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Carleman Estimates for the Lamé System with Stress Boundary Condition

By

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Abstract

In this paper, for functions without compact supports, we established Carleman estimates for the two-dimensional non-stationary Lamé system with the stress boundary condition.

§1. Introduction and the Main Carleman Estimates

In this paper, for functions without compact supports, we establish Carleman estimates for the two-dimensional non-stationary Lamé system with stress boundary condition:

$$
\begin{aligned}
(1.1)\qquad & P(x,D)\mathbf{u} \equiv (P_1(x,D)\mathbf{u}, P_2(x,D)\mathbf{u})^T \\
& = \rho(\widetilde{x})\frac{\partial^2 \mathbf{u}}{\partial x_0^2} - \mu(\widetilde{x})\Delta\mathbf{u} - (\mu(\widetilde{x}) + \lambda(\widetilde{x}))\nabla_{\widetilde{x}}\mathrm{div}\,\mathbf{u} \\
& - (\mathrm{div}\,\mathbf{u})\nabla_{\widetilde{x}}\lambda(\widetilde{x}) - (\nabla_{\widetilde{x}}\mathbf{u} + (\nabla_{\widetilde{x}}\mathbf{u})^T)\nabla_{\widetilde{x}}\mu(\widetilde{x}) = \mathbf{f} \quad \text{in } Q = (0,T)\times\Omega,
\end{aligned}
$$

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$$\text{(1.2)}\qquad \begin{cases} \mathbb{B}(x,D)\mathbf{u} \equiv \left(\sum_{j=1}^{2} n_j\sigma_{j1}, \sum_{j=1}^{2} n_j\sigma_{j2}\right)^T = \mathbf{g} \quad \text{on } (0,T)\times\partial\Omega, \\ \mathbf{u}(T,\widetilde{x}) = \dfrac{\partial \mathbf{u}}{\partial x_0}(T,\widetilde{x}) = \mathbf{u}(0,\widetilde{x}) = \dfrac{\partial \mathbf{u}}{\partial x_0}(0,\widetilde{x}) = 0, \end{cases}$$

where $\mathbf{u} = (u_1, u_2)^T$, $\mathbf{f} = (f_1, f_2)^T$ are the vector functions, $\mathbf{u}^T$ denotes the transpose of the vector $\mathbf{u}$, Ω is a bounded domain in $\mathbb{R}^2$ with $\partial\Omega \in C^3, x = (x_0, \widetilde{x}), \widetilde{x} = (x_1, x_2)$ and $(n_1, n_2)^T$ is the unit outward normal vector to $\partial\Omega$,

$$\sigma_{jk} = \lambda(\widetilde{x})\delta_{jk}\mathrm{div}\,\mathbf{u} + \mu(\widetilde{x})\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right).$$

The boundary condition in (1.2) describes the surface stress. In (1.1), the coefficients $\rho, \mu, \lambda \in C^2(\overline{\Omega})$ are assumed to satisfy

(1.3)

$$\rho(\widetilde{x}) > 0, \quad \mu(\widetilde{x}) > 0, \quad \mu(\widetilde{x}) + \lambda(\widetilde{x}) > 0, \quad \forall \widetilde{x} \in \overline{\Omega}, \quad \lambda(\tilde{x}) \neq 0, \quad \forall \widetilde{x} \in \partial\Omega.$$

Physically λ and μ are the Lamé coefficients of the isotropic medium occupying the domain Ω, and ρ is the density. A Carleman estimate is an inequality for solutions to a partial differential equation with weighted L^2-norm and is a strong tool for proving the uniqueness for Cauchy problems or the unique continuation of partial differential equations with non-analytic coefficients. Moreover Carleman estimates have been applied successfully to estimation of energy of solutions (e.g., [KK]) and to inverse problems of determining coefficients by boundary observations (e.g., [BuK], [K] as initiating works). As a pioneering work, we refer to Carleman [Ca] which derived a Carleman estimate and used it to prove the uniqueness in the Cauchy problem for a two-dimensional elliptic equation. Since [Ca], the theory of Carleman estimates has been studied extensively. We refer to Hörmander [Hö] in the case where the symbol of a partial differential equation is isotropic and functions under consideration have compact supports (that is, they and their derivatives of suitable orders vanish on the boundary of a domain). Later Carleman estimates for functions with compact supports have been obtained for partial differential operators with anisotropic symbols by Isakov ([Is]). For general results in the case of functions without compact supports, see [Ta] and for hyperbolic equations, see [Im]. Our main task of establishing a Carleman estimate for (1.1)–(1.2) is difficult twofold: Firstly, in (1.1) the highest order derivatives are coupled and secondly (1.2) contains a boundary condition of the non-Dirichlet type. *First difficulty*: As for Carleman estimates for strongly coupled systems, there are not many works. In fact, all the above-mentioned works discuss single partial differential equations. As long as the unique continuation is concerned, to our best knowledge, the most general result for such systems of partial differential equations is Calderón's uniqueness theorem (see e.g., [E], [Zui]). However, the non-stationary Lamé system does not satisfy all the conditions of that theorem. More precisely, the eigenvalues of the matrix associated with the principal symbol of the Lamé system change the multiplicities and at some points of cotangent bundle, they are not smooth, which break the assumptions in the known Caledrón's uniqueness theorem. On the other hand, for proving the unique continuation, the Lamé system can be decoupled (modulo low order terms) for example by introducing a new function div $\mathbf{u}$ and applying to the new system the technique developed for the scalar partial differential equations (see e.g., [EINT]). This method may produce a Carleman estimate for the Lamé system, but the displacement function $\mathbf{u}$ is required to have a compact support, so that the method does not work for (1.1) and (1.2) if $\mathbf{u}$ does not have a compact support. In [IY1] and [IY3], we have established Carleman estimates for the Dirichlet case where the stress boundary condition in (1.2) is replaced by $\mathbf{u} = \mathbf{g}$ on $(0, T) \times \partial\Omega$. It is known that there are two types of the interior waves for the Lamé system: the longitudinal wave with the velocity $\sqrt{\frac{\lambda+2\mu}{\rho}}$ and the transverse wave with the velocity $\sqrt{\frac{\mu}{\rho}}$. Thus a weight function in the Carleman estimate is assumed to be pseudoconvex with respect to the two symbols (see Condition 1.1). *Second difficulty*: The essential difference between the stress boundary condition and the Dirichlet boundary condition which was studied by the authors in [IY3] and [IY4], is that the stress boundary condition requires us to deal with the new phenomena - the Rayleigh boundary waves. In order to treat the boundary waves, we have to additionally assume that a weight function is strictly pseudoconvex with respect to the pseudodifferential operator whose principal symbol is given by the Lopatinskii determinant (see Condition 1.2). Furthermore, from the practical point of view (e.g., in view of the seismology), the stress boundary condition is very important and well describes the reality such as the surface wave, so that the associated inverse problems and energy estimation are highly requested to be studied. Under Conditions 1.1 and 1.2, we state our main results - the Carleman estimates (Theorem 1.1 and Corollaries 1.1 and 1.2). Among applications of the Carleman estimates obtained in this paper, we mention the sharp unique continuation/conditional stability results for the Cauchy problem for (1.1), the exact controllability of the Lamé system with stress boundary conditions by means of controls in a subdomain or on a subboundary, and an inverse problem of determining the Lamé coefficients and the density by measurements in a subdomain. For the inverse problems, the method in [BuK] and [K] can be validated by means of our Carleman estimates. Thanks to our Carleman estimate for functions without compact supports, we can establish the exact controllability and the stability over the whole domain Ω in the inverse problem with controls or measurements in a subdomain satisfying a related geometric optics condition (e.g., [BLR]). Those are longstanding open problems in spite of the physical significance. However we will postpone such applications to our forthcoming papers and we exclusively consider Carleman estimates in the two-dimensional spatial case. The higher dimensional case is more difficult. Really, as is shown in [Y], in the case where the spatial dimension is greater than two, the Lopatinskii determinant equals zero at some point. Among related papers, we refer to Bellassoued [B1]–[B3], Dehman and Robbiano [DR], and Imanuvilov and Yamamoto [IY2], where Carleman estimates for the stationary Lamé system were obtained. Also see Weck [W] for the unique continuation for the stationary Lamé system.

Throughout this paper, we use:

Notations. $i=\sqrt{-1}$, $\overline{z}$: the complex conjugate of $z\in\mathbb{C}$, $\vec{e}_1=(1,0)$, $\vec{e}_2=(0,1)$, $\vec{n}=(n_1,n_2)$, $x=(x_0,x_1,x_2)=(x_0,\widetilde{x})$, $\widetilde{x}=(x_1,x_2)$, $y=(y_0,y_1,y_2)$, $y'=(y_0,y_1)$, $\xi=(\xi_0,\xi_1,\xi_2)$, $\xi'=(\xi_0,\xi_1)$, $\partial_{y_j}\phi=\phi_{y_j}=\frac{\partial\phi}{\partial y_j}$, $\partial_{x_j}\phi=\phi_{x_j}=\frac{\partial\phi}{\partial x_j}$, $\phi_{x_jx_k}=\partial^2_{x_jx_k}\phi=\partial_{x_j}\partial_{x_k}\phi$, $\nabla=(\partial_{x_0},\partial_{x_1},\partial_{x_2})$ or $\nabla=(\partial_{y_0},\partial_{y_1},\partial_{y_2})$ if there is no fear of confusion (Otherwise we will add the subscript x or y). $\nabla_{\widetilde{x}}=(\partial_{x_1},\partial_{x_2})$, $\operatorname{div}\mathbf{u}=\partial_{x_1}u_1+\partial_{x_2}u_2$ for $\mathbf{u}=(u_1,u_2)^T$, $\mathbf{D}_{y_j}=\frac{1}{i}\frac{\partial}{\partial y_j}+is\partial_{y_j}\phi$, $\mathbf{D}'=(\mathbf{D}_{y_0},\mathbf{D}_{y_1})$, $\mathbf{D}=(\mathbf{D}_{y_0},\mathbf{D}_{y_1},\mathbf{D}_{y_2})$, $\nabla_{y'}=(\partial_{y_0},\partial_{y_1})$, $D'=(D_{y_0},D_{y_1})$, $D_{y_j}=\frac{1}{i}\frac{\partial}{\partial y_j}$, $\alpha=(\alpha_0,\alpha_1,\alpha_2)$, $\alpha_j\in\mathbb{N}_+\cup\{0\}$, $\partial_x^\alpha=\partial_{x_0}^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$, $\zeta=(s,\xi_0,\xi_1)$, S^2- the two dimensional sphere: $S^2=\{\zeta;|\zeta|=1\}$. For a domain Q in the $x-$ space, $H^{m,s}(Q)$ is the Sobolev space of scalar-valued functions equipped with the norm

$$\|u\|_{H^{m,s}(Q)}=\left(\sum_{|\alpha|\le m}s^{2m-2|\alpha|}\|\partial_x^\alpha u\|^2_{L^2(Q)}\right)^{\frac{1}{2}},$$

$\mathbf{H}^{m,s}(Q)=H^{m,s}(Q)\times\cdots\times H^{m,s}(Q)$ is the corresponding space of vector-valued functions $\mathbf{u}$. Also we use the space

$$W^m_q(Q)=\{u;D^\alpha u\in L^q(Q),|\alpha|\le m\},\quad \|u\|_{W^m_q(Q)}=\sum_{|\alpha|\le m}\|D^\alpha u\|_{L^q(Q)}.$$

For a domain Ω in the $\widetilde{x}$-space, we will similarly define the Sobolev spaces $H^{1,s}(\Omega)$ and $\mathbf{H}^{1,s}(\Omega)$. Let $[A,B]=AB-BA$, and let $\epsilon(\delta)$ be a nonnegative function such that $\epsilon(\delta)\to+0$ as $\delta\to+0$. By $\mathcal{O}(\delta_1)$, we denote the conic neighbourhood of the point ζ^* with $|\zeta^*| = 1$: $\mathcal{O}(\delta_1) = \left\{\zeta; \left|\frac{\zeta}{|\zeta|} - \zeta^*\right| \le \delta_1\right\}$, $B_\delta(y^*) = \{y; |y - y^*| < \delta\}$ is the ball centered at y^* with radius δ. $\mathcal{L}(X_1, X_2)$ is the space of linear continuous operators from a normed space X_1 to a normed space X_2, E_k is the $k \times k$ unit matrix.

Our main purpose is to establish Carleman estimates for system (1.1)–(1.2) for $\mathbf{u}$ having non-compact supports. Let $\omega \subset \Omega$ be an arbitrarily fixed open set which is not necessarily connected. Denote by $\vec{n}$ and $\vec{t}$, the outward unit normal vector and the unit counterclockwise oriented tangential vector on $\partial\Omega$, and we set $\frac{\partial u}{\partial \vec{n}} = \nabla_{\widetilde{x}} u \cdot \vec{n}$ and $\frac{\partial u}{\partial \vec{t}} = \nabla_{\widetilde{x}} u \cdot \vec{t}$. By Q_ω we denote the cylindrical domain $Q_\omega = (0, T) \times \omega$. We set

$$p_1(x, \xi) = \rho(\widetilde{x})\xi_0^2 - \mu(\widetilde{x})(\xi_1^2 + \xi_2^2), \quad p_2(x, \xi) = \rho(\widetilde{x})\xi_0^2 - (\lambda(\widetilde{x}) + 2\mu(\widetilde{x}))(\xi_1^2 + \xi_2^2).$$

For arbitrary smooth functions $\phi(x, \xi)$ and $\psi(x, \xi)$, we define the Poisson bracket by $\{\phi, \psi\} = \sum_{j=0}^2 \left(\frac{\partial \phi}{\partial \xi_j}\frac{\partial \psi}{\partial x_j} - \frac{\partial \phi}{\partial x_j}\frac{\partial \psi}{\partial \xi_j}\right)$. We assume that the coefficients μ, λ, ρ and Ω, ω satisfy the following conditions:

Condition 1.1. *There exists a function* $\psi \in C^2(\overline{Q})$ *such that* $|\nabla_x \psi(x)| \ne 0$ *for* $x \in \overline{Q \setminus Q_\omega}$, *and* (i), (ii) *and* (1.6) *hold*:
(i)

$$\{p_k, \{p_k, \psi\}\}(x, \xi) > 0, \quad \forall k \in \{1, 2\} \tag{1.4}$$

if $\xi \in \mathbb{R}^3 \setminus \{0\}$ *and* $x \in \overline{Q \setminus Q_\omega}$ *satisfy* $p_k(x, \xi) = \langle \nabla_\xi p_k, \nabla_x \psi \rangle = 0$.
(ii)

$$\frac{1}{2is}\{p_k(x, \xi - is\nabla_x\psi(x)), p_k(x, \xi + is\nabla_x\psi(x))\} > 0, \quad \forall k \in \{1, 2\} \tag{1.5}$$

if $\xi \in \mathbb{R}^3 \setminus \{0\}$, $s > 0$ *and* $x \in \overline{Q \setminus Q_\omega}$ *satisfy* $p_k(x, \xi + is\nabla_x\psi(x)) = \langle \nabla_\xi p_k(x, \xi + is\nabla_x\psi(x)), \nabla_x\psi \rangle = 0$.

On the lateral boundary we assume

$$\begin{cases} \sqrt{\rho}|\psi_{x_0}| < \frac{\mu}{\sqrt{\lambda+2\mu}}\left|\frac{\partial \psi}{\partial \vec{t}}\right| + \frac{\sqrt{\mu}\sqrt{\lambda+\mu}}{\sqrt{\lambda+2\mu}}\left|\frac{\partial \psi}{\partial \vec{n}}\right|, & \forall x \in [0, T] \times (\overline{\partial\Omega \setminus \partial\omega}) \\ p_1(x, \nabla\psi) < 0, \quad \forall x \in [0, T] \times (\overline{\partial\Omega \setminus \partial\omega}), \\ \frac{\partial \psi}{\partial \vec{n}} < 0, \quad \frac{\partial \psi}{\partial \vec{t}} \ne 0 \quad \text{on } [0, T] \times (\overline{\partial\Omega \setminus \partial\omega}). \end{cases} \tag{1.6}$$

Let ψ satisfy Condition 1.1. We introduce the function $\phi(x)$ by

$$\phi(x) = e^{\tau\psi(x)} \quad \tau > 1, \tag{1.7}$$

where the parameter τ will be fixed below. In order to deal with surface waves, we additionally need Condition 1.2 on the function ψ. We formulate that assumptions below as (1.23), and for the statement, we need to introduce some boundary differential operators by means of a new local coordinate. For an arbitrarily fixed point $(x_1^0, x_2^0) \in \partial\Omega$, we set $\widehat{x}_1 = x_1 - x_1^0$ and $\widehat{x}_2 = x_2 - x_2^0$. We consider (1.1) and (1.2) in the new coordinates $(\widehat{x_1}, \widehat{x_2})$. Since (1.1) and (1.2) are invariant with respect to the translation by the constant vector (x_1^0, x_2^0), we use the same notations x_1, x_2 instead of $\widehat{x_1}$, $\widehat{x_2}$. Therefore we may assume that $(0,0) \in \partial\Omega$ and that locally near $(0,0)$, the boundary $\partial\Omega$ is given by an equation $x_2 - \ell(x_1) = 0$, where $\ell = \ell(x_1)$ is a C^3-function. Moreover, since the function $\widetilde{\mathbf{u}} = \mathcal{O}\mathbf{u}(x_0, \mathcal{O}^{-1}\widetilde{x})$ satisfies system (1.1) and (1.2) with $\widetilde{\mathbf{f}} = \mathcal{O}\mathbf{f}(x_0, \mathcal{O}^{-1}\widetilde{x})$ for any orthogonal matrix $\mathcal{O}$, we may assume that

$$\ell'(0) \equiv \frac{d\ell}{dx_1}(0) = 0.$$

We make the change of variables $y = (y_0, y_1, y_2) = Y(x) \equiv (x_0, x_1, x_2 - \ell(x_1))$. Then we reduce equations (1.1) to

(1.8)
$$P_1(y,D)\mathbf{u} \equiv \rho\frac{\partial^2 u_1}{\partial y_0^2} - \mu\left\{\frac{\partial^2 u_1}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 u_1}{\partial y_1 \partial y_2} + (1 + |\ell'(y_1)|^2)\frac{\partial^2 u_1}{\partial y_2^2}\right\}$$
$$+\mu\ell''(y_1)\frac{\partial u_1}{\partial y_2} - (\lambda+\mu)\frac{\partial}{\partial y_1}\left(\operatorname{div}\mathbf{u} - \frac{\partial u_1}{\partial y_2}\ell'\right) + (\lambda+\mu)\frac{\partial}{\partial y_2}\left(\operatorname{div}\mathbf{u} - \frac{\partial u_1}{\partial y_2}\ell'\right)\ell'$$
$$+\widetilde{K}_1(y,D)\mathbf{u} = f_1 \qquad \text{in } \mathcal{G},$$

(1.9)
$$P_2(y,D)\mathbf{u} \equiv \rho\frac{\partial^2 u_2}{\partial y_0^2} - \mu\left\{\frac{\partial^2 u_2}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 u_2}{\partial y_1 \partial y_2} + (1 + |\ell'(y_1)|^2)\frac{\partial^2 u_2}{\partial y_2^2}\right\}$$
$$+\mu\ell''(y_1)\frac{\partial u_2}{\partial y_2} - (\lambda+\mu)\frac{\partial}{\partial y_2}\left(\operatorname{div}\mathbf{u} - \frac{\partial u_1}{\partial y_2}\ell'\right) + \widetilde{K}_2(y,D)\mathbf{u} = f_2 \qquad \text{in } \mathcal{G},$$

where we set

$$\mathcal{G} = \{y;\, y_2 \ge 0, y \in Y((0,T) \times B_\varepsilon(0,0))\}$$

with some $\varepsilon > 0$, and we keep the same notations P_1, P_2, $\mathbf{u}, \mathbf{f}$ after the change of variables, and $\widetilde{K}_j(y,D)$ are first order differential operators with C^1-coefficients. We set $P(y,D) = (P_1(y,D), P_2(y,D))$. In the new coordinates, the stress boundary condition (1.2) has the form

(1.10)
$$n_1(\widetilde{x})\left\{\lambda(\widetilde{x})\left(\frac{\partial u_1}{\partial y_1} + \frac{\partial u_1}{\partial y_2}(-\ell') + \frac{\partial u_2}{\partial y_2}\right) + 2\mu(\widetilde{x})\left(\frac{\partial u_1}{\partial y_1} + \frac{\partial u_1}{\partial y_2}(-\ell')\right)\right\}$$
$$+n_2(\widetilde{x})\mu(\widetilde{x})\left(\frac{\partial u_2}{\partial y_1} + \frac{\partial u_2}{\partial y_2}(-\ell') + \frac{\partial u_1}{\partial y_2}\right) = g_1,$$

$$(1.11)\qquad \begin{aligned} &n_1(\widetilde{x})\mu(\widetilde{x})\left\{\frac{\partial u_1}{\partial y_2}+\frac{\partial u_2}{\partial y_1}+\frac{\partial u_2}{\partial y_2}(-\ell')\right\}\\ &+n_2(\widetilde{x})\left\{\lambda\left(\frac{\partial u_1}{\partial y_1}+\frac{\partial u_1}{\partial y_2}(-\ell')+\frac{\partial u_2}{\partial y_2}\right)+2\mu(\widetilde{x})\frac{\partial u_2}{\partial y_2}\right\}=g_2. \end{aligned}$$

Here we use the same notations n_1, n_2 after the change of the variables. We can solve system (1.10) and (1.11) with respect to $\left(\frac{\partial u_1}{\partial y_2}, \frac{\partial u_2}{\partial y_2}\right)$ in the form:

(1.12)

$$\begin{pmatrix} \frac{\partial u_1}{\partial y_2} \\ \frac{\partial u_2}{\partial y_2} \end{pmatrix} = A(y_1)\begin{pmatrix} \frac{\partial u_1}{\partial y_1} \\ \frac{\partial u_2}{\partial y_1} \end{pmatrix} + \widetilde{A}(y_1)\mathbf{g}, \quad A(0) = \begin{pmatrix} 0 & -1 \\ -\frac{\lambda}{\lambda+2\mu}(0) & 0 \end{pmatrix}, \quad y \in \partial\mathcal{G},$$

and $\widetilde{A}(y_1)$ is a C^2 matrix-valued function. By A_1 and $\widetilde{A_1}$, we denote the first rows of the matrices A and $\widetilde{A}$ respectively, and the second by A_2 and $\widetilde{A_2}$: $A_j = (a_{j1}, a_{j2})$ and $\widetilde{A_j} = (\widetilde{a_{j1}}, \widetilde{a_{j2}})$, $j = 1, 2$.

System (1.1) can be decoupled (up to lower order terms) if we consider as a new unknown functions $\operatorname{rot}\mathbf{u}$ and $\operatorname{div}\mathbf{u}$. The great advantage of dealing with $\operatorname{rot}\mathbf{u}$, $\operatorname{div}\mathbf{u}$ instead of $\mathbf{u}$ is that the divergence and the rotation solve the scalar second order wave equations for which the theory of Carleman estimates- the main machinery used in this paper- is well developed.

Below we need a formulae for $\operatorname{rot}\mathbf{u}$ and $\operatorname{div}\mathbf{u}$ in new coordinates. After the change of variables, the functions $z_1 \equiv \operatorname{rot}\mathbf{u} = \partial_{x_1}u_2 - \partial_{x_2}u_1$ and $z_2 \equiv \operatorname{div}\mathbf{u}$ have the form

$$z_1(y) = \frac{\partial u_2}{\partial y_1} - \frac{\partial u_2}{\partial y_2}\ell'(y_1) - \frac{\partial u_1}{\partial y_2}, \quad z_2(y) = \frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2} - \frac{\partial u_1}{\partial y_2}\ell'(y_1).$$

Using (1.12), we can transform these functions on the boundary as follows:

$$(1.13)\qquad \begin{aligned} &(\operatorname{rot}\mathbf{u})(y) = z_1(y)\\ &= \frac{\partial u_2}{\partial y_1} - \ell'(y_1)A_2(y_1)\frac{\partial \mathbf{u}}{\partial y_1} - A_1(y_1)\frac{\partial \mathbf{u}}{\partial y_1} - \ell'(y_1)\widetilde{A}_2(y_1)\mathbf{g} - \widetilde{A}_1(y_1)\mathbf{g}\\ &\equiv b_{11}(y_1, D')u_1 + b_{12}(y_1, D')u_2 + \widetilde{C}_1(y_1)\mathbf{g}, \quad y \in \partial\mathcal{G}, \end{aligned}$$

where

$$(1.14)\qquad \begin{aligned} b_{11}(y_1, \xi) &= i(-\ell'(y_1)a_{21}(y_1) - a_{11}(y_1))\xi_1,\\ b_{12}(y_1, \xi) &= i(1 - a_{22}(y_1)\ell'(y_1) - a_{12}(y_1))\xi_1. \end{aligned}$$

For the function $z_2(y)$, we have

$$(1.15)\qquad \begin{aligned} &(\operatorname{div}\mathbf{u})(y) = z_2(y) = \frac{\partial u_1}{\partial y_1} + A_2(y_1)\frac{\partial \mathbf{u}}{\partial y_1} - \ell'(y_1)A_1(y_1)\frac{\partial \mathbf{u}}{\partial y_1}\\ &+ \widetilde{A}_2(y_1)\mathbf{g} - \widetilde{A}_1(y_1)\mathbf{g}\ell'(y_1)\\ &\equiv b_{21}(y_1, D')u_1 + b_{22}(y_1, D')u_2 + \widetilde{C}_2(y_1)\mathbf{g}, \quad y \in \partial\mathcal{G}, \end{aligned}$$

where

$$b_{21}(y_1,\xi)=i(\xi_1+a_{21}(y_1)\xi_1-\ell'(y_1)a_{11}(y_1)\xi_1),\qquad (1.16)$$
$$b_{22}(y_1,\xi)=i(a_{22}(y_1)\xi_1-a_{12}(y_1)\xi_1\ell'(y_1)),$$

and $\widetilde{C}_j$ are C^2 matrix-valued functions. Denote

(1.17)
$$b_1(y_1,D')=(b_{11}(y_1,D'),b_{12}(y_1,D')),$$
$$b_2(y_1,D')=(b_{21}(y_1,D'),b_{22}(y_1,D')),$$
$$p_\beta(y,s,\xi_0,\xi_1,\xi_2)=-\rho(\xi_0+is\partial_{y_0}\phi)^2$$
$$+\beta[(\xi_1+is\partial_{y_1}\phi)^2-2\ell'(\xi_1+is\partial_{y_1}\phi)(\xi_2+is\partial_{y_2}\phi)+(\xi_2+is\partial_{y_2}\phi)^2|G|],$$

where $|G|=1+(\ell'(y_1))^2$, $\beta\in\{\mu,\lambda+2\mu\}$ and s is a positive parameter. The roots of polynomial p_β with respect to the variable ξ_2 are

$$\Gamma^{\pm}_\beta(y,s,\xi_0,\xi_1)=-is\partial_{y_2}\phi+\alpha^{\pm}_\beta(y,s,\xi_0,\xi_1),\qquad (1.18)$$

$$\alpha^{\pm}_\beta(y,s,\xi_0,\xi_1)=\frac{(\xi_1+is\partial_{y_1}\phi)\ell'(y_1)}{|G|}\pm\sqrt{r_\beta(y,s,\xi_0,\xi_1)},\qquad (1.19)$$

(1.20)
$$r_\beta(y,s,\xi_0,\xi_1)$$
$$=\frac{(\rho(\xi_0+is\partial_{y_0}\phi)^2-\beta(\xi_1+is\partial_{y_1}\phi)^2)|G|+\beta(\xi_1+is\partial_{y_1}\phi)^2(\ell')^2}{\beta|G|^2}.$$

Henceforth, fix $\zeta^*\in\mathbb{R}^3$ such that $|\zeta^*|=1$ arbitrarily, and set $\mathbf{y}^*=(y_0,0,0)$ and $\gamma=(\mathbf{y}^*,\zeta^*)$. Suppose that $|r_\beta(\gamma)|\ge 2\widehat{\delta}>0$. In [IY3], it was shown that there exists $\delta_0(\widehat{\delta})>0$ such that for all $\delta,\delta_1\in(0,\delta_0)$, there exists a constant $C_1>0$ such that for one of the roots of the polynomial (1.17), which we denote by Γ^-_β, we have

(1.21)
$$-\mathrm{Im}\,\Gamma^-_\beta(y,s,\xi_0,\xi_1)\ge sC_1,\quad \forall y\in B_\delta(y_0,0,0),\ (s,\xi_0,\xi_1)\in\mathcal{O}(\delta_1),\ |\zeta|\ge 1.$$

Set

$$\mathcal{B}(y',s,D')=\begin{pmatrix}\mathcal{B}_{11}(y',s,D') & \mathcal{B}_{12}(y',s,D')\\ \mathcal{B}_{21}(y',s,D') & \mathcal{B}_{22}(y',s,D')\end{pmatrix},\quad y\in\partial\mathcal{G},\qquad (1.22)$$

where

$$\mathcal{B}_{11}(y',s,D')=-\rho\mathbf{D}^2_{y_0}+\mu i\alpha^+_\mu(y',0,s,D')b_{11}(y_1,\mathbf{D}')$$
$$-(\lambda+2\mu)\{i\mathbf{D}_{y_1}-\ell'(y_1)i\alpha^+_{\lambda+2\mu}(y',0,s,D')\}b_{21}(y_1,\mathbf{D}'),$$

$$\mathcal{B}_{12}(y', s, D') = -(\lambda + 2\mu)\{i\mathbf{D}_{y_1} - \ell'(y_1) i\alpha^+_{\lambda+2\mu}(y', 0, s, D')\}b_{22}(y_1, \mathbf{D}')$$
$$+\mu i\alpha^+_\mu(y', 0, s, D')b_{12}(y_1, \mathbf{D}'),$$

$$\mathcal{B}_{21}(y', s, D') = -(\lambda + 2\mu)i\alpha^+_{\lambda+2\mu}(y', 0, s, D')b_{21}(y_1, \mathbf{D}')$$
$$-\mu(i\mathbf{D}_{y_1} - \ell'(y_1)i\alpha^+_\mu(y', 0, s, D'))b_{11}(y_1, \mathbf{D}'),$$

$$\mathcal{B}_{22}(y', s, D') = -\rho\mathbf{D}^2_{y_0} - (\lambda + 2\mu)i\alpha^+_{\lambda+2\mu}(y', 0, s, D')b_{22}(y_1, \mathbf{D}')$$
$$-\mu(i\mathbf{D}_{y_1} - \ell'(y_1)i\alpha^+_\mu(y', 0, s, D'))b_{12}(y_1, \mathbf{D}').$$

Remark 1. For readers' convenience we derive the boundary operator $\mathcal{B}(y', s, D')$. We rewrite equations (1.1) on the boundary in the form

$$\rho\frac{\partial^2 u_1}{\partial y_0^2} + \mu\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) - (\lambda + 2\mu)\frac{\partial}{\partial x_1}\operatorname{div}\mathbf{u} + l.o.t. = f_1$$

$$\rho\frac{\partial^2 u_2}{\partial y_0^2} - \mu\frac{\partial}{\partial x_1}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) - (\lambda + 2\mu)\frac{\partial}{\partial x_2}\operatorname{div}\mathbf{u} + l.o.t. = f_2$$

Next we make the change of variables in the above equations. Observing that $\frac{\partial}{\partial x_1} \to \frac{\partial}{\partial y_1} - \ell'(y_2)\frac{\partial}{\partial y_2}$, $\frac{\partial}{\partial x_2} \to \frac{\partial}{\partial y_2}$ we obtain

$$\rho\frac{\partial^2 u_1}{\partial y_0^2} + \mu\frac{\partial z_1}{\partial y_2} - (\lambda + 2\mu)\left(\frac{\partial z_2}{\partial y_1} - \ell'\frac{\partial z_2}{\partial y_2}\right) + l.o.t. = f_1$$

and

$$\rho\frac{\partial^2 u_2}{\partial y_0^2} + \mu\left(\frac{\partial z_1}{\partial y_1} - \ell'\frac{\partial z_1}{\partial y_2}\right) - (\lambda + 2\mu)\frac{\partial z_2}{\partial y_2} + l.o.t. = f_2.$$

By (1.13) and (1.15), we have

$$\rho\frac{\partial^2 u_1}{\partial y_0^2}+\mu\frac{\partial z_1}{\partial y_2}-(\lambda+2\mu)\left(\frac{\partial}{\partial y_1}(b_{21}(y_1, D')u_1 + b_{22}(y_1, D')u_2) - \ell'\frac{\partial z_2}{\partial y_2}\right)+l.o.t. = \tilde{f}_1$$

and

$$\rho\frac{\partial^2 u_2}{\partial y_0^2}+\mu\left(\frac{\partial}{\partial y_1}(b_{11}(y_1, D')u_1 + b_{12}(y_1, D')u_2) - \ell'\frac{\partial z_1}{\partial y_2}\right)-(\lambda+2\mu)\frac{\partial z_2}{\partial y_2}+l.o.t. = \tilde{f}_2.$$

Setting $\mathbf{v} = \mathbf{u}e^{s\phi}, \mathbf{w} = \mathbf{z}e^{s\phi}$ we obtain

$$-\rho\mathbf{D}^2_{y_0}v_1 + \mu i\mathbf{D}_{y_2}w_1 - i(\lambda + 2\mu)(\mathbf{D}_{y_1}(b_{21}(y_1, \mathbf{D}')v_1$$
$$+ b_{22}(y_1, \mathbf{D}')v_2) - \ell'\mathbf{D}_{y_2}w_2) + l.o.t. = F_1$$

and

$$
\begin{aligned}
&-\rho \mathbf{D}_{y_0}^2 v_2 + \mu(\mathbf{D}_{y_1}(b_{11}(y_1,\mathbf{D}')v_1 + b_{12}(y_1,\mathbf{D}')v_2) - \ell' \mathbf{D}_{y_2} w_1) \\
&-(\lambda+2\mu)\mathbf{D}_{y_2} w_2 + l.o.t. = F_2.
\end{aligned}
$$

Later, provided that symbols $\alpha_\mu^\pm$ and $\alpha_{\lambda+2\mu}^\pm$ are smooth at a small conic neighbourhood, we will be able to prove that the functions $\mathbf{D}_{y_2} w_1 - \alpha_\mu^+(y',0,s,D')w_1$ and $\mathbf{D}_{y_2} w_2 - \alpha_{\lambda+2\mu}^+(y',0,s,D')w_2$ are bounded in terms of the right hand side of (1.24). Thus substituting in the above equations instead of $\mathbf{D}_{y_2} w_1$ and $\mathbf{D}_{y_2} w_2$ functions $\alpha_\mu^+(y',0,s,D')(b_{11}(y_1,\mathbf{D}')v_1 + b_{12}(y_1,\mathbf{D}')v_2)$ and $\alpha_{\lambda+2\mu}^+(y',0,s,D')$ $\cdot(b_{21}(y_1,\mathbf{D}')v_1 + b_{22}(y_1,\mathbf{D}')v_2)$, we obtain the operator $\mathcal{B}(y',s,D')$.

Now we formulate a condition which allows us to observe the surface waves. For this purpose, we use the operator $\mathcal{B}$ which was introduced in the local coordinates. For an arbitrary point $x^0 \equiv (x_0^0, x_1^0, x_2^0) \in [0,T] \times \overline{(\partial\Omega \setminus \partial\omega)}$, we rotate and translate Ω such that after the rotation and the translation, the normal vector to the boundary at x^0 is $(0,0,-1)$. Then by $\mathcal{Y}(x)$, we denote the transform involved with the rotation and the translation. Now we are ready to state the condition:

Condition 1.2. *Let $x \in [0,T] \times \overline{(\partial\Omega \setminus \partial\omega)}$ be an arbitrary point and $y = \mathcal{Y}(x)$. We assume that*

$$
Im \frac{1}{s} \sum_{j=0}^{1} \frac{\partial det \mathcal{B}(y',s,\xi_0,\xi_1)}{\partial y_j} \overline{\frac{\partial det \mathcal{B}(y',s,\xi_0,\xi_1)}{\partial \xi_j}} > 0 \tag{1.23}
$$

for any $(y,s,\xi_0,\xi_1) \in \{(y,s,\xi_0,\xi_1) \in \partial\mathcal{G} \times S^2;\ det\, \mathcal{B}(y',s,\xi_0,\xi_1) = 0,\ s > 0,\ y_0 \in (0,T),\ Im \Gamma_\beta^+(y',0,s,\xi_0,\xi_1)/s \ge 0,\ \forall \beta \in \{\mu, \lambda+2\mu\},\ \xi_0 \ne 0\}$.

Now, under Conditions 1.1 and 1.2, we are ready to state our Carleman estimates:

Theorem 1.1. *We assume* (1.3), *Conditions* 1.1 *and* 1.2. *Let* $\mathbf{f} \in \mathbf{H}^1(Q)$, $\mathbf{g} \in \mathbf{H}^{\frac{3}{2}}(\partial Q)$ *and let the function ϕ be given by* (1.7). *Then there exists $\widehat{\tau} > 0$ such that for any $\tau > \widehat{\tau}$, we can choose $s_0(\tau) > 0$ such that for any solution $\mathbf{u} \in \mathbf{H}^1(Q) \cap L^2(0,T;\mathbf{H}^2(\Omega))$ to problem* (1.1)–(1.2), *the following* *estimate holds true*:

(1.24)

$$\int_Q \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi}dx + s\left\|\left(\mathbf{u}e^{s\phi}, \frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right)\right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)\times \mathbf{H}^{\frac{1}{2},s}(\partial Q)}$$

$$\leq C\left(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + \|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)} + \int_{Q_\omega}\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi}dx\right),$$

$$\forall s \geq s_0(\tau),$$

where the constant $C = C(\tau) > 0$ *is independent of* s.

Remark 2. In (1.3), the final condition is relaxed as

$$\lambda(\widetilde{x}) \neq 0, \quad \forall \widetilde{x} \in (\overline{\partial\Omega \setminus \partial\omega}).$$

Assume in addition that

$$\partial_{x_0}\phi(0,\cdot) > 0 \quad \text{and} \quad \partial_{x_0}\phi(T,\cdot) < 0 \qquad \text{on } \overline{\Omega}. \tag{1.25}$$

Then we can formulate Carleman estimates in the situations when the right hand side of equation (1.1) belongs to the spaces $\mathbf{L}^2(Q)$ or $\mathbf{H}^{-1}(Q)$.

Corollary 1.1. *We assume* (1.3), (1.25), *Conditions* 1.1 *and* 1.2. *Let* $\mathbf{f} \in \mathbf{L}^2(Q)$, $\mathbf{g} = 0$ *and let the function* ϕ *be given by* (1.7). *Then there exists* $\widehat{\tau} > 0$ *such that for any* $\tau > \widehat{\tau}$, *we can choose* $s_0(\tau) > 0$ *such that for any solution* $\mathbf{u} \in \mathbf{H}^1(Q)$ *to problem* (1.1)–(1.2), *the following estimate holds true*:

$$\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(Q)} \leq C(\|\mathbf{f}e^{s\phi}\|_{\mathbf{L}^2(Q)} + \|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(Q_\omega)}), \quad \forall s \geq s_0(\tau).$$

Here $C = C(\tau) > 0$ *is independent of* s.

Corollary 1.2. *We assume* (1.3), (1.25), *Conditions* 1.1 *and* 1.2. *Let* $\mathbf{f} = \mathbf{f}_{-1} + \sum_{j=0}^2 \partial_{x_j}\mathbf{f}_j$ *where* $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2 \in \mathbf{L}^2(Q)$, $\mathbf{f}_{-1} \in \mathbf{H}^{-1}(Q)$, $\operatorname{supp}\mathbf{f}_{-1} \subset Q$, $\mathbf{g} = 0$, *and let the function* ϕ *be given by* (1.7). *Then there exists* $\widehat{\tau} > 0$ *such that for any* $\tau > \widehat{\tau}$, *we can choose* $s_0(\tau) > 0$ *such that for any solution* $\mathbf{u} \in \mathbf{H}^1(Q)$ *to problem* (1.1)–(1.2), *the following estimate holds true*:

$$\|\mathbf{u}e^{s\phi}\|_{L^2(Q)} \leq C\left(\|\mathbf{f}_{-1}e^{s\phi}\|_{\mathbf{H}^{-1}(Q)} + \sum_{j=0}^{2}\|\mathbf{f}_j e^{s\phi}\|_{\mathbf{L}^2(Q)} + \|\mathbf{u}e^{s\phi}\|_{L^2(Q_\omega)}\right),$$

$$\forall s \geq s_0(\tau).$$

Here $C = C(\tau) > 0$ *is independent of* s.

Similarly to Theorems 2.2 and 2.3 in [IY3], we can derive Corollaries 1.1 and 1.2 from Theorem 1.1, and we omit the arguments. The rest of this section is devoted to describing a sufficient condition for inequality (1.23) in Condition 1.2 which is convenient for the applications to inverse problems, etc.

For any fixed $\widetilde{x} \in \partial\Omega$, we define a cubic polynomial in t by

(1.26)
$$H(t) = t^3 - t^2 \left(8\frac{\mu}{\rho}\right)(\widetilde{x}) + t\left(\frac{24\mu^2}{\rho^2} - \frac{16\mu^3}{\rho^2(\lambda+2\mu)}\right)(\widetilde{x}) - \left(\frac{16\mu^3(\lambda+\mu)}{\rho^3(\lambda+2\mu)}\right)(\widetilde{x}).$$

Then we can directly verify that $H'(t) > 0$ if $t \le 0$, $H(0) < 0$, $H\left(\frac{\mu}{\rho}(\tilde{x})\right) = \frac{\mu^3}{\rho^3}(\tilde{x}) > 0$ and $H''\left(\frac{8\mu}{3\rho}(\tilde{x})\right) = 0$. Therefore we can prove that $H(t) = 0$ possesses a unique simple root t in the interval $\left(0, \left(\frac{\mu}{\rho}\right)(\widetilde{x})\right)$ for any $\widetilde{x} \in \partial\Omega$, and by $\mathcal{C} = \mathcal{C}(\widetilde{x})$ we denote this root. Moreover if there exists another real root, then it is greater than $\left(\frac{\mu}{\rho}\right)(\widetilde{x})$.

Remark 3. By means of the Cardano formula, we can compute $\mathcal{C} = \mathcal{C}(\widetilde{x})$ explicitly. We set

$$\widetilde{a_1} = -8\frac{\mu}{\rho}(\tilde{x}), \quad \widetilde{a_2} = 24\frac{\mu^2}{\rho^2} - \frac{16\mu^3}{\rho^2(\lambda+2\mu)}(\tilde{x}),$$
$$\widetilde{a_3} = -\frac{16\mu^3(\lambda+\mu)}{\rho^3(\lambda+2\mu)}(\tilde{x}).$$

That is, $H(t) = t^3 + \widetilde{a_1}t^2 + \widetilde{a_2}t + \widetilde{a_3}$. Moreover we put

$$\widetilde{b_1} = \frac{\widetilde{a_1}^3}{27} - \frac{\widetilde{a_1}\widetilde{a_2}}{6} + \frac{\widetilde{a_3}}{2}, \quad \widetilde{b_2} = \frac{1}{9}(3\widetilde{a_2} - \widetilde{a_1}^2),$$
$$\widetilde{b_3} = \widetilde{b_1}^2 + \widetilde{b_2}^3, \quad \widetilde{b_4} = \mathrm{sign}\,(\widetilde{b_1})|\widetilde{b_2}|^{\frac{1}{2}}.$$

Then we have:
$$\mathcal{C} = -\frac{\widetilde{a_1}}{3} - 2\widetilde{b_4}\cosh\left(\frac{\theta}{3}\right)$$
if $\widetilde{b_3} > 0$ and $\widetilde{b_2} < 0$, where θ solves the equation: $\cosh\theta = \frac{\widetilde{b_1}}{\widetilde{b_4}^3}$.

$$\mathcal{C} = -\frac{\widetilde{a_1}}{3} - 2\widetilde{b_4}\sinh\left(\frac{\theta}{3}\right)$$

if $\widetilde{b_2} > 0$, where θ solves the equation: $\sinh\theta = \frac{\widetilde{b_1}}{\widetilde{b_4}^3}$. If $\widetilde{b_2} < 0$ and $\widetilde{b_3} \le 0$, then we define $\mathcal{C}$ by the one of the three zeros of the polynomial H which belongs to the interval $[0,\mu/\rho(\widetilde{x})]$: $t_1 = -\frac{\widetilde{a_1}}{3} - 2\widetilde{b_4}\cos\left(\frac{\theta}{3}\right)$, $t_2 = -\frac{\widetilde{a_1}}{3} + 2\widetilde{b_4}\cos\left(\frac{\pi}{3} - \frac{\theta}{3}\right)$, $t_3 = -\frac{\widetilde{a_1}}{3} + 2\widetilde{b_4}\cos\left(\frac{\pi}{3} + \frac{\theta}{3}\right)$, where θ solves the equation: $\cos\theta = \frac{\widetilde{b_1}}{\widetilde{b_4}^3}$. In terms of $\mathcal{C}(\widetilde{x})$, we can state one sufficient condition:

Proposition 1.1. *Let $\psi \in C^2(\overline{Q})$, and*

$$\partial_{x_0}\psi(x) \pm \sqrt{\mathcal{C}(\widetilde{x})}\frac{\partial\psi}{\partial\vec{t}}(x) \neq 0 \tag{1.27}$$

for any $x \in [0,T] \times \overline{(\partial\Omega \setminus \partial\omega)}$. Then there exists $\tau_0 > 0$ such that Condition 1.2 holds for $\phi = e^{\tau\psi}$ if $\tau > \tau_0$.

Proof of Proposition 1.1. For this, it suffices to prove : *Let $\psi \in C^2(\overline{Q})$ satisfy $\partial_{y_1}\psi \neq 0$ on $\overline{Q}$, and for $(x_1^0, x_2^0) \in \overline{\partial\Omega \setminus \partial\omega}$, let the local coordinate $\widetilde{y} = (y_1, y_2)$ be introduced by the local representation $x_2 = \ell(x_1)$ of $\partial\Omega$. We assume*

$$\partial_{y_0}\psi(\mathbf{y}^*) \pm \sqrt{\mathcal{C}(0)}\partial_{y_1}\psi(\mathbf{y}^*) \neq 0$$

for any $(x_1^0, x_2^0) \in \overline{\partial\Omega \setminus \partial\omega}$ and $y_0 \in (0,T)$. Then there exists $\tau_0 > 0$ such that Condition 1.2 holds for the function $\phi = e^{\tau\psi}$ if $\tau > \tau_0$.

We recall that $\mathbf{y}^{*\prime} = (y_0, 0)$. The principal symbol of the operator $\mathcal{B}$ at the point $\mathbf{y}^{*\prime}$ is

$$\mathcal{B}(\mathbf{y}^{*\prime}, \zeta') = \begin{pmatrix} -\rho(0)\widetilde{\zeta}_0^2 + 2\mu(0)\widetilde{\zeta}_1^2 & -2\mu(0)\alpha_\mu^+(\mathbf{y}^*, \zeta)\widetilde{\zeta}_1 \\ 2\mu(0)\alpha_{\lambda+2\mu}^+(\mathbf{y}^*, \zeta)\widetilde{\zeta}_1 & -\rho(0)\widetilde{\zeta}_0^2 + 2\mu(0)\widetilde{\zeta}_1^2 \end{pmatrix},$$

where $\zeta = (s, \xi_0, \xi_1) \in S^2$ and $\widetilde{\zeta}_j = \xi_j + is\phi_{y_j}(\mathbf{y}^*)$. Obviously

$$\det\mathcal{B}(\mathbf{y}^{*\prime}, \zeta) = \rho^2(0)\left(-\widetilde{\zeta}_0^2 + 2\frac{\mu}{\rho}(0)\widetilde{\zeta}_1^2\right)^2 + 4\mu^2(0)\alpha_{\lambda+2\mu}^+(\mathbf{y}^*, \zeta)\alpha_\mu^+(\mathbf{y}^*, \zeta)\widetilde{\zeta}_1^2. \tag{1.28}$$

We study the structure of the set

$$\Psi = \Bigg\{\zeta \in \mathbb{R}^3 \setminus \{0\}; \det\mathcal{B}(\mathbf{y}^{*\prime}, \zeta) = 0, \quad \mathrm{Im}\frac{\Gamma_\mu^+(\mathbf{y}^*, \zeta)}{s} \geq 0,\ \mathrm{Im}\frac{\Gamma_{\lambda+2\mu}^+(\mathbf{y}^*, \zeta)}{s} \geq 0\Bigg\}. \tag{1.29}$$

We have

Lemma 1.1. *Let* (1.3) *hold true and let $\partial_{y_1}\psi(\mathbf{y}^*) \neq 0$. Then*

$$\Psi \subset \Psi_1 \cup \Psi_2, \qquad \mathrm{dist}\,(\Psi_1, \Psi_2) > 0,$$

where $\Psi_1 = \{\zeta = (s, \xi_0, \xi_1) \in S^2;\ \xi_0 + is\phi_{y_0}(\mathbf{y}^*) = 0\}$ *and* $\Psi_2 = \{\zeta \in S^2;\ \xi_0 + is\phi_{y_0}(\mathbf{y}^*) = \pm\sqrt{\mathcal{C}(0)}(\xi_1 + is\phi_{y_1}(\mathbf{y}^*)), \mathrm{Im}\,\frac{\Gamma^+_\mu(\mathbf{y}^*,\zeta)}{s} \ge 0,\ \ \mathrm{Im}\,\frac{\Gamma^+_{\lambda+2\mu}(\mathbf{y}^*,\zeta)}{s} \ge 0\}$.

Proof. We can directly see from the definition that $\mathrm{dist}\,(\Psi_1, \Psi_2) > 0$. Taking into account that $(\alpha^+_\beta(\mathbf{y}^*, \zeta))^2 = \frac{\rho}{\beta}(0)\widetilde{\zeta}_0^2 - \widetilde{\zeta}_1^2$, we obtain

$$\Psi \subset \Bigg\{\zeta \in S^2;\ \mathcal{F}(\widetilde{\zeta}_0^2, \widetilde{\zeta}_1^2) \equiv \left(\widetilde{\zeta}_0^2 - 2\frac{\mu}{\rho}(0)\widetilde{\zeta}_1^2\right)^4$$
$$-16\frac{\mu^3(0)}{\rho^2(0)(\lambda+2\mu)(0)}\widetilde{\zeta}_1^4\left(\widetilde{\zeta}_0^2 - \frac{\mu}{\rho}(0)\widetilde{\zeta}_1^2\right)\left(\widetilde{\zeta}_0^2 - \frac{\lambda+2\mu}{\rho}(0)\widetilde{\zeta}_1^2\right) = 0\Bigg\}.$$

We fix $\rho(0)$ and by $t_2 = t_2(\lambda(0), \mu(0))$ and $t_3 = t_3(\lambda(0), \mu(0))$, we denote the roots of $H(t)$ with $\widetilde{x} = (0, 0)$ which are distinct from $\mathcal{C}(0)$. Then we have

$$t_2, t_3 > \frac{\mu(0)}{\rho(0)}$$

if they are real. Therefore, noting that $\mathcal{F}(\widetilde{\zeta}_0^2, \widetilde{\zeta}_1^2) = \widetilde{\zeta}_0^2\widetilde{\zeta}_1^6 H(t)$ with $\widetilde{\zeta}_0^2 = t\widetilde{\zeta}_1^2$, we have only to prove that

$$\det \mathcal{B}(\mathbf{y}^{*\prime}, \zeta) \ne 0 \quad \text{if} \quad (\xi_0 + is\phi_{y_0}(\mathbf{y}^*))^2 = t_j(\xi_1 + is\phi_{y_1}(\mathbf{y}^*))^2, \qquad \forall j \in \{2, 3\}. \tag{1.30}$$

Moreover we have only the two cases: $t_2, t_3 \in \mathbb{R}$ or $t_2, t_3 \in \mathbb{C}\setminus\mathbb{R}$. First we consider the case of $t_2, t_3 \in \mathbb{R}$. Really $\left(\widetilde{\zeta}_0^2 - 2\frac{\mu}{\rho}(0)\widetilde{\zeta}_1^2\right)^2 = \widetilde{\zeta}_1^4\left(t_j - 2\frac{\mu}{\rho}(0)\right)^2$ and

$$\alpha^+_\mu(\mathbf{y}^*, \zeta) = \sqrt{\widetilde{\zeta}_1^2(t_j(\rho/\mu)(0) - 1)} = \mathrm{sign}(\phi_{y_1}(\mathbf{y}^*))\widetilde{\zeta}_1\sqrt{t_j(\rho/\mu)(0) - 1},$$

where we used the fact that $t_j(\rho/\mu)(0) - 1 > 0$ and a part of the assumption (1.6) which guarantee that $\phi_{y_1}(\mathbf{y}^*) \ne 0$. If $t_j\rho(0)/(\lambda+2\mu)(0) - 1 > 0$, then

$$\alpha^+_{\lambda+2\mu}(\mathbf{y}^*, \zeta) = \mathrm{sign}(\phi_{y_1}(\mathbf{y}^*))\widetilde{\zeta}_1\sqrt{t_j\rho(0)/(\lambda+2\mu)(0) - 1}$$

and we have

$$\det \mathcal{B}(\mathbf{y}^{*\prime}, \zeta) = \widetilde{\zeta}_1^4\Bigg\{\rho^2(0)\left(t_j - 2\frac{\mu}{\rho}(0)\right)^2$$
$$+4\mu^2(0)\sqrt{t_j(\rho/\mu)(0) - 1} \times \sqrt{t_j\rho(0)/(\lambda+2\mu)(0) - 1}\Bigg\} \ne 0.$$

If $t_j\rho(0)/(\lambda+2\mu)(0)-1<0$, then $\alpha^+_{\lambda+2\mu}(\mathbf{y}^*,\zeta)=i\text{sign}(\xi_1)\widetilde{\zeta_1}\sqrt{-t_j\rho(0)/(\lambda+2\mu)(0)+1}$ and

$$\det\mathcal{B}(\mathbf{y}^{*\prime},\zeta)=\widetilde{\zeta_1}^4\left\{\rho^2(0)\left(t_j-2\frac{\mu}{\rho}(0)\right)^2\right.$$
$$\left.+\,4i\mu^2(0)\text{sign}(\xi_1)\text{sign}(\phi_{y_1}(\mathbf{y}^*))\sqrt{t_j\rho(0)/\mu(0)-1}\sqrt{-t_j\rho(0)/(\lambda+2\mu)(0)+1}\right\}$$
$$\neq 0.$$

Note that if $\xi_1=0$ then $\text{Im}\frac{\Gamma^{\pm}_{\lambda+2\mu}}{s}(\mathbf{y}^*,\zeta)<0$. Next we will consider the case of $t_j\notin\mathbb{R}$, $j=2,3$. We set $\rho_0=\rho(0)$. Henceforth in place of $H(t)$ defined by (1.26), we consider

$$H(t)=t^3-\frac{8b}{\rho_0}t^2+t\left(\frac{24b^2}{\rho_0^2}-\frac{16b^3}{\rho_0^2(a+2b)}\right)-\frac{16b^3(a+b)}{\rho_0^3(a+2b)} \tag{1.31}$$

for all $(a,b)\in\mathbb{R}^2$ such that $a,b>0$. We note that if we set $a=\lambda(0)$ and $b=\mu(0)$, then this coincides with $H(t)$ by (1.26). Moreover, without fear of confusion, by $t_2=t_2(a,b)$ and $t_3=t_3(a,b)$, we may denote the roots which are not in the interval $\left[0,\frac{b}{\rho_0}\right]$ of $H(t)$ defined by (1.31). We set $\mathbf{z}=(z_0,z_1)\in\mathbb{C}^2$ and for factorizing $\mathcal{F}(\widetilde{\zeta_0}^2,\widetilde{\zeta_1}^2)$ with $\lambda(0)=a$, $\mu(0)=b$ and $\rho(0)=\rho_0$, we introduce two functions

$$\mathcal{H}^{\pm}(\mathbf{z})=\left(z_0^2-2\frac{b}{\rho_0}z_1^2\right)^2\pm4\frac{b^2}{\rho_0^2}z_1^2\alpha^+_{a+2b}(\mathbf{z})\alpha^+_b(\mathbf{z}),$$

and $\alpha^+_\beta(\mathbf{z})=\sqrt{\frac{\rho_0}{\beta}z_0^2-z_1^2}$ with $\beta\in\{a+2b,b\}$. Henceforth we set

$$Dom\,\alpha^+_\beta=\left\{\mathbf{z}=(z_0,z_1);\ \frac{\rho_0}{\beta}z_0^2-z_1^2\notin\mathbb{R}_+\right\}.$$

For $\mathbf{z}\in Dom\,\alpha^+_\beta$, we take the complex root $\sqrt{\frac{\rho_0}{\beta}z_0^2-z_1^2}$ in such a way that $\text{Im}\,\alpha^+_\beta(\mathbf{z})>0$. We set $Dom\,\mathcal{H}^+=Dom\,\mathcal{H}^-=Dom\,\alpha^+_b\cap Dom\,\alpha^+_{a+2b}$. In order to prove Lemma 1.1 it suffices to show that

$$\mathcal{H}^-(\mathbf{z}^{(j)}(\lambda(0),\mu(0),z_1))=0,\qquad\forall j\in\{2,3\} \tag{1.32}$$

if

$$\mathbf{z}^{(j)}(a,b,z_1)\equiv(\pm\sqrt{t_j(a,b)}z_1,z_1)\in Dom\,\mathcal{H}^+\quad\text{for } j\in\{2,3\},\ z_1\in\mathbb{C}.$$

In fact, if (1.32) will be proved, then we have (1.30) in the following manner: Assume contrarily that $\det \mathcal{B}(\mathbf{y}^{*\prime}, \zeta) = 0$. Then, in terms of (1.28), we obtain $\widetilde{\zeta_0}^2 = \frac{2\mu(0)}{\rho_0}\widetilde{\zeta_1}^2$, which contradicts that $\widetilde{\zeta_0}^2 = t_j\widetilde{\zeta_1}^2$, $j = 2, 3$ where $t_j \in \mathbb{C} \setminus \mathbb{R}$.

Proof of (1.32). Let $\delta_2 > 0$ be an arbitrary but fixed number. We introduce the sets

$$\begin{aligned}&\Pi_1 = \{(a,b);\, a,b > \delta_2, \quad \text{there exists } z_1 \in \mathbb{C} \text{ with } |z_1| = 1 \text{ such that}\\ &\mathcal{H}^+(\mathbf{z}^{(j)}(a,b,z_1)) = 0 \text{ for } j = 2 \text{ or } j = 3\}\end{aligned}$$

and

$$\begin{aligned}&\Pi_2 = \{(a,b);\, a,b > \delta_2,\\ &\mathcal{H}^+(\mathbf{z}^{(j)}(a,b,z_1)) \ne 0 \quad \text{for any } z_1 \in \mathbb{C} \text{ with } |z_1| = 1 \text{ and any } j \in \{2,3\}\\ &\text{such that } \mathbf{z}^{(j)}(a,b,z_1) \in Dom\mathcal{H}^+\}.\end{aligned}$$

In order to prove (1.32) it suffices to prove that $\Pi_1 = \emptyset$ because we have either $(a,b) \in \Pi_1$ or $(a,b) \in \Pi_2$. Let $(\widetilde{a}, \widetilde{b}) \in \Pi_2$. Such a point exists because there exist a_0, b_0 such that $t_j(a_0,b_0) \in \mathbb{R}$ for $j = 2, 3$ and then we have already shown that $\det \mathcal{B}(\mathbf{y}^{*\prime}, \zeta') \ne 0$, so that $(a_0, b_0) \in \Pi_2$ in terms of (1.28) and the definition of $\mathcal{H}^\pm$. Assume contrarily that the set $\Pi_1 \ne \emptyset$. Then $\operatorname{dist}((\widetilde{a}, \widetilde{b}), \Pi_1) > 0$. There exist sequences $\{(a_n, b_n)\}_{n=1}^\infty \subset \Pi_1$ and $\{z_{1,n}\}_{n=1}^\infty \in \mathbb{C}$ such that $|z_{1,n}| = 1$, $\lim_{n\to\infty}(a_n, b_n) = (\widehat{a}, \widehat{b})$, $\lim_{n\to\infty} z_{1,n} = \widehat{z}_1$,

$$\operatorname{dist}((\widetilde{a}, \widetilde{b}), (\widehat{a}, \widehat{b})) = \operatorname{dist}((\widetilde{a}, \widetilde{b}), \Pi_1)$$

and $\mathcal{H}^+(\mathbf{z}^{j_1}{}_n) = 0$, where we set

$$\begin{aligned}&\mathbf{z}^{j_1}{}_n = (\pm\sqrt{t_{j_1}(a_n, b_n)}z_{1,n}, z_{1,n}),\\ &\text{for some } j_1 \in \{2,3\} \text{ and some } z_{1,n} \in \mathbb{C} \text{ with } |z_{1,n}| = 1.\end{aligned}$$

Let us show that there exists $\widetilde{z}_1 \in \mathbb{C}$ such that $|\widetilde{z}_1| = 1$ and $\widetilde{\mathbf{z}} \equiv (\pm\sqrt{t_{j_1}(\hat{a}, \hat{b})}\widetilde{z}_1, \widetilde{z}_1) \in Dom\mathcal{H}^+$. Really if $\widehat{\mathbf{z}} \equiv (\pm\sqrt{t_{j_1}(\widehat{a}, \widehat{b})}\widehat{z}_1, \widehat{z}_1) \in Dom\mathcal{H}^+$, then we can take $\widetilde{\mathbf{z}} = \widehat{\mathbf{z}}$. On the other hand, if $\widehat{\mathbf{z}} \notin Dom\mathcal{H}^\pm$, then

$$\widehat{\mathbf{z}} \notin Dom\, \alpha^+_{\widehat{b}} \quad \text{or} \quad \widehat{\mathbf{z}} \notin Dom\, \alpha^+_{\widehat{a}+2\widehat{b}}.$$

Let us assume for example that

$$\widehat{\mathbf{z}} \notin Dom\, \alpha^+_{\widehat{b}}.$$

Then $\sqrt{t_{j_1}(\widehat{a},\widehat{b})\rho_0/\widehat{b}-1} = re^{i\widehat{\theta}}$ and $\sqrt{t_{j_1}(\widehat{a},\widehat{b})\rho_0/(\widehat{a}+2\widehat{b})-1} = r_0e^{i\widehat{\theta}_0}$. Since the imaginary part of t_{j_1} is not zero, we have $\widehat{\theta}_0 \neq \widehat{\theta}$ (mod 2π). Then either

$$\mathcal{H}^+(\widetilde{\mathbf{z}}) = 0 \quad \text{for } \beta \in \{\widehat{b}, \widehat{a}+2\widehat{b}\} \text{ with } \widetilde{\mathbf{z}} = (\pm\sqrt{t_{j_1}(\widehat{a},\widehat{b})}\widetilde{z}_1, \widetilde{z}_1),\ \widetilde{z}_1 = e^{\frac{-i}{2}(\widehat{\theta}+\widehat{\theta}_0)},$$

or

$$\begin{aligned}&\mathcal{H}^+(\widetilde{\mathbf{z}}) = 0 \text{ for } \beta \in \{\widehat{b}, \widehat{a}+2\widehat{b}\} \text{ with } \widetilde{\mathbf{z}} = (\pm\sqrt{t_{j_1}(\widehat{a},\widehat{b})}\widetilde{z}_1, \widetilde{z}_1),\\ &\widetilde{z}_1 = e^{\frac{-i}{2}(\widehat{\theta}+\widehat{\theta}_0)+i\pi}.\end{aligned}$$

On the other hand, we have

$$[(\widetilde{a},\widetilde{b}),(\widehat{a},\widehat{b})) \equiv \{t(\widetilde{a},\widetilde{b})+(1-t)(\widehat{a},\widehat{b});\ 0<t\le 1\} \subset \Pi_2. \tag{1.33}$$

In fact, noting that $\{(a,b);\ a,b>\delta_2\}$ is convex and is contained in $\Pi_1\cup\Pi_2$, we see that $[(\widetilde{a},\widetilde{b}),(\widehat{a},\widehat{b})) \subset \Pi_1\cup\Pi_2$. Assume contrarily that there exists $(a^*,b^*) \in \Pi_1$ such that (a^*,b^*) is in the open segment $((\widetilde{a},\widetilde{b}),(\widehat{a},\widehat{b}))$. Then $\mathrm{dist}\,((\widetilde{a},\widetilde{b}),(a^*,b^*)) < \mathrm{dist}\,((\widetilde{a},\widetilde{b}),(\widehat{a},\widehat{b})) = \mathrm{dist}\,((\widetilde{a},\widetilde{b}),\Pi_1)$, which is a contradiction. Thus we have proved that $[(\widetilde{a},\widetilde{b}),(\widehat{a},\widehat{b})) \subset \Pi_2$. We set $(a_\varepsilon,b_\varepsilon) = \varepsilon(\widetilde{a},\widetilde{b})+(1-\varepsilon)(\widehat{a},\widehat{b})$. Then, for sufficiently small $\varepsilon>0$, we have

$$\widetilde{\mathbf{z}}_\varepsilon = (\pm\sqrt{t_{\widehat{j}}(a_\varepsilon,b_\varepsilon)}\widetilde{z}_1, \widetilde{z}_1) \in Dom\mathcal{H}^+.$$

Then, by (1.33), we have $\lim_{\varepsilon\to+0}\mathcal{H}^-(\widetilde{\mathbf{z}}_\varepsilon) = \mathcal{H}^-(\widetilde{\mathbf{z}}) = 0$. Moreover by the choice of $\widetilde{\mathbf{z}}$, we have $\mathcal{H}^+(\widetilde{\mathbf{z}}) = 0$. Hence $\mathcal{H}^\pm(\widetilde{\mathbf{z}}) = 0$. This implies that $t_{j_1}(\widehat{a},\widehat{b}) = \frac{2\widehat{b}}{\rho_0}$ but this is impossible because the left hand side is not real and the right hand side is real. Thus we have a contradiction. Thus the proof of Lemma 1.1 is complete. ☐

Now we proceed to

Completion of Proof of Proposition 1.1. Let $\mathcal{C}(0) \in [0,(\mu/\rho)(\mathbf{y}^*)]$ be the zero of the polynomial H. By Lemma 1.1, for any $\zeta \in S^2$, the set of all the possible solutions to the equation

$$\det\mathcal{B}(\mathbf{y}^{*\prime},\zeta) = 0, \quad \mathrm{Im}\,\frac{\Gamma_\beta^+}{s}(\mathbf{y}^*,\zeta) \ge 0,\ \forall\beta\in\{\mu,\lambda+2\mu\},\ \xi_0 \neq 0$$

is given by the formula

$$\xi_0 + is\phi_{y_0}(\mathbf{y}^*) \pm \sqrt{\mathcal{C}(0)}(\xi_1 + is\phi_{y_1}(\mathbf{y}^*)) = 0.$$

Let $(\xi_1^*, s^*) \in S^1$ be an arbitrary but fixed point. Let $z_0^* = \xi_0^* + is^*\phi_{y_0}(\mathbf{y}^*)$ with some $\xi_0^* \neq 0$ and $z_1^* = \xi_1^* + is^*\phi_{y_1}(\mathbf{y}^*)$ satisfy $z_0^* = \pm\sqrt{\mathcal{C}}z_1^*$ and $\mathrm{Im}\,\frac{\Gamma_\beta^+}{s}(\mathbf{y}^*, s^*, \xi_0^*, \xi_1^*) \geq 0$ for $\beta \in \{\mu, \lambda + 2\mu\}$. Consider the following function $J(y', z_0, z_1) \equiv \det \mathcal{B}(y', s, \xi_0, \xi_1), z_0 = \xi_0 + is\phi_{y_0}(\mathbf{y}^*), z_1 = \xi_1 + is\phi_{y_1}(\mathbf{y}^*)$. Applying the implicit function theorem (keeping in mind that $J_{z_0}(0, z_0^*, z_1^*) \neq 0$), we see that there exists a function $q(y', z_1)$ which is defined in a neighbourhood of $(\mathbf{y}^*, z_1^*)$ and analytic in z_1 such that $(y', q(y', z_1), z_1)$ is a solution to the equation $J(y', z_0, z_1) = 0$. Note that

$$q(\mathbf{y}^*, z_1) = \pm\sqrt{\mathcal{C}}(0)z_1. \tag{1.34}$$

Set $r(y', s, \xi_0, \xi_1) = \xi_0 + is\phi_{y_0}(y) - q(y', \xi_1 + is\phi_{y_1}(y))$. Since

$$\begin{aligned}\det \mathcal{B}(y', s, \xi_0, \xi_1) &= r(y', s, \xi_0, \xi_1) \times \frac{\det \mathcal{B}(y', s, \xi_0, \xi_1)}{r(y', s, \xi_0, \xi_1)}\\ &\equiv r(y', s, \xi_0, \xi_1)\widehat{r}(y', s, \xi_0, \xi_1),\end{aligned}$$

where $\hat{r}$ is smooth and not equal to zero, Condition 1.2 is equivalent to

$$\mathrm{Im}\frac{1}{s}\sum_{k=0}^{1}\overline{\frac{\partial r(y', s, \xi_0, \xi_1)}{\partial \xi_k}}\frac{\partial r(y', s, \xi_0, \xi_1)}{\partial y_k} > 0.$$

Computing the left hand side of this inequality, we obtain

$$\begin{aligned}&\sum_{k=0}^{1}\mathrm{Im}\frac{1}{s}\overline{\frac{\partial r(y', s, \xi_0, \xi_1)}{\partial \xi_k}}\frac{\partial r(y', s, \xi_0, \xi_1)}{\partial y_k}|_{(\mathbf{y}^*, s, \xi_0, \xi_1)}\\ =&\mathrm{Im}\,\frac{1}{s}\{is\phi_{y_0y_0}(\mathbf{y}^*) - q_{y_0}(y_0, 0, \xi_1 + is\phi_{y_1}(\mathbf{y}^*)) \pm\sqrt{\mathcal{C}(0)}is\phi_{y_0y_1}(\mathbf{y}^*)\\ &\mp\sqrt{\mathcal{C}(0)}(is\phi_{y_0y_1}(\mathbf{y}^*) - q_{y_1}(y_0, 0, \xi_1 + is\phi_{y_1}(\mathbf{y}^*)) \mp\sqrt{\mathcal{C}(0)}is\phi_{y_1y_1}(\mathbf{y}^*))\}\\ =&\phi_{y_0y_0}(\mathbf{y}^*) \pm\sqrt{\mathcal{C}(0)}\phi_{y_0y_1}(\mathbf{y}^*) + \mathcal{C}(0)\phi_{y_1y_1}(\mathbf{y}^*)\\ &-\mathrm{Im}\,\frac{1}{s}(q_{y_0}(y_0, 0, \xi_1 + is\phi_{y_1}(\mathbf{y}^*)) \pm\sqrt{\mathcal{C}(0)}q_{y_1}(y_0, 0, \xi_1 + is\phi_{y_1}(\mathbf{y}^*))).\end{aligned}$$

By the implicit function theorem, $q_{y_k}(y_0, 0, z_1^*) = -\frac{J_{y_k}(0, z_0^*, z_1^*)}{J_{z_0}(0, z_0^*, z_1^*)} = m_k(\xi_1^* + is^*\phi_{y_1}(\mathbf{y}^*))$, $k = 0, 1$, where a number m_k depends on the sign in formula (1.34). Using this formula we obtain

$$\begin{aligned}&\sum_{k=0}^{1}\mathrm{Im}\,\frac{1}{s}\overline{\frac{\partial r(y', s, \xi_0, \xi_1)}{\partial \xi_k}}\frac{\partial r(y', s, \xi_0, \xi_1)}{\partial y_k}|_{(\mathbf{y}^*, s, \xi_0, \xi_1)}\\ =&\tau\phi(\tau(\psi_{y_0}(\mathbf{y}^*) \pm\sqrt{\mathcal{C}(0)}\psi_{y_1}(\mathbf{y}^*))^2 + \psi_{y_0y_0}(\mathbf{y}^*)\\ &\pm\sqrt{\mathcal{C}(0)}\psi_{y_0y_1}(\mathbf{y}^*) + \mathcal{C}(0)\psi_{y_1y_1}(\mathbf{y}^*) - (m_0\psi_{y_0}(\mathbf{y}^*) \pm\sqrt{\mathcal{C}(0)}m_1\psi_{y_1}(\mathbf{y}^*))).\end{aligned}$$

Obviously under condition (1.27), for all sufficiently large $\tau > 0$, inequality (1.23) holds true at the point $\mathbf{y}^{*\prime}$. The proof of Proposition 1.1 is completed. $\square$

§2. Proof of Theorem 1.1 (the beginning)

We shall prove Theorem 1.1 in several steps. Our proof is based on decoupling of the Lamé system into the scalar acoustic equations for $\operatorname{rot} \mathbf{u}$ and $\operatorname{div} \mathbf{u}$. Then we apply the standard procedure (e.g., [Hö]) to these equations for obtaining a Carleman estimate. Finally we analyze the boundary integrals, which appear in the previous steps by means of the microlocalization technique.

First we show that it suffices to consider the case when displacement has a support in a ball of a small radius (Lemma 2.1). The strategic goal is a priori estimates for the traces of displacement and its normal derivative on the boundary. For overcoming the difficulties caused by non-compact supports of $\mathbf{u}$, we shall argue microlocally and the argument is lengthy but based on standard source books ([Ku], [T1], [T2]). The estimates for $\mathbf{u}$ and $\partial_n \mathbf{u}$ are of two types. Outside of the set Ψ_2 (see Lemma 1.1) in the cotangent bundle we have estimate (3.1) while in a neighbourhood of this set we have a weaker estimates (5.2), (5.3).

In order to prove estimate (3.1) we shall separate several cases corresponding to that the roots $r_\mu(\gamma)$ and $r_{\lambda+2\mu}(\gamma)$ defined by (1.20) are zero or non-zero. In Section 3 we consider the case $r_\mu(\gamma) = 0$ and Section 4 is devoted to the case $r_{\lambda+2\mu}(\gamma) = 0$. Finally in Section 5 we consider the remaining case $r_\mu(\gamma) \neq 0$ and $r_{\lambda+2\mu}(\gamma) \neq 0$ and obtain estimate (1.24).

First we show that it suffices to consider only the case where the support of $\mathbf{u}$ is located in a small neighbourhood of an arbitrary point $\mathbf{y}^* = (y_0^*, 0, 0) \in Q$.

Lemma 2.1. *Under the conditions of Theorem* 1.1 *it suffices to prove* (1.24) *under the assumption that*

$$\operatorname{supp} \mathbf{u} \subset B_\delta(\mathbf{y}^*), \tag{2.1}$$

where $\delta > 0$ *is an arbitrary small number and* $\mathbf{y}^*$ *is an arbitrary point in* Q.

Proof of Lemma 2.1. Let us consider the finite covering of $\overline{Q}$ by balls $B_\delta(\mathbf{y}_j^*)$. Let $e \in C_0^\infty(\overline{B_{2\delta}(0)})$ be a non-negative function such that $e|_{B_{\frac{5}{4}\delta}(0)} = 1$ and $e(x) < 1$ for all $x \in B_{2\delta}(0) \setminus \overline{B_{\frac{5}{4}\delta}(0)}$ and let $\widetilde{e} \in C_0^\infty(\overline{B_{2\delta}(0)})$ be a non-negative function such that $\widetilde{e}|_{B_{\frac{9}{8}\delta}(0)} = 1$ and $\widetilde{e}(x) < 1$ for all $x \in B_{2\delta}(0) \setminus \overline{B_{\frac{9}{8}\delta}(0)}$.

We set $e_j(x) = e(x - \mathbf{y}_j^*)$ and $\widetilde{e}_j(x) = \widetilde{e}(x - \mathbf{y}_j^*)$. For the function $e_j\mathbf{u}$ we have the following boundary condition

$$\mathbb{B}(x, D)e_j\mathbf{u} = -[e_j, \mathbb{B}]\mathbf{u} + e_j\mathbf{g}. \tag{2.2}$$

Let $\psi_j(x) = \psi(x) + \epsilon(\widetilde{e}_j(x) - 1)$, $\phi_j(x) = e^{\tau\psi_j(x)}$ and $\epsilon \in (0, 1)$. The function ψ_j satisfies Condition 1.1 and Condition 1.2 for all sufficiently small ϵ. Applying Carleman estimate (1.24) to the equation $P(x, D)e_j\mathbf{u} = e_j\mathbf{f} - [e_j, P]\mathbf{u}$, we have

(2.3)
$$\begin{aligned}
&\int_Q \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi}dx + s\left\|\left(\mathbf{u}e^{s\phi}, \frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right)\right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)\times \mathbf{H}^{\frac{1}{2},s}(\partial Q)}\\
\le &C_1 \sum_j \int_{B_\delta(\mathbf{y}_j)} \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}e_j|^2 e^{2s\phi}dx\\
+&s\left\|\left(\mathbf{u}e^{s\phi}e_j, \frac{\partial \mathbf{u}}{\partial \vec{n}}e_j e^{s\phi}\right)\right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)\times \mathbf{H}^{\frac{1}{2},s}(\partial Q)}\\
\le &C_2 \sum_j \int_{B_{2\delta}(\mathbf{y}_j)} \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}e_j|^2 e^{2s\phi_j}dx\\
+&s\left\|\left(\mathbf{u}e^{s\phi_j}e_j, \frac{\partial \mathbf{u}}{\partial \vec{n}}e_j e^{s\phi_j}\right)\right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)\times \mathbf{H}^{\frac{1}{2},s}(\partial Q)}\\
\le &C_3\Bigg(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + \|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)} + \sum_j \|[e_j, \mathbb{B}(x, D)]\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)}\\
+&\sum_j \|[e_j, P]\mathbf{u}e^{s\phi_j}\|^2_{\mathbf{H}^{1,s}(Q)} + \int_{Q_\omega} \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi}dx\Bigg), \quad \forall s \ge s_0.
\end{aligned}$$

Note that $\|[e_j, P]\mathbf{u}e^{s\phi_j}\|^2_{\mathbf{H}^{1,s}(Q)} = \|[e_j, P]\mathbf{u}e^{s\phi_j}\|^2_{\mathbf{H}^{1,s}(B_{2\delta}(\mathbf{y}_j^*)\setminus B_{\frac{5}{4}\delta}(\mathbf{y}_j^*))}$. Moreover thanks to our choice of the functions ψ_j, we have $\phi_j(x) < \phi(x)$ for all $x \in \overline{B_{2\delta}(\mathbf{y}_j^*) \setminus B_{\frac{5}{4}\delta}(\mathbf{y}_j^*)}$. Therefore increasing s_0 if necessary, we have

$$\sum_j \|[e_j, P]\mathbf{u}e^{s\phi_j}\|^2_{\mathbf{H}^{1,s}(Q)} \le \frac{C}{s}\int_Q \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi}dx, \qquad \forall s \ge s_0.$$

Hence the fourth term on the right hand side of (2.3) can be absorbed into the left hand side, so that we obtain (1.24). Thus the proof of Lemma 2.1 is completed. $\square$

Without loss of generality, we may assume that $\rho \equiv 1$. Otherwise we introduce new coefficients $\mu_1 = \mu/\rho, \lambda_1 = \lambda/\rho$. We can directly prove that the functions $\mathrm{rot}\,\mathbf{u} \equiv \partial_{x_1} u_2 - \partial_{x_2} u_1$ and $\mathrm{div}\,\mathbf{u} \equiv \partial_{x_1} u_1 + \partial_{x_2} u_2$ satisfy the equations

$$\partial^2_{x_0} \mathrm{rot}\,\mathbf{u} - \mu\Delta \mathrm{rot}\,\mathbf{u} = m_1 \quad \text{in}\, Q, \quad \partial^2_{x_0} \mathrm{div}\,\mathbf{u} - (\lambda+2\mu)\Delta \mathrm{div}\,\mathbf{u} = m_2 \quad \text{in}\, Q, \tag{2.4}$$

$$m_1 = M_1(x,D)\mathrm{rot}\,\mathbf{u} + M_2(x,D)\mathrm{div}\,\mathbf{u} + \mathcal{M}_1(x,D)\mathbf{u} + \mathrm{rot}\,\mathbf{f},$$
$$m_2 = M_3(x,D)\mathrm{rot}\,\mathbf{u} + M_4(x,D)\mathrm{div}\,\mathbf{u} + \mathcal{M}_2(x,D)\mathbf{u} + \mathrm{div}\,\mathbf{f}.$$

Here and henceforth $\mathcal{M}_1(x,D)$, $\mathcal{M}_2(x,D)$, $M_j(x,D)$, $j = 1,2,3,4$, $\widetilde{\mathcal{K}}_j(y,D)$, $\widetilde{K}_j(y,D)$, etc. denote first order differential operators with L^∞-coefficients, if they are not specified.

If $\mathbf{y}^* \in Q$, then we take $\delta > 0$ sufficiently small and we can assume $B_\delta(\mathbf{y}^*) \cap \partial Q = \emptyset$. In that case, directly by means of (2.4), we have (1.24). Therefore, thanks to Lemma 2.1, we have to concentrate on the case $\mathbf{y}^* \in \partial Q$. Henceforth without loss of generality, we assume that

$$\mathbf{y}^* = (y_0^*, 0, 0), \quad \mathbf{y}^{*\prime} = (y_0^*, 0).$$

Below we will apply to equations (2.4) a Carleman estimate for the scalar equations. Since the functions $\mathrm{div}\,\mathbf{u}$ and $\mathrm{rot}\,\mathbf{u}$ do not have compact supports, some boundary integrals in this estimates will appear. In order to estimate these boundary integrals, it is convenient to use a weight function φ such that $\varphi|_{\partial\Omega} = \phi|_{\partial\Omega}$ and $\varphi(x) < \phi(x)$ for all x in a neighborhood of ∂Q. We construct such a function φ locally near the boundary $\partial\Omega$:

$$\varphi(x) = e^{\tau\widetilde{\psi}(x)}, \quad \widetilde{\psi}(x) = \psi(x) - \frac{1}{\sqrt{N}}\ell_1(x) + N\ell_1^2(x), \tag{2.5}$$

where $N > 0$ is a large positive parameter and $\ell_1 \in C^3(\overline{\Omega})$ satisfies

$$\ell_1(x') > 0 \quad \forall x' \in \Omega, \quad \ell_1|_{\partial\Omega} = 0, \ \nabla\ell_1|_{\partial\Omega} \neq 0.$$

Denote $\Omega_{\frac{1}{N^2}} = \{x' \in \Omega; 0 < \mathrm{dist}(x', \partial\Omega) < \frac{1}{N^2}\}$. Obviously for any fixed $\widehat{\epsilon} > 0$, there exists $N_0 > 0$ such that

$$\varphi(x) < \phi(x), \qquad \forall x \in [0,T] \times \Omega_{\frac{1}{N^2}}, \ N \in (N_0, \infty). \tag{2.6}$$

Our goal is to prove an analogue of (1.24) for the weight function φ instead of ϕ. We make the additional assumption

$$\mathrm{supp}\,\mathbf{u} \subset B_\delta(\mathbf{y}^*) \cap \mathcal{G} = \mathbb{R}^2 \times [0, 1/N^2]. \tag{2.7}$$

Below in order to simplify the notations we denote the functions $\psi(x(y))$, $\phi(x(y)), \varphi(x(y))$ as $\psi(y), \phi(y), \varphi(y)$. Therefore

$$\phi(y) = \varphi(y) \quad \text{on } \partial\mathcal{G}$$

and for some $N > N_1$ we have

$$\varphi(y) < \phi(y) \quad \forall y \in \mathcal{G}.$$

We need the following proposition:

Proposition 2.1. *Let* (2.7) *holds true. There exist* $\widehat{\tau} > 1$ *and* $N_1 > 1$ *such that for any* $\tau > \widehat{\tau}$, *there exists* $s_0(\tau, N) > 0$ *such that for any function* $\mathbf{u} \in \mathbf{H}^2(Q)$ *satisfying* (1.8) *and* (1.9), *we have*

$$N\|\mathbf{u}e^{s\varphi}\|^2_{\mathbf{H}^{2,s}(Q)} \le C_4 \Bigg(N\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{L}^2(Q)} + s\|(rot\,\mathbf{u}, div\,\mathbf{u})e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(Q)}$$

$$+N\left\|\left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial \vec{n}}\right)e^{s\varphi}\right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)\times \mathbf{H}^{\frac{1}{2},s}(\partial Q)}\Bigg), \quad s \ge s_0(\tau, N), \quad N \ge N_1$$

and C_4 *is independent of* N, s.

We give the proof of this proposition in Appendix I. Thanks to Lemma 2.1 we can work with the variable y instead of x. By (1.8) and (1.9) on the boundary $\partial\mathcal{G}$, we have

(2.8)
$$\frac{\partial^2 u_1}{\partial y_0^2} - \mu\left(\frac{\partial^2 u_1}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 u_1}{\partial y_1 \partial y_2}\right) + \mu\ell''(y_1)\frac{\partial u_1}{\partial y_2}$$
$$-(\lambda+\mu)\frac{\partial}{\partial y_1}\left(\mathrm{div}\,\mathbf{u} - \frac{\partial u_1}{\partial y_2}\ell'(y_1)\right) + (\lambda+\mu)\frac{\partial^2 u_1}{\partial y_1 \partial y_2}\ell'(y_1)$$
$$= f_1 + \mu(1+|\ell'|^2)\frac{\partial^2 u_1}{\partial y_2^2} - \frac{\partial^2 u_2}{\partial y_2^2}(\lambda+\mu)\ell' + (\lambda+\mu)\frac{\partial^2 u_1}{\partial y_2^2}|\ell'|^2 - \widetilde{K}_3(y, D)\mathbf{u}$$

and

$$\frac{\partial^2 u_2}{\partial y_0^2} - \mu\left(\frac{\partial^2 u_2}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 u_2}{\partial y_1 \partial y_2}\right) + \mu\ell''(y_1)\frac{\partial u_2}{\partial y_2} - (\lambda+\mu)\frac{\partial^2 u_1}{\partial y_1 \partial y_2}$$
$$= f_2 + \mu(1+|\ell'|^2)\frac{\partial^2 u_2}{\partial y_2^2} + (\lambda+\mu)\left(\frac{\partial^2 u_2}{\partial y_2^2} - \frac{\partial^2 u_1}{\partial y_2^2}\ell'\right) - \widetilde{K}_4(y, D)\mathbf{u}. \tag{2.9}$$

Here and below by $K_j(y, D)$ we denote a general first order differential operator with C^1 coefficient. By (1.12) we know that

(2.10)
$$\frac{\partial u_1}{\partial y_2} = \left(A_1(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) + (\widetilde{A}_1(y_1), \mathbf{g}), \quad \frac{\partial u_2}{\partial y_2} = \left(A_2(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) + (\widetilde{A}_2(y_1), \mathbf{g}).$$

Hence

$$\frac{\partial^2 u_1}{\partial y_2 \partial y_1} = \left(A_1(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_1'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) + \frac{\partial}{\partial y_1}(\widetilde{A}_1(y_1), \mathbf{g}), \tag{2.11}$$
$$\frac{\partial^2 u_2}{\partial y_2 \partial y_1} = \left(A_2(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_2'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) + \frac{\partial}{\partial y_1}(\widetilde{A}_2(y_1), \mathbf{g}).$$

Using these equations we may transform (2.8) and (2.9) to

$$\begin{aligned}
(2.12)\quad & B_1(y', D')\mathbf{u} \\
&= \frac{\partial^2 u_1}{\partial y_0^2} - \mu \left\{ \frac{\partial^2 u_1}{\partial y_1^2} - 2\ell'(y_1) \left(\left(A_1(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_1'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) \right) \right\} \\
&+ \mu \ell''(y_1) \left(A_1(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) - (\lambda+\mu) \frac{\partial^2 u_1}{\partial y_1^2} \\
&- (\lambda+\mu) \left\{ \left(A_2(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_2'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) \right\} \\
&+ (\lambda+\mu) \left(A_1(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) \ell''(y_1) \\
&+ (\lambda+\mu) \left\{ \left(A_1(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_1'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) \right\} \ell'(y_1) \\
&+ (\lambda+\mu)\ell'(y_1) \left\{ \left(A_1(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_1'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) \right\} \\
&\equiv f_1 + \mu(1+|\ell'|^2) \frac{\partial^2 u_1}{\partial y_2^2} - \frac{\partial^2 u_2}{\partial y_2^2}(\lambda+\mu)\ell' \\
&+ (\lambda+\mu) \frac{\partial^2 u_1}{\partial y_2^2} |\ell'|^2 + \widetilde{\mathcal{K}_5}(y, D')\mathbf{u} + \mathfrak{R}_1(y, D)\mathbf{g}
\end{aligned}$$

and

$$\begin{aligned}
(2.13)\quad & \\
& B_2(y', D')\mathbf{u} \\
&= \frac{\partial^2 u_2}{\partial y_0^2} - \mu \left\{ \frac{\partial^2 u_2}{\partial y_1^2} - 2\ell'(y_1) \left(\left(A_2(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_2'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) \right) \right\} \\
&- \mu \ell''(y_1) \left(A_2(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) - (\lambda+\mu) \left\{ \left(A_1(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_1'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) \right\} \\
&\equiv f_2 + \mu(1+|\ell'|^2) \frac{\partial^2 u_2}{\partial y_2^2} + (\lambda+\mu) \left(\frac{\partial^2 u_2}{\partial y_2^2} - \frac{\partial^2 u_1}{\partial y_2^2} \ell' \right) \\
&+ \widetilde{\mathcal{K}_6}(y, D')\mathbf{u} + \mathfrak{R}_2(y, D)\mathbf{g}.
\end{aligned}$$

Here $\mathfrak{R}_1(y, D), \mathfrak{R}_2(y, D)$ are first order differential operators with C^1 coefficients. Set

$$B(y',\xi)=\begin{pmatrix} B_{11}(y,\xi) & B_{12}(y,\xi)\\ B_{21}(y,\xi) & B_{22}(y,\xi)\end{pmatrix},$$

$$Q(y')=\begin{pmatrix} \mu+(\lambda+2\mu)|\ell'(y_1)|^2 & -(\lambda+\mu)\ell'(y_1)\\ -(\lambda+\mu)\ell'(y_1) & (\lambda+2\mu)+\mu|\ell'(y_1)|^2\end{pmatrix}.$$

In terms of the new notations, we may rewrite (2.12) and (2.13) as

(2.14)
$$\widetilde{B}(y',D')\mathbf{u}\equiv Q^{-1}B(y',D')\mathbf{u}$$
$$=-D_{y_2}^2\mathbf{u}+Q^{-1}\mathbf{f}(y',0)+Q^{-1}(\mathfrak{R}_1(y',D),\mathfrak{R}_2(y,D))\mathbf{g},\quad y\in\partial\mathcal{G}.$$

Note by (1.12) that the principal symbol of the operator $\widetilde{B}$ is given by
(2.15)
$$\widetilde{B}(\mathbf{y}^{*\prime},\xi)=\begin{pmatrix} \frac{-\xi_0^2+\left(\lambda(0)+2\mu(0)-\frac{\lambda(\lambda+\mu)}{\lambda+2\mu}(0)\right)\xi_1^2}{\mu(0)} & 0\\ 0 & \frac{-\xi_0^2-\lambda(0)\xi_1^2}{(\lambda+2\mu)(0)}\end{pmatrix}\quad \forall\xi\in\mathbb{R}^n\setminus 0,$$

and that

$$Q(0)=\begin{pmatrix}\mu(0) & 0\\ 0 & (\lambda+2\mu)(0)\end{pmatrix}.$$

In the y-coordinate, equations (2.4) for $z_1\equiv\operatorname{rot}\mathbf{u}$ and $z_2\equiv\operatorname{div}\mathbf{u}$ have the form

$$\text{(2.16)}\quad P_\mu(y,D)z_1=D_0^2z_1-\mu(D_2^2z_1-2\ell'(y_1)D_1D_2z_1+(1+|\ell'(y_1)|^2)D_2^2z_1)$$
$$-\mu i\ell''(y_1)D_2z_1=m_1,$$

$$\text{(2.17)}\quad P_{\lambda+2\mu}(y,D)z_2=D_0^2z_2-(\lambda+2\mu)(D_2^2z_2-2\ell'(y_1)D_1D_2z_2$$
$$+(1+|\ell'(y_1)|^2)D_2^2z_2)-(\lambda+2\mu)i\ell''(y_1)D_2z_2=m_2.$$

After the change of the coordinates, we use the same letters m_1,m_2 as in (2.4).

We consider a finite covering of the unit sphere $S^2\equiv\{(s,\xi_0,\xi_1);s^2+\xi_0^2+\xi_1^2=1\}$. That is, $S^2\subset\cup_{\nu=1}^{K(\delta_1)}\{(s,\xi_0,\xi_1)\in S^2;|\zeta-\zeta_\nu^*|<\delta_1\}$ where $\zeta_\nu^*\in S^2$, and by $\{\chi_\nu(\zeta)\}_{1\le\nu\le K(\delta_1)}$ we denote the corresponding partition of unity: $\sum_{\nu=1}^{K(\delta_1)}\chi_\nu(\zeta)=1$ for any $\zeta\in S^2$ and $\operatorname{supp}\chi_\nu\subset\{\zeta\in S^2;|\zeta-\zeta_\nu^*|<\delta_1\}$. Henceforth we extend χ_ν to the set $\{\zeta;|\zeta|\ne 1\}$ as the homogeneous function of the order zero in $C^\infty(\mathbb{R}^3\setminus\{0\})$ such that

$$\operatorname{supp}\chi_\nu\subset\mathcal{O}(\delta_1)\equiv\left\{\zeta;\left|\frac{\zeta}{|\zeta|}-\zeta^*\right|<\delta_1\right\}.$$

Moreover we set

$$P_{\mu,s}(y,s,D) = P_\mu(y,\mathbf{D}) = \mathbf{D}_{y_0}^2 - \mu(\mathbf{D}_{y_2}^2 - 2\ell'(y_1)\mathbf{D}_{y_1}\mathbf{D}_{y_2} + (1+|\ell'(y_1)|^2)\mathbf{D}_{y_2}^2) \\ -\mu i\ell''(y_1)\mathbf{D}_{y_2}$$

$$P_{\lambda+2\mu,s}(y,s,D) = P_{\lambda+2\mu}(y,\mathbf{D}) = \mathbf{D}_{y_0}^2 - (\lambda+2\mu)(\mathbf{D}_{y_2}^2 - 2\ell'(y_1)\mathbf{D}_{y_1}\mathbf{D}_{y_2} \\ +(1+|\ell'(y_1)|^2)\mathbf{D}_{y_2}^2) - (\lambda+2\mu)i\ell''(y_1)\mathbf{D}_{y_2}.$$

Under some condition, we can factor the operator $P_{\beta,s}$ as a product of two pseudodifferential operators.

Proposition 2.2. *Let $\beta \in \{\mu, \lambda+2\mu\}$ and $|r_\beta(y,\zeta)| \ge \widehat{\delta} > 0$ for all $(y,\zeta) \in B_\delta(\mathbf{y}^*) \times \mathcal{O}(2\delta_1)$. Then we can factor the operator $P_{\beta,s}$ into the product of two pseudodifferential operators*:

(2.18)
$$P_{\beta,s}\chi_\nu(s,D')V = \beta|G|(D_{y_2} - \Gamma_\beta^-(y,s,D'))(D_{y_2} - \Gamma_\beta^+(y,s,D'))\chi_\nu(s,D')V \\ +T_\beta V,$$

where $\operatorname{supp} V \subset B_\delta(\mathbf{y}^*) \cap \mathcal{G}$ *and*

$$T_\beta \in \mathcal{L}(L^2(0,1;H^{1,s}(\mathbb{R}^3)); L^2(0,1;L^2(\mathbb{R}^3))).$$

Let us consider the equation

$$(D_{y_2} - \Gamma_\beta^-(y,s,D'))\chi_\nu(s,D')V = q, \quad V|_{y_2=\frac{1}{N^2}} = 0, \quad \operatorname{supp}\ V \subset B_\delta(\mathbf{y}^*) \cap \mathcal{G}.$$

For solutions of this problem we have an a priori estimate:

Proposition 2.3. *Let $\beta \in \{\mu, \lambda+2\mu\}$ and $|r_\beta(y,\zeta)| \ge \widehat{\delta} > 0$ for all $(y,\zeta) \in B_\delta(\mathbf{y}^*) \times \mathcal{O}(2\delta_1)$. Then there exists a constant $C_5 > 0$ independent of N such that*

$$\|\sqrt{s}\chi_\nu(s,D')V|_{y_2=0}\|_{L^2(\mathbb{R}^2)} \le C_5\|q\|_{L^2(\mathcal{G})}. \tag{2.19}$$

The proofs of Propositions 2.2 and 2.3 can be found for example in [IY3]. Next we consider the equation

$$(D_{y_2} - \Gamma_\beta^+(y,s,D'))\chi_\nu(s,D')w = g, \quad w|_{y_2=1/N^2} = 0, \quad \operatorname{supp}\ w \subset B_\delta(\mathbf{y}^*) \cap \mathcal{G}.$$

We have

Proposition 2.4. *Let* $\beta \in \{\mu, \lambda + 2\mu\}$ *and* $|r_\beta(y,\zeta)| \ge \widehat{\delta} > 0$ *for all* $(y,\zeta) \in B_\delta(\mathbf{y}^*) \times \mathcal{O}(\delta_1)$, $s^* \ne 0$, *or* $\operatorname{Im}\alpha^+_\beta(\gamma) \ne 0$ *and* $\operatorname{supp}\chi_\nu \subset \mathcal{O}(\delta_1)$. *Then for sufficiently small* δ, δ_1 *there exists a constant* $C_6 > 0$ *independent of* N *such that*

$$\|\chi_\nu(s,D')w\|_{H^{1,s}(\mathcal{G})} \le C_6 \left(\frac{1}{\sqrt{s}}\|g\|_{H^{1,s}(\mathcal{G})} + s^{\frac14}\|\chi_\nu w(\cdot,0)\|_{H^{\frac12,s}(\partial\mathcal{G})}\right). \tag{2.20}$$

For the proof of this proposition, we can use the exactly same arguments as in [IP], and we give it in Appendix II for completeness.

Let $\beta \in \{\mu, \lambda+2\mu\}$ and $\widetilde{w} = \widetilde{w}(y)$ satisfy a scalar second order hyperbolic equation

$$P_{\beta,s}(y,s,D)\widetilde{w} = q \quad \text{in } \mathcal{G}, \quad \frac{\partial \widetilde{w}}{\partial y_2}|_{y_2=1/N^2} = \widetilde{w}|_{y_2=1/N^2} = 0, \quad \operatorname{supp}\widetilde{w} \subset B_\delta(\mathbf{y}^*).$$

Next we remind main facts related to the Carleman estimates with boundary for the operator $P_{\beta,s}(y,s,D)$. Set $L_{+,\beta}(y,s,D) = \frac{P_{\beta,s}(y,s,D)+P^*_{\beta,s}(y,s,D)}{2}$ and $L_{-,\beta}(y,s,D) = \frac{P_{\beta,s}(y,s,D)-P^*_{\beta,s}(y,s,D)}{2}$, where $P^*_{\beta,s}$ is the formally adjoint operator to $P_{\beta,s}$. One can easily check that the principal part of the operator $L_{-,\beta}$ is given by formula

$$\begin{aligned} &L_{-,\beta}(y,s,D)\widetilde{w} = -2s\varphi_{y_0}\frac{\partial \widetilde{w}}{\partial y_0} \\ &+\beta\left\{2s\varphi_{y_1}\frac{\partial \widetilde{w}}{\partial y_1} - 2s\ell'(y_1)\left(\varphi_{y_2}\frac{\partial \widetilde{w}}{\partial y_1} + \varphi_{y_1}\frac{\partial \widetilde{w}}{\partial y_2}\right) + 2s(1+(\ell'(y_1))^2)\varphi_{y_2}\frac{\partial \widetilde{w}}{\partial y_2}\right\} \\ &+l.o.t. \end{aligned}$$

Obviously $L_{+,\beta}(y,s,D)\tilde{w} + L_{-,\beta}(y,s,D)\tilde{w} = q$. For all $s \in \mathbb{R}^1$ the following equality holds true:

$$\begin{aligned} \Sigma_\beta(\widetilde{w}) + \|L_{-,\beta}(y,s,D)\widetilde{w}\|^2_{L^2(\mathcal{G})} + \|L_{+,\beta}(y,s,D)\widetilde{w}\|^2_{L^2(\mathcal{G})} \\ + \operatorname{Re}\int_{\mathcal{G}}([L_{+,\beta}, L_{-,\beta}]\widetilde{w}, \overline{\widetilde{w}})dy = \|q\|^2_{L^2(\mathcal{G})}, \end{aligned} \tag{2.21}$$

where

(2.22)

$$\begin{aligned} \Sigma_\beta(\widetilde{w}) = &\int_{\partial\mathcal{G}} \widetilde{p}_\beta(y,\nabla\varphi,(0,-\vec{e}_2,-1,0))(s\widetilde{p}_\beta(y,\nabla\widetilde{w},\overline{\nabla\widetilde{w}}) \\ &- s^3\tilde{p}_\beta(y,\nabla\varphi,\nabla\varphi)|\widetilde{w}|^2)dy_0dy_1 + \operatorname{Re}\int_{\partial\mathcal{G}}\widetilde{p}_\beta(y,\nabla\widetilde{w},-\vec{e}_2)\overline{L_{-,\beta}(y,s,D)\widetilde{w}}dy_0dy_1, \end{aligned}$$

and

$$\widetilde{p}_\beta(y,\xi,\widetilde{\xi}) = \xi_0\widetilde{\xi}_0 - \beta(\xi_1\widetilde{\xi}_1 - \ell'(y_1)(\xi_1\widetilde{\xi}_2 + \xi_2\widetilde{\xi}_1) + (1+|\ell'(y_1)|^2)\xi_2\widetilde{\xi}_2).$$

We note that by (2.5) $\phi_{y_k}|_{\partial\mathcal{G}} = \varphi_{y_k}|_{\partial\mathcal{G}}$ for $k \in \{0,1\}$. Therefore on $\partial\mathcal{G}$ the function $\nabla_{y'}\varphi$ is independent of N and $|\nabla\phi(y') - \nabla\varphi(y')| \le C_7/\sqrt{N}$ where the constant C_7 is independent of N. In particular, for all sufficiently large N, we have (1.6) for the function $\widetilde{\psi}$.

It is convenient for us to rewrite (2.21) in the form

$$\Sigma_\beta(\widetilde{w}) = \Sigma^{(1)}_\beta(\widetilde{w}) + \Sigma^{(2)}_\beta(\widetilde{w}),$$

$$\begin{aligned}
&\Sigma^{(1)}_\beta(\widetilde{w}) = \mathrm{Re}\int_{y_2=0} 2s\beta(\mathbf{y}^*)\frac{\partial\widetilde{w}}{\partial y_2} \\
&\times\left\{(\beta(\mathbf{y}^*)\frac{\partial\widetilde{w}}{\partial y_1}\varphi_{y_1}(\mathbf{y}^*) + \beta(\mathbf{y}^*)\frac{\partial\widetilde{w}}{\partial y_2}\varphi_{y_2}(\mathbf{y}^*) - \frac{\partial\widetilde{w}}{\partial y_0}\varphi_{y_0}(\mathbf{y}^*))\right\}dy_0dy_1 \\
&+\int_{y_2=0} s\beta(\mathbf{y}^*)\varphi_{y_2}(\mathbf{y}^*)\left\{\left|\frac{\partial\widetilde{w}}{\partial y_0}\right|^2 - \beta(\mathbf{y}^*)\left(\left|\frac{\partial\widetilde{w}}{\partial y_1}\right|^2 + \left|\frac{\partial\widetilde{w}}{\partial y_2}\right|^2\right)\right. \\
&\left. - s^2(\varphi^2_{y_0}(\mathbf{y}^*) - \beta(\mathbf{y}^*)(\varphi^2_{y_1}(\mathbf{y}^*) + \varphi^2_{y_2}(\mathbf{y}^*)))|\widetilde{w}|^2\right\}dy_0dy_1.
\end{aligned}$$

Then

$$|\Sigma^{(2)}_\beta(\widetilde{w})| \le \epsilon(\delta)s\left\|\left(\frac{\partial\widetilde{w}}{\partial y_2},\widetilde{w}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}, \tag{2.23}$$

where $\epsilon(\delta) \to 0$ as $\delta \to +0$. It is known (see e.g., [Im]) that there exists a parameter $\widehat{\tau} > 1$ such that for any $\tau > \widehat{\tau}$ there exists $s_0(\tau)$ such that

$$\begin{aligned}
&\frac{1}{4}\|L_{-,\beta}(y,s,D)\widetilde{w}\|^2_{L^2(\mathcal{G})} + \frac{1}{4}\|L_{+,\beta}(y,s,D)\widetilde{w}\|^2_{L^2(\mathcal{G})} \\
&+\mathrm{Re}([L_{+,\beta},L_{-,\beta}]\widetilde{w},\overline{\widetilde{w}})_{L^2(\mathcal{G})} + C_8 s\|\widetilde{w}\|_{L^2(\partial\mathcal{G})}\|\partial_{y_2}\widetilde{w}\|_{L^2(\partial\mathcal{G})} \\
&\ge C_9 s\|\widetilde{w}\|^2_{H^{1,s}(\mathcal{G})} \quad \forall s \ge s_0(\tau),
\end{aligned} \tag{2.24}$$

where $C_9 > 0$ is independent of s. Combining (2.21) and (2.24), we arrive at

$$\begin{aligned}
&\frac{1}{4}\|L_{-,\beta}(y,s,D)\widetilde{w}\|^2_{L^2(\mathcal{G})} + \frac{1}{4}\|L_{+,\beta}(y,s,D)\widetilde{w}\|^2_{L^2(\mathcal{G})} \\
&+C_9 s\|\widetilde{w}\|^2_{H^{1,s}(\mathcal{G})} + \Sigma_\beta(\widetilde{w}) \\
&\le C_{10}(\|q\|^2_{L^2(\mathcal{G})} + s\|\widetilde{w}\|_{L^2(\partial\mathcal{G})}\|\partial_{y_2}\widetilde{w}\|_{L^2(\partial\mathcal{G})} + \|\widetilde{w}\|^2_{H^{1,s}(\partial\mathcal{G})}), \quad \forall s \ge s_0(\tau).
\end{aligned} \tag{2.25}$$

We finish this section by recalling some results of calculus of pseudodifferential operators with symbols of limited smoothness. Let

$$C^k S^p_{1,0} = \{a(x,s,\xi); |\partial_x^\alpha \partial_\xi^\beta a(x,s,\xi)| \le c_{\alpha,\beta}(1+|(\xi,s)|^{p-|\beta|}), \quad |\alpha| \le k\}$$

for some compact set $K \subset \Omega$ we have $a(x,s,\xi) = 0$ for $x \notin K$.

For every symbol $a(x,s,\xi) \in C^k S^p_{1,0}$ we introduce the operator $a(x,s,D)u = \int_{\mathbb{R}^n} a(x,s,\xi)\hat{u}(\xi)e^{i(x,\xi)}d\xi$. It is known (see [T2]) that $a(x,s,D) \in \mathcal{L}(H_0^{m+p,s}(\Omega), H^{m,s}(\mathbb{R}))$ for $m \in (-k,k)$. Here $\widehat{u}$ is the Fourier transform.

For the operators with nonsmooth symbols, we have the Gårding inequality: Let $p(x,s,\xi) \in C^\ell S^{2m}_{1,0}$, $\ell > m$ and there exists $C > 0$ such that if $\mathrm{Re}\, p(x,s,\xi) \ge C|(\xi,s)|^m$ for all large $|(\xi,s)|$, then

$$Re(p(x,s,D)u,u)_{L^2(\Omega)} \le C_0\|u\|^2_{H^{m/2,s}(\Omega)} - C_1\|u\|^2_{L^2(\Omega)}$$

for any u such that supp $u \subset K$.

We say that a symbol $a(x,s,\xi) \in C^k S^p_{cl}$ if $a(x,s,\xi) \in C^k S^p_{1,0}$ and

$$a(x,s,\xi) = \sum_{j\ge 0} a_j(x,s,\xi), \quad \forall|(s,\xi)| \ge 1$$

where $a(x,s,\xi) - \sum_{j\ge 0}^N a_j(x,s,\xi) \in C^k S^{p-N}_{1,0}$.

The following proposition will be very useful.

Proposition 2.5 [T2]. *Let* $A_j(x,s,\xi) \in C^1 S^j_{cl}$, $B(x,s,\xi) \in C^1 S^\mu_{cl}$. *Then*

$$A_j(x,s,D)B(x,s,D) = C_j(x,s,D) + R_j \quad j \in \{0,1\},$$
$$C_j(x,s,\xi) = A_j(x,s,\xi)B(x,s,\xi)$$

and

$$R_0 : H^{\mu+\ell,s} \to H^{\ell+1,s}; \quad R_1 : H^{\mu,s} \to L^2.$$

We argue microlocally to obtain the Carleman estimate for the function $\chi_\nu(D',s)(\mathbf{u}e^{s\varphi})$. In Section 3, we consider the case where the support of the function χ_ν is in a neighbourhood of ζ^* such that $r_\mu(\mathbf{y}^*,\zeta^*) = 0$. The case $r_{\lambda+2\mu}(\mathbf{y}^*,\zeta^*) = 0$ is discussed in Section 4. In Section 5, we consider the case of $r_\mu(\mathbf{y}^*,\zeta^*) \ne 0$ and $r_{\lambda+2\mu}(\mathbf{y}^*,\zeta^*) \ne 0$. Hence all the possible cases are covered. Finally, for completing the proof of Theorem 1.1, we combine all these microlocal estimates.

§3. Case $r_\mu(\gamma) = 0$

In this section we treat the case where $\operatorname{supp}\chi_\nu \subset \mathcal{O}(\delta_1)$ and $r_\mu(\gamma) = 0$ for $\gamma = (\mathbf{y}^*, \zeta^*) \in \partial\mathcal{G} \times S^2$. Throughout this paper, we use the following notations:

$$\mathbf{v} = \mathbf{u}e^{s\varphi}, \quad \mathbf{w} = \mathbf{z}e^{s\varphi},$$

$$\mathbf{z}(y) = (z_1(y), z_2(y)) = (\operatorname{rot}\mathbf{u}, \operatorname{div}\mathbf{u})(x), \quad \mathbf{u} = (u_1, u_2), \quad \mathbf{v} = (v_1, v_2).$$

Henceforth $\widehat{v}(s, \xi', y_2)$ is the Fourier transform of $v(s, y_0, y_1, y_2)$ with respect to y_0, y_1, and we set $\mathbf{w}_\nu \equiv (w_{1,\nu}, w_{2,\nu}) = \chi_\nu(s, D')\mathbf{w}$. This section is devoted to the proof of the following lemma.

Lemma 3.1. *Let $\gamma = (\mathbf{y}^*, \zeta^*) \in \partial\mathcal{G} \times S^2$ be a point such that $r_\mu(\gamma) = 0$ and $\operatorname{supp}\chi_\nu \subset \mathcal{O}(\delta_1)$. Then for all sufficiently small $\delta_1 > 0$, we have*

(3.1)
$$N \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|} \|\partial_y^\alpha \mathbf{v}_\nu\|^2_{\mathbf{L}^2(\mathcal{G})} + s \left\| \left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2} \right) \right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G}) \times \mathbf{H}^{1,s}(\partial\mathcal{G}) \times \mathbf{L}^2(\partial\mathcal{G})}$$
$$\le C(\|\mathbf{v}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}),$$

where C is independent of s, N.

Proof. There exists a constant $C_1 > 0$ such that

$$|\xi_0^2 - s^2\varphi_{y_0}^2(\mathbf{y}^*) - \mu(0)\xi_1^2 + \mu(0)s^2\varphi_{y_1}^2(\mathbf{y}^*)| \le C_1\delta_1(|\xi_0|^2 + |\xi_1|^2 + s^2), \quad \forall \zeta \in \mathcal{O}(\delta_1). \tag{3.2}$$

We recall that by (2.21)–(2.25) there exist constants $C_2, C_3 > 0$ such that

$$C_2 s\|w_{1,\nu}\|^2_{H^{1,s}(\mathcal{G})} + \Sigma_\mu^{(1)} \le C_3\|P_{\mu,s}w_1\|^2_{L^2(\mathcal{G})} + \epsilon(\delta)s \left\| \left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu} \right) \right\|^2_{L^2(\partial\mathcal{G}) \times H^{1,s}(\partial\mathcal{G})}, \tag{3.3}$$

where $\epsilon(\delta) \to 0$ as $\delta \to +0$. Note that $\Sigma_\mu^{(1)}$ can be written in the form

(3.4)
$$\Sigma_\mu^{(1)}(w_{1,\nu}) = \int_{\partial\mathcal{G}} \left(s\mu^2(0)\varphi_{y_2}(\mathbf{y}^*) \left| \frac{\partial w_{1,\nu}}{\partial y_2} \right|^2 + s^3\mu^2(0)\varphi_{y_2}^3(\mathbf{y}^*)|w_{1,\nu}|^2 \right) d\Sigma$$
$$+ \operatorname{Re} \int_{\partial\mathcal{G}} 2s\mu(0)\frac{\partial w_{1,\nu}}{\partial y_2} \overline{\left(\mu(0)\varphi_{y_1}(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_1} - \varphi_{y_0}(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_0} \right)} d\Sigma$$
$$+ \int_{\partial\mathcal{G}} s\mu(0)\varphi_{y_2}(\mathbf{y}^*)(\xi_0^2 - \mu(0)\xi_1^2 - s^2\varphi_{y_0}^2(\mathbf{y}^*) + s^2\mu(0)\varphi_{y_1}^2(\mathbf{y}^*))|\widehat{w}_{1,\nu}|^2 d\Sigma$$
$$\equiv J_1 + J_2 + J_3.$$

Note that $r_{\lambda+2\mu}(\gamma) \neq 0$. Really if $r_{\lambda+2\mu}(\gamma) = 0$ then $\frac{\rho(0)}{(\lambda+2\mu)(0)}(\xi_0^*+is^*\varphi_{y_0}(\mathbf{y}^*))^2 - (\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*))^2 = 0$. On the other hand since $r_\mu(\gamma) = 0$ we have $\frac{\rho(0)}{\mu(0)}(\xi_0^* + is^*\varphi_{y_0}(\mathbf{y}^*))^2 - (\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*))^2 = 0$. Hence $\xi_0^* + is^*\varphi_{y_0}(\mathbf{y}^*) = 0$ and $\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*) = 0$. Since $|\zeta^*| = 1$, these equalities imply

$$\varphi_{y_0}(\mathbf{y}^*) = 0, \quad \varphi_{y_1}(\mathbf{y}^*) = 0, \quad \xi_0^* = \xi_1^* = 0, \;\; s^* = 1.$$

This contradicts assumption (1.6) which states $\frac{\partial\psi}{\partial t} \neq 0$. Therefore $r_{\lambda+2\mu}(\gamma) \neq 0$ and factorization (2.18) holds true. We set $V^+_{\lambda+2\mu} = (D_{y_2} - \Gamma^+_{\lambda+2\mu}(y,s,D'))w_{2,\nu}$. Then

$$P_{\lambda+2\mu,s}(y,s,D)w_{2,\nu} = (\lambda+2\mu)|G|(D_{y_2} - \Gamma^-_{\lambda+2\mu}(y,s,D'))V^+_{\lambda+2\mu} + T_{\lambda+2\mu}w_{2,\nu},$$

where $T_{\lambda+2\mu} \in \mathcal{L}(H^{1,s}(\mathcal{G}), L^2(\mathcal{G}))$. Therefore Proposition 2.3 immediately yields

$$\begin{aligned} &\sqrt{s}\|(D_{y_2} - \Gamma^+_{\lambda+2\mu}(y,s,D'))w_{2,\nu}|_{y_2=0}\|_{L^2(\partial\mathcal{G})} \\ &\le C_4(\|P_{\lambda+2\mu,s}w_{2,\nu}\|_{L^2(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \end{aligned} \tag{3.5}$$

Now we have to estimate $\Sigma_\mu^{(1)}$. First we note that

$$\begin{aligned} \frac{\partial z_1}{\partial y_2}|_{y_2=0} &= \frac{\partial^2 u_2}{\partial y_1\partial y_2} - \frac{\partial^2 u_1}{\partial y_2^2} - \frac{\partial^2 u_2}{\partial y_2^2}\ell'(y_1) \\ &= \left(A_2(y_1), \frac{\partial^2\mathbf{u}}{\partial y_1^2}\right) + \left(A_2'(y_1), \frac{\partial\mathbf{u}}{\partial y_1}\right) \\ &+ \frac{\partial}{\partial y_1}(\widetilde{A}_2(y_1), \mathbf{g}) - \frac{\partial^2 u_1}{\partial y_2^2} - \frac{\partial^2 u_2}{\partial y_2^2}\ell'(y_1), \end{aligned} \tag{3.6}$$

$$\begin{aligned} \frac{\partial z_2}{\partial y_2}|_{y_2=0} &= \frac{\partial^2 u_1}{\partial y_1\partial y_2} + \frac{\partial^2 u_2}{\partial y_2^2} - \frac{\partial^2 u_1}{\partial y_2^2}\ell'(y_1) \\ &= \left(A_1(y_1), \frac{\partial^2\mathbf{u}}{\partial y_1^2}\right) + \left(A_1'(y_1), \frac{\partial\mathbf{u}}{\partial y_1}\right) \\ &+ \frac{\partial}{\partial y_1}(\widetilde{A}_1(y_1), \mathbf{g}) + \frac{\partial^2 u_2}{\partial y_2^2} - \frac{\partial^2 u_1}{\partial y_2^2}\ell'(y_1). \end{aligned} \tag{3.7}$$

We may rewrite (3.6) and (3.7) as

$$e^{s\varphi}\begin{pmatrix} \frac{\partial z_1}{\partial y_2} \\ \frac{\partial z_2}{\partial y_2} \end{pmatrix} = -\mathcal{A}(y_1)\mathbf{D}^2_{y_1}\mathbf{v} - i\mathcal{A}'(y_1)\mathbf{D}_{y_1}\mathbf{v} + \mathfrak{K}(y', s, D')(\mathbf{g}e^{s\varphi}) - I(y_1)\mathbf{D}^2_{y_2}\mathbf{v}.$$

where we used the notations

$$\mathcal{A}(y_1) = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix}, \quad I(y_1) = \begin{pmatrix} -1 & -\ell'(y_1) \\ -\ell'(y_1) & 1 \end{pmatrix},$$

$$\mathcal{A}(0) = \begin{pmatrix} -\frac{\lambda}{\lambda+2\mu}(0) & 0 \\ 0 & -1 \end{pmatrix},$$

a_{ij} are the elements of the matrix A introduced in (1.12) and $\mathfrak{K}(y', s, D')$ stands for some first order differential operator. Therefore

$$(3.8) \qquad I^{-1}(y_1)e^{s\varphi}\begin{pmatrix} \frac{\partial z_1}{\partial y_2} \\ \frac{\partial z_2}{\partial y_2} \end{pmatrix} = -I^{-1}(y_1)\mathcal{A}(y_1)\mathbf{D}^2_{y_1}\mathbf{v} - iI^{-1}(y_1)\mathcal{A}'(y_1)\mathbf{D}_{y_1}\mathbf{v}$$
$$+ I^{-1}(y_1)\widetilde{\mathfrak{K}}(y', s, D')(\mathbf{g}e^{s\varphi}) - \mathbf{D}^2_{y_2}\mathbf{v}.$$

Using the definition of the operator $\Gamma^+_{\lambda+2\mu}(y, s, D')$, we have

$$(3.9) \quad \chi_\nu(s, D')\left(\frac{\partial z_2}{\partial y_2}e^{s\varphi}\right) = \chi_\nu(s, D')i\mathbf{D}_{y_2}w_2 = i\mathbf{D}_{y_2}w_{2,\nu} + i[\chi_\nu, \mathbf{D}_{y_2}]z_2$$
$$= iV^+_{\lambda+2\mu}(y', 0) - [\chi_\nu, s\varphi_{y_2}]w_2 + i\alpha^+_{\lambda+2\mu}(y', 0, s, D')(b_{21}(y_1, \mathbf{D}_{y_1})v_{1,\nu}$$
$$+ b_{22}(y_1, \mathbf{D}_{y_1})v_{2,\nu} + [\chi_\nu, b_2(y, \mathbf{D})]\mathbf{v} + \chi_\nu(\widetilde{C}_2(y_1)\mathbf{g}e^{s\varphi})).$$

Here we recall that b_{21} and b_{22} are defined by (1.16). Applying to the both sides of (3.8) the operator χ_ν and substituting (3.9) into (3.8), we can obtain

$$(3.10) \qquad \mathbf{D}^2_{y_2}\mathbf{v}_\nu = \widetilde{\mathbf{f}} - iI^{-1}(y_1)\mathbf{D}_{y_2}w_{1,\nu}\vec{e_1} + R(y', s, D')\mathbf{v}_\nu,$$

where we recall that $\vec{e_1} = (1, 0)$, and we set

$$\widetilde{\mathbf{f}} = -\chi_\nu I^{-1}$$
$$\times \begin{pmatrix} 0 \\ iV^+_{\lambda+2\mu}(y', 0) + i\alpha^+_{\lambda+2\mu}(y', 0, s, D')\chi_\nu(\widetilde{C}_2(y_1)\mathbf{g}e^{s\varphi}) - [\chi_\nu, s\varphi_{y_2}]w_2 + [\chi_\nu, b_2]\mathbf{v} \end{pmatrix}$$
$$+[\chi_\nu, -I^{-1}\mathcal{A}(y_1)\mathbf{D}^2_{y_1} - iI^{-1}(y_1)\widetilde{A}'(y_1)\mathbf{D}_{y_1}]\mathbf{v} + \chi_\nu(I^{-1}(y_1)\tilde{\mathfrak{K}}(y', s, D)(\mathbf{g}e^{s\varphi}))$$
$$-i[\chi_\nu, I^{-1}\mathbf{D}_{y_2}]\mathbf{w} - [\chi_\nu, \mathbf{D}^2_{y_2}]\mathbf{v},$$

$$R(y', s, D')\mathbf{v}_\nu$$
$$= I^{-1}(y_1)$$
$$\times \begin{pmatrix} 0 & 0 \\ -i\alpha^+_{\lambda+2\mu}(y', 0, s, D')b_{21}(y_1, \mathbf{D}_{y_1}) & -i\alpha^+_{\lambda+2\mu}(y', 0, s, D')b_{22}(y_1, \mathbf{D}_{y_1}) \end{pmatrix}\mathbf{v}_\nu$$
$$-I^{-1}(y_1)\mathcal{A}(y_1)\mathbf{D}^2_{y_1}\mathbf{v}_\nu - I^{-1}(y_1)\mathcal{A}'(y_1)\mathbf{D}_{y_1}\mathbf{v}_\nu,$$

$$\mathbf{w}_\nu = (w_{1,\nu}, w_{2,\nu}), \qquad \mathbf{v}_\nu = (v_{1,\nu}, v_{2,\nu}), \qquad \mathbf{v}_\nu = \chi_\nu(s, D')\mathbf{v}.$$

By (2.14) and (3.10), we obtain

$$
\begin{aligned}
(3.11)\qquad & \chi_\nu(s,D')\widetilde{B}(y',\mathbf{D}')\mathbf{v} \\
&= \chi_\nu(s,D')(Q^{-1}\mathbf{f}e^{s\varphi}) + \chi_\nu(s,D')(Q^{-1}e^{s\varphi}(\mathfrak{R}_1(y,D),\mathfrak{R}_2(y,D))\mathbf{g}) \\
&- [\chi_\nu,\mathbf{D}^2_{y_2}]\mathbf{v} - \widetilde{\mathbf{f}} + iI^{-1}(y_1)\mathbf{D}_{y_2}w_{1,\nu}\vec{e}_1 - R(y',s,D')\mathbf{v}_\nu \quad y\in\partial\mathcal{G}.
\end{aligned}
$$

Next we note that

(3.12)

$$
\det(\widetilde{B}(\mathbf{y}^{*\prime},\xi^{*\prime}+is^*\nabla'\varphi(\mathbf{y}^*))+R(\mathbf{y}^{*\prime},s^*,\xi^*)) = \frac{\mu(0)}{(\lambda+2\mu)(0)}(\xi_1^*+is^*\varphi_{y_1}(\mathbf{y}^*))^4.
$$

Really by (2.15)

$$
\widetilde{B}(\mathbf{y}^{*\prime},\xi^*+is^*\nabla'\varphi(\mathbf{y}^*)) =
$$

$$
\begin{pmatrix}
\frac{-(\xi_0^*+is^*\varphi_{y_0}(\mathbf{y}^*))^2+\left(\lambda(0)+2\mu(0)-\frac{\lambda(\lambda+\mu)}{\lambda+2\mu}(0)\right)(\xi_1+is^*\varphi_{y_1}(\mathbf{y}^*))^2}{\mu(0)} & 0 \\
0 & \frac{-(\xi_0^*+is^*\varphi_{y_0}(\mathbf{y}^*))^2-\lambda(0)(\xi_1+is^*\varphi_{y_1}(\mathbf{y}^*))^2}{(\lambda+2\mu)(0)}
\end{pmatrix}.
$$

Since $r_\mu(\gamma)=0$ we have $(\xi_0^*+is^*\varphi_{y_0}(\mathbf{y}^*))^2=\mu(0)(\xi_1+is^*\varphi_{y_1}(\mathbf{y}^*))^2$ which implies

$$
(3.13)\quad \widetilde{B}(\mathbf{y}^{*\prime},\xi^*+is^*\nabla'\varphi(\mathbf{y}^*)) = (\xi_1^*+is^*\varphi_{y_1}(\mathbf{y}^*))^2\begin{pmatrix}\frac{2(\lambda+\mu)}{\lambda+2\mu}(0) & 0\\ 0 & -\frac{(\mu+\lambda)}{(\lambda+2\mu)}(0)\end{pmatrix}
$$

and

$$
\alpha_{\lambda+2\mu}(\gamma) = i\mathrm{sign}(\xi_1^*)\sqrt{\frac{\lambda+\mu}{\lambda+2\mu}}(0)(\xi_1^*+is^*\varphi_{y_1}(\mathbf{y}^*))^2.
$$

Moreover by (1.16)

$$
\begin{pmatrix}
0 & 0\\
-i\alpha^+_{\lambda+2\mu}(\mathbf{y}^*,s^*,\xi^{*\prime})b_{21}(\mathbf{y}^*,\xi^*+is^*\nabla'\varphi(\mathbf{y}^*)) & -i\alpha^+_{\lambda+2\mu}(\mathbf{y}^*,s^*,\xi^{*\prime})b_{22}(\mathbf{y}^*,\xi^*+is^*\nabla'\varphi(\mathbf{y}^*))
\end{pmatrix}
$$

$$
=\begin{pmatrix}
0 & 0\\
i\mathrm{sign}(\xi_1^*)\sqrt{\frac{\lambda+\mu}{\lambda+2\mu}}(0)\frac{2\mu}{\lambda+2\mu}(0)(\xi_1^*+is^*\varphi_{y_1}(\mathbf{y}^*))^2 & 0
\end{pmatrix}
$$

and

$$
I^{-1}(0)\mathcal{A}(0) = \begin{pmatrix}\frac{\lambda}{\lambda+2\mu}(0) & 0\\ 0 & -1\end{pmatrix}.
$$

Thus

$$(3.14)\quad R(\mathbf{y}^{*\prime},\zeta^*)=\begin{pmatrix} -\frac{\lambda}{\lambda+2\mu}(0) & 0\\ i\mathrm{sign}(\xi_1^*)\frac{2\mu(0)}{(\lambda+2\mu)(0)}\sqrt{\frac{\lambda+\mu}{\lambda+2\mu}(0)} & 1\end{pmatrix}(\xi_1^*+is^*\varphi_{y_1}(\mathbf{y}^*))^2.$$

In terms of (3.13) and (3.14), we easily obtain (3.12). Set

$$S(y',s,\xi_0,\xi_1)=\widetilde{B}(y',\xi'+is\nabla'\varphi(y))+R(y',s,\xi').$$

Let $S(y',s,D')$ be the corresponding pseudodifferential operator. Then

$$\begin{aligned}(3.15)\quad & S(y',s,D')\mathbf{v}_\nu+[\chi_\nu,\widetilde{B}(y,\mathbf{D})]\mathbf{v}\\ &=\chi_\nu(Q^{-1}\mathbf{f}e^{s\varphi})-[\chi_\nu,\mathbf{D}^2_{y_2}]\mathbf{v}+\chi_\nu(Q^{-1}e^{s\varphi}(\mathfrak{R}_1(y,D),\mathfrak{R}_2(y,D))\mathbf{g})\\ &-\widetilde{\mathbf{f}}+iI^{-1}(y_1)\mathbf{D}_{y_2}w_{1,\nu}\vec{e}_1.\end{aligned}$$

Since we can directly verify that $\det S(\mathbf{y}^{*\prime},s^*,\xi^{*\prime})\ne 0$, we have

$$\begin{aligned}(3.16)\qquad & s\left\|\left(\mathbf{v}_\nu,\frac{\partial\mathbf{v}_\nu}{\partial y_2},\frac{\partial^2\mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}\\ &\le C_5(s\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{L}^2(\partial\mathcal{G})}+s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}\\ &+J_1+s\|V^+_{\lambda+2\mu}(\cdot,0)\|^2_{\mathbf{L}^2(\partial\mathcal{G})}).\end{aligned}$$

Note by (3.2) that for any $\epsilon>0$ there exists $\delta_2(\epsilon)>0$ such that

$$\begin{aligned}(3.17)\qquad & J_3\le\epsilon s\left\|\left(\mathbf{v}_\nu,\frac{\partial\mathbf{v}_\nu}{\partial y_2},\frac{\partial^2\mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}\\ &+C_6 s(\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}),\quad\forall\zeta\in\mathcal{O}(\delta_2(\epsilon)).\end{aligned}$$

In order to estimate the term J_2, we consider two cases. First we assume that $s^*\ne 0$. Then by (1.18)–(1.20) and $r_\mu(\gamma)=0$, for given $\epsilon>0$, there exists δ_0 such that if $\delta\in(0,\delta_0)$, then

$$(3.18)\qquad |\mu(0)\xi_1\varphi_{y_1}(\mathbf{y}^*)-\xi_0\varphi_{y_0}(\mathbf{y}^*)|\le\epsilon|\zeta|,\quad\forall\zeta\in\mathcal{O}(\delta).$$

By this inequality, we obtain

$$\begin{aligned}(3.19)\qquad & J_2\le\epsilon s\left\|\left(\mathbf{v}_\nu,\frac{\partial\mathbf{v}_\nu}{\partial y_2},\frac{\partial^2\mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}\\ &+C_7 s(\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}).\end{aligned}$$

Second let us assume that $s^*=0$. We solve equation (3.15) with respect to the variable $\mathbf{v}_\nu$:

(3.20) $$\mathbf{v}_\nu = S(y', s, D')^{-1}(-[\chi_\nu, \tilde B(y,\mathbf{D})]\mathbf{v} - [\chi_\nu, \mathbf{D}^2_{y_2}]\mathbf{v} + \chi_\nu(Q^{-1}\mathbf{f}e^{s\varphi}) + \chi_\nu(Q^{-1}(\mathfrak{R}_1(y,D), \mathfrak{R}_2(y,D))\mathbf{g}e^{s\varphi}) - \widetilde{\mathbf{f}} + iI^{-1}(y_1)\mathbf{D}_{y_2}w_{1,\nu}\vec e_1) + T\mathbf{v},$$

where $S(y', s, D')^{-1}$ is the bounded operator from $L^2(\partial\mathcal{G})$ into $H^{2,s}(\partial\mathcal{G})$ such that the principal symbol of the operator $S(y', s, D')^{-1}$ is the inverse matrix to the matrix $S(y', s, \xi_0, \xi_1)$ and $T \in \mathcal{L}(H^{1,s}(\partial\mathcal{G}), H^{2,s}(\partial\mathcal{G}))$. Therefore on $\partial\mathcal{G}$ we have

(3.21)
$$\mu(0)\varphi_{y_1}(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_1} - \varphi_{y_0}(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_0}$$
$$= i(\mu(0)\varphi_{y_1}(\mathbf{y}^*)D_{y_1} - \varphi_{y_0}(\mathbf{y}^*)D_{y_0})b_1(y_1, \mathbf{D}') \times \{S(y', s, D')^{-1}(-[\chi_\nu, \tilde B(y,\mathbf{D})]\mathbf{v}$$
$$+\chi_\nu(Q^{-1}\mathbf{f}e^{s\varphi}) - [\chi_\nu, \mathbf{D}^2_{y_2}]\mathbf{v} + \chi_\nu(Q^{-1}(\mathfrak{R}_1(y,D), \mathfrak{R}_2(y,D))\mathbf{g}e^{s\varphi})$$
$$-\widetilde{\mathbf{f}} + iI^{-1}(y_1)\mathbf{D}_{y_2}w_{1,\nu}\vec e_1) + T\mathbf{v}\}$$
$$+i(\mu(0)\varphi_{y_1}(\mathbf{y}^*)D_{y_1} - \varphi_{y_0}(\mathbf{y}^*)D_{y_0})([\chi_\nu, b_1]\mathbf{v} - \chi_\nu(\widetilde{C}_1\mathbf{g}e^{s\varphi})).$$

By $\mathcal{M}$ we denote the pseudodifferential operator with the symbol

$$\mathcal{M}(y', s, \xi') = (\varphi_{y_0}(\mathbf{y}^*)\xi_0 - \mu(0)\varphi_{y_1}(\mathbf{y}^*)\xi_1)b_1(y', \xi' + is\nabla'\varphi)S(y', s, \xi')^{-1}I^{-1}(y_1)\vec e_1.$$

Since $b_{11}(\mathbf{y}^{*\prime}, \xi') = 0$ and $b_{12}(\mathbf{y}^{*\prime}, \xi') = 2i\xi_1$, we have $\mathrm{Re}\,\mathcal{M}(\mathbf{y}^{*\prime}, s^*, \xi_0^*, \xi_1^*) = 0$. Therefore, by Gårding's inequality, we see

(3.22)
$$\mathrm{Re}\int_{\partial\mathcal{G}} \frac{\partial w_{1,\nu}}{\partial y_2}\overline{(\mu(0)\varphi_{y_1}(\mathbf{y}^*)\partial_{y_1} - \varphi_{y_0}(\mathbf{y}^*)\partial_{y_0})b_1(y', \mathbf{D}')S(y', s, D')^{-1}iI^{-1}\vec e_1\frac{\partial w_{1,\nu}}{\partial y_2}}d\Sigma$$
$$\ge -\epsilon\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}.$$

On the other hand

(3.23)
$$
\begin{aligned}
&s\|(\mu(0)\varphi_{y_1}(\mathbf{y}^*)\partial_{y_1} - \varphi_{y_0}(\mathbf{y}^*)\partial_{y_0})b_1(y',\mathbf{D}')\\
&\times\{S(y',s,D')^{-1}(-[\chi_\nu,\tilde{B}(y,\mathbf{D}) - \mathbf{D}^2_{y_2}]\mathbf{v}\\
&+\chi_\nu(Q^{-1}(\Re_1(y,D),\Re_2(y,D))\mathbf{g}e^{s\varphi}) + \tilde{\mathbf{f}} + \chi_\nu(Q^{-1}\mathbf{f}e^{s\varphi})) + T\mathbf{v}\}\\
&+i(\mu(0)\varphi_{y_1}(\mathbf{y}^*)D_{y_1} - \varphi_{y_0}(\mathbf{y}^*)D_{y_0})([\chi_\nu,b_1]\mathbf{v} - \chi_\nu(\widetilde{C}_1\mathbf{g}e^{s\varphi}))\}\|^2_{L^2(\partial\mathcal{G})}\\
&\quad\le s\|S(y',s,D')^{-1}(-[\chi_\nu,\tilde{B}(y,\mathbf{D}) - \mathbf{D}^2_{y_2}]\mathbf{v}\\
&+\chi_\nu(Q^{-1}(\Re_1(y,D),\Re_2(y,D))\mathbf{g}e^{s\varphi}) + \tilde{\mathbf{f}} + \chi_\nu(Q^{-1}\mathbf{f}e^{s\varphi})) + T\mathbf{v}\|^2_{H^{2,s}(\partial\mathcal{G})}\\
&+s\|(\mu(0)\varphi_{y_1}(\mathbf{y}^*)D_{y_1} - \varphi_{y_0}(\mathbf{y}^*)D_{y_0})([\chi_\nu,b_1]\mathbf{v} - \chi_\nu(\widetilde{C}_1\mathbf{g}e^{s\varphi}))\}\|^2_{L^2(\partial\mathcal{G})}\\
&\quad\le s\| - [\chi_\nu,\tilde{B}(y,\mathbf{D}) - \mathbf{D}^2_{y_2}]\mathbf{v} + \chi_\nu(Q^{-1}(\Re_1(y,D),\Re_2(y,D))\mathbf{g}e^{s\varphi})\\
&+\tilde{\mathbf{f}} + \chi_\nu(Q^{-1}\mathbf{f}e^{s\varphi}) + T\mathbf{v}\|^2_{L^2(\partial\mathcal{G})}\\
&+s\|(\mu(0)\varphi_{y_1}(\mathbf{y}^*)D_{y_1} - \varphi_{y_0}(\mathbf{y}^*)D_{y_0})([\chi_\nu,b_1]\mathbf{v} - \chi_\nu(\widetilde{C}_1\mathbf{g}e^{s\varphi}))\}\|^2_{L^2(\partial\mathcal{G})}\\
&\le C_8(\|P_{\lambda+2\mu,s}w_{2,\nu}\|^2_{L^2(\mathcal{G})} + s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + s\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{L}^2(\partial\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}).
\end{aligned}
$$

Here we recall that $\widetilde{C}_1$ is defined in (1.13). Inequalities (3.22) and (3.23) imply

$$
\begin{aligned}
J_2 \le \epsilon s\left\|\left(\mathbf{v}_\nu,\frac{\partial\mathbf{v}_\nu}{\partial y_2},\frac{\partial^2\mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}\\
+ C_9(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}).
\end{aligned}
\tag{3.24}
$$

By (3.5), (3.16), (3.17), (3.19) and (3.24), there exist constants $C_{10} > 0$ and $C_{11} > 0$ such that

$$
\begin{aligned}
\Sigma^{(1)}_\mu(w_{1,\nu}) \ge C_{10}s\left\|\left(\mathbf{v}_\nu,\frac{\partial\mathbf{v}_\nu}{\partial y_2},\frac{\partial^2\mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}\\
- C_{11}(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}).
\end{aligned}
\tag{3.25}
$$

By (3.3) and (3.25), we obtain

(3.26)
$$
\begin{aligned}
&\sqrt{s}\|w_{1,\nu}\|_{H^{1,s}(\mathcal{G})} + \sqrt{s}\left\|\left(\mathbf{v}_\nu,\frac{\partial\mathbf{v}_\nu}{\partial y_2},\frac{\partial^2\mathbf{v}_\nu}{\partial y_2^2}\right)\right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}\\
&\le C_{12}(\sqrt{s}\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}).
\end{aligned}
$$

Note that

$$(3.27)\ \sqrt{s}\left\|\left(w_{2,\nu},\frac{\partial w_{2,\nu}}{\partial y_2}\right)\right\|_{H^{1,s}(\partial\mathcal{G})\times L^2(\partial\mathcal{G})}$$
$$\le C_{13}\left(\left\|\left(\mathbf{v}_\nu,\frac{\partial \mathbf{v}_\nu}{\partial y_2},\frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}+\sqrt{s}\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}\right)$$
$$\le C_{14}(\sqrt{s}\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\sqrt{s}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

By (3.27), (2.24) and (2.25) with $\beta=\lambda+2\mu$, we have

$$(3.28)\ \sqrt{s}\|w_{2,\nu}\|_{H^{1,s}(\mathcal{G})}$$
$$\le C_{15}(\sqrt{s}\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\sqrt{s}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

Therefore combining (3.26) and (3.28), we obtain

$$\sqrt{s}\|\mathbf{w}_\nu\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\sqrt{s}\left\|\left(\mathbf{v}_\nu,\frac{\partial \mathbf{v}_\nu}{\partial y_2},\frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}$$
$$\le C_{16}(\sqrt{s}\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\sqrt{s}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}),$$
$$\forall s\ge s_0(\tau,N),\ N\ge N_1.$$

This inequality and Proposition 2.1 imply (3.1). Thus the proof of Lemma 3.1 is completed. □

§4. Case $r_{\lambda+2\mu}(\gamma)=0$

In this section, we will prove

Lemma 4.1. *Let $\gamma=(\mathbf{y}^*,\zeta^*)\in\partial\mathcal{G}\times S^2$ be a point such that $r_{\lambda+2\mu}(\gamma)=0$ and supp $\chi_\nu\subset\mathcal{O}(\delta_1)$. Then for all sufficiently small $\delta_1>0$, estimate* (3.1) *holds true.*

Proof. By (1.19) and (1.20), there exist $\delta_0>0$ and $C_1>0$ such that for all $\delta_1\in(0,\delta_0)$ we have

$$(4.1)\qquad \xi_0^2\le C_1(\xi_1^2+s^2),\qquad \forall\zeta\in\mathcal{O}(\delta_1).$$

We note that if $r_\mu(\gamma)=0$, then $\xi_0^*=\xi_1^*=0$, $s^*=1$ and $\varphi_{y_0}(\mathbf{y}^*)=\varphi_{y_1}(\mathbf{y}^*)=0$. By (1.6) this is impossible. Therefore $r_\mu(\gamma)\ne0$ must hold true.

Case A. Assume that $s^*=0$ and $\varphi_{y_2}(\mathbf{y}^*)>\frac{|\frac{1}{\mu(0)}\xi_0^*\varphi_{y_0}(\mathbf{y}^*)-\xi_1^*\varphi_{y_1}(\mathbf{y}^*)|}{\sqrt{\frac{\lambda+\mu}{\mu}(0)}|\xi_1^*|}$. Then by decreasing the parameter δ_1, we can assume that for some constant $C_2>0$

$$(4.2)\qquad \xi_0^2+s^2\le C_2\xi_1^2,\qquad \forall\zeta\in\mathcal{O}(\delta_1).$$

Then there exists a constant $C_3 > 0$ such that

$$-\mathrm{Im}\,\Gamma_\mu^\pm(y,\zeta) \ge C_3 s, \qquad \forall (y,\zeta)\in B_\delta(\mathbf{y}^*)\times \mathcal{O}(\delta_1),$$

provided that $|\delta| + |\delta_1|$ is sufficiently small. We set $V_\mu^\pm = (D_{y_2} - \Gamma_\mu^\pm(y,s,D'))w_{1,\nu}$. Then we can represent $P_{\mu,s}$ as

$$P_{\mu,s}(y,s,D)w_{1,\nu} = \mu|G|(D_{y_2} - \Gamma_\mu^\mp(y,s,D'))V_\mu^\pm + T_\mu^\pm w_{1,\nu},$$

where $T_\mu^\pm \in \mathcal{L}(H^{1,s}(\mathcal{G}), L^2(\mathcal{G}))$. This decomposition and Proposition 2.3 immediately imply

$$\begin{aligned} &\|\sqrt{s}(D_{y_2} - \Gamma_\mu^\mp(y,s,D'))w_{1,\nu}|_{y_2=0}\|_{L^2(\partial\mathcal{G})}\\ \le &C_4(\|P_{\mu,s}w_{1,\nu}\|_{L^2(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \end{aligned} \tag{4.3}$$

This inequality implies

$$\sqrt{s}\|D_{y_2}w_{1,\nu}\|_{L^2(\partial\mathcal{G})} + \sqrt{s}\|w_{1,\nu}\|_{H^{1,s}(\partial\mathcal{G})} \le C_5(\|P_{\mu,s}w_{1,\nu}\|_{L^2(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{4.4}$$

Next we need the estimate for $\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)$. We may rewrite equations (2.8) and (2.9) as

$$-\mathbf{D}_{y_0}^2 v_1 - (\lambda+2\mu)(i\mathbf{D}_{y_1} - i\ell'(y_1)\mathbf{D}_{y_2})w_2 + \mu i\mathbf{D}_{y_2}w_1 + \widetilde{K}_7(y,\mathbf{D})\mathbf{v} = f_1 e^{s\varphi}, \tag{4.5}$$

$$-\mathbf{D}_{y_0}^2 v_2 - (\lambda+2\mu)i\mathbf{D}_{y_2}w_2 - \mu(i\mathbf{D}_{y_1} - \ell'(y_1)i\mathbf{D}_{y_2})w_1 + \widetilde{K}_8(y,\mathbf{D})\mathbf{v} = f_2 e^{s\varphi}, \tag{4.6}$$

where $\widetilde{K}_7$ and $\widetilde{K}_8$ are first order differential operators. Furthermore, setting

$$\begin{aligned} q_1 &= f_1 e^{s\varphi} - \mu i\mathbf{D}_{y_2}w_1 - \widetilde{K}_7(y,\mathbf{D})\mathbf{v},\\ q_2 &= f_2 e^{s\varphi} + \mu(i\mathbf{D}_{y_1} - \ell'(y_1)i\mathbf{D}_{y_2})w_1 - \widetilde{K}_8(y,\mathbf{D})\mathbf{v}, \end{aligned}$$

we rewrite (4.5) and (4.6) as

$$\begin{cases} -\mathbf{D}_{y_0}^2 v_1 - (\lambda+2\mu)(i\mathbf{D}_{y_1} - i\ell'(y_1)\mathbf{D}_{y_2})w_2 = q_1,\\ -\mathbf{D}_{y_0}^2 v_2 - (\lambda+2\mu)i\mathbf{D}_{y_2}w_2 = q_2. \end{cases} \tag{4.7}$$

Using (4.7) we get rid of the term $\mathbf{D}_{y_2}w_2$ in (4.6):

$$\begin{aligned} &-\mathbf{D}_{y_0}^2 v_1 - (\lambda+2\mu)i\mathbf{D}_{y_1}(b_{21}(y_1,\mathbf{D}_{y_1})v_1 + b_{22}(y_1,\mathbf{D}_{y_1})v_2) - \ell'(y_1)\mathbf{D}_{y_0}^2 v_2\\ =&q_1 + \ell'(y_1)q_2 + \widetilde{K}_9(y',\mathbf{D}')\mathbf{g}. \end{aligned} \tag{4.8}$$

Using (1.13), we obtain
(4.9)

$$\mathbf{D}_{y_1} b_{11}(y', \mathbf{D}_{y_1}) v_1 + \mathbf{D}_{y_1} b_{12}(y', \mathbf{D}_{y_1}) v_2 = \mathbf{D}_{y_1} w_1|_{y_2=0} + \widetilde{K}_{10}(y', \mathbf{D}')(\mathbf{g}e^{s\varphi}).$$

Here and henceforth $\widetilde{K}_9$ and $\widetilde{K}_{10}$ are first order differential operators.
Set

$$\mathcal{K}(y', \mathbf{D}') = \begin{pmatrix} -\mathbf{D}^2_{y_0} - (\lambda+2\mu) i \mathbf{D}_{y_1} b_{21}(y', \mathbf{D}_{y_1}) & -(\lambda+2\mu) i \mathbf{D}_{y_1} b_{22}(y', \mathbf{D}_{y_1}) - \ell'(y_1)\mathbf{D}^2_{y_0} \\ \mathbf{D}_{y_1} b_{11}(y', \mathbf{D}_{y_1}) & \mathbf{D}_{y_1} b_{12}(y', \mathbf{D}_{y_1}) \end{pmatrix}.$$

By (4.8) and (4.9), we have

$$\mathcal{K}(y', \mathbf{D}')\mathbf{v} = \mathbf{m},$$

where $\mathbf{m} = (q_1 + \ell'(y_1) q_2 + \tilde{K}_9(y', \mathbf{D}')\mathbf{g}, \mathbf{D}_{y_1} w_1|_{y_2=0} + \tilde{K}_{10}(y', \mathbf{D}')(\mathbf{g}e^{s\varphi}))$. Therefore

$$\mathcal{K}(y', \mathbf{D}')\mathbf{v}_\nu = \chi_\nu(s, D')\mathbf{m} - [\chi_\nu(s, D'), \mathcal{K}]\mathbf{v},$$

and since $\det \mathcal{K}(0, \zeta^*) \neq 0$ if $\lambda(0) \neq 0$, we have

$$\mathbf{v}_\nu = \mathcal{K}^{-1}(y', \mathbf{D}')(\chi_\nu(s, D')\mathbf{m} - [\chi_\nu(s, D'), \mathcal{K}]\mathbf{v}) + T(y', s, D')\mathbf{v}_\nu,$$

where $T \in \mathcal{L}(H^{1,s}(\mathbb{R}^2), H^{2,s}(\mathbb{R}^2))$. Hence

$$\begin{aligned} (4.10) \qquad & \sqrt{s} \left\| \left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2} \right) \right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G}) \times \mathbf{H}^{1,s}(\partial\mathcal{G}) \times \mathbf{L}^2(\partial\mathcal{G})} \\ & \le C_6 \Bigg(\sqrt{s} \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \\ & + \sqrt{s} \left\| \left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu} \right) \right\|_{L^2(\partial\mathcal{G}) \times H^{1,s}(\partial\mathcal{G})} \Bigg). \end{aligned}$$

By (2.25) with $\beta = \lambda + 2\mu$ and $\beta = 2\mu$, we obtain

$$\begin{aligned} (4.11) \quad & s\|\mathbf{w}_\nu\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s \left\| \left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2} \right) \right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G}) \times \mathbf{H}^{1,s}(\partial\mathcal{G}) \times \mathbf{L}^2(\partial\mathcal{G})} \\ & \le C_7 (s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \end{aligned}$$

Applying Proposition 2.1 we obtain (3.1).

Next we need to consider two more cases. Using the definition of the operator $\Gamma^+_{\lambda+2\mu}(y,s,D')$, we have

$$\chi_\nu(s,D')\left(\frac{\partial z_1}{\partial y_2}e^{s\varphi}\right) = iV^+_\mu(y',0) - [\chi_\nu, s\varphi_{y_2}]w_1 \tag{4.12}$$
$$+ i\alpha^+_\mu(y',0,s,D')(b_{11}(y',\mathbf{D}_{y_1})v_{1,\nu} + b_{12}(y',\mathbf{D}_{y_1})v_{2,\nu}$$
$$+ [\chi_\nu, b_1]\mathbf{v} + \chi_\nu(\widetilde{C}_1(y_1)\mathbf{g}e^{s\varphi})).$$

Here we recall that the operators b_{21} and b_{22} are defined by (1.16). Substituting (4.12) into (3.8), we can obtain

$$\mathbf{D}^2_{y_2}\mathbf{v}_\nu = \widetilde{\mathbf{f}}_0 - iI^{-1}(y_1)\mathbf{D}_{y_2}w_{2,\nu}\vec{e}_2 + \widetilde{R}(y',s,D')\mathbf{v}_\nu, \tag{4.13}$$

where we recall that $\vec{e_2} = (0,1)$, and we set

$$\widetilde{\mathbf{f}}_0 =$$
$$\chi_\nu I^{-1}(y_1)\begin{pmatrix} iV^+_\mu(y',0) + i\alpha^+_\mu(y',0,s,D')\widetilde{C}_1(y_1)\mathbf{g}e^{s\varphi} - [\chi_\nu, s\varphi_{y_2}]w_1 + [\chi_\nu, b_1]\mathbf{v} \\ 0 \end{pmatrix}$$
$$+[\chi_\nu, -I^{-1}\mathcal{A}(y_1)\mathbf{D}^2_{y_1} - I^{-1}(y_1)\mathcal{A}'(y_1)\mathbf{D}_{y_1}]\mathbf{v}$$
$$+\chi_\nu(I^{-1}(y_1)\mathfrak{K}(y',s,D)\mathbf{g}e^{s\varphi}) - i[\chi_\nu, I^{-1}\mathbf{D}_{y_2}]w_1\vec{e}_1 - [\chi_\nu, \mathbf{D}^2_{y_2}]\mathbf{v},$$

and

$$\widetilde{R}(y',s,D')\mathbf{v}_\nu$$
$$=I^{-1}(y_1)\begin{pmatrix} -i\alpha^+_\mu(y',0,s,D')b_{11}(y',\mathbf{D}_{y_1}) & -i\alpha^+_\mu(y',0,s,D')b_{12}(y',\mathbf{D}_{y_1}) \\ 0 & 0 \end{pmatrix}\mathbf{v}_\nu$$
$$-I^{-1}(y_1)\mathcal{A}(y_1)\mathbf{D}^2_{y_1}\mathbf{v}_\nu - I^{-1}(y_1)\mathcal{A}'(y_1)\mathbf{D}_{y_1}\mathbf{v}_\nu.$$

By (2.14) and (4.13), we obtain

$$\chi_\nu(s,D')\widetilde{B}(y',\mathbf{D}')\mathbf{v} = \chi_\nu(s,D')(Q^{-1}\mathbf{f}e^{s\varphi}) \tag{4.14}$$
$$+ \chi_\nu(Q^{-1}(\mathfrak{R}_1(y,D),\mathfrak{R}_2(y,D))\mathbf{g}e^{s\varphi})$$
$$- [\chi_\nu, \mathbf{D}^2_{y_2}]\mathbf{v} - (\widetilde{\mathbf{f}}_0 - iI^{-1}(y_1)\mathbf{D}_{y_2}w_{2,\nu}\vec{e}_2 + \tilde{R}(y',s,D')\mathbf{v}_\nu).$$

Next we note that for the principal symbol of the operator $\Pi(y',s,D') = \widetilde{B}(y',s,D') + \widetilde{R}(y',s,D')$ we have

$$\Pi(\gamma) = \Pi(\gamma) = \begin{pmatrix} \frac{-\lambda}{\mu}(0)(\xi_1^*)^2 & -2\sqrt{\frac{\lambda+\mu}{\mu}(0)}\mathbf{I}\xi_1^*|\xi_1^*| \\ 0 & \frac{-\lambda}{(\lambda+2\mu)}(0)(\xi_1^*)^2 \end{pmatrix}, \tag{4.15}$$

where $\mathbf{I}=\operatorname{sign}(\xi_0^*\varphi_{y_0}(\mathbf{y}^*)-\mu(0)\xi_1^*\varphi_{y_1}(\mathbf{y}^*))$. Really, since $b_{11}(\mathbf{y}^{*\prime},\xi_1^*+is^*\varphi_{y_1}(\mathbf{y}^*)) = 0$ and $b_{12}(\mathbf{y}^{*\prime},\xi_1^*+is^*\varphi_{y_1}(\mathbf{y}^*)) = 2i$ we have

$$-I^{-1}(\mathbf{y}^*)i\begin{pmatrix}\alpha_\mu(\gamma)w_1\\ 0\end{pmatrix} = \begin{pmatrix}0 & -2\sqrt{\frac{\lambda+\mu}{\mu}(0)}\mathbf{I}\\ 0 & 0\end{pmatrix}\xi_1^*|\xi_1^*| \tag{4.16}$$

and

$$\begin{aligned}(4.17)\qquad &(-I^{-1}(0)\mathcal{A}(0)\xi_1^2+\tilde{B}(\gamma))\\ &=(\xi_1^*)^2\left[\begin{pmatrix}-\frac{\lambda}{\lambda+2\mu}(0) & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}-\frac{\lambda(\lambda+\mu)}{(\lambda+2\mu)\mu}(0) & 0\\ 0 & -\frac{2(\lambda+\mu)}{(\lambda+2\mu)}(0)\end{pmatrix}\right]\\ &=(\xi_1^*)^2\begin{pmatrix}-\frac{\lambda}{\mu}(0) & 0\\ 0 & \frac{-\lambda}{(\lambda+2\mu)}(0)\end{pmatrix}.\end{aligned}$$

By (4.16) and (4.17), formula (4.15) follows immediately.

Case B. Assume that $s^*=0$ and

$$\varphi_{y_2}(\mathbf{y}^*) \le \frac{|\frac{1}{\mu(0)}\xi_0^*\varphi_{y_0}(\mathbf{y}^*)-\xi_1^*\varphi_{y_1}(\mathbf{y}^*)|}{\sqrt{\frac{\lambda+\mu}{\mu}(0)}|\xi_1^*|}. \tag{4.18}$$

Note that in this case we still have (4.2). Then, since $s^*=0$, we see that $\operatorname{Re} r_\mu(\mathbf{y}^*,\zeta^*)>0$. Hence

$$\Gamma_\mu^+(\mathbf{y}^*,\zeta^*)=\mathbf{I}\,|\sqrt{\operatorname{Re} r_\mu(\mathbf{y}^*,\zeta^*)}|,\quad \mathbf{I}=\operatorname{sign}(\xi_0^*\varphi_{y_0}(\mathbf{y}^*)-\mu(0)\xi_1^*\varphi_{y_1}(\mathbf{y}^*)).$$

Therefore

$$\begin{aligned}&\Gamma_\mu^+(\mathbf{y}^*,\zeta^*)(\mu(0)\varphi_{y_1}(\mathbf{y}^*)\xi_1^*-\varphi_{y_0}(\mathbf{y}^*)\xi_0^*)\\ &=\left|\sqrt{\operatorname{Re} r_\mu(\mathbf{y}^*,\zeta^*)}\right|(\mu(0)\varphi_{y_1}(\mathbf{y}^*)\xi_1^*-\varphi_{y_0}(\mathbf{y}^*)\xi_0^*)<0.\end{aligned} \tag{4.19}$$

We note that by (4.15)

$$w_{1,\nu}=R_1(y',s,D')\frac{\partial w_{2,\nu}}{\partial y_2}+q_3,\, w_{2,\nu}=R_2(y',s,D')\frac{\partial w_{2,\nu}}{\partial y_2}+q_4, \tag{4.20}$$

where

$$R_1(\gamma)=-\frac{2i\lambda(0)\mu(0)}{(\lambda+2\mu)^2(0)\xi_1^*}(\mathbf{y}^*),\quad R_2(\gamma)=\frac{2i\lambda(0)\mathbf{I}}{(\lambda+2\mu)(0)|\xi_1^*|}(\mathbf{y}^*) \tag{4.21}$$

and

$$\sqrt{s}\|(q_3,q_4)\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_8(\sqrt{s}\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{4.22}$$

Consider two cases. First let

$$(\varphi_{y_1}(\mathbf{y}^*)\xi_1^*)\Gamma_\mu^+(\mathbf{y}^*,\zeta^*) > 0.$$

This inequality and (4.18) yield that $|\xi_0^*\varphi_{y_0}(\mathbf{y}^*)| > |\xi_1^*\varphi_{y_1}(\mathbf{y}^*)|$. If $\xi_0^*\varphi_{y_0}(\mathbf{y}^*) > 0$, then $\Gamma_\mu^+(\mathbf{y}^*,\zeta^*) = |\sqrt{r_\mu(\gamma)}|$ and $\xi_1^*\varphi_{y_1}(\mathbf{y}^*) > 0$. Hence by (1.6)

$$\varphi_{y_2}(\mathbf{y}^*) > \frac{\left|\varphi_{y_1}(\mathbf{y}^*)\xi_1^* - \frac{\varphi_{y_0}(\mathbf{y}^*)}{\mu(0)}\xi_0^*\right|}{\sqrt{\frac{\lambda+\mu}{\mu}(0)}|\xi_1^*|} = \frac{-\varphi_{y_1}(\mathbf{y}^*)\xi_1^* + \frac{\varphi_{y_0}(\mathbf{y}^*)}{\mu(0)}\xi_0^*}{\sqrt{\frac{\lambda+\mu}{\mu}(0)}|\xi_1^*|}.$$

This contradicts (4.18). If $\xi_0^*\varphi_{y_0}(\mathbf{y}^*) < 0$, then $\Gamma_\mu^+(\mathbf{y}^*,\zeta^*) = -|\sqrt{r_\mu(\gamma)}|$ and $\xi_1^*\varphi_{y_1}(\mathbf{y}^*) < 0$. Therefore

$$\varphi_{y_2}(\mathbf{y}^*) > \frac{|\varphi_{y_1}(\mathbf{y}^*)\xi_1^* - \frac{\varphi_{y_0}(\mathbf{y}^*)\xi_0^*}{\mu(0)}|}{\sqrt{\frac{\lambda+\mu}{\mu}(0)}|\xi_1^*|} = \frac{\varphi_{y_1}(\mathbf{y}^*)\xi_1^* - \frac{\varphi_{y_0}(\mathbf{y}^*)}{\mu(0)}\xi_0^*}{\sqrt{\frac{\lambda+\mu}{\mu}(0)}|\xi_1^*|}.$$

By (1.6) this again contradicts (4.18). As the second case, one has to consider $(\varphi_{y_1}(\mathbf{y}^*)\xi_1^*)\Gamma_\mu^+(\mathbf{y}^*,\zeta^*) < 0$. Similarly to (3.4), we note that $\Sigma^{(1)}_{\lambda+2\mu}$ can be written in the form

(4.23)

$$\begin{aligned}
&\Sigma^{(1)}_{\lambda+2\mu}(w_{2,\nu}) \\
&= \int_{\partial\mathcal{G}} \left\{ s(\lambda+2\mu)^2(0)\varphi_{y_2}(\mathbf{y}^*)\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2 + s^3(\lambda+2\mu)^2(0)\varphi^3_{y_2}(\mathbf{y}^*)w^2_{2,\nu}\right\} d\Sigma \\
&+\mathrm{Re}\int_{\partial\mathcal{G}} 2s(\lambda+2\mu)(0)\frac{\partial w_{2,\nu}}{\partial y_2}\overline{\left\{(\lambda+2\mu)(0)\varphi_{y_1}(\mathbf{y}^*)\frac{\partial w_{2,\nu}}{\partial y_1} - \varphi_{y_0}(\mathbf{y}^*)\frac{\partial w_{2,\nu}}{\partial y_0}\right\}} d\Sigma \\
&+\int_{\partial\mathcal{G}} s(\lambda+2\mu)(0)\varphi_{y_2}(\mathbf{y}^*)\{\xi_0^2 - (\lambda+2\mu)(0)\xi_1^2 - s^2\varphi^2_{y_0}(\mathbf{y}^*) \\
&+s^2(\lambda+2\mu)(0)\varphi^2_{y_1}(\mathbf{y}^*)\}|\widehat{w}_{2,\nu}|^2 d\Sigma \\
&\equiv \widetilde{J}_1 + \widetilde{J}_2 + \widetilde{J}_3.
\end{aligned}$$

Using (4.20) and (4.22), we can transform $\widetilde{J}_2$ as

$$\widetilde{J}_2 = \mathrm{Re}\int_{\partial\mathcal{G}} 2s(\lambda+2\mu)(0)\frac{\partial w_{2,\nu}}{\partial y_2}(-i)\times \overline{\{(\lambda+2\mu)(0)\varphi_{y_1}(\mathbf{y}^*)D_{y_1} - \varphi_{y_0}(\mathbf{y}^*)D_{y_0}\}\, R_2(y,s,D')\frac{\partial w_{2,\nu}}{\partial y_2}} d\Sigma + I_3, \tag{4.24}$$

where

$$|I_3| \le \epsilon s \left\| \left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2} \right) \right\|^2_{\mathbf{H}^{2,s}(\partial \mathcal{G}) \times \mathbf{H}^{1,s}(\partial \mathcal{G}) \times \mathbf{L}^2(\partial \mathcal{G})}$$
$$+ C_9(\|\mathbf{f} e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g} e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial \mathcal{G})}).$$

Note that for the symbol of the operator we have

$$\begin{aligned}(4.25)\qquad & (-i)\left\{(\lambda+2\mu)(0)\varphi_{y_1}(\mathbf{y}^*)\xi_1^* - \varphi_{y_0}(\mathbf{y}^*)\xi_0^*\right\} R_2(\gamma)^* \\ &= (-i)\left\{(\lambda+2\mu)(0)\varphi_{y_1}(\mathbf{y}^*)\xi_1^* - \varphi_{y_0}(\mathbf{y}^*)\xi_0^*\right\} \frac{-2i\lambda(0)\mathbf{I}}{(\lambda+2\mu)(0)|\xi_1^*|} \\ &= -\left\{(\lambda+2\mu)(0)\varphi_{y_1}(\mathbf{y}^*)\xi_1^* - \varphi_{y_0}(\mathbf{y}^*)\xi_0^*\right\} \frac{2\lambda(0)\mathbf{I}}{(\lambda+2\mu)(0)|\xi_1^*|} \\ &= -\left\{\mu(0)\varphi_{y_1}(\mathbf{y}^*)\xi_1^* - \varphi_{y_0}(\mathbf{y}^*)\xi_0^*\right\} \frac{2\lambda(0)\mathbf{I}}{(\lambda+2\mu)(0)|\xi_1^*|} \\ & \quad -(\lambda+\mu)(0)(\varphi_{y_1}(\mathbf{y}^*)\xi_1^*)\Gamma_\mu^+(\mathbf{y}^*,\zeta^*) > 0.\end{aligned}$$

By (4.25), (4.24) and Gårding's inequality, we obtain

$$\begin{aligned}(4.26)\qquad \widetilde{J}_2 \ge{} & C_{10} s \left\| \left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu}, w_{2,\nu} \right) \right\|^2_{L^2(\partial \mathcal{G}) \times H^{1,s}(\partial \mathcal{G}) \times H^{1,s}(\partial \mathcal{G})} \\ & - C_{11}(\|\mathbf{f} e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}) \\ & - \epsilon s \left\| \left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2} \right) \right\|^2_{\mathbf{H}^{2,s}(\partial \mathcal{G}) \times \mathbf{H}^{1,s}(\partial \mathcal{G}) \times \mathbf{L}^2(\partial \mathcal{G})}.\end{aligned}$$

Now we estimate $\widetilde{J}_3$. By (1.18)–(1.20) there exists a constant $C_{12} > 0$ such that

$$\begin{aligned}(4.27)\qquad & |\xi_0^2 - s^2\varphi_{y_0}^2(\mathbf{y}^*) - (\lambda+2\mu)(0)\xi_1^2 + s^2(\lambda+2\mu)(0)\varphi_{y_1}^2(\mathbf{y}^*)| \\ & \le C_{12}\delta_1(\xi_0^2 + \xi_1^2 + s^2), \quad \forall \zeta \in \mathcal{O}(\delta_1),\end{aligned}$$

which yields

$$(4.28)\qquad |\widetilde{J}_3| \le C_{13}\epsilon(\delta_1) \left\| \left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|^2_{L^2(\partial \mathcal{G}) \times H^{1,s}(\partial \mathcal{G})},$$

where $\epsilon(\delta_1) \to 0$ as $\delta_1 \to 0$. By (4.26) and (4.28), we can choose constants $C_{14}, C_{15} > 0$ such that

$$\Sigma^{(1)}_{\lambda+2\mu}(w_{2,\nu}) \ge C_{14} s \left\| \left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2} \right) \right\|^2_{\mathbf{H}^{2,s}(\partial \mathcal{G}) \times \mathbf{H}^{1,s}(\partial \mathcal{G}) \times \mathbf{L}^2(\partial \mathcal{G})}$$
$$- C_{15}(\|\mathbf{f} e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g} e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial \mathcal{G})} + \|\mathbf{v}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}).$$

This inequality immediately implies (4.11). Applying Proposition 2.1 we obtain (3.1).

Case C. Assume that $s^* \neq 0$. If $\delta_1 > 0$ is small enough, then there exists a constant $C_{16} > 0$ such that

$$|\xi_0 \varphi_{y_1}(\mathbf{y}^*) - (\lambda + 2\mu)(0)\xi_1 \varphi_{y_1}(\mathbf{y}^*)|^2 \leq \delta_1^2 C_{16}(\xi_1^2 + s^2). \tag{4.29}$$

By (2.24) and (2.25), there exist constants $C_{17} > 0$ and $C_{18} > 0$ such that

$$\begin{aligned}\Sigma^{(1)}_{\lambda+2\mu}(w_{2,\nu}) + C_{17}s\|w_{2,\nu}\|^2_{H^{1,s}(\mathcal{G})} &\leq C_{18}(\|P_{\lambda+2\mu,s}w_2\|^2_{L^2(\mathcal{G})} + \|\mathbf{v}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}) \\ &+\epsilon \left\| \left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})},\end{aligned} \tag{4.30}$$

where $\epsilon(\delta) \to 0$ as $\delta \to +0$. By (4.27) and (4.29), we have

$$|\widetilde{J}_2 + \widetilde{J}_3| \leq C_{19}\epsilon(\delta_1) \left\| \left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}, \tag{4.31}$$

where $\epsilon(\delta_1) \to 0$ as $\delta_1 \to 0$. By (4.31) we obtain from (4.23) that there exists a constant $C_{20} > 0$ such that

$$\begin{aligned}\Sigma^{(1)}_{\lambda+2\mu}(w_{2,\nu}) \geq C_{20} \int_{\partial\mathcal{G}} &\left\{ s(\lambda + 2\mu)^2(0)\varphi_{y_2}(\mathbf{y}^*) \left| \frac{\partial w_{2,\nu}}{\partial y_2} \right|^2 \right. \\ &\left. + s^3(\lambda + 2\mu)^2(0)\varphi^3_{y_2}(\mathbf{y}^*)|w_{2,\nu}|^2 \right\} d\Sigma \\ &- \epsilon \left\| \left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu} \right) \right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}.\end{aligned} \tag{4.32}$$

In terms of (1.9), we can represent

$$\begin{aligned}&\mathbf{D}^2_{y_0} v_{2,\nu} - 2\mu \mathbf{D}^2_{y_1} v_{2,\nu} + T_5\mathbf{v}_\nu + T_6\mathbf{v} + T_7(\mathbf{f}e^{s\phi}, \mathbf{g}e^{s\phi}) \\ &= (\lambda + 2\mu)\mathbf{D}_{y_2} w_{2,\nu} + \chi_\nu(s, D')(f_2 e^{s\varphi}), \quad y \in \partial\mathcal{G},\end{aligned} \tag{4.33}$$

where we have estimates:

$$\|T_5\mathbf{v}_\nu\|_{\mathbf{L}^2(\partial\mathcal{G})} \leq \epsilon(\delta)\|\mathbf{v}_\nu\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})}, \qquad \|T_6\mathbf{v}\|_{\mathbf{L}^2(\partial\mathcal{G})} \leq C_{21}\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})},$$
$$\|T_7(\mathbf{f}e^{s\phi}, \mathbf{g}e^{s\phi})\|_{\mathbf{L}^2(\partial\mathcal{G})} \leq C_{21}\|(\mathbf{f}e^{s\phi}, \mathbf{g}e^{s\phi})\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})},$$

where $\epsilon(\delta) \to +0$ as $\delta \to +0$. On the other hand, by $r_{\lambda+2\mu}(\gamma) = 0$, we have $(\xi_0^* + i\varphi_{y_0}(\mathbf{y}^*)s^*)^2 - (\lambda+2\mu)(0)(\xi_1^* + i\varphi_{y_1}(\mathbf{y}^*)s^*)^2 = 0$, that is,

$$(\xi_0^* + i\varphi_{y_0}(\mathbf{y}^*)s^*)^2 - 2\mu(0)(\xi_1^* + i\varphi_{y_1}(\mathbf{y}^*)s^*)^2 = \lambda(0)(\xi_1^* + i\varphi_{y_1}(\mathbf{y}^*)s^*)^2 \neq 0$$

by the final condition in (1.3). Therefore, from (4.1) and (4.33) we obtain

$$\text{(4.34)} \qquad \|\sqrt{s}v_{2,\nu}\|_{H^{2,s}(\partial\mathcal{G})} \le C_{22}\left\{\left\|\sqrt{s}\frac{\partial w_{2,\nu}}{\partial y_2}\right\|_{L^2(\partial\mathcal{G})} + \|\sqrt{s}\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\sqrt{s}f_2e^{s\varphi}\|_{L^2(\partial\mathcal{G})}\right\} + \epsilon\sqrt{s}\left\|\left(\mathbf{v}_\nu, \frac{\partial\mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2\mathbf{v}_\nu}{\partial y_2^2}\right)\right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}.$$

Thanks to (1.12), (1.15) and the fact that $\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*) \neq 0$, we have

$$\text{(4.35)} \quad \sqrt{s}\|v_{1,\nu}\|_{H^{2,s}(\partial\mathcal{G})} \le C_{23}\left\{\sqrt{s}\left\|\frac{\partial w_{2,\nu}}{\partial y_2}\right\|_{L^2(\partial\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}\|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \epsilon(\delta,\delta_1)\sqrt{s}\left\|\left(\mathbf{v}_\nu, \frac{\partial\mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2\mathbf{v}_\nu}{\partial y_2^2}\right)\right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}\right\}.$$

Consequently (4.34), (4.35), (1.12), (1.8) and (1.9) imply

$$\text{(4.36)} \qquad \sqrt{s}\left\|\left(\mathbf{v}_\nu, \frac{\partial\mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2\mathbf{v}_\nu}{\partial y_2^2}\right)\right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \le C_{24}\left\{\sqrt{s}\left\|\frac{\partial w_{1,\nu}}{\partial y_2}\right\|_{L^2(\partial\mathcal{G})} + \sqrt{s}\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}\right\}.$$

By (4.32) and (4.36)

$$\Sigma^{(1)}_{\lambda+2\mu}(w_{2,\nu}) \ge C_{25}s\left\|\left(\mathbf{v}_\nu, \frac{\partial\mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2\mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} - C_{25}(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}).$$

Hence, from this inequality and (2.25) with $\beta = \lambda + 2\mu$, we obtain

$$\sqrt{s}\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} + \sqrt{s}\|w_{2,\nu}\|_{H^{1,s}(\mathcal{G})} \le C_{26}(\sqrt{s}\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{4.37}$$

By (4.37) and (2.25) with $\beta = \mu$, we have

$$\sqrt{s}\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} + \sqrt{s}\|\mathbf{w}_\nu\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le C_{27}(\sqrt{s}\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}).$$

Finally using Proposition 2.1, we obtain (3.1). The proof of Lemma 4.1 is completed. □

§5. Case $r_\mu(\gamma) \ne 0$ and $r_{\lambda+2\mu}(\gamma) \ne 0$

In this section, we consider the case where

$$|r_\mu(\mathbf{y}^*, \zeta^*)| \ne 0 \quad \text{and } |r_{\lambda+2\mu}(\mathbf{y}^*, \zeta^*)| \ne 0. \tag{5.1}$$

Denote

$$H^{\frac{3}{2},s}(\partial\mathcal{G}) = \{u(y_0, y_1) \in L^2(\mathbb{R}^2); \|u\|^2_{H^{\frac{3}{2},s}(\partial\mathcal{G})} = \int_{\mathbb{R}^2} (1 + s^3 + |\xi_0|^3 + |\xi_1|^3)|\hat{u}|^2 d\xi_0 d\xi_1 < \infty\},$$

where we set $\hat{u}(\xi_0, \xi_1) = \int_{\mathbb{R}^2} u(y_0, y_1)e^{-i(\xi_0 y_0 + \xi_1 y_1)} dy_0 dy_1$. We have

Lemma 5.1. *Let* (5.1) *hold at* $\gamma \equiv (\mathbf{y}^*, \zeta^*)$ *and let* supp $\chi_\nu \subset \mathcal{O}(\delta_1)$ *where* δ_1 *is a sufficiently small positive number. If* $\zeta^* \in \Psi_2$*, then we have*

$$\|\mathbf{v}_\nu\|_{\mathbf{H}^{2,s}(\mathcal{G})} \le C_1\left\{\frac{1}{s^{\frac{1}{4}}}(\|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})}) + s^{\frac{1}{4}}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \frac{1}{s^{\frac{1}{4}}}\|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}\right\} \tag{5.2}$$

and

$$\sqrt{s}\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}\right)\right\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})\times\mathbf{H}^{\frac{1}{2},s}(\partial\mathcal{G})} \le C_2(\sqrt{s}\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})}). \tag{5.3}$$

If $\zeta^* \notin \Psi_2$*, then estimate* (3.1) *hold true.*

We recall that the sets Ψ, Ψ_1 and Ψ_2 are defined by (1.29) and Lemma 1.1.

Proof. Thanks to (5.1) and Proposition 2.2, decomposition (2.18) holds true for $\beta = \mu$ and $\beta = \lambda + 2\mu$. Therefore we have

$$(D_{y_2} - \Gamma^+_\mu(y, s, D'))w_{1,\nu}|_{y_2=0} = V^+_\mu(\cdot, 0), \tag{5.4}$$

$$(D_{y_2} - \Gamma^+_{\lambda+2\mu}(y, s, D'))w_{2,\nu}|_{y_2=0} = V^+_{\lambda+2\mu}(\cdot, 0). \tag{5.5}$$

By Proposition 2.3 we have the a priori estimate:

$$\begin{aligned} \sqrt{s}\|V^+_\mu(\cdot, 0)\|_{L^2(\partial\mathcal{G})} &+ \sqrt{s}\|V^+_{\lambda+2\mu}(\cdot, 0)\|_{L^2(\partial\mathcal{G})} \\ &\le C_3(\|P_{\lambda+2\mu,s}w_2\|_{L^2(\mathcal{G})} + \|P_{\mu,s}w_1\|_{L^2(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \end{aligned} \tag{5.6}$$

System (1.1) can be written in the following form:

$$\frac{\partial^2 u_1}{\partial x_0^2} + \mu\frac{\partial \,\mathrm{rot}\,\mathbf{u}}{\partial x_2} - (\lambda + 2\mu)\frac{\partial \,\mathrm{div}\,\mathbf{u}}{\partial x_1} = m_1,$$

$$\frac{\partial^2 u_2}{\partial x_0^2} - \mu\frac{\partial \,\mathrm{rot}\,\mathbf{u}}{\partial x_1} - (\lambda + 2\mu)\frac{\partial \,\mathrm{div}\,\mathbf{u}}{\partial x_2} = m_2,$$

where $\mathbf{m} = (m_1, m_2) = (\nabla_{\widetilde{x}}\mathbf{u} + (\nabla_{\widetilde{x}}\mathbf{u})^T)\nabla_{\widetilde{x}}\mu(\widetilde{x}) + \mathbf{f}$. In $y-$ coordinates these equations can be written as:

$$\begin{aligned} \frac{\partial^2 u_1}{\partial y_0^2} + \mu\frac{\partial z_1}{\partial y_2} - (\lambda + 2\mu)\left(\frac{\partial}{\partial y_1} - \ell'\frac{\partial}{\partial y_2}\right)z_2 = m_1, \\ \frac{\partial^2 u_2}{\partial y_0^2} - \mu\left(\frac{\partial}{\partial y_1} - \ell'\frac{\partial}{\partial y_2}\right)z_1 - (\lambda + 2\mu)\frac{\partial z_2}{\partial y_2} = m_2. \end{aligned}$$

Here we still keep the same notations for $\mathbf{u}$ and $\mathbf{m}$ in the new coordinates. The above equations can be written in the form:

$$\begin{aligned} -\mathbf{D}^2_{y_0}v_1 + \mu i\mathbf{D}_{y_2}w_1 - (\lambda + 2\mu)i(\mathbf{D}_{y_1} - \ell'\mathbf{D}_{y_2})w_2 = m_1e^{s\varphi}, \\ -\mathbf{D}^2_{y_0}v_2 - i\mu(\mathbf{D}_{y_1} - \ell'\mathbf{D}_{y_2})w_1 - (\lambda + 2\mu)i\mathbf{D}_{y_2}w_2 = m_2e^{s\varphi}. \end{aligned}$$

Applying to these equations the operator $\chi_\nu(s, D')$ we obtain

$$-\mathbf{D}^2_{y_0}v_{1,\nu} + \mu i\mathbf{D}_{y_2}w_{1,\nu} - (\lambda + 2\mu)i(\mathbf{D}_{y_1} - \ell'\mathbf{D}_{y_2})w_{2,\nu} = h_1, \tag{5.7}$$

$$-\mathbf{D}^2_{y_0}v_{2,\nu} - i\mu(\mathbf{D}_{y_1} - \ell'\mathbf{D}_{y_2})w_{1,\nu} - (\lambda + 2\mu)i\mathbf{D}_{y_2}w_{2,\nu} = h_2. \tag{5.8}$$

Here $h_1 = \chi_\nu(s, D')(m_1 e^{s\varphi}) + [\chi_\nu, \mathbf{D}_{y_0}^2]v_1 + [\mu i \mathbf{D}_{y_2}, \chi_\nu]w_1 - [(\lambda+2\mu)i(\mathbf{D}_{y_1} - \ell'\mathbf{D}_{y_2}), \chi_\nu]w_2, h_2 = \chi_\nu(s, D')(m_2 e^{s\varphi}) + [\chi_\nu, \mathbf{D}_{y_0}^2]v_2 - i[\mu(\mathbf{D}_{y_1} - \ell'\mathbf{D}_{y_2}), \chi_\nu]w_1 - [(\lambda+2\mu)i\mathbf{D}_{y_2}, \chi_\nu]w_2$. Finally using (5.4) and (5.5) we arrive at the system

(5.9)
$$-\mathbf{D}_{y_0}^2 v_{1,\nu} + \mu i \alpha_\mu^+(y,s,D')w_{1,\nu} - (\lambda+2\mu)i(\mathbf{D}_{y_1} - \ell'\alpha_{\lambda+2\mu}^+(y,s,D'))w_{2,\nu}$$
$$= h_1 - i\mu V_\mu^+ - i(\lambda+2\mu)\ell' V_{\lambda+2\mu}^+ \quad \text{on } \mathcal{G},$$

(5.10)
$$-\mathbf{D}_{y_0}^2 v_{2,\nu} - i\mu(\mathbf{D}_{y_1} - \ell'\alpha_\mu^+(y,s,D'))w_{1,\nu} - (\lambda+2\mu)i\alpha_{\lambda+2\mu}^+(y,s,D')w_{2,\nu}$$
$$= h_2 - i\mu\ell' V_\mu^+ - (\lambda+2\mu)V_{\lambda+2\mu}^+ \quad \text{on } \mathcal{G}.$$

We set $\tilde{h}_1 = h_1 - i\mu V_\mu^+ - i(\lambda+2\mu)\ell' V_{\lambda+2\mu}^+, \tilde{h}_2 = h_2 - i\mu\ell' V_\mu^+ - (\lambda+2\mu)iV_{\lambda+2\mu}^+$. By (1.13)–(1.16) we have

(5.11)
$$\begin{aligned} &-\mathbf{D}_{y_0}^2 v_{1,\nu} + \mu i \alpha_\mu^+(y,s,D')b_{11}(y',\mathbf{D}')v_{1,\nu} \\ &-(\lambda+2\mu)i(\mathbf{D}_{y_1} - \ell'\alpha_{\lambda+2\mu}^+(y,s,D'))b_{21}(y',\mathbf{D}')v_{1,\nu} \\ &+\mu i\alpha_\mu^+(y,s,D')b_{21}(y',\mathbf{D}')v_{2,\nu} \\ &-(\lambda+2\mu)i(\mathbf{D}_{y_1} - \ell'\alpha_{\lambda+2\mu}^+(y,s,D'))b_{22}(y',\mathbf{D}')v_{2,\nu} = q_1 \quad \text{on } \mathcal{G}, \end{aligned}$$

(5.12)
$$\begin{aligned} &-\mathbf{D}_{y_0}^2 v_{2,\nu} - i\mu(\mathbf{D}_{y_1} - \ell'\alpha_\mu^+(y,s,D'))b_{11}(y',\mathbf{D}')v_{1,\nu} \\ &-(\lambda+2\mu)i\alpha_{\lambda+2\mu}^+(y,s,D')b_{21}(y',\mathbf{D}')v_{1,\nu} \\ &-i\mu(\mathbf{D}_{y_1} - \ell'\alpha_\mu^+(y,s,D'))b_{21}(y',\mathbf{D}')v_{2,\nu} \\ &-(\lambda+2\mu)i\alpha_{\lambda+2\mu}^+(y,s,D')b_{22}(y',\mathbf{D}')v_{2,\nu} = q_2 \quad \text{on } \mathcal{G}, \end{aligned}$$

with

$$\begin{aligned} q_1 = {}&\tilde{h}_1 + \mu i\alpha_\mu^+(y,s,D')([\chi_\nu, b_1(y',\mathbf{D})]\tilde{C}_1(y)\mathbf{g}e^{s\varphi}) \\ &-(\lambda+2\mu)i(\mathbf{D}_{y_1} - \ell'\alpha_{\lambda+2\mu}^+(y,s,D'))([\chi_\nu, b_2(y',\mathbf{D})]\tilde{C}_2(y)\mathbf{g}e^{s\varphi}), \\ q_2 = {}&\tilde{h}_2 - i\mu(\mathbf{D}_{y_1} - \ell'\alpha_\mu^+(y,s,D'))([\chi_\nu, b_1(y',\mathbf{D})]\tilde{C}_1(y)\mathbf{g}e^{s\varphi}) \\ &-(\lambda+2\mu)i\alpha_{\lambda+2\mu}^+(y,s,D')([\chi_\nu, b_2(y',\mathbf{D})]\tilde{C}_2(y)\mathbf{g}e^{s\varphi}). \end{aligned}$$

By (5.11), (5.12) and (1.22) we have :

$$\mathcal{B}(y',s,D')\mathbf{v}_\nu = \mathbf{q}, \quad y' \in \partial\mathcal{G}, \tag{5.13}$$

where we recall that the operator $\mathcal{B}(y',s,D')$ is defined by (1.22) and we set

$$\mathbf{q} = T_1(\mathbf{g}e^{s\varphi}) + T_2(\mathbf{f}(y',0)e^{s\varphi}) + T_3\mathbf{v} + \mathbb{G}(y')(V_\mu^+(\cdot,0), V_{\lambda+2\mu}^+(\cdot,0)), \tag{5.14}$$

$T_1, T_3 \in \mathcal{L}(\mathbf{H}^{1,s}(\partial\mathcal{G}), \mathbf{L}^2(\partial\mathcal{G})), T_2 \in \mathcal{L}(\mathbf{L}^2(\partial\mathcal{G}), \mathbf{L}^2(\partial\mathcal{G}))$ and $\mathbb{G}(y')$ is a C^1 matrix-valued function.

Now we consider the following three cases.

Case A. Let $\det \mathcal{B}(\gamma) \neq 0$ and $\zeta^* \notin \Psi$. In that case, there exists a parametrix of the operator $\mathcal{B}(y', s, D')$ which we denote by $\mathcal{B}^{-1}(y', s, D')$, and we have

$$\mathbf{v}_\nu = \mathcal{B}^{-1}(y', s, D')\mathbf{q} + K\mathbf{v}_\nu, \tag{5.15}$$

where

$$K \in \mathcal{L}(\mathbf{H}^{1,s}(\partial\mathcal{G}), \mathbf{H}^{2,s}(\partial\mathcal{G})) \text{ and } \quad \mathcal{B}^{-1}(y', s, D') \in \mathcal{L}(\mathbf{L}^2(\partial\mathcal{G}), \mathbf{H}^{2,s}(\partial\mathcal{G})).$$

Then

$$\|\mathbf{v}_\nu\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})} \le C_4(\|\mathbf{q}\|_{L^2(\partial\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{5.16}$$

This estimate and (2.10) imply

$$\left\|\frac{\partial \mathbf{v}_\nu}{\partial y_2}\right\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_5(\|\mathbf{q}\|_{\mathbf{L}^2(\partial\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{5.17}$$

Next (2.14) and (5.16), (5.17) give us

$$\left\|\frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right\|_{\mathbf{L}^2(\partial\mathcal{G})} \le C_6(\|\mathbf{q}\|_{\mathbf{L}^2(\partial\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\phi}\|_{\mathbf{L}^2(\partial\mathcal{G})}). \tag{5.18}$$

Finally taking into account

$$\|\mathbf{q}\|_{\mathbf{L}^2(\partial\mathcal{G})} \le C_7(\|\mathbf{f}e^{s\phi}\|_{\mathbf{L}^2(\partial\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})})$$

we obtain from (5.16)–(5.18)

$$|\Sigma_\mu(w_{1,\nu})| + |\Sigma_{\lambda+2\mu}(w_{2,\nu})| \le C_8(\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{5.19}$$

Inequalities (2.25) and (5.19) yield

$$s\|\mathbf{w}_\nu\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \tag{5.20}$$
$$\le C_9(\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}).$$

By (5.20) and Proposition 2.1 we obtain (3.1).

If $\det \mathcal{B}(\gamma) = 0$ and $\zeta^* \notin \Psi$ then either $\operatorname{Im}\frac{\Gamma^+_\mu}{s}(\gamma) < 0$ or $\operatorname{Im}\frac{\Gamma^+_{\lambda+2\mu}}{s}(\gamma) < 0$. In the first case: $\operatorname{Im}\frac{\Gamma^+_\mu}{s}(\gamma) < 0$, we have

$$(5.21)\qquad \sqrt{s}\|w_{1,\nu}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \sqrt{s}\|(b_{11}+b_{12})(y_1,s,D')\mathbf{v}_\nu\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_{10}(\|P_{\lambda+2\mu,s}w_2\|_{L^2(\mathcal{G})} + \|P_{\mu,s}w_1\|_{L^2(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

Next we consider the operator $\Phi(y',s,D')$ given by

$$\Phi(y',s,D')\mathbf{v}_\nu = (-\mathbf{D}^2_{y_0}v_{1,\nu} - (\lambda+2\mu)i(\mathbf{D}_{y_1} - \ell'\alpha^+_{\lambda+2\mu}(y,s,D'))b_2(y',\mathbf{D})v_\nu, \Lambda(s,D')b_1(y',\mathbf{D})v_\nu) = (q_3,q_4),$$

where Λ is the pseudodifferential operator with the symbol $\sqrt{1+s^2+\xi_0^2+\xi_1^2}$. By (5.21) and (5.14), we have

$$(5.22)\qquad \sqrt{s}\|(q_3,q_4)\|_{\mathbf{L}^2(\partial\mathcal{G})} \le C_{11}(\|P_{\lambda+2\mu,s}w_2\|_{L^2(\mathcal{G})} + \|P_{\mu,s}w_1\|_{L^2(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}(\|\mathbf{f}e^{s\phi}\|_{\mathbf{L}^2(\partial\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})})).$$

Next we observe that since the principal symbol of the operator $\Phi(\gamma) \ne 0$ we have

$$(5.23)\qquad \|\mathbf{v}_\nu\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})} \le C_{12}(\|\mathbf{q}\|_{\mathbf{L}^2(\partial\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}).$$

Then repeating the arguments (5.17)–(5.20) we obtain (3.1).

In the second case: $\operatorname{Im}\frac{\Gamma^+_{\lambda+2\mu}}{s}(\gamma) < 0$, we have

$$(5.24)\qquad \sqrt{s}\|w_{2,\nu}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \sqrt{s}\|(b_{21}+b_{22})(y_1,s,D')\mathbf{v}_\nu\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \le C_{13}(\|P_{\lambda+2\mu,s}w_2\|_{L^2(\mathcal{G})} + \|P_{\mu,s}w_1\|_{L^2(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

Next we consider the operator $\tilde{\Phi}(y',s,D')$ given by

$$\tilde{\Phi}(y',s,D')\mathbf{v}_\nu = (-\mathbf{D}^2_{y_0}v_{2,\nu} - i\mu(\mathbf{D}_{y_1} - \ell'\alpha^+_\mu(y,s,D'))b_1(y',\mathbf{D})v_\nu, \Lambda(s,D')b_2(y',\mathbf{D})v_\nu) = (q_5,q_6).$$

By (5.24) and (5.14) we have

$$(5.25)\qquad \sqrt{s}\|(q_5,q_6)\|_{\mathbf{L}^2(\partial\mathcal{G})} \le C_{14}(\|P_{\lambda+2\mu,s}w_2\|_{L^2(\mathcal{G})} + \|P_{\mu,s}w_1\|_{L^2(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}(\|\mathbf{f}e^{s\phi}\|_{\mathbf{L}^2(\partial\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})})).$$

Next we observe that since the principal symbol of the operator $\tilde{\Phi}(\gamma) \neq 0$ we have

$$\|\mathbf{v}_\nu\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})} \le C_{15}(\|\mathbf{q}\|_{\mathbf{L}^2(\partial\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{5.26}$$

Then repeating the arguments (5.17)–(5.20), we obtain (3.1).

Now we have consider the case where $\zeta^* \in \Psi$. In order to treat this case, we will use the Calderón method. First we introduce the new variables $U = (U_1, U_2, U_3, U_4)$, where $(U_1, U_2) = \Lambda(s, D')\mathbf{v}, (U_3, U_4) = \mathbf{D}_{y_2}\mathbf{v}$. Then problem (1.8)–(1.9) can be written in the form:

(5.27)

$$D_{y_2}U = M(y, s, D')U + F \quad \text{in } \mathbb{R}^2 \times [0,1], \quad \mathbb{B}(y', s, D')U(y)|_{y_2=0} = \mathbf{g}e^{s\phi},$$

where $F = (0, P(y, \mathbf{D})\mathbf{v})$ and we set $\mathbf{D} = (\mathbf{D}_{y_0}, \mathbf{D}_{y_1}, \mathbf{D}_{y_2})$, $\mathbf{D}_{y_j} = \frac{1}{i}\frac{\partial}{\partial y_j} + is\phi_{y_j}$ and $M(y, s, D')$ is the matrix pseudodifferential operator whose principal symbol $M_1(y, \zeta)$ is given by formula (see [Y]):

$$M_1(y,\zeta) = \begin{pmatrix} 0 & \Lambda_1 E_2 \\ \mathbb{A}^{-1}M_{21}\Lambda_1^{-1} & \mathbb{A}^{-1}M_{22} \end{pmatrix} - is\varphi_{y_2}E_4,$$

where $\Lambda_1 = |\zeta|$, $M_{21}(y, \xi' + is\nabla_{y'}\varphi) = ((\xi_0 + is\varphi_{y_0})^2 - \mu(\xi_1 + is\varphi_{y_1})^2)E_2 - (\lambda+\mu)\vec{\theta}^T\vec{\theta}$, $M_{22}(y,\xi') = -(\lambda+\mu)(\vec{\theta}^T G + G^T\vec{\theta}) - 2\mu\vec{\theta}G^T E_2$, $\mathbb{A} = \mathbb{A}(y) = (\lambda+\mu)G^T G + \mu|G|^2 E_2$, $G(y_1) = (-\ell'(y_1), 1)$, $\vec{\theta} = (\xi_1 + is\varphi_{y_1}, 0)$. Hence and henceforth, $\vec{\theta}^T$ denotes the transpose of the vector $\vec{\theta}$. For the stress boundary conditions, we have

$$\mathbb{B}(y', s, D')U = (\mathbb{B}_1(y', s, D'), \mathbb{B}_2(y', s, D'))U = \mathbf{g}e^{s\varphi},$$

where

$$\mathbb{B}_1(y', s, \xi') = \frac{\lambda G^T\vec{\theta} + \mu\vec{\theta}^T G + \mu\vec{\theta}G^T E_2}{\sqrt{s^2 + \xi_1^2 + \xi_0^2}}, \quad \mathbb{B}_2(y', s, D') = \mathbb{A}(y', 0).$$

Case B. Let $\det\mathcal{B}(\gamma) = 0$ and $\zeta^* \in \Psi_2$. We introduce the matrix symbol $\mathcal{K}(y', s, \xi_0, \xi_1)$ by formula

$$\mathcal{K}(y',\zeta) = \frac{1}{1 + s^2 + \xi_0^2 + \xi_1^2}\begin{pmatrix} B_{22}(y',\zeta) & -B_{12}(y',\zeta) \\ -B_{21}(y',\zeta) & B_{11}(y',\zeta) \end{pmatrix}. \tag{5.28}$$

Applying the pseudodifferential operator $\mathcal{K}(y', s, D')$ to equation (5.13), we have

$$\mathcal{K}(y', s, D')\mathcal{B}(y', s, D')\mathbf{v}_\nu = \mathcal{K}(y', s, D')\mathbf{q}. \tag{5.29}$$

The principal symbol of the operator $\mathcal{K}(y', s, D')\mathcal{B}(y', s, D')$ is given by the formula $\mathcal{K}(y', \zeta)\mathcal{B}(y', \zeta) = \det \mathcal{B}(y', \zeta)E_4/|\zeta|^2$. Note that if $\zeta^* \in \Psi_2$, then $\alpha^2_\beta(\gamma) = (\frac{\rho(0)}{\beta}\mathcal{C}(0) - 1)(\xi_1^* + is^*\varphi(\mathbf{y}^*))^2 \neq \mathbb{R}^1_+$ for $\beta \in \{\mu(0), (\lambda+2\mu)(0)\}$. Therefore Im $\alpha^-_\mu(\gamma) < 0$ and Im $\alpha^-_{\lambda+2\mu}(\gamma) < 0$. Hence we may rewrite estimate (5.6) in the form

$$\begin{aligned}(5.30)\qquad & \|V^+_\mu(\cdot,0)\|_{H^{\frac{1}{2},s}(\partial\mathcal{G})} + \|V^+_{\lambda+2\mu}(\cdot,0)\|_{H^{\frac{1}{2},s}(\partial\mathcal{G})} \\ & \le C_{16}(\|P_{\lambda+2\mu,s}w_2\|_{L^2(\mathcal{G})} + \|P_{\mu,s}w_1\|_{L^2(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}).\end{aligned}$$

Thanks to (5.14), (5.30) and Condition 1.2, applying Theorem 1.1 (Chapter 8) from [E] we have estimate (5.3).

Now we need to show that estimate (5.2) holds true. By $\zeta^* \in \Psi_2$, the matrix $M_1(\gamma)$ has four distinct eigenvalues given by (1.18)–(1.20). Following [T1], in terms of the change of variables $W = S^{-1}(y, s, D')U$, we can transform system (5.27) to

$$(5.31)\qquad D_{y_2}W = \widetilde{M}(y, s, D')W + T(y, s, D')W + \widetilde{\mathbf{F}},$$

where

$$(5.32)\qquad \|\widetilde{\mathbf{F}}\|_{L^2(\mathbb{R}^1;\mathbf{H}^{1,s}(\partial\mathcal{G}))} \le C_{17}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{L^2(\mathbb{R}^1;\mathbf{H}^{1,s}(\partial\mathcal{G}))}).$$

Here the matrix $\widetilde{M}$ has the form

$$\widetilde{M}(y,\zeta) = \begin{pmatrix} M_+(y,\zeta) & 0 \\ 0 & M_-(y,\zeta)\end{pmatrix}, \quad M_\pm(y,\zeta) = \begin{pmatrix} \Gamma^\pm_{\lambda+2\mu}(y,\zeta) & 0 \\ 0 & \Gamma^\pm_\mu(y,\zeta)\end{pmatrix},$$

and the operator $T(y, s, D') \in L^\infty(0,1;\mathcal{L}(\mathbf{L}^2(\mathcal{G}), \mathbf{L}^2(\mathcal{G})))$. We represent the symbol S in the form $S = (s_1^+, s_2^+, s_1^-, s_2^-)$. Here $\Lambda_1 = |\,(s, \zeta_0, \zeta_1)\,|$,

$$\begin{aligned} s_1^\pm &= ((\vec{\theta} + \alpha^\pm_{\lambda+2\mu}G)\Lambda_1^{-1}, \alpha^\pm_{\lambda+2\mu}(\vec{\theta} + \alpha^\pm_{\lambda+2\mu}G)\Lambda_1^{-2}), \\ s_2^\pm &= ((\alpha^\pm_\mu(\xi_1 + is\varphi_{y_1} - \alpha^\pm_\mu\ell'), -(\xi_1 + is\varphi_{y_1} - \alpha^\pm_\mu\ell')^2)\Lambda_1^{-2}, \\ &\qquad \alpha^\pm_\mu\Lambda_1^{-1}(\alpha^\pm_\mu(\xi_1 + is\varphi_{y_1} - \alpha^\pm_\mu\ell'), -(\xi_1 + is\varphi_{y_1} - \alpha^\pm_\mu\ell')^2)\Lambda_1^{-2}),\end{aligned}$$

are the eigenvectors of the matrix $M_1(y,\zeta)$, $\zeta \in S^2$, which corresponds to the eigenvalues $\Gamma^\pm_{\lambda+2\mu}$ and $\Gamma^\pm_\mu$. Now using the standard arguments (see e.g., §4 of Chapter 7 in [Ku]), we can estimate the last two components of W as follows

$$(5.33)\qquad \|(W_3, W_4)\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le C_{18}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{L}^2(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})}),$$

where the constant C_{18} is independent of N. Now we need the estimate for the first two components of vector W. Henceforth we set $j(\beta) = 2$ if $\beta = \mu$ and $j(\beta) = 1$ if $\beta = \lambda + 2\mu$.

There are the following two possibilities (i) and (ii): (i) $\operatorname{Im}\Gamma^+_\beta(\gamma) > 0$ for any $\beta \in \{\mu, \lambda + 2\mu\}$.
Then, by the same argument (see e.g., [Ku], pp. 241-247), we have

(5.34)
$$\|W_{j(\beta)}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le C_{19}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{L}^2(\mathcal{G})} + \|W_{j(\beta)}(\cdot,0)\|_{\mathbf{H}^{1/2,s}(\partial\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

Combining this inequality with a priori estimate (5.3), we have

$$\|W_{j(\beta)}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le C_{20}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{L}^2(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \frac{1}{\sqrt{s}}(\|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}) + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{5.35}$$

(ii) *There exists* $\beta \in \{\mu, \lambda + 2\mu\}$ *such that* $\operatorname{Im}\Gamma^+_\beta(\gamma) = 0$.
Applying Proposition 2.4, we obtain

$$\|W_{j(\beta)}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le C_{21}\left(\frac{1}{\sqrt{s}}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}) + s^{\frac14}\|W_{j(\beta)}(\cdot,0)\|_{\mathbf{H}^{\frac12,s}(\partial\mathcal{G})}\right).$$

Combining this inequality with a priori estimate (5.3), we have

(5.36)
$$\|W_{j(\beta)}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le C_{22}\Bigg\{\frac{1}{\sqrt{s}}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}) + s^{-\frac14}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{\frac32,s}(\partial\mathcal{G})} + s^{\frac14}(\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \frac{1}{\sqrt{s}}\|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \frac{1}{\sqrt{s}}\|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})})\Bigg\}.$$

In view of (5.33), (5.34) and (5.36), we obtain (5.2).

Case C. $\det\mathcal{B}(\gamma) = 0$ and $\zeta^* \in \Psi_1$. In that case we have

$$\xi_0^* = 0, \ \ s^*\varphi_{y_0}(\mathbf{y}^*) = 0. \tag{5.37}$$

Then we can assume that

$$\operatorname{Im}\Gamma^+_\mu(\gamma) = \operatorname{Im}\Gamma^+_{\lambda+2\mu}(\gamma) \ge 0. \tag{5.38}$$

In fact, if

$$\operatorname{Im}\Gamma^+_\mu(\gamma) = \operatorname{Im}\Gamma^+_{\lambda+2\mu}(\gamma) < 0, \tag{5.39}$$

then the situation is simple because we have the decomposition

$$P_{\beta,s}(y,s,D)w_{j(\beta),\nu} = \beta|G|(D_{y_2} - \Gamma^{\mp}_{\beta}(y,s,D'))V^{\pm}_{\beta} + T^{\pm}_{\mu} w_{j(\beta),\nu},$$

where $T^{\pm}_{\beta} \in \mathcal{L}(H^{1,s}(\mathcal{G}), L^2(\mathcal{G}))$, $\beta \in \{\mu, \lambda+2\mu\}$, $j(\beta) = 1$ for $\beta = \mu$ and $j(\beta) = 2$ for $\beta = \lambda + 2\mu$. This decomposition, (5.39) and Proposition 2.3 imply

$$\begin{aligned} &\|\sqrt{s}(D_{y_2} - \Gamma^{\pm}_{\beta}(y,s,D'))w_{j(\beta),\nu}|_{y_2=0}\|_{L^2(\partial\mathcal{G})} \\ &\le C_{23}(\|P_{\beta,s}w_{j(\beta),\nu}\|_{L^2(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}). \end{aligned} \tag{5.40}$$

Obviously

$$\begin{aligned} &-V^{+}_{\mu}(\cdot,0) + V^{-}_{\mu}(\cdot,0) = (\alpha^{+}_{\mu}(y',0,s,D') \\ &-\alpha^{-}_{\mu}(y',0,s,D'))\{(b_{11}(y',\mathbf{D}_{y_1}) + b_{12}(y',\mathbf{D}_{y_1}))\mathbf{v}_{\nu} + \widetilde{C}_1(y_1)\mathbf{g}e^{s\phi}\} \quad \text{on } \partial\mathcal{G} \end{aligned}$$

and

$$\begin{aligned} &-V^{+}_{\lambda+2\mu}(\cdot,0) + V^{-}_{\lambda+2\mu}(\cdot,0) = (\alpha^{+}_{\lambda+2\mu}(y',0,s,D') \\ &-\alpha^{-}_{\lambda+2\mu}(y',0,s,D'))\{(b_{21}(y',\mathbf{D}_{y_1}) + b_{22}(y',\mathbf{D}_{y_1}))\mathbf{v}_{\nu} + \widetilde{C}_2(y_1)\mathbf{g}e^{s\phi}\} \quad \text{on } \partial\mathcal{G}. \end{aligned}$$

Since $\alpha^{+}_{\beta}(\mathbf{y}^*,\zeta^*) - \alpha^{-}_{\beta}(\mathbf{y}^*,\zeta^*) = 2\sqrt{r_{\mu}(\mathbf{y}^*,\zeta^*)} \ne 0$ and the determinant of the matrix

$$\begin{pmatrix} 2\sqrt{r_{\mu}(\mathbf{y}^*,\zeta^*)}b_{11}(\mathbf{y}^{*\prime},\xi^*+is^*\varphi_{y_1}(\mathbf{y}^*)) & 2\sqrt{r_{\mu}(\mathbf{y}^*,\zeta^*)}b_{12}(\mathbf{y}^{*\prime},\xi^*+is^*\varphi_{y_1}(\mathbf{y}^*)) \\ 2\sqrt{r_{\mu}(\mathbf{y}^*,\zeta^*)}b_{21}(\mathbf{y}^{*\prime},\xi^*+is^*\varphi_{y_1}(\mathbf{y}^*)) & 2\sqrt{r_{\mu}(\mathbf{y}^*,\zeta^*)}b_{22}(\mathbf{y}^{*\prime},\xi^*+is^*\varphi_{y_1}(\mathbf{y}^*)) \end{pmatrix}$$

is not equal to zero, by (5.38) and Gårding's inequality, we obtain

$$\begin{aligned} &\sqrt{s}\left\|\left(\mathbf{v}_{\nu}, \frac{\partial \mathbf{v}_{\nu}}{\partial y_2}, \frac{\partial^2 \mathbf{v}_{\nu}}{\partial y_2^2}\right)\right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \\ &\le C_{24}(\|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}\|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}). \end{aligned} \tag{5.41}$$

In terms of (5.41) and (2.25), we obtain (3.1).

The matrix $M_1(\gamma)$ has only two eigenvalues given by (1.18)–(1.20). Moreover it is known that the Jordan form of the matrix $M_1(\gamma)$ has two Jordan blocks of the form:

$$M^{\pm} = \begin{pmatrix} \Gamma^{\pm}_{\mu}(\gamma) & 1 \\ 0 & \Gamma^{\pm}_{\mu}(\gamma) \end{pmatrix}.$$

Following [T1], in terms of the change of variables $W = S^{-1}(y,s,D')U$, we can transform system (5.27) to the form

$$D_{y_2}W = \widetilde{M}(y,s,D')W + T(y,s,D')W + \widetilde{\mathbf{F}}, \tag{5.42}$$

(5.43) $$\widetilde{\mathbb{B}}(y', s, D')W = \mathbf{g}e^{s\phi},$$

where

$$\|\widetilde{\mathbf{F}}\|_{L^2(\mathbb{R}^1;\mathbf{H}^{1,s}(\partial\mathcal{G}))} \le C_{25}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{L^2(\mathbb{R}^1;\mathbf{H}^{1,s}(\partial\mathcal{G}))})$$

and the principal symbol of the operator $\widetilde{\mathbb{B}}$ is defined by formula

(5.44) $$\widetilde{\mathbb{B}}(y', s, \xi') = \mathbb{B}(y', s, \xi')S(y', 0, s, \xi').$$

Here the matrix $\widetilde{M}$ has the form

$$\widetilde{M}(y,\zeta) = \begin{pmatrix} M_+(y,\zeta) & 0 \\ 0 & M_-(y,\zeta) \end{pmatrix}, \quad M_\pm(y,\zeta) = \begin{pmatrix} \Gamma^\pm_{\lambda+2\mu}(y,\zeta) & m^\pm_{12}(y,\zeta) \\ 0 & \Gamma^\pm_\mu(y,\zeta) \end{pmatrix},$$

and the operator $T(y,s,D') \in L^\infty(0,1;\mathcal{L}(\mathbf{H}^{1,s}(\mathcal{G}), \mathbf{H}^{1,s}(\mathcal{G})))$. We describe the construction of the pseudodifferential operator S. We write the symbol S in the form $S(y,s,\xi') = (s^+_1, s^+_2, s^-_1, s^-_2)$. Here $s^\pm_1 = ((\vec{\theta} + \alpha^\pm_{\lambda+2\mu}G)\Lambda_1^{-1}, \alpha^\pm_{\lambda+2\mu}(\vec{\theta} + \alpha^\pm_{\lambda+2\mu}G)\Lambda_1^{-2})$ are the eigenvectors of the matrix $M_1(y,\zeta)$ on the sphere $\zeta \in S^2$ which corresponds to the eigenvalue $\Gamma^\pm_{\lambda+2\mu}(y,s,\xi')$ and the vectors $s^\pm_2$ are given by

$$s^\pm_2(y,s,\xi') = E_\pm s^\pm, \quad E_\pm = \frac{1}{2\pi i}\int_{C^\pm}(z - M_1(y,\zeta))^{-1}dz,$$

where $C^\pm$ are small circles oriented counterclockwise and centered at $\Gamma^\pm_\mu(\gamma)$, and $s^\pm$ solves the equation $M_1(\gamma)s^\pm - \Gamma^\pm_\mu(\gamma)s^\pm = s^\pm_1(\gamma)$. By (5.38) the circles $C^\pm$ can be taken such that the disks bounded by these circles do not intersect. Note that the vectors $s^\pm_j \in C^2(B_\delta \times \mathcal{O}_{\delta_1})$ are homogeneous functions of the order zero in (s, ξ_0, ξ_1). Now, similarly to (5.34), we can estimate the last two components of W as follows

(5.45) $$\|(W_3, W_4)\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} \le C_{26}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}),$$

where the constant C_{26} is independent of N. Now we need to estimate the first two components of the vector W on $\partial\mathcal{G}$. We can decompose the boundary operator $\widetilde{\mathbb{B}}(y', s, D') = (\widetilde{\mathbb{B}}^+(y', s, D'), \widetilde{\mathbb{B}}^-(y', s, D'))$ such that

(5.46) $$\widetilde{\mathbb{B}}^+(y', s, D')(W_1, W_2) = -\widetilde{\mathbb{B}}^-(y', s, D')(W_3, W_4) + \mathbf{g}e^{s\phi},$$

where $\widetilde{\mathbb{B}}^+(y',\zeta) = (\widetilde{\mathbb{B}}_1(y',\zeta), \mathbb{A}(y',0))S_+(y',0,\zeta)$, $\widetilde{\mathbb{B}}_1$ is the principal symbol of the boundary operator $\widetilde{\mathbb{B}}$ and $S_+ = (s^+_1, s^+_2)$. At the point γ the vectors s_1, s_2 are given explicitly by

$$\widetilde{\eta} = (\xi^*_1 + is^*\varphi_{y_1}(\mathbf{y}^*), i\operatorname{sign}(\xi^*_1)(\xi^*_1 + is^*\varphi_{y_1}(\mathbf{y}^*)))$$

$$s_1^+(\gamma) = \left(\widetilde{\eta}, i\frac{\mathrm{sign}(\xi_1^*)(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*))}{\sqrt{(\xi_1^*)^2 + (s^*)^2}}\widetilde{\eta}\right),$$

$$\vec{\varsigma} = -\frac{(\lambda + 3\mu)(0)}{2\sqrt{(\xi_1^*)^2 + (s^*)^2}(\lambda+\mu)(0)}(i\mathrm{sign}(\xi_1^*), 1),$$

$$s_2^+(\gamma) = \left(\vec{\varsigma}, \frac{1}{\sqrt{(\xi_1^*)^2 + (s^*)^2}}\{i\mathrm{sign}(\xi_1^*)(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*))\vec{\varsigma} + \widetilde{\eta}\}\right).$$

Therefore

(5.47)
$$\det \widetilde{\mathbb{B}}^+(\gamma) = \begin{pmatrix} 2\mu(0)i(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*))^2 & \mu(0)(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*)) \\ 2\mu(0)(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*))^2 & -i\mu\frac{\lambda+2\mu}{\lambda+\mu}(0)(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*)) \end{pmatrix}$$
$$= (\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*))^3\left(\mu^2\frac{2\lambda+4\mu}{\lambda+\mu}\right)(0) \neq 0.$$

By (5.44)–(5.47) and Gårding's inequality, we obtain

$$\sqrt{s}\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \le C_{27}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}). \tag{5.48}$$

This inequality implies (5.19). Then from (2.25) and Proposition 2.1 we obtain (3.1). □

Now we will proceed to

Completion of the proof of Theorem 1.1. Microlocally we obtain two types of estimates: If $\zeta^* \in \Psi_2$, then we have estimate (5.2), while if $\zeta^* \notin \Psi_2$, then we have estimate (3.1). By $\mathcal{O}\Psi_2(\delta_2)$, we denote the δ_2-neighbourhood of the set Ψ_2 in S^2. We take the parameter δ_2 sufficiently small. From the covering of the set $\mathcal{O}\Psi_2(\frac{15}{4}\delta_2)$ by balls with radius $4\delta_2$, we take the finite subcovering $\{B_{4\delta_2}(\zeta_j)\}_{j\in\Upsilon_1}$, $\zeta_j \in \Psi_2$. Let $\{\chi_\nu\}_{\nu\in\Upsilon_1}$ be a partition of unity associated with this subcovering. For the set $\overline{S^2 \setminus \mathcal{O}\Psi_2(3\delta_2)}$, we take the finite covering by balls with radius δ_2. Let $\{\chi_\nu\}_{\nu\in\Upsilon_2}$ be a partition of unity associated with this subcovering. We extend the functions χ_ν as a homogeneous functions of order zero to a function in $C^\infty(\mathbb{R}^3)$. Since $\Psi_2 \subset \mathcal{O}\Psi_2(\frac{15}{4}\delta_2) \cup \overline{S^2 \setminus \mathcal{O}\Psi_2(3\delta_2)}$, it follows from (5.2) and (3.1) that

$$\begin{aligned}(5.49)\quad \|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})} &\le C_{28}' \sum_{\nu\in\Upsilon_1\cup\Upsilon_2} \|\chi_\nu \mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})} \\ &\le C_{28}\Bigg\{\sqrt{\left(\frac{1}{N}+\frac{1}{s^{\frac14}}\right)}\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac32,s}(\partial\mathcal{G})} \\ &\quad + \sqrt{\left(\frac{1}{N}+\frac{1}{s^{\frac14}}\right)} \sum_{|\alpha|=0}^{2} s^{2-|\alpha|}\|\partial_y^\alpha \mathbf{v}\|_{\mathbf{L}^2(\mathcal{G})}\Bigg\}, \quad \forall N\ge \widehat{N}, s\ge s_0(\widehat{N}).\end{aligned}$$

Fixing the parameters $\widehat{N}$ and $s_0(N)$ sufficiently large, we obtain

$$(5.50)\qquad \|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})} \le C_{29}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac32,s}(\partial\mathcal{G})}), \quad \forall s\ge s_0.$$

Combination of (5.50) with estimates (5.3) and (3.1), yields

$$\begin{aligned}(5.51)\qquad &\sqrt{s}\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \\ &\le C_{30}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac32,s}(\partial\mathcal{G})}), \quad \forall \nu\in\Upsilon_2\end{aligned}$$

and

$$(5.52)\quad \sqrt{s}\|\mathbf{v}_\nu\|_{\mathbf{H}^{\frac32,s}(\partial\mathcal{G})} \le C_{31}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac32,s}(\partial\mathcal{G})}), \quad \forall \nu\in\Upsilon_1.$$

Estimates (5.51) and (5.52) yield

$$(5.53)\qquad \sqrt{s}\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{\frac32,s}(\partial\mathcal{G})} \le C_{32}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac32,s}(\partial\mathcal{G})}),$$

where we used $\phi=\varphi$ on $\partial\Omega$. We note that estimate (5.53) is obtained under additional assumption (2.7). Now we will get rid of (2.7). For this, we consider the function $\theta\mathbf{u}$ instead of the function $\mathbf{u}$, where θ is a smooth cut-off function such that $\theta|_{\Omega_{\frac{1}{4N^2}}}=1$ and $\theta|_{\Omega_{\frac{1}{N^2}}\setminus\Omega_{\frac{1}{2N^2}}}=0$. Then it suffices to modify (5.53) and prove

$$\begin{aligned}(5.54)\qquad &\sqrt{s}\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{\frac32,s}(\partial\mathcal{G})} \\ &\le C_{33}\left(\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac32,s}(\partial\mathcal{G})} + \frac{1}{\sqrt{s}}\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}\right), \\ &\forall s\ge s_0,\end{aligned}$$

where we used the fact that $\varphi(x) \le \phi(x)$ in $(0,T)\times(\Omega_{\frac{1}{N^2}} \setminus \Omega_{\frac{1}{4N^2}})$ (see (2.6)). Next by Lemma 5.1 we note that the analogues to estimates (5.3) and (5.2) hold true for the weight function ϕ instead of φ:

$$(5.55)\quad \begin{aligned}\sqrt{s}\|\chi_\nu(\mathbf{u}e^{s\phi})\|_{\mathbf{H}^{2,s}(\mathcal{G})} &+\sqrt{s}\|\chi_\nu(\mathbf{u}e^{s\phi})\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})}\\ \le C_{34}(\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})} &+ \|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})}), \quad \forall \nu\in\Upsilon_1.\end{aligned}$$

Let χ_{-1} and χ_{-2} be C^∞-functions on the sphere S^2 such that $\chi_{-1}\in C_0^\infty(\mathcal{O}\Psi_2(\frac{15}{4}\delta_2))$, $\chi_{-2}\in C_0^\infty(S^2\setminus\mathcal{O}\Psi_2(3\delta_2))$ and $\chi_{-1}|_{\mathcal{O}\Psi_2(\frac{15}{16}\delta_2)}=1$, $\chi_{-2}|_{S^2\setminus\mathcal{O}\Psi_2(\frac{8}{9}\delta_2)}=1$. Hence we have $|\chi_{-1}(s,\xi')|+|\chi_{-2}(s,\xi')|\ge 1$ for all $\zeta\in S^2$. We extend the functions χ_{-1} and χ_{-2} as homogeneous functions of order zero to functions in $C^\infty(\mathbb{R}^3)$. By (1.8) and (1.9), we have

$$(5.56)\qquad \mathbb{A}^{-1}P(y,\mathbf{D})\chi_{-1}\mathbf{v}+[\chi_{-1},\mathbb{A}^{-1}P(y,\mathbf{D})]\mathbf{v}=\chi_{-1}\mathbb{A}^{-1}\mathbf{f}e^{s\phi}.$$

Note that we can estimate

$$(5.57)\quad \|[\chi_{-1},\mathbb{A}^{-1}P(y,\mathbf{D})]\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\le C_{35}(\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}+\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

Hence applying estimate (5.55) and (5.57) to (5.56), we obtain.

$$(5.58)\qquad \begin{aligned}&\sqrt{s}\|\chi_{-1}(s,D')(\mathbf{u}e^{s\phi})\|_{\mathbf{H}^{2,s}(\mathcal{G})}\\ &\le C_{36}(\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}+\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})}).\end{aligned}$$

Setting $\widetilde{P}_{\beta,s}(y,s,D)=\frac{1}{\beta(1+|\ell'(y_1)|^2)}P_{\beta,s}(y,s,D)$, we have

$$\widetilde{P}_{\mu,s}(y,s,D)\chi_{-2}(s,D')w_1+[\chi_{-2},\widetilde{P}_{\mu,s}]w_1=\chi_{-2}\left(\frac{1}{\mu(1+|\ell'(y_1)|^2)}q_1\right)$$

and

$$\begin{aligned}&\widetilde{P}_{\lambda+2\mu,s}(y,s,D)\chi_{-2}(s,D')w_2+[\chi_{-2},\widetilde{P}_{\lambda+2\mu,s}]w_2\\ =&\chi_{-2}\left(\frac{1}{(\lambda+2\mu)(1+|\ell'(y_1)|^2)}q_2\right).\end{aligned}$$

Note that

$$\begin{aligned}&|\Sigma_\mu(\chi_{-2}(s,D')w_1)|+|\Sigma_{\lambda+2\mu}(\chi_{-2}(s,D')w_2)|\\ \le& C_{37}\sqrt{s}\left\|\chi_{-2}(s,D')\left(\frac{\partial\mathbf{w}}{\partial y_2},\mathbf{w}\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}.\end{aligned}$$

Combining this inequality with (2.25) we have

$$\sqrt{s}\|\chi_{-2}(s,D')\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le C_{38}\Bigg\{\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} + \|[\chi_{-2},\widetilde{P}_{\mu,s}]w_1\|_{L^2(\mathcal{G})} + \|[\chi_{-2},\widetilde{P}_{\lambda+2\mu,s}]w_2\|_{L^2(\mathcal{G})} + \sqrt{s}\left\|\chi_{-2}(s,D')\left(\frac{\partial \mathbf{w}}{\partial y_2},\mathbf{w}\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\Bigg\}. \tag{5.59}$$

Thanks to the estimates

$$\begin{aligned}&\|[\chi_{-2},\widetilde{P}_{\mu,s}]w_1\|_{L^2(\mathcal{G})} + \|[\chi_{-2},\widetilde{P}_{\lambda+2\mu,s}]w_2\|_{L^2(\mathcal{G})}\\ \le& C_{39}(\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})})\end{aligned}$$

and

$$\begin{aligned}&\left\|\chi_{-2}(s,D')\left(\frac{\partial \mathbf{w}}{\partial y_2},\mathbf{w}\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\\ \le& C_{40}\left\{\frac{1}{\sqrt{s}}\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\chi_{-2}(s,D')(\mathbf{u}e^{s\phi})\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})}\right\},\end{aligned}$$

we obtain

$$\begin{aligned}&\sqrt{s}\|\chi_{-2}(s,D')\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\\ &\le C_{41}(\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} + \|(1-\chi_{-2}(s,D'))\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\\ &+\sqrt{s}(\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\chi_{-2}(s,D')\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})})).\end{aligned} \tag{5.60}$$

Estimating the last term on the right hand side of (5.60), we obtain

$$\begin{aligned}&\sqrt{s}\|\chi_{-2}(s,D')\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\\ &\le C_{42}(\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} + \|(1-\chi_{-2}(s,D'))\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}).\end{aligned} \tag{5.61}$$

Since $\chi_{-1}|_{\mathrm{supp}(1-\chi_{-2})} = 1$, by (5.55) we have

$$\begin{aligned}&\|(1-\chi_{-2}(s,D'))\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le \|\chi_{-1}(s,D')\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\\ \le& C_{42}(\|\chi_{-1}(s,D')(\mathbf{u}e^{s\phi})\|_{\mathbf{H}^{2,s}(\mathcal{G})} + \|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})})\\ \le& C_{43}\frac{1}{\sqrt{s}}\left(\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})} + \|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})}\right).\end{aligned}$$

Hence, using this estimate and (5.58), we obtain from (5.61)

$$\begin{aligned}&\sqrt{s}\|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\\ &\le C_{44}\left(\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} + \frac{1}{\sqrt{s}}\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}\right).\end{aligned} \tag{5.62}$$

We can rewrite equation (1.1) as

$$\rho(\widetilde{x})\frac{\partial^2 \mathbf{u}}{\partial x_0^2} - \mu(\widetilde{x})\Delta \mathbf{u}$$
$$=(\mu(\widetilde{x}) + \lambda(\widetilde{x}))\nabla_{\widetilde{x}}\mathrm{div}\,\mathbf{u} + (\mathrm{div}\,\mathbf{u})\nabla_{\widetilde{x}}\lambda(\widetilde{x}) + (\nabla_{\widetilde{x}}\mathbf{u} + (\nabla_{\widetilde{x}}\mathbf{u})^T)\nabla_{\widetilde{x}}\mu(\widetilde{x}) + \mathbf{f}.$$

Applying to each component of this system the Carleman estimate for hyperbolic equations we have

$$s\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le C_{45}(\sqrt{s}\|\mathbf{f}e^{s\phi}\|_{\mathbf{L}^2(\mathcal{G})}$$
$$+\sqrt{s}\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}\|(\mathrm{div}\ \mathbf{u})e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{u}e^{s\phi}\|_{H^{1,s}(\partial\mathcal{G})}).$$

Estimating the term with $\mathrm{div}\,\mathbf{u}$ on the right hand side of this inequality by (5.62), we obtain

(5.63)
$$s\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le C_{46}(\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} + s\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}).$$

We set $\mathrm{rot}^* v = (\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1})$. Then we have

$$-\Delta_{\tilde{x}}\mathbf{u} \equiv -\Delta\mathbf{u} = \mathrm{rot}^*\mathrm{rot}\ \mathbf{u} - \nabla_{\widetilde{x}}(\mathrm{div}\ \mathbf{u}).$$

Therefore we obtain

(5.64)
$$\sum_{i,j=1}^{2} \|(\partial^2_{x_i x_j}\mathbf{u})e^{s\phi}\|_{\mathbf{L}^2(Q)} \le C_{47}(\|\mathbf{u}e^{s\phi}\|_{L^2(0,T;\mathbf{H}^{\frac{3}{2},s}(\partial\Omega))} + s\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(Q)}).$$

In view of equations (1.8) and (1.9), we can estimate $\partial^2_{x_0}(\mathbf{u}e^{s\phi})$:

(5.65)
$$\|\partial^2_{x_0}(\mathbf{u}e^{s\phi})\|_{\mathbf{L}^2(\mathcal{G})} \le C_{48}\left(\sum_{|\alpha|=0,\alpha=(0,\alpha_1,\alpha_2)}^{2} s^{2-|\alpha|}\|\partial^\alpha(\mathbf{u}e^{s\phi})\|_{\mathbf{L}^2(\mathcal{G})} + \|\mathbf{f}e^{s\phi}\|_{\mathbf{L}^2(\mathcal{G})}\right).$$

Hence (5.54) and (5.63)–(5.65) yield

(5.66)
$$\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})} \le C_{49}\left\{\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} + \frac{1}{\sqrt{s}}\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}\right\}.$$

Estimate (5.66) implies (1.24). Thus the proof of Theorem 1.1 is completed. □

Appendix I. Proof of Proposition 2.1

Let $x_0 \in [0,T]$ be arbitrary fixed.

First we choose $N_0 > 0$ sufficiently large such that

$$\nabla_{\widetilde{x}}\psi(x) \neq 0, \quad \forall \widetilde{x} \in \Omega_{\frac{1}{N^2}},\ x_0 \in [0,T]. \tag{1}$$

The existence of such N_0 follows from condition (1.6).

Let $\mathrm{rot}^*v = (\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1})$. Using the well-known formula $\mathrm{rot}^*\mathrm{rot} = -\Delta_{\widetilde{x}} + \nabla_{\widetilde{x}}\mathrm{div}$, we obtain

$$\Delta_{\widetilde{x}}\mathbf{u} = -\mathrm{rot}^*(\mathrm{rot}\,\mathbf{u}) + \nabla_{\widetilde{x}}(\mathrm{div}\,\mathbf{u}) \quad \text{in } \Omega_{\frac{1}{N^2}}.$$

The function $\mathbf{v} = \mathbf{u}e^{s\varphi}$ satisfies equations

$$L_1\mathbf{v} + L_2\mathbf{v} = q_s \quad \text{in } \Omega_{\frac{1}{N^2}}, \qquad \mathbf{v}|_{\partial\Omega_{\frac{1}{N^2}}} = 0, \tag{2}$$

where $L_1\mathbf{v} = -\Delta_{\widetilde{x}}\mathbf{v} - s^2|\nabla_{\widetilde{x}}\varphi|^2\mathbf{v}$, $L_2\mathbf{v} = 2s\sum_{k=1}^2 \mathbf{v}_{x_k}\varphi_{x_k} + s(\Delta_{\widetilde{x}}\varphi)\mathbf{v}$ and $q_s = (\mathrm{rot}^*\mathrm{rot}\,\mathbf{u} - \nabla_{\widetilde{x}}(\mathrm{div}\,\mathbf{u}))e^{s\varphi}$. Taking the L^2 norms of the right and the left hand sides of the first equation in (2), we obtain

$$\|L_1\mathbf{v}\|^2_{\mathbf{L}^2\left(\Omega_{\frac{1}{N^2}}\right)} + \|L_2\mathbf{v}\|^2_{\mathbf{L}^2\left(\Omega_{\frac{1}{N^2}}\right)} + 2(L_1\mathbf{v}, L_2\mathbf{v})_{\mathbf{L}^2\left(\Omega_{\frac{1}{N^2}}\right)} = \|q_s\|^2_{\mathbf{L}^2\left(\Omega_{\frac{1}{N^2}}\right)}.$$

After integrations, we will arrive at the formula:

$$\begin{aligned}(L_1\mathbf{v}, L_2\mathbf{v})_{\mathbf{L}^2\left(\Omega_{\frac{1}{N^2}}\right)} &= \int_{\Omega_{\frac{1}{N^2}}} \Bigg\{ 2s\sum_{k,j=1}^2 \mathbf{v}_{x_j}\mathbf{v}_{x_k}\varphi_{x_jx_k} \\
&+ s^3(\mathrm{div}(|\nabla_{\widetilde{x}}\varphi|^2\nabla_{\widetilde{x}}\varphi) - |\nabla_{\widetilde{x}}\varphi|^2\Delta_{\widetilde{x}}\varphi)|\mathbf{v}|^2 - \frac{s}{2}\sum_{j=1}^2 \frac{\partial^2\Delta_{\widetilde{x}}\varphi}{\partial x_j^2}|\mathbf{v}|^2 \Bigg\} d\widetilde{x} \\
&- \int_{\partial\Omega} \left(\frac{\partial\mathbf{v}}{\partial\vec{n}}, L_2\mathbf{v}\right) d\sigma + s\int_{\partial\Omega} \left|\frac{\partial\mathbf{v}}{\partial\vec{n}}\right|^2 (\nabla_{\widetilde{x}}\varphi, \vec{n}) d\sigma \\
&+ s\int_{\partial\Omega} (\nabla_{\widetilde{x}}\varphi, \vec{n}) s^3|\nabla_{\widetilde{x}}\varphi|^2|\mathbf{v}|^2 d\sigma.\end{aligned} \tag{3}$$

Denote $\psi_1(x) = \psi(x) - \widehat{\epsilon}\ell_1(x)$. Then

$$\mathrm{div}(|\nabla_{\widetilde{x}}\varphi|^2\nabla_{\widetilde{x}}\varphi) - |\nabla_{\widetilde{x}}\varphi|^2\Delta_{\widetilde{x}}\varphi = 2\sum_{k,j=1}^2 \varphi_{x_k}\varphi_{x_j}\varphi_{x_kx_j}$$

$$\begin{aligned}=&2\varphi^3\sum_{k,j=1}^2 \tau^4(\partial_{x_k}\psi_1 + 2N\ell_1\partial_{x_k}\ell_1)^2(\partial_{x_j}\psi_1 + 2N\ell_1\partial_{x_j}\ell_1)^2 \\
&+\tau^3(\partial_{x_k}\psi_1 + 2N\ell_1\partial_{x_k}\ell_1)(\partial_{x_j}\psi_1 + 2N\ell_1\partial_{x_j}\ell_1)(\partial^2_{x_kx_j}\psi_1 \\
&+2N(\partial_{x_k}\ell_1)\partial_{x_j}\ell_1 + 2N\ell_1\partial^2_{x_kx_j}\ell_1).\end{aligned}$$

Since $(\nabla_{\widetilde{x}}\psi_1, \nabla_{\widetilde{x}}\ell_1) > 0$ on $\partial\Omega$, there exists a constant $C_1 > 0$ independent of $N, \widetilde{\tau}, s$, such that

$$\text{div}(|\nabla_{\widetilde{x}}\varphi|^2\nabla_{\widetilde{x}}\varphi) - |\nabla_{\widetilde{x}}\varphi|^2\Delta_{\widetilde{x}}\varphi \geq 2\varphi^3\tau^4|\nabla_{\widetilde{x}}\psi_1|^4 + C_1N\tau^3\varphi^3 + \varphi^2 O(\tau^3). \tag{4}$$

On the other hand,

$$\begin{aligned}\sum_{k,j=1}^{2} \mathbf{v}_{x_j}\mathbf{v}_{x_k}\varphi_{x_jx_k} &= \tau^2(\nabla_{\widetilde{x}}\mathbf{v}, \nabla_{\widetilde{x}}\widetilde{\psi})^2\varphi \\ &+ 2\tau\sum_{k,j=1}^{2}\mathbf{v}_{x_j}\mathbf{v}_{x_k}(\psi_{x_jx_k} + N\ell_1\partial^2_{x_jx_k}\ell_1)\varphi - \frac{\tau\varphi}{\sqrt{N}}\sum_{j,k=1}^{2}\mathbf{v}_{x_j}\mathbf{v}_{x_k}\partial^2_{x_jx_k}\ell_1 \\ &+ N\tau(\nabla_{\widetilde{x}}\mathbf{v}, \nabla_{\widetilde{x}}\ell_1)^2\varphi.\end{aligned} \tag{5}$$

Note that there exists a constant $C_2 > 0$, independent of N, such that

$$\|N\ell_1\partial^2_{x_jx_k}\ell_1\|_{C^0\left(\overline{\Omega_{\frac{1}{N^2}}}\right)} \leq \frac{C_2}{N}. \tag{6}$$

By (3)–(6) we obtain

(7)
$$\begin{aligned}&\|L_1\mathbf{v}\|^2_{\mathbf{L}^2\left(\Omega_{\frac{1}{N^2}}\right)} + \|L_2\mathbf{v}\|^2_{\mathbf{L}^2\left(\Omega_{\frac{1}{N^2}}\right)} + \int_{\Omega_{\frac{1}{N^2}}}(2\varphi^3\tau^4|\nabla_{\widetilde{x}}\psi_1|^4 + C_3N\tau^3\varphi^3)|\mathbf{v}|^2d\widetilde{x} \\ &-s\tau C_4\int_{\Omega_{\frac{1}{N^2}}}\varphi|\nabla_{\widetilde{x}}\mathbf{v}|^2d\widetilde{x} \\ &\leq \|q_s\|^2_{\mathbf{L}^2\left(\Omega_{\frac{1}{N^2}}\right)} + C_5\left(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\Omega)} + s\left\|\frac{\partial\mathbf{v}}{\partial\vec{n}}\right\|^2_{\mathbf{L}^2(\partial\Omega)}\right).\end{aligned}$$

Multiplying the first equation in (2) by $sN\varphi\mathbf{v}$ and integrating by parts, we obtain

(8)
$$\begin{aligned}&\int_{\Omega_{\frac{1}{N^2}}}\left\{sN\varphi|\nabla_{\widetilde{x}}\mathbf{v}|^2 + s^2N(\Delta_{\widetilde{x}}\varphi)\varphi|\mathbf{v}|^2 - s^3\varphi^3|\nabla_{\widetilde{x}}\varphi|^2|\mathbf{v}|^2 - \frac{sN}{2}\text{div}\varphi|\mathbf{v}|^2\right\}d\widetilde{x} \\ &+\int_{\partial\Omega}\left\{-\left(\frac{\partial\mathbf{v}}{\partial\vec{n}}, sN\varphi\mathbf{v}\right) + \left(s^2\varphi N + \frac{sN}{2}\right)(\nabla_{\widetilde{x}}\varphi, \vec{n})|\mathbf{v}|^2\right\}d\sigma \\ &= \int_{\Omega_{\frac{1}{N^2}}} q_s sN\varphi\mathbf{v}d\widetilde{x}.\end{aligned}$$

Next we note that

$$\Delta_{\widetilde{x}}\varphi = (|\nabla_{\widetilde{x}}\widetilde{\psi}|^2\tau^2 + \tau\Delta_{\widetilde{x}}\psi_1 + 2\tau N|\nabla_{\widetilde{x}}\ell_1|^2 + 2\tau N\ell_1\Delta_{\widetilde{x}}\ell_1)\varphi \geq C_6\tau N\varphi.$$

This inequality and (8) imply

$$\int_{\Omega_{\frac{1}{N^2}}} \left(sN\varphi|\nabla_{\widetilde{x}}\mathbf{v}|^2 + \frac{1}{2}s^2N(\Delta_{\widetilde{x}}\varphi)\varphi|\mathbf{v}|^2 - s^3\varphi^3|\nabla_{\widetilde{x}}\varphi|^2|\mathbf{v}|^2 \right) d\widetilde{x} \\ \leq C_6\|q_s\|^2_{L^2\left(\Omega_{\frac{1}{N^2}}\right)} + C_6\left(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\Omega)} + s\left\|\frac{\partial\mathbf{v}}{\partial\vec{n}}\right\|^2_{\mathbf{L}^2(\partial\Omega)}\right). \tag{9}$$

In view of (7) and (9), we obtain

$$\|L_1\mathbf{v}\|^2_{\mathbf{L}^2\left(\Omega_{\frac{1}{N^2}}\right)} + \|L_2\mathbf{v}\|^2_{\mathbf{L}^2\left(\Omega_{\frac{1}{N^2}}\right)} + \int_{\Omega_{\frac{1}{N^2}}} \left(\frac{1}{2}\varphi^3\tau^4|\nabla_{\widetilde{x}}\psi_1|^4 \\ + C_3N\tau^3\varphi^3\right)|\mathbf{v}|^2 d\widetilde{x} + sN\int_{\Omega_{\frac{1}{N^2}}} \varphi|\nabla_{\widetilde{x}}\mathbf{v}|^2 d\widetilde{x} \\ \leq C_7\|q_s\|^2_{L^2\left(\Omega_{\frac{1}{N^2}}\right)} + C_8\left(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\Omega)} + s\left\|\frac{\partial\mathbf{v}}{\partial\vec{n}}\right\|^2_{\mathbf{L}^2(\partial\Omega)}\right). \tag{10}$$

Let $\mathbf{v} = \widetilde{\mathbf{v}}_1 + \widetilde{\mathbf{v}}_2$ where

$$-\Delta_{\widetilde{x}}\widetilde{\mathbf{v}}_1 = L_1\mathbf{v} \text{ in } \Omega_{\frac{1}{N^2}},\ \widetilde{\mathbf{v}}_1|_{\partial\Omega_{\frac{1}{N^2}}} = \mathbf{v},$$
$$-\Delta_{\widetilde{x}}\widetilde{\mathbf{v}}_2 = s^2|\nabla_{\widetilde{x}}\varphi|^2\mathbf{v} \quad \text{in} \quad \Omega_{\frac{1}{N^2}},\ \widetilde{\mathbf{v}}_2|_{\partial\Omega_{\frac{1}{N^2}}} = 0.$$

By standard a priori estimates for the Laplace operator, we have

$$\|\widetilde{\mathbf{v}}_1\|_{\mathbf{H}^2\left(\Omega_{\frac{1}{N^2}}\right)} \leq C_9(\|L_1\mathbf{v}\|_{\mathbf{L}^2\left(\Omega_{\frac{1}{N^2}}\right)} + \|\mathbf{v}\|_{\mathbf{H}^{\frac{3}{2}}(\partial\Omega)}), \tag{11}$$

$$\frac{\sqrt{N}}{\sqrt{s}}\|\widetilde{\mathbf{v}}_2\|_{\mathbf{H}^2\left(\Omega_{\frac{1}{N^2}}\right)} \leq C_{10}\sqrt{N}\|s^{\frac{3}{2}}|\nabla_{\widetilde{x}}\varphi|^2\mathbf{v}\|_{\mathbf{L}^2(\Omega_{\frac{1}{N^2}})}, \tag{12}$$

where $C_9, C_{10} > 0$ are independent of N. Taking $s_0(\tau, N) \geq N$, from (9)–(12) we obtain

$$N\sum_{|\alpha|=0,\alpha=(\alpha_0,\alpha_1,0)}^{2} s^{4-2|\alpha|}\|(\partial^\alpha\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)} \\ \leq C_{11}\left\{ \|(\operatorname{div}\mathbf{u})e^{s\varphi}\|^2_{L^2(Q)} + s^2\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial Q)} + s^2\left\|\frac{\partial\mathbf{v}}{\partial\vec{n}}\right\|^2_{\mathbf{L}^2(\partial Q)} \\ + N\|\mathbf{v}\|^2_{L^2(0,T;\mathbf{H}^{\frac{3}{2}}(\partial\Omega))}\right\}. \tag{13}$$

By (1.8) and (1.9), we estimate the norm of $\partial_{x_0}^2 \mathbf{u}$:

$$N\|(\partial_{x_0}^2 \mathbf{u})e^{s\varphi}\|^2_{\mathbf{L}^2(Q)} \le C_{12}\Bigg\{ \|(\operatorname{div}\mathbf{u})e^{s\varphi}\|^2_{L^2(Q)} + s^2\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial Q)} + s^2 \left\|\frac{\partial \mathbf{v}}{\partial \vec{n}}\right\|^2_{\mathbf{L}^2(\partial Q)} + N\|\mathbf{v}\|^2_{L^2(0,T;\mathbf{H}^{\frac{3}{2}}(\partial\Omega))} + N\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{L}^2(Q)} \Bigg\}. \tag{14}$$

Finally we note

$$Ns^2\|(\partial_{x_0}\mathbf{u})e^{s\varphi}\|^2_{\mathbf{L}^2(Q)} \le C_{13}(N\|(\partial_{x_0}^2\mathbf{u})e^{s\varphi}\|^2_{\mathbf{L}^2(Q)} + s^4 N\|\mathbf{u}e^{s\varphi}\|^2_{\mathbf{L}^2(Q)}) \tag{15}$$

and

$$N\|\partial_{x_k x_j}\mathbf{v}\|^2_{\mathbf{L}^2(Q)} \le C_{14}\left(N\|\partial^2_{x_k}\mathbf{v}\|^2_{\mathbf{L}^2(Q)} + \|\partial^2_{x_j}\mathbf{v}\|^2_{\mathbf{L}^2(Q)} + \left\|\left(\mathbf{v}, \frac{\partial \mathbf{v}}{\partial \vec{n}}\right)\right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)\times \mathbf{H}^{\frac{1}{2},s}(\partial Q)} \right). \tag{16}$$

Thus the proof of Proposition 2.1 is completed. □

Appendix II. Proof of Proposition 2.4

Let us consider the following problem

$$L^* p = \left(-\frac{\partial}{\partial y_2} - \widetilde{\Gamma}^{+,*}_{\beta}(y,s,D') \right) p = i\chi_\nu w \quad \text{in } \mathcal{G}, \tag{1}$$

where $\beta \in \{\mu, \lambda + 2\mu\}$ and $\widetilde{\Gamma}^{+,*}_{\beta}$ is the operator which is formally adjoint to $\widetilde{\Gamma}^{+}_{\beta} = i\Gamma^{+}_{\beta}$. We have

Lemma 1. *There exist constants $C_1 > 0$ and $s_0 > 0$ such that for every $s \ge s_0$, there exists p satisfying* (1) *and*

$$s\int_{\mathcal{G}} |p|^2 dy + \sqrt{s}\int_{\mathcal{G}\cap\{y_2=0\}} |p(y',0)|^2 dy' \le C_1 \int_{\mathcal{G}} |\chi_\nu w|^2 dy. \tag{2}$$

Proof of Lemma 1. For $\epsilon > 0$, let us consider the functional:

$$J_\epsilon(p) = \frac{1}{2}\|p\|^2_{L^2(\mathcal{G})} + \frac{1}{2\epsilon}\left\| \frac{\partial p}{\partial y_2} + \widetilde{\Gamma}^{+,*}_{\beta}(y,s,D')p + i\chi_\nu w \right\|^2_{L^2(\mathcal{G})}. \tag{3}$$

Notice that there exists p such that $J_\epsilon(p)$ is finite and, for example, we can set $p = 0$. We consider the minimization problem

$$\min_{p\in U} J_\epsilon(p), \tag{4}$$

where

$$U = \left\{ p \in L^2(\mathcal{G});\ \frac{\partial p}{\partial y_2} + \widetilde{\Gamma}_\beta^{+,*}(y,s,D')p + i\chi_\nu w \in L^2(\mathcal{G}) \right\}. \tag{5}$$

There exists a minimizing sequence $\{p_n\}_{n=1}^\infty$ such that $p_n \in U$ and

$$\lim_{n\to\infty} J_\epsilon(p_n) = \inf_{p\in U} J_\epsilon(p). \tag{6}$$

Then the sequences $\{p_n\}$ and $\{\frac{\partial p_n}{\partial y_2} + \widetilde{\Gamma}_\beta^{+,*}(y,s,D')p_n + \chi_\nu w\}$ are bounded in $L^2(\mathcal{G})$. Therefore $\widetilde{\Gamma}_\beta^{+,*}(y,s,D')p_n$ is bounded in $L^2\left(0,\frac{1}{N^2}; H^{-1,s}(\mathbb{R}^2)\right)$ and $\frac{\partial p_n}{\partial y_2}$, $n \in \mathbb{N}$ are bounded in $L^2\left(0,\frac{1}{N^2}; H^{-1,s}(\mathbb{R}^2)\right)$. Consequently we can extract a subsequence, still denoted by $\{p_n\}_{n=1}^\infty$ such that

$$\begin{aligned} & p_n \rightharpoonup p_\epsilon \text{ weakly in } L^2(\mathcal{G}),\\ & \frac{\partial p_n}{\partial y_2} \rightharpoonup \frac{\partial p_\epsilon}{\partial y_2} \quad \text{weakly in} \quad L^2\left(0,\frac{1}{N^2}; H^{-1,s}(\mathbb{R}^2)\right),\\ & \frac{\partial p_n}{\partial y_2} + \widetilde{\Gamma}_\beta^{+,*}(y,s,D')p_n + i\chi_\nu w \rightharpoonup \frac{\partial p_\epsilon}{\partial y_2} + \widetilde{\Gamma}_\beta^{+,*}(y,s,D')p_\epsilon + i\chi_\nu w\\ & \qquad \text{weakly in } L^2\left(0,\frac{1}{N^2}; H^{-1}(\mathbb{R}^2)\right). \end{aligned} \tag{7}$$

On the other hand, as $\left\|\frac{\partial p_n}{\partial y_2} + \widetilde{\Gamma}_\beta^{+,*}(y,s,D')p_n + i\chi_\nu w\right\|_{L^2(\mathcal{G})}$ remains bounded, we have

$$\frac{\partial p_n}{\partial y_2} + \widetilde{\Gamma}_\beta^{+,*}(y,s,D')p_n + i\chi_\nu w \rightharpoonup \frac{\partial p_\epsilon}{\partial y_2} + \widetilde{\Gamma}_\beta^{+,*}(y,s,D')p_\epsilon + i\chi_\nu w$$

weakly in $L^2(\mathcal{G})$. Then p_ϵ is a minimizer of J_ϵ, that is, $p_\epsilon \in U$ and

$$J_\epsilon(p_\epsilon) = \min_{p\in U} J_\epsilon(p). \tag{8}$$

Writing the first order optimality conditions, we have for every $r \in H^1(\mathcal{G})$:

$$\langle J'_\epsilon(p_\epsilon), r\rangle = 0. \tag{9}$$

Let us define q_ϵ by

$$q_\epsilon = \frac{1}{\epsilon}\left(\frac{\partial p_\epsilon}{\partial y_2} + \widetilde{\Gamma}_\beta^{+,*}(y,s,D')p_\epsilon + i\chi_\nu w\right). \tag{10}$$

In view of (9), for every $r \in H^1(\mathcal{G})$, we see:

$$\int_{\mathcal{G}} p_\epsilon \overline{r} dy + \int_{\mathcal{G}} q_\epsilon \overline{\left(\frac{\partial r}{\partial y_2} + \widetilde{\Gamma}_\beta^{+,*}(y,s,D')r\right)} dy = 0. \tag{11}$$

Then q_ϵ satisfies the following problem

$$\frac{\partial q_\epsilon}{\partial y_2} - \widetilde{\Gamma}_\beta^{+}(y,s,D')q_\epsilon = p_\epsilon \quad \text{in } \mathcal{G}, \tag{12}$$

$$q_\epsilon(y',0) = 0, \ q_\epsilon\left(y', \frac{1}{N^2}\right) = 0, \quad \forall y' \in \mathbb{R}^2. \tag{13}$$

Denote $L_1 = \frac{1}{2}(-\widetilde{\Gamma}_\beta^{+} - \widetilde{\Gamma}_\beta^{+,*})$ and $L_2 = \frac{\partial}{\partial y_2} + \frac{1}{2}(\widetilde{\Gamma}_\beta^{+,*} - \widetilde{\Gamma}_\beta^{+})$. Then we can rewrite (12) and (13) as follows.

$$Lq_\epsilon \equiv (L_1 + L_2)q_\epsilon = p_\epsilon \text{ in } \mathcal{G}, \ q_\epsilon(y',0) = 0, \ q_\epsilon\left(y', \frac{1}{N^2}\right) = 0, \ y' \in \mathbb{R}^2. \tag{14}$$

There exist constants $C_2 > 0$ and $s_0 > 0$ such that

$$\|L_1 q_\epsilon\|^2_{L^2(\mathcal{G})} + \|L_2 q_\epsilon\|^2_{L^2(\mathcal{G})} + sN\int_{\mathcal{G}} |q_\epsilon|^2 dy \le C_2 \|p_\epsilon\|^2_{L^2(\mathcal{G})}, \quad \forall s \ge s_0. \tag{15}$$

Notice that $L_1 q_\epsilon \in L^2(\mathcal{G})$ implies $q_\epsilon \in L^2\left(0, \frac{1}{N^2}; H^1(\mathbb{R}^2)\right)$, which implies $\frac{\partial q_\epsilon}{\partial y_2} \in L^2(\mathcal{G})$ from (12). Now it follows from definition (10) of q_ϵ that p_ϵ satisfies

$$\frac{\partial p_\epsilon}{\partial y_2} + \widetilde{\Gamma}_\beta^{+,*}(y,s,D')p_\epsilon = \epsilon q_\epsilon - i\chi_\nu w, \tag{16}$$

which can be written as

$$(L_2 - L_1)p_\epsilon = \frac{\partial p_\epsilon}{\partial y_2} + \widetilde{\Gamma}_\beta^{+,*}(y,s,D')p_\epsilon = \epsilon q_\epsilon - i\chi_\nu w. \tag{17}$$

Multiplying (17) by q_ϵ in $L^2(\mathcal{G})$ and using the boundary conditions for q_ϵ, we obtain

$$-\int_{\mathcal{G}} p_\epsilon \overline{(L_1 + L_2)q_\epsilon} dy = \epsilon \int_{\mathcal{G}} |q_\epsilon|^2 dy - \int_{\mathcal{G}} i\chi_\nu w \overline{q_\epsilon} dy, \tag{18}$$

so that

$$\int_{\mathcal{G}} |p_\epsilon|^2 dy + \epsilon \int_{\mathcal{G}} |q_\epsilon|^2 dy = \int_{\mathcal{G}} i\chi_\nu w \overline{q_\epsilon} dy$$

and

$$\begin{aligned} \int_{\mathcal{G}} |p_\epsilon|^2 dy &\le \left(\int_{\mathcal{G}} |\chi_\nu w|^2 dy\right)^{\frac{1}{2}} \left(\int_{\mathcal{G}} |q_\epsilon|^2 dy\right)^{\frac{1}{2}} \\ &\le \frac{C_3}{\sqrt{s}} \left(\int_{\mathcal{G}} |\chi_\nu w|^2 dy\right)^{\frac{1}{2}} \left(\int_{\mathcal{G}} |p_\epsilon|^2 dy\right)^{\frac{1}{2}}. \end{aligned} \tag{19}$$

Therefore we obtain the first estimate on p_ϵ

$$s\int_{\mathcal{G}} |p_\epsilon|^2 dy \le C_4 \int_{\mathcal{G}} |\chi_\nu w|^2 dy. \tag{20}$$

Let us now notice from (12) and (13) that we have

$$p_\epsilon(y',0) = \frac{\partial q_\epsilon}{\partial y_2}(y',0),\ y' \in \mathbb{R}^2.$$

Let $\theta = \theta(y_2) \in C^\infty[0, \frac{1}{N^2}]$ such that $0 \le \theta \le 1$, $\theta(0) = 1$ and $\theta(\frac{1}{N^2}) = 0$. We have

$$(L_1 + L_2)(\theta q_\epsilon) = \theta(L_1 + L_2)(q_\epsilon) + \frac{\partial \theta}{\partial y_2} q_\epsilon = \theta p_\epsilon + \frac{\partial \theta}{\partial y_2} q_\epsilon \text{ in } \mathcal{G}, \tag{21}$$

$$(\theta q_\epsilon)(y',0) = 0,\ (\theta q_\epsilon)\left(y', \frac{1}{N^2}\right) = 0,\ \frac{\partial(\theta q_\epsilon)}{\partial y_2}\left(y', \frac{1}{N^2}\right) = 0, \quad y' \in \mathbb{R}^2. \tag{22}$$

Now we apply the operator $(L_2 - L_1)$ to the first equation, so that

$$\begin{aligned}(L_2 - L_1)(L_2 + L_1)(\theta q_\epsilon) &= (L_2 - L_1)(\theta p_\epsilon) + (L_2 - L_1)\left(\frac{\partial \theta}{\partial y_2} q_\epsilon\right) \\ &= \theta(L_2 - L_1)(p_\epsilon) + \frac{\partial \theta}{\partial y_2} p_\epsilon + \frac{\partial \theta}{\partial y_2}(L_2 - L_1) q_\epsilon + \frac{\partial^2 \theta}{\partial y_2^2} q_\epsilon.\end{aligned} \tag{23}$$

Then we have

$$\begin{aligned}&L_2^2(\theta q_\epsilon) - L_1^2(\theta q_\epsilon) + [L_2, L_1](\theta q_\epsilon) \\ &= \epsilon \theta q_\epsilon - i\theta \chi_\nu w + \frac{\partial \theta}{\partial y_2} p_\epsilon + \frac{\partial \theta}{\partial y_2}(L_2 - L_1) q_\epsilon + \frac{\partial^2 \theta}{\partial y_2^2} q_\epsilon.\end{aligned} \tag{24}$$

We now take the scalar product in $L^2(\mathcal{G})$ of this equation with $L_2(\theta q_\epsilon)$ and take the real part. Henceforth we give the computations of the successive terms. We can calculate the first term as follows.

$$\begin{aligned}(L_2^2(\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})} &= \left(\frac{\partial}{\partial y_2}(L_2(\theta q_\epsilon)), L_2(\theta q_\epsilon)\right)_{L^2(\mathcal{G})} \\ &- \frac{1}{2}((\widetilde{\Gamma}^+_\beta(y,s,D') - \widetilde{\Gamma}^{+,*}_\beta(y,s,D'))L_2(\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})} \\ &= -\|L_2(\theta q_\epsilon)(\cdot,0)\|^2_{L^2(\mathbb{R}^2)} - (L_2(\theta q_\epsilon), L_2^2(\theta q_\epsilon))_{L^2(\mathcal{G})} \\ &= -\left\|\frac{\partial q_\epsilon}{\partial y_2}(\cdot,0)\right\|^2_{L^2(\mathbb{R}^2)} - (L_2(\theta q_\epsilon), L_2^2(\theta q_\epsilon))_{L^2(\mathcal{G})}.\end{aligned} \tag{25}$$

Therefore we have

$$\mathrm{Re}\,(L_2^2(\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})} = -\frac{1}{2}\left\|\frac{\partial q_\epsilon}{\partial y_2}(\cdot,0)\right\|^2_{L^2(\mathbb{R}^2)}. \tag{26}$$

Now for the second term

(27)

$$\begin{aligned}
&2\mathrm{Re}\,(L_1^2(\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})} = (L_1^2(\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})} + (L_2(\theta q_\epsilon), L_1^2\theta q_\epsilon)_{L^2(\mathcal{G})}\\
=&-(L_2L_1^2(\theta q_\epsilon), (\theta q_\epsilon))_{L^2(\mathcal{G})} + (L_1^2L_2(\theta q_\epsilon), (\theta q_\epsilon))_{L^2(\mathcal{G})}\\
=&(L_1[L_1,L_2](\theta q_\epsilon), (\theta q_\epsilon))_{L^2(\mathcal{G})} + ([L_1,L_2]L_1(\theta q_\epsilon), (\theta q_\epsilon))_{L^2(\mathcal{G})}\\
=&2\mathrm{Re}\,([L_1,L_2](\theta q_\epsilon), L_1(\theta q_\epsilon))_{L^2(\mathcal{G})}.
\end{aligned}$$

Here we notice that $([L_1,L_2]u,v)_{L^2(\mathcal{G})} = (u,[L_1,L_2]v)_{L^2(\mathcal{G})}$.
We have already seen that

$$[L_1,L_2] = (N+1)K(s), \tag{28}$$

where $K \in C([0,1];\mathcal{L}(H^{1,s}(\mathbb{R}^2), L^2(\mathbb{R}^2)))$ is some operator. Therefore

$$|\mathrm{Re}\,(L_1^2(\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}| \le \|\theta q_\epsilon\|_{L^2(0,T;H^{1,s}(\mathbb{R}^2))}\|L_1(\theta q_\epsilon)\|_{L^2(\mathcal{G})}. \tag{29}$$

We already know from (15) that

$$\|L_1(\theta q_\epsilon)\|_{L^2(\mathcal{G})} \le \|L_1 q_\epsilon\|_{L^2(\mathcal{G})} \le C_5\|p_\epsilon\|_{L^2(\mathcal{G})} \tag{30}$$

and from the definition of L_1

(31)

$$\|\theta q_\epsilon\|_{L^2\left(0,\frac{1}{N^2};H^{1,s}(\mathbb{R}^2)\right)} \le \|q_\epsilon\|_{L^2\left(0,\frac{1}{N^2};H^{1,s}(\mathbb{R}^2)\right)} \le C_6(\|L_1q_\epsilon\|_{L^2(\mathcal{G})} + s\|q_\epsilon\|_{L^2(\mathcal{G})}).$$

Using again (15), we obtain

$$|\mathrm{Re}\,(L_1^2(\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}| \le C_7\sqrt{s}\|p_\epsilon\|^2_{L^2(\mathcal{G})} + C_8\|p_\epsilon\|^2_{L^2(\mathcal{G})}, \tag{32}$$

so that we have

$$|\mathrm{Re}\,(L_1^2(\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}| \le C_8\sqrt{s}\|p_\epsilon\|^2_{L^2(\mathcal{G})} \tag{33}$$

for $s \ge s_0$.

Concerning the third term, we have

$$|\mathrm{Re}\,([L_2,L_1](\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}| \le \|[L_2,L_1](\theta q_\epsilon)\|_{L^2(\mathcal{G})}\|L_2(\theta q_\epsilon)\|_{L^2(\mathcal{G})}. \tag{34}$$

Using the form of $[L_2, L_1]$ we obtain

$$
\begin{aligned}
&|\mathrm{Re}\,([L_2, L_1](\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}| \\
&\le C_9 N \|\theta q_\epsilon\|_{L^2\left(0,\frac{1}{N^2}; H^{1,s}(\mathbb{R}^2)\right)} \|L_2(\theta q_\epsilon)\|_{L^2(\mathcal{G})}
\end{aligned}
\tag{35}
$$

and, since $L_2(\theta q_\epsilon) = \theta L_2 q_\epsilon + \frac{\partial\theta}{\partial y_2} q_\epsilon$, from (15) and (31) we have

$$
\begin{aligned}
&|\mathrm{Re}\,([L_2, L_1](\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}| \le C_{10}\sqrt{s}\|p_\epsilon\|^2_{L^2(\mathcal{G})} + C_{10}N^3\|p_\epsilon\|^2_{L^2(\mathcal{G})} \\
&\le C_{11}\sqrt{s}\|p_\epsilon\|^2_{L^2(\mathcal{G})}
\end{aligned}
\tag{36}
$$

for $s \ge s_0$ and $s_0 \ge N^8 + 1$.

On the other hand, for the right hand side of (24), we have

$$
\begin{aligned}
&\left|\mathrm{Re}\left(\epsilon\theta q_\epsilon - i\chi_\nu\varphi w + \frac{\partial\theta}{\partial y_2}p_\epsilon + \frac{\partial\theta}{\partial y_2}(L_2 - L_1)q_\epsilon + \frac{\partial^2\theta}{\partial y_2^2}q_\epsilon, L_2(\theta q_\epsilon)\right)_{L^2(\mathcal{G})}\right| \\
&\le C_{12}\epsilon\|q_\epsilon\|_{L^2(\mathcal{G})}\|p_\epsilon\|_{L^2(\mathcal{G})} + C_{12}\|\chi_\nu w\|_{L^2(\mathcal{G})}\|p_\epsilon\|_{L^2(\mathcal{G})} + C_{12}\|p_\epsilon\|^2_{L^2(\mathcal{G})}.
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
&\left|\mathrm{Re}\left(\epsilon\theta q_\epsilon - \theta i\chi_\nu w + \frac{\partial\theta}{\partial y_2}p_\epsilon + \frac{\partial\theta}{\partial y_2}(L_2 - L_1)q_\epsilon + \frac{\partial^2\theta}{\partial y_2^2}q_\epsilon, L_2(\theta q_\epsilon)\right)_{L^2(\mathcal{G})}\right| \\
&\le C_{13}\|p_\epsilon\|^2_{L^2(\mathcal{G})} + C_{13}\|\chi_\nu w\|_{L^2(\mathcal{G})}\|p_\epsilon\|_{L^2(\mathcal{G})}.
\end{aligned}
\tag{37}
$$

Putting together (28), (33), (36) and (37), we obtain the following estimate:

$$
\left\|\frac{\partial q_\epsilon}{\partial y_2}(\cdot, 0)\right\|^2_{L^2(\mathcal{G}\cap\{y_2=0\})} \le C_{14}\sqrt{s}\|p_\epsilon\|^2_{L^2(\mathcal{G})} + C_{14}\|\chi_\nu w\|_{L^2(\mathcal{G})}\|p_\epsilon\|_{L^2(\mathcal{G})}.
\tag{38}
$$

Now combining (15), (20) and (38), we can easily obtain

$$
s\int_{\mathcal{G}} |p_\epsilon|^2 dy + \sqrt{s}\int_{\mathbb{R}^2} |p_\epsilon(y', 0)|^2 dy' \le C_{15}\int_{\mathcal{G}} |\chi_\nu w|^2 dy,
\tag{39}
$$

which is estimate (2) for p_ϵ.

Now p_ϵ and $\frac{\partial p_\epsilon}{\partial y_2} + \Gamma^{+,*}_\beta(y, s, D')p_\epsilon$ are bounded in $L^2(\mathcal{G})$ uniformly in ϵ. After extraction of a subsequence (still denoted by p_ϵ), we can assume that

$$
\begin{aligned}
&p_\epsilon \rightharpoonup p \quad \text{weakly in } L^2(\mathcal{G}), \\
&\frac{\partial p_\epsilon}{\partial y_2} \rightharpoonup \frac{\partial p}{\partial y_2} \text{ weakly in } L^2\left(0, \frac{1}{N^2}; H^{-1}(\mathbb{R}^2)\right),
\end{aligned}
\tag{40}
$$

so that

$$p_\epsilon(\cdot, 0) \rightharpoonup p(\cdot, 0) \quad \text{weakly in} \quad H^{-\frac{1}{2}}(\mathbb{R}^2).$$

Since $p_\epsilon(\cdot, 0)$ remains bounded in $L^2(\mathbb{R}^2)$, we also have $p_\epsilon(\cdot, 0) \rightharpoonup p(\cdot, 0)$ weakly in $L^2(\mathcal{G} \cap \{y_2 = 0\})$. By (15) and (40), we easily see that p satisfies

$$L^* p = \left(-\frac{\partial}{\partial y_2} - \widetilde{\Gamma}^{+,*}_{\beta}(y, s, D') \right) p = i \chi_\nu w \quad \text{in } \mathcal{G},$$

which is (1), and from (39) we see that

$$s \int_{\mathcal{G}} |p|^2 dy + \sqrt{s} \int_{\mathcal{G} \cap \{y_2 = 0\}} |p(y', 0)|^2 dy' \le C_{16} \int_{\mathcal{G}} |\chi_\nu w|^2 dy,$$

which is (2). The proof of Lemma 1 is now completed. □

We take the scalar product of equation (1) and the function $\chi_\nu w$ in $L^2(\mathcal{G})$

$$\|\chi_\nu w\|^2_{L^2(\mathcal{G})} = (g, p)_{L^2(\mathcal{G})} + (p(\cdot, 0), \chi_\nu w(\cdot, 0))_{L^2(\mathbb{R}^2)}.$$

Applying estimate (2) to this equality, we have

$$\|\chi_\nu w\|^2_{L^2(\mathcal{G})} \le C_{17} \left(\frac{1}{\sqrt{s}} \|g\|_{L^2(\mathcal{G})} \|\chi_\nu w\|_{L^2(\mathcal{G})} + \frac{1}{s^{\frac{1}{4}}} \|\chi_\nu w(\cdot, 0)\|_{L^2(\mathbb{R}^2)} \|\chi_\nu w\|_{L^2(\mathcal{G})} \right).$$

Therefore

$$\sqrt{s} \|\chi_\nu w\|_{L^2(\mathcal{G})} \le C_{17} (\|g\|_{L^2(\mathcal{G})} + s^{\frac{1}{4}} \|\chi_\nu w(\cdot, 0)\|_{L^2(\mathbb{R}^2)}), \quad \forall s \ge s_0. \tag{41}$$

If $s^* \neq 0$, then estimate (41) imply (2.20) immediately. If Im $\alpha^+(\gamma) \neq 0$, then we obtain

$$\|\chi_\nu w\|_{L^2\left(0, \frac{1}{N^2}; H^{1,s}(\mathbb{R}^2)\right)} \le C_{18} \left(\frac{1}{\sqrt{s}} \|g\|_{H^{1,s}(\mathcal{G})} + s^{\frac{1}{4}} \|\chi_\nu w(\cdot, 0)\|_{H^{\frac{1}{2},s}(\mathbb{R}^2)} \right).$$

Thus the proof of Proposition 2.4 is finished. □

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References

[BLR] C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, SIAM J. Control Optim. **30** (1992), no. 5, 1024–1065.

[B1] M. Bellassoued, Distribution of resonances and decay rate of the local energy for the elastic wave equation, Comm. Math. Phys. **215** (2000), no. 2, 375–408.

[B2] ———, Carleman estimate and decay rate of the local energy for the Neumann problem of elasticity, in *Carleman estimates and applications to uniqueness and control theory (Cortona, 1999)*, 15–36, Birkhäuser, Boston, Boston, MA.

[B3] ———, Unicité et contrôle pour le système de Lamé, ESAIM Control Optim. Calc. Var. **6** (2001), 561–592 (electronic).

[BuK] A. L. Bukhgeim and M. V. Klibanov, Global uniqueness of a class of multidimensional inverse problems, Soviet Math. Dokl. **24** (1981), 244–247

[Ca] T. Carleman, Sur un problème d'unicité pour les systèmes d'équations aux derivées partielles à deux variables independents, Ark. Mat. Astr. Fys. **2B** (1939), 1–9

[DR] B. Dehman and L. Robbiano, La propriété du prolongement unique pour un système elliptique. Le système de Lamé, J. Math. Pures Appl. (9) **72** (1993), no. 5, 475–492.

[E] Yu. V. Egorov, *Linear differential equations of principal type*, Translated from the Russian by Dang Prem Kumar, Consultants Bureau, New York, 1986.

[EINT] M. Eller, V. Isakov, G. Nakamura and D. Tataru, Uniqueness and stability in the Cauchy problem for Maxwell and elasticity systems, in *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XIV (Paris, 1997/1998)*, 329–349, North-Holland, Amsterdam.

[Hö] L. Hörmander, *Linear partial differential operators*, Springer, Berlin, 1976.

[Im] O. Yu. Imanuvilov, On Carleman estimates for hyperbolic equations, Asymptot. Anal. **32** (2002), no. 3-4, 185–220.

[IY1] O. Yu. Imanuvilov and M. Yamamoto, Remarks on Carleman estimates and exact controllability of the Lamé system, in *Journées "Équations aux Dérivées Partielles" (Forges-les-Eaux, 2002)*, Exp. V, 19 pp, Univ. Nantes, Nantes.

[IY2] ———, Carleman estimate for a stationary isotropic Lamé system and the applications, Appl. Anal. **83** (2004), no. 3, 243–270.

[IY3] ———, Carleman estimates for the non-stationary Lamé system and the application to an inverse problem, ESIAM: COCV **11** (2005), 1–56

[IY4] ———, Carleman estimates for the three-dimensional non-stationary Lamé system and application to an inverse problem, in "Control Theory of Partial Differential Equations" (2005), Chapman & Hall, Bosa Ralton, 337–374

[IP] O. Yu. Imanuvilov and J.-P. Puel, Global Carleman estimates for weak solutions of elliptic nonhomogeneous Dirichlet problems, Int. Math. Res. Not. **2003**, no. 16, 883–913.

[Is] V. Isakov, Carleman type estimates in an anisotropic case and applications, J. Differential Equations **105** (1993), no. 2, 217–238.

[K] M. V. Klibanov, Inverse problems and Carleman estimates, Inverse Problems **8** (1992), no. 4, 575–596.

[Ku] H. Kumano-go, Pseudo-differential Operators, MIT Press, Cambridge, 1981.

[T1] M. Taylor, *Pseudodifferential Operators*, Princeton University Press, Princeton, New Jersey, 1981.

[T2] ———, *Pseudodifferential operators and nonlinear PDE*, Progr. Math., 100, Birkhäuser Boston, Boston, MA, 1991.

[Ta] D. Tataru, Carleman estimates and unique continuation for solutions to boundary value problems, J. Math. Pures Appl. (9) **75** (1996), no. 4, 367–408.

[W] N. Weck, Unique continuation for systems with Lamé principal part, Math. Methods Appl. Sci. **24** (2001), no. 9, 595–605.

[Y] K. Yamamoto, Singularities of solutions to the boundary value problems for elastic and Maxwell's equations, Japan J. Math. **14** (1988), 119–163.

[Zui] C. Zuily, *Uniqueness and nonuniqueness in the Cauchy problem*, Progr. Math., 33, Birkhäuser Boston, Boston, MA, 1983.