)$ $(\ell \geq 1)$, *then there exists a constant* $C > 0$ *such that*

$$\sup_{t\in(0,T^*)} ||u(\cdot,t)||_{L^\infty(\mathbf{R})} \leq C\delta^{-\frac{1}{6\ell-m-1}}. \tag{3.3}$$

Proof. From the inequality (3.2) and an assumption (I"), we have

$$\begin{aligned}
&\delta \int_{\mathbf{R}} u_x^{2\ell}(x,T)dx + 2\ell(2\ell-1)\varepsilon\delta \int_0^T \int_{\mathbf{R}} u_x^{2(2\ell-2)}u_{xx}^2 dxdt \\
&\leq C_0 + C \sup_{t\in[0,T]} ||u(\cdot,t)||_{L^\infty(\mathbf{R})}^{m-1} \left(||u(\cdot,T)||_{L^2(\mathbf{R})} + \varepsilon \int_0^T \int_{\mathbf{R}} u_x^{2\ell} dxdt \right)
\end{aligned}$$

with some $C > 0$. In view of an estimate (3.1), we get

$$\delta \int_{\mathbf{R}} u_x^{2\ell}(x,T)dx + 2\ell(2\ell-1)\varepsilon\delta \int_0^T \int_{\mathbf{R}} u_x^{2(2\ell-2)} u_{xx}^2 dxdt \le C\left(1 + \sup_{t\in[0,T]} ||u(\cdot,t)||_{L^\infty(\mathbf{R})}^{m-1}\right) \tag{3.4}$$

which implies for every $T \in (0, T^*)$ that

$$\delta^{\frac{1}{2\ell}} ||u_x(\cdot,T)||_{L^{2\ell}(\mathbf{R})} \le C\left(1 + \sup_{t\in[0,T]} ||u(\cdot,t)||_{L^\infty(\mathbf{R})}^{m-1}\right)^{\frac{1}{2\ell}} \quad \text{with some } C > 0.$$

Hence by the Hölder's inequality and the estimate (3.1) again, we have for $\forall t_1 \in [0,T]$

$$\begin{aligned}
|u(x,t_1)|^3 &\le 3\int_{-\infty}^{x} |u^2(y,t_1)u_x(y,t_1)|dy \\
&\le 3\left(\int_{\mathbf{R}} |u|^{2p}dy\right)^{\frac{1}{p}} \left(\int_{\mathbf{R}} |u_x|^{2\ell}dy\right)^{\frac{1}{2\ell}} \\
&\le 3\delta^{-\frac{1}{2\ell}} \left(\sup_{t_1\in[0,T]} ||u(\cdot,t_1)||_{L^\infty(\mathbf{R})}^{2p-2} \int_{\mathbf{R}} |u|^2 dy\right)^{\frac{1}{p}} \delta^{\frac{1}{2\ell}} ||u_x(\cdot,t_1)||_{L^{2\ell}(\mathbf{R})} \\
&\le C\delta^{-\frac{1}{2\ell}} \sup_{t_1\in[0,T]} ||u(\cdot,t_1)||_{L^\infty(\mathbf{R})}^{\frac{2p-2}{p}} \left(1 + \sup_{t_1\in[0,T]} ||u(\cdot,t_1)||_{L^\infty(\mathbf{R})}^{m-1}\right)^{\frac{1}{2\ell}} \\
&\le C\delta^{-\frac{1}{2\ell}} \sup_{t_1\in[0,T]} ||u(\cdot,t_1)||_{L^\infty(\mathbf{R})}^{\frac{1}{\ell}} \left(1 + \sup_{t_1\in[0,T]} ||u(\cdot,t_1)||_{L^\infty(\mathbf{R})}^{m-1}\right)^{\frac{1}{2\ell}}
\end{aligned}$$

with some $C > 0$ where $p = \frac{2\ell}{2\ell-1}$. Therefore, for $\forall t \in (0, T^*)$, we have

$$\sup_{t\in(0,T^*)} ||u(\cdot,t)||_{L^\infty(\mathbf{R})}^{6\ell} \le C\delta^{-1} \sup_{t\in(0,T^*)} ||u(\cdot,t)||_{L^\infty(\mathbf{R})}^{2} \left(1 + \sup_{t\in(0,T^*)} ||u(\cdot,t)||_{L^\infty(\mathbf{R})}^{m-1}\right)$$

with some $C > 0$. Here we describe $h := \sup_{t\in(0,T^*)} ||u(\cdot,t)||_{L^\infty(\mathbf{R})}$, and consider the algebraic inequality $h^{6\ell} \le C\delta^{-1}h^2(1 + h^{m-1})$. Therefore we obtain the uniform estimate (3.3). □

Substituting the uniform boundedness (3.3) of u in the L^∞ norm into the inequality (3.4), we can easily obtain

Lemma 3.3. *For any $T \in (0, T^*)$ and $m \in (1, 6\ell - 1)$ $(\ell \geq 1)$, there exists a constant $C > 0$ such that*

$$\int_{\mathbf{R}} u_x(x,T)^{2\ell} dx + 2\ell(2\ell-1)\varepsilon \int_0^T \int_{\mathbf{R}} u_x^{2(2\ell-2)} u_{xx}^2 dxdt \leq C\delta^{-\frac{2(3\ell-1)}{6\ell-m-1}}.$$

We remark that this inequality implies

$$\int_{\mathbf{R}} u_x(x,T)^{2\ell} dx + \varepsilon \int_0^T \int_{\mathbf{R}} u_x^{2(2\ell-2)} u_{xx}^2 dxdt \leq C\delta^{-\frac{2(3\ell-1)}{6\ell-m-1}} \tag{3.5}$$

with some $C > 0$.

Utilizing some estimates obtained in this section, the uniform boundedness of the sequence $\{u^\varepsilon\}$ in $L^q(\mathbf{R})$ for $q \in \left(\frac{3\ell-1}{\ell}, \frac{3\ell^2+2\ell-1}{\ell}\right)$ $(\ell \geq 1)$ is established. For some technical reasons, we divide our proof into $\ell > 1$ and $\ell = 1$.

Proposition 3.1. *Suppose the condition* (I) *for $m < \frac{3\ell^2 - q\ell + 3\ell - 1}{\ell}$ $\left(q \in \left(\frac{3\ell-1}{\ell}, \frac{3\ell^2+2\ell-1}{\ell}\right), \ \ell > 1\right)$ and the uniform bound* (1.6) *for the initial data, then the sequence $\{u^\varepsilon\}$ of solutions to Eqs.* (1.1), (1.2) *is uniformly bounded in $L^q(\mathbf{R})$ with respect to $t \in (0, T^*)$ provided that $\delta = O\left(\varepsilon^{\frac{(\ell+1)(6\ell-m-1)}{2(3\ell^2 - m\ell - q\ell + 3\ell - 1)}}\right)$.*

Proof. To show the uniform boundedness of the sequence $\{u^\varepsilon\}$ in $L^q(\mathbf{R})$, we obtain a priori estimate of the solutions in $L^q(\mathbf{R})$. We set $\rho(u) := |u|^q$ for $q \in \left(\frac{3\ell-1}{\ell}, \frac{3\ell^2+2\ell-1}{\ell}\right)$ $(\ell > 1)$. Multiplying Eq. (1.1) by $\rho'(u)$ and integrating in space and time, we find

$$\begin{aligned} &\int_{\mathbf{R}} \rho(u(x,T)) dx + \varepsilon \int_0^T \int_{\mathbf{R}} \rho''(u) u_x^{2\ell} dxdt \\ &\quad = \int_{\mathbf{R}} \rho(u_0(x)) dx + \delta \int_0^T \int_{\mathbf{R}} \rho'(u)_x \left(u_x^{2\ell-1}\right)_x dxdt. \end{aligned} \tag{3.6}$$

Applying inequalities (3.1), (3.3) and (3.5), we estimate the second term in the right-hand side of Eq. (3.6):

$$\begin{aligned} &\left|\delta \int_0^T \int_{\mathbf{R}} \rho'(u)_x \left(u_x^{2\ell-1}\right)_x dxdt\right| \\ &\leq \left|(2\ell-1)\delta \int_0^T \int_{\mathbf{R}} \rho''(u) u_x \cdot u_x^{2\ell-2} u_{xx} dxdt\right| \\ &\leq C\delta \int_0^T \int_{\mathbf{R}} |u|^{q-2} |u_x| |u_x^{2\ell-2} u_{xx}| dxdt \end{aligned}$$

$$\le C\delta\left(\int_0^T\int_{\mathbf{R}}|u|^{p_1(q-2)}dxdt\right)^{\frac{1}{p_1}}\left(\int_0^T\int_{\mathbf{R}}|u_x|^{2\ell}dxdt\right)^{\frac{1}{2\ell}}\left(\int_0^T\int_{\mathbf{R}}|u_x^{2\ell-2}u_{xx}|^2dxdt\right)^{\frac{1}{2}}$$
$$\le C\delta\left(\sup_{t'\in(0,T)}||u(\cdot,t')||_{L^\infty(\mathbf{R})}^{p_1(q-2)-2}\int_0^T\int_{\mathbf{R}}|u|^2dxdt\right)^{\frac{1}{p_1}}\cdot\varepsilon^{-\frac{1}{2\ell}}\cdot\varepsilon^{-\frac{1}{2}}\delta^{-\frac{3\ell-1}{6\ell-m-1}}$$
$$\le C\delta\sup_{t'\in(0,T)}||u(\cdot,t')||_{L^\infty(\mathbf{R})}^{\frac{\ell(q-3)+1}{\ell}}\cdot T^{\frac{1}{p_1}}\cdot\varepsilon^{-\frac{\ell+1}{2\ell}}\delta^{-\frac{3\ell-1}{6\ell-m-1}}$$
$$\le CT^{\frac{\ell-1}{2\ell}}\varepsilon^{-\frac{\ell+1}{2\ell}}\delta^{1-\frac{\ell(q-3)+1}{\ell(6\ell-m-1)}-\frac{3\ell-1}{6\ell-m-1}}$$
$$\le CT^{\frac{\ell-1}{2\ell}}\varepsilon^{-\frac{\ell+1}{2\ell}}\delta^{\frac{3\ell^2-m\ell-q\ell+3\ell-1}{\ell(6\ell-m-1)}}$$

with some $C>0$ where $\frac{1}{p_1}+\frac{1}{2\ell}+\frac{1}{2}=1$ $\left(\text{i.e. } p_1=\frac{2\ell}{\ell-1}\right)$ for $\ell>1$. Substituting this estimate into Eq. (3.6), we obtain the uniform estimate in the $L^q(\mathbf{R})$ under the condition (I) for $m<\frac{3\ell^2-q\ell+3\ell-1}{\ell}$. Namely there exists a constant $C>0$ such that, for any $q\in\left(\frac{3\ell-1}{\ell},\frac{3\ell^2+2\ell-1}{\ell}\right)$ $(\ell>1)$,

$$\sup_{t\in(0,T^*)}||u(\cdot,t)||_{L^q(\mathbf{R})}^q\le C\left(1+T^{\frac{\ell-1}{2\ell}}\varepsilon^{-\frac{\ell+1}{2\ell}}\delta^{\frac{3\ell^2-m\ell-q\ell+3\ell-1}{\ell(6\ell-m-1)}}\right). \tag{3.7}$$

From the structure of this a priori estimate (3.7), it follows Proposition 3.1 directly. □

When $\ell=1$, we can replace the estimates (3.1), (3.3), (3.5) by following estimates respectively:

$$\int_{\mathbf{R}}u^2(x,T)dx+2\varepsilon\int_0^T\int_{\mathbf{R}}u_x^2(x,t)dxdt\le C_0, \tag{3.1'}$$

$$\sup_{t\in(0,T^*)}||u(\cdot,t)||_{L^\infty(\mathbf{R})}\le C\delta^{-\frac{1}{5-m}}, \tag{3.3'}$$

$$\int_{\mathbf{R}}u_x(x,T)^2dx+\varepsilon\int_0^T\int_{\mathbf{R}}u_{xx}^2dxdt\le C\delta^{-\frac{4}{5-m}}. \tag{3.5'}$$

Hence for $q'\in(2,4)$, the uniform boundedness of the sequence $\{u^\varepsilon\}$ of solutions to

$$\partial_t u+\partial_x f(u)=\varepsilon\partial_x^2u-\delta\partial_x^3u,\quad (x,t)\in\mathbf{R}\times(0,\infty) \tag{1.1'}$$

in $L^{q'}(\mathbf{R})$ is obtained as follows:

Proposition 3.2. *Suppose the condition* (I) *for* $m < 5 - q'$ $(q' \in (2,4))$ *and the uniform bound* (1.6) *for the initial data, then the sequence* $\{u^\varepsilon\}$ *of solutions to Eqs.* (1.1)', (1.2) *is uniformly bounded in* $L^{q'}(\mathbf{R})$ *with respect to* $t \in (0, T^*)$ *provided that* $\delta = O\left(\varepsilon^{\frac{5-m}{5-m-q'}}\right)$.

Proof. In the same way of proof of Proposition 3.1, we set $\widetilde{\rho}(u) := |u|^{q'}$ for $q' \in (2,4)$, and multiply Eq. (1.1)' by $\widetilde{\rho}'(u)$. Integrating it in space and time, we get

$$
\begin{aligned}
(3.8) \qquad & \int_{\mathbf{R}} \widetilde{\rho}(u(x,T))dx + \varepsilon \int_0^T \int_{\mathbf{R}} \widetilde{\rho}''(u) u_x^2 dxdt \\
&= \int_{\mathbf{R}} \widetilde{\rho}(u_0(x))dx + \delta \int_0^T \int_{\mathbf{R}} \widetilde{\rho}''(u) u_x u_{xx} dxdt.
\end{aligned}
$$

Thus using (3.1)', (3.3)' and (3.5)', we can obtain the estimate for the second term in the right-hand side of Eq. (3.8):

$$
\begin{aligned}
&\left| \delta \int_0^T \int_{\mathbf{R}} \widetilde{\rho}''(u) u_x u_{xx} dxdt \right| \\
&\le C\delta \int_0^T \int_{\mathbf{R}} |u|^{q'-2} |u_x||u_{xx}| dxdt \\
&\le C\delta \sup_{t' \in (0,T)} ||u(\cdot, t')||_{L^\infty(\mathbf{R})}^{q'-2} \left(\int_0^T \int_{\mathbf{R}} |u_x|^2 \right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbf{R}} |u_{xx}|^2 \right)^{\frac{1}{2}} \\
&\le C\delta \cdot \delta^{-\frac{q'-2}{5-m}} \cdot \varepsilon^{-\frac{1}{2}} \cdot \varepsilon^{-\frac{1}{2}} \delta^{-\frac{2}{5-m}} \\
&\le C\varepsilon^{-1} \delta^{\frac{5-m-q'}{5-m}}
\end{aligned}
$$

with some $C > 0$. Therefore substituting this estimate into Eq. (3.8), it follows that there exists a constant $C > 0$ such that, for any $q' \in (2,4)$,

$$
(3.9) \qquad \sup_{t \in (0,T^*)} ||u(\cdot, t)||_{L^{q'}(\mathbf{R})}^{q'} \le C \left(1 + \varepsilon^{-1} \delta^{\frac{5-m-q'}{5-m}} \right).
$$

which gives Proposition 3.2 for $m < 5 - q'$. □

Combining Propositions 3.1 and 3.2, for any $\ell \ge 1$, we arrive at the uniform boundedness of the sequence $\{u^\varepsilon\}$ in $L^q(\mathbf{R})$.

Corollary 3.1. *Suppose the condition* (I) *for* $m < \frac{3\ell^2 - q\ell + 3\ell - 1}{\ell}$ $\left(q \in \left(\frac{3\ell-1}{\ell}, \frac{3\ell^2+2\ell-1}{\ell}\right),\ \ell \ge 1\right)$ *and the uniform bound* (1.6) *for the initial data,* *then the sequence* $\{u^\varepsilon\}$ *of solutions to Eqs.* (1.1), (1.2) *is uniformly bounded in* $L^q(\mathbf{R})$ *with respect to* $t \in (0, T^*)$ *provided that* $\delta = O\left(\varepsilon^{\frac{(\ell+1)(6\ell-m-1)}{2(3\ell^2-m\ell-q\ell+3\ell-1)}}\right)$.

Due to the fact that $\left[\frac{3\ell^2+3\ell-1}{2\ell}, 3\ell\right] \subset \left(\frac{3\ell-1}{\ell}, \frac{3\ell^2+2\ell-1}{\ell}\right)$ $(\ell \geq 1)$ and $q > \frac{3\ell^2-q\ell+3\ell-1}{\ell}$ $(> m)$ for $q \in \left[\frac{3\ell^2+3\ell-1}{2\ell}, 3\ell\right]$, Corollary 3.1 holds for $m < q$ $\left(q \in \left[\frac{3\ell^2+3\ell-1}{2\ell}, 3\ell\right], \ell \geq 1\right)$.

§4. Proof of the Main Result

Due to the uniform boundedness of a sequence $\{u^\varepsilon\}$ in $L^\infty(0, T^*; L^q(\mathbf{R}))$ (Corollary 3.1) in the previous section, we can apply Theorem 2.1 as a convergence tool if it is obtained that a Young measure ν associated with a sequence $\{u^\varepsilon\}$ is an entropy m.-v. solution of the Cauchy problem (1.3), (1.4). To accomplish the objective, we show the proof of the main Theorem 1.1 by using several uniform estimates for the sequence $\{u^\varepsilon\}$ under the growth condition (I) for $m < q$ $\left(q \in \left[\frac{3\ell^2+3\ell-1}{2\ell}, 3\ell\right]\right)$ and the assumptions (1.5), (1.6) for the initial data u_0^ε.

Proof of Theorem 1.1. To apply the convergence Theorem 2.1, we will show that a Young measure ν associated with a sequence $\{u^\varepsilon\}$ is an entropy m.-v. solution. In other words, it is necessary to establish that a Young measure ν satisfies the entropy inequality (2.2) and the initial condition (2.3).

As first step, we consider for the entropy inequality (2.2). For any convex smooth function $\eta(u) : \mathbf{R} \to \mathbf{R}$ such that η' and η'' are uniformly bounded on $\mathbf{R}$, we consider the distribution

$$\Lambda^\varepsilon := \partial_t \eta(u^\varepsilon) + \partial_x \sigma(u^\varepsilon), \tag{4.1}$$

where the flux $\sigma : \mathbf{R} \to \mathbf{R}$ is defined by $\sigma'(u) = f'(u)\eta'(u)$ for $u \in \mathbf{R}$. Then Λ^ε converges to a nonpositive measure in $\mathcal{D}'(\mathbf{R} \times (0, T^*))$. In fact, we observe that Λ^ε is decomposed as follows:

$$\begin{aligned}
\Lambda^\varepsilon &= \eta(u^\varepsilon)_t + \sigma(u^\varepsilon)_x \\
&= \eta'(u^\varepsilon)u_t^\varepsilon + \sigma'(u^\varepsilon)u_x^\varepsilon \\
&= \eta'(u^\varepsilon)\left\{\varepsilon\left((u_x^\varepsilon)^{2\ell-1}\right)_x - \delta\left((u_x^\varepsilon)^{2\ell-1}\right)_{xx} - f(u^\varepsilon)_x\right\} + f'(u^\varepsilon)\eta'(u^\varepsilon)u_x^\varepsilon \\
&= \varepsilon\eta'(u^\varepsilon)\left((u_x^\varepsilon)^{2\ell-1}\right)_x - \delta\eta'(u^\varepsilon)\left((u_x^\varepsilon)^{2\ell-1}\right)_{xx} \\
&= \varepsilon\left\{\left(\eta'(u^\varepsilon)(u_x^\varepsilon)^{2\ell-1}\right)_x - \eta''(u^\varepsilon)(u_x^\varepsilon)^{2\ell}\right\} \\
&\quad -\delta\left\{\left(\eta'(u^\varepsilon)\left((u_x^\varepsilon)^{2\ell-1}\right)_x\right)_x - \eta''(u^\varepsilon)\left((u_x^\varepsilon)^{2\ell-1}\right)_x u_x^\varepsilon\right\}
\end{aligned}$$

$$\begin{aligned}
&= \varepsilon\left(\eta'(u^\varepsilon)(u_x^\varepsilon)^{2\ell-1}\right)_x - \varepsilon\eta''(u^\varepsilon)(u_x^\varepsilon)^{2\ell} \\
&\quad -\delta\left(\eta'(u^\varepsilon)(u_x^\varepsilon)^{2\ell-1}\right)_{xx} + \delta\left(\eta''(u^\varepsilon)(u_x^\varepsilon)^{2\ell}\right)_x + \delta\eta''(u^\varepsilon)\left((u_x^\varepsilon)^{2\ell-1}\right)_x u_x^\varepsilon \\
&=: \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4 + \Lambda_5.
\end{aligned}$$

The estimate of each terms in Λ^ε hold for all smooth function $\theta \in C_0^\infty(\mathbf{R} \times (0, T^*))$ $(\theta \geq 0)$ below. Throughout a process of calculation, we omit the upper-index ε for simplicity similarly to Section 3.

To begin with, consider the term Λ_1. By an estimate (3.1) and Hölder's inequality, we have

$$(4.2)\qquad \begin{aligned}
|\langle \Lambda_1, \theta\rangle| &= \left|\varepsilon \int_0^{T^*}\int_{\mathbf{R}} \eta'(u) u_x^{2\ell-1}\theta_x dxdt\right| \\
&\leq C\varepsilon\left(\iint_{\operatorname{supp}\theta} |u_x|^{p_1(2\ell-1)} dxdt\right)^{\frac{1}{p_1}} ||\theta_x||_{L^{p_1'}(\mathbf{R}\times(0,T^*))} \\
&\leq C\varepsilon\left(\iint_{\operatorname{supp}\theta} |u_x|^{2\ell} dxdt\right)^{\frac{2\ell-1}{2\ell}} ||\theta_x||_{L^{p_1'}(\mathbf{R}\times(0,T^*))} \\
&\leq C\varepsilon\cdot\varepsilon^{-\frac{2\ell-1}{2\ell}} \\
&\leq C\varepsilon^{\frac{1}{2\ell}} \to 0 \ (\varepsilon \to 0)
\end{aligned}$$

with some $C > 0$ where $\frac{1}{p_1} + \frac{1}{p_1'} = 1$ with $p_1(2\ell - 1) = 2\ell$. We denotes by $\operatorname{supp}\theta$ the support of θ in $\mathbf{R} \times (0, T^*)$.

Next the second term Λ_2 is nonpositive:

$$(4.3)\qquad \langle \Lambda_2, \theta\rangle = -\varepsilon \int_0^{T^*}\int_{\mathbf{R}} \eta''(u) u_x^{2\ell}\theta dxdt \leq 0.$$

Using an estimate (3.1) again, we estimate the term Λ_3:

$$(4.4)\qquad \begin{aligned}
|\langle \Lambda_3, \theta\rangle| &= \left|\delta \int_0^{T^*}\int_{\mathbf{R}} \eta'(u) u_x^{2\ell-1}\theta_{xx} dxdt\right| \\
&\leq C\delta\left(\iint_{\operatorname{supp}\theta} |u_x|^{p_2(2\ell-1)} dxdt\right)^{\frac{1}{p_2}} ||\theta_{xx}||_{L^{p_2'}(\mathbf{R}\times(0,T^*))} \\
&\leq C\delta\left(\iint_{\operatorname{supp}\theta} |u_x|^{2\ell} dxdt\right)^{\frac{2\ell-1}{2\ell}} ||\theta_{xx}||_{L^{p_2'}(\mathbf{R}\times(0,T^*))} \\
&\leq C\varepsilon^{-\frac{2\ell-1}{2\ell}}\delta
\end{aligned}$$

with some $C > 0$ where $\frac{1}{p_2} + \frac{1}{p_2'} = 1$ with $p_2(2\ell - 1) = 2\ell$. In this case, when $\delta = o\left(\varepsilon^{\frac{2\ell-1}{2\ell}}\right)$, $\Lambda_3 \to 0$ in $\mathcal{D}'(\mathbf{R} \times (0, T^*))$ as $\varepsilon \to 0$.

On the other hand, by applying an estimate (3.1) to Λ_4, we find:

$$
\begin{aligned}
|\langle\Lambda_4,\theta\rangle| &= \left|\delta\int_0^{T^*}\int_{\mathbf{R}}\eta''(u)u_x^{2\ell}\theta_x dxdt\right| \\
&\le C\delta||\theta_{xx}||_{L^\infty(\mathbf{R}\times(0,T^*))}\int_0^{T^*}\int_{\mathbf{R}}|u_x|^{2\ell}dxdt \\
&\le C\varepsilon^{-1}\delta
\end{aligned}
\tag{4.5}
$$

with some $C>0$ which includes $\delta=o(\varepsilon)$.

To deal with the last term Λ_5, we divide into $\ell>1$ and $\ell=1$. In the case that $\ell>1$, remarking that $\left(u_x^{2\ell-1}\right)_x=u_x^{2\ell-2}u_{xx}$, we combine the estimates (3.1) and (3.5) as follows:

$$
\begin{aligned}
|\langle\Lambda_5,\theta\rangle| &= \left|\delta\int_0^{T^*}\int_{\mathbf{R}}\eta''(u)\left(u_x^{2\ell-1}\right)_x u_x\theta dxdt\right| \\
&\le C\delta||\theta||_{L^{p_3}(\mathbf{R}\times(0,T^*))} \\
&\quad\times\left(\iint_{\operatorname{supp}\theta}|u_x^{2\ell-2}u_{xx}|^2dxdt\right)^{\frac{1}{2}}\left(\iint_{\operatorname{supp}\theta}|u_x|^{2\ell}dxdt\right)^{\frac{1}{2\ell}} \\
&\le C\delta\cdot\varepsilon^{-\frac{1}{2}}\delta^{-\frac{3\ell-1}{6\ell-m-1}}\cdot\varepsilon^{-\frac{1}{2\ell}} \\
&\le C\varepsilon^{-\frac{\ell+1}{2\ell}}\delta^{\frac{3\ell-m}{6\ell-m-1}}
\end{aligned}
\tag{4.6}
$$

with some $C>0$ where $\frac{1}{2}+\frac{1}{2\ell}+\frac{1}{p_3}=1$ hence $p_3=\frac{2\ell}{\ell-1}$ $(\ell>1)$. In the case that $\ell=1$, using the estimates (3.1)' and (3.5)', it follows that

$$
\begin{aligned}
|\langle\Lambda_5,\theta\rangle| &= \left|\delta\int_0^{T^*}\int_{\mathbf{R}}\eta''(u)u_xu_{xx}\theta dxdt\right| \\
&\le C\delta||\theta||_{L^\infty(\mathbf{R}\times(0,T^*))} \\
&\quad\times\left(\iint_{\operatorname{supp}\theta}|u_x|^2dxdt\right)^{\frac{1}{2}}\left(\iint_{\operatorname{supp}\theta}|u_{xx}|^2dxdt\right)^{\frac{1}{2}} \\
&\le C\delta\cdot\varepsilon^{-\frac{1}{2}}\cdot\varepsilon^{-\frac{1}{2}}\delta^{-\frac{2}{5-m}} \\
&\le C\varepsilon^{-1}\delta^{\frac{3-m}{5-m}}
\end{aligned}
\tag{4.7}
$$

with some $C>0$. Now paying attention to an exponent of δ which are yielded from inequalities (4.6) and (4.7), it holds that $\frac{3\ell-m}{6\ell-m-1}>0$ for $m<q$ $\left(q\in\left[\frac{3\ell^2+3\ell-1}{2\ell},3\ell\right],\ \ell\ge 1\right)$. Hence inequalities (4.6) and (4.7) imply the condition $\delta=o\left(\varepsilon^{\frac{(\ell+1)(6\ell-m-1)}{2\ell(3\ell-m)}}\right)$.

By the estimates (4.2)–(4.7), if $\delta = o\left(\varepsilon^{\frac{(\ell+1)(6\ell-m-1)}{2\ell(3\ell-m)}}\right)$, then Λ^ε converges to a nonpositive measure in $\mathcal{D}'(\mathbf{R}\times(0,T^*))$ as $\varepsilon \to 0$. In particular, one can verify that $\delta = O\left(\varepsilon^{\frac{(\ell+1)(6\ell-m-1)}{2(3\ell^2-m\ell-q\ell+3\ell-1)}}\right)$ implies $\delta = o\left(\varepsilon^{\frac{(\ell+1)(6\ell-m-1)}{2\ell(3\ell-m)}}\right)$. Combining the convergence property of Λ^ε and that $\eta(u) \to \langle\nu,\eta\rangle$, $\sigma(u) \to \langle\nu,\sigma\rangle$ in $\mathcal{D}'(\mathbf{R}\times(0,T^*))$ as $\varepsilon \to 0$ which are obtained by owing to Proposition 2.1, it follows that

$$\partial_t\langle\nu_{(x,t)}(\lambda),\, \eta(\lambda)\rangle + \partial_x\langle\nu_{(x,t)}(\lambda),\, \sigma(\lambda)\rangle \le 0 \tag{4.8}$$

for any convex entropy pairs such that η' and η'' are uniformly bounded on $\mathbf{R}$. Therefore, by the regularization of $|u-k|$ (for all $k \in \mathbf{R}$), the inequality (2.2) follows.

Next, in the rest of this paper, we give a proof that the initial condition (2.3) is satisfied by the argument due to DiPerna [8] and Szepessy [27].

Let g be a function $g(\lambda) = |\lambda|^r$ for $r \in (1,2)$ and $\{\phi_n\} \subseteq C_0^\infty(\mathbf{R})$ be a sequence of test functions such that

$$\lim_{n\to\infty} \phi_n = g'(u_0) \quad \text{in} \quad L^{r'}(\mathbf{R})$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. Furthermore we set

$$G(\lambda,\lambda_0) := g(\lambda) - g(\lambda_0) - g'(\lambda_0)(\lambda-\lambda_0).$$

Following a detailed discussion in [9, 22], by the Cauchy-Schwarz inequality and the Jensen inequality, we can easily check

$$\begin{aligned}
&\left(\frac{1}{T}\int_0^T\int_K \langle\nu_{(x,t)}(\lambda), |\lambda-u_0(x)|\rangle dxdt\right)^2 \\
&\le \frac{C_K}{T}\int_0^T\int_K \langle\nu_{(x,t)}(\lambda), G(\lambda,u_0(x))\rangle dxdt \\
&\le \frac{C_K}{T}\int_0^T\int_{\mathbf{R}} \langle\nu_{(x,t)}(\lambda), u_0(x)-\lambda\rangle\phi_n dxdt \\
&\quad +C_K||u_0||_{L^r(\mathbf{R})}||g'(u_0)-\phi_n||_{L^{r'}(\mathbf{R})}
\end{aligned} \tag{4.9}$$

for any compact set $K \subseteq \mathbf{R}$. From the definition of ϕ_n, it follows that

$$||g'(u_0)-\phi_n||_{L^{r'}(\mathbf{R})} \to 0 \text{ as } n \to \infty$$

which indicates that the second term in the right-hand side of the inequality (4.9) tends to zero as $n \to \infty$. Consequently, it is sufficient to show that the first term of the right-hand side of Eq. (4.9) tends to zero as the upper bound at $t = 0$ i.e.

$$\lim_{T\to 0^+} \frac{1}{T}\int_0^T \int_{\mathbf{R}} \langle \nu_{(x,t)}(\lambda), u_0(x) - \lambda\rangle \phi_n dxdt \leq 0 \tag{4.10}$$

so as to prove the initial condition (2.3). From the definition of the Young measure ν, it holds that

$$\begin{aligned}
&\frac{1}{T}\int_0^T \int_{\mathbf{R}} \langle \nu_{(x,t)}(\lambda), u_0(x) - \lambda\rangle \phi_n dxdt \\
&= \lim_{\varepsilon\to 0} \frac{1}{T}\int_0^T \int_{\mathbf{R}} (u_0(x) - u^\varepsilon(x,t))\phi_n dxdt \\
&= \lim_{\varepsilon\to 0} \frac{1}{T}\int_0^T \int_{\mathbf{R}} u_0(x)\phi_n dxdt - \lim_{\varepsilon\to 0} \frac{1}{T}\int_0^T \int_{\mathbf{R}} u^\varepsilon(x,t)\phi_n dxdt \\
&= \lim_{\varepsilon\to 0} \left(\int_{\mathbf{R}} (u_0(x) - u_0^\varepsilon(x))\phi_n dx + \int_{\mathbf{R}} u_0^\varepsilon(x)\phi_n dx\right) \\
&\quad - \lim_{\varepsilon\to 0} \frac{1}{T}\int_0^T \int_{\mathbf{R}} u^\varepsilon(x,t)\phi_n dxdt \\
&= \lim_{\varepsilon\to 0} \int_{\mathbf{R}} (u_0(x) - u_0^\varepsilon(x))\phi_n dx - \lim_{\varepsilon\to 0} \frac{1}{T}\int_0^T \int_{\mathbf{R}} (u^\varepsilon(x,t) - u_0^\varepsilon(x))\phi_n dxdt \\
&= -\lim_{\varepsilon\to 0} \frac{1}{T}\int_0^T \int_{\mathbf{R}} \left(\int_0^t \partial_s u^\varepsilon(x,s)ds\right)\phi_n(x)dxdt.
\end{aligned}$$

where we use an assumption (1.5) for the initial data. Here, by the growth condition (I) and the definition of ϕ_n, we remark that $|f(u)| \leq C(|u| + |u|^m)$ $(C > 0)$ and $\int_{\mathbf{R}} \phi_n dx < C_n$, and set

$$\Gamma^\varepsilon := \frac{1}{T}\int_0^T \int_{\mathbf{R}} \left(\int_0^t \partial_s u^\varepsilon(x,s)ds\right)\phi_n(x)dxdt.$$

Owing to the uniform boundedness of a sequence $\{u^\varepsilon\}$ in $L^\infty(0,T^*;L^2(\mathbf{R})\cap L^q(\mathbf{R}))$ for $q > m$ (an estimate (3.1), Corollary 3.1) and the same argument as the inequalities (4.2), (4.4), we can estimate Γ^ε as follows:

$$\begin{aligned}
&|\Gamma^\varepsilon| \\
&= \left|\frac{1}{T}\int_0^T\int_{\mathbf{R}}\left(\int_0^t \partial_s u^\varepsilon(x,s)ds\right)\phi_n(x)dxdt\right| \\
&= \left|\frac{1}{T}\int_0^T\int_{\mathbf{R}}\left(\int_0^t \left(-\partial_x f(u^\varepsilon) + \varepsilon\partial_x (u_x^\varepsilon)^{2\ell-1} - \delta\partial_x^2 (u_x^\varepsilon)^{2\ell-1}\right) ds\right)\phi_n(x)dxdt\right|
\end{aligned}$$

$$\begin{aligned}
&= \left| \frac{1}{T}\int_0^T\int_{\mathbf{R}}\int_0^t \big(f(u^\varepsilon)\partial_x\phi_n - \varepsilon(u_x^\varepsilon)^{2\ell-1}\partial_x\phi_n - \delta(u_x^\varepsilon)^{2\ell-1}\partial_x^2\phi_n\big)\, dsdxdt \right| \\
&\le \frac{C}{T}\int_0^T\int_{\mathbf{R}}\int_0^t (|u^\varepsilon| + |u^\varepsilon|^m|)\,|\partial_x\phi_n| dsdxdt + C\varepsilon^{\frac{1}{2\ell}} + C\varepsilon^{-\frac{2\ell-1}{2\ell}}\delta \\
&\le \frac{C}{T}\int_0^T dt \int_0^t ds \left(\int_{\mathbf{R}} |u^\varepsilon|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}} |\partial_x\phi_n|^2 dx\right)^{\frac{1}{2}} \\
&\quad + \frac{C}{T}\int_0^T dt \int_0^t ds \left(\int_{\mathbf{R}} |u^\varepsilon|^{m\tilde{q}} dx\right)^{\frac{1}{\tilde{q}}} \left(\int_{\mathbf{R}} |\partial_x\phi_n|^{\tilde{q}'} dx\right)^{\frac{1}{\tilde{q}'}} \\
&\quad + C\varepsilon^{\frac{1}{2\ell}} + C\varepsilon^{-\frac{2\ell-1}{2\ell}}\delta \\
&\le C_{n_1}T + \frac{C}{T}\cdot C_{n_2}\int_0^T dt\int_0^t ds \left(\int_{\mathbf{R}} |u^\varepsilon|^q dx\right)^{\frac{1}{q}} + C\varepsilon^{\frac{1}{2\ell}} + C\varepsilon^{-\frac{2\ell-1}{2\ell}}\delta \\
&\le C_nT + C\varepsilon^{\frac{1}{2\ell}} + C\varepsilon^{-\frac{2\ell-1}{2\ell}}\delta
\end{aligned}$$

with some $C > 0$ where $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$ with $m\tilde{q} = q$ $(> m)$. When $\varepsilon \to 0$ with $\delta = o(\varepsilon^{\frac{2\ell-1}{2\ell}})$, we obtain that $\limsup_{\varepsilon\to 0} |\Gamma^\varepsilon| \le C_nT$. Hence we arrive at

$$\frac{1}{T}\int_0^T \int_{\mathbf{R}} \langle \nu_{(x,t)}(\lambda), u_0(x) - \lambda\rangle \phi_n dxdt \le C_nT,$$

which implies the inequality (4.10), accordingly, we establish that the initial condition (2.3) is satisfied.

Consequently Young measure ν is an entropy m.-v. solution to Eqs. (1.3) and (1.4). Applying Theorem 2.1, the sequence $\{u^\varepsilon\}$ of solutions to Eqs. (1.1) and (1.2) converges to the unique entropy solution $u \in L^\infty(0,T^*;L^q(\mathbf{R}))$ to Eqs. (1.3) and (1.4) in $L^k(0,T^*;L^p(\mathbf{R}))$ ($\forall k < \infty$ and $\forall p < q$). This completes the proof of Theorem 1.1. □

References

[1] P. Baiti, P. G. LeFloch and B. Piccoli, Uniqueness of classical and nonclassical solutions for nonlinear hyperbolic systems, J. Differential Equations **172** (2001), no. 1, 59–82.

[2] N. Bedjaoui and P. G. LeFloch, Diffusive-dispersive traveling waves and kinetic relations. I. Nonconvex hyperbolic conservation laws, J. Differential Equations **178** (2002), no. 2, 574–607.

[3] J. L. Bona and M. E. Schonbek, Travelling-wave solutions to the Korteweg-de Vries-Burgers equation, Proc. Roy. Soc. Edinburgh Sect. A **101** (1985), no. 3-4, 207–226.

[4] J. L. Bona and R. Smith, The initial-value problem for the Korteweg-de Vries equation, Philos. Trans. Roy. Soc. London Ser. A **278** (1975), no. 1287, 555–601.

[5] G. Q. Chen and Y. G. Lu, A study of approaches to applying the theory of compensated compactness, Kexue Tongbao (Chinese) **33** (1988), no. 9, 641–644.

[6] G. Q. Chen and Y. G. Lu, Convergence of the approximate solutions to isentropic gas dynamics, Acta Math. Sci., (English Ed.) **10** (1990), no. 1, 39–45.
[7] G. M. Coclite and K. H. Karlsen, A Singular limit problem for conservation laws related to the Camassa-Holm shallow water equation, Comm. Partial Differential Equations **31** (2006), no. 8, 1253–1272.
[8] R. J. DiPerna, Measure-valued solutions to conservation laws, Arch. Rational Mech. Anal. **88** (1985), no. 3, 223–270.
[9] N. Fujino and M. Yamazaki, Hyperbolic conservation laws with nonlinear diffusion and nonlinear dispersion, J. Differential Equations **228** (2006), no. 1, 171–190.
[10] B. T. Hayes and P. G. LeFloch, Non-classical shocks and kinetic relations: scalar conservation laws, Arch. Rational Mech. Anal. **139** (1997), no. 1, 1–56.
[11] ———, Nonclassical shocks and kinetic relations: strictly hyperbolic systems, SIAM J. Math. Anal. **31** (2000), no. 5, 941–991 (electronic).
[12] ———, Nonclassical shocks and kinetic relations: finite difference schemes, SIAM J. Numer. Anal. **35** (1998), no. 6, 2169–2194 (electronic).
[13] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math. **3** (1950), 201–230.
[14] D. Jacobs, B. McKinney and M. Shearer, Travelling wave solutions of the modified Korteweg-de Vries-Burgers equation, J. Differential Equations **116** (1995), no. 2, 448–467.
[15] C. I. Kondo and P. G. Lefloch, Zero diffusion-dispersion limits for scalar conservation laws, SIAM J. Math. Anal. **33** (2002), no. 6, 1320–1329 (electronic).
[16] D. J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, Philosophical Magazine **39** (1895), 422–443.
[17] S. N. Kružkov, First order quasilinear equations in several independent variables, Mat. Sb. (N.S.) **81 (123)** (1970), 228-255; Math. USSR Sb. **10** (1970), 217–243.
[18] P. D. Lax, The zero dispersion limit, a deterministic analogue of turbulence, Comm. Pure Appl. Math. **44** (1991), no. 8-9, 1047–1056.
[19] P. D. Lax and C. D. Levermore, The small dispersion limit of the Korteweg-de Vries equation. II–III, Comm. Pure Appl. Math. **36** (1983), no. 3, 253–290, no, 5, 571–593, no. 6, 809–829.
[20] ———, The zero dispersion limit for the Korteweg-de Vries KdV equation, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), no. 8, 3602–3606.
[21] P. G. LeFloch, An introduction to nonclassical shocks of systems of conservation laws, in *An introduction to recent developments in theory and numerics for conservation laws (Freiburg/Littenweiler, 1997)*, 28–72, Springer, Berlin.
[22] P. G. LeFloch and R. Natalini, Conservation laws with vanishing nonlinear diffusion and dispersion, Nonlinear Anal. **36** (1999), no. 2, Ser. A: Theory Methods, 213–230.
[23] P. G. Lefloch and C. Rohde, High-order schemes, entropy inequalities, and nonclassical shocks, SIAM J. Numer. Anal. **37** (2000), no. 6, 2023–2060 (electronic).
[24] P. G. LeFloch and M. D. Thanh, Nonclassical Riemann solvers and kinetic relations. III. A nonconvex hyperbolic model for van der Waals fluids, Electron. J. Differential Equations **2000**, No. 72, 1–19. (electronic).
[25] Y. Lu, *Hyperbolic conservation laws and the compensated compactness method*, Chapman & Hall/CRC, Boca Raton, FL, 2003.
[26] M. E. Schonbek, Convergence of solutions to nonlinear dispersive equations, Comm. Partial Differential Equations **7** (1982), no. 8, 959–1000.
[27] A. Szepessy, An existence result for scalar conservation laws using measure valued solutions, Comm. Partial Differential Equations **14** (1989), no. 10, 1329–1350.