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Simultaneous Linearization of Holomorphic Maps with Hyperbolic and Parabolic Fixed Points

By

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Abstract

We study local holomorphic mappings of one complex variable with parabolic fixed points as a limit of a families of mappings with attracting fixed points. We show that the Fatou coordinate for a parabolic fixed point can be obtained as a limit of some linear function of the solutions to Schröder equation for perturbed mappings with attracting fixed points.

§1. Introduction

Let $g(w)$ be a holomorphic function of one variable of the form

$$g(w) = \lambda w + \sum_{\nu=2}^{\infty} b_\nu w^\nu$$

defined in a neighborhood of the origin 0. If $0 < |\lambda| < 1$, then there are a neighborhood V of 0 such that $g(V) \subset V$ and an injective holomorphic function $\rho(w)$ on V satisfying the Schröder equation

$$\rho(g(w)) = \lambda \rho(w).$$

If $\lambda = 1$ and $b_2 \neq 0$, then there are a domain V whose boundary contains 0 and an injective function $\varphi(w)$ (Fatou coordinate) satisfying the Abel equation

$$\varphi(g(w)) = \varphi(w) + 1,$$

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which is unique up to an additive constant. (See Schröder [9], Koenigs [5], Leau [6], Fatou [3] and Milnor [8], for these classical results.)

In this paper, we consider families of functions $g_\lambda(w)$ which have λ as a parameter and show that, when λ tends non-tangentially to 1 from inside of the unit disk, some linear function of $\rho_\lambda(w)$ converges to $\varphi(w)$.

To do this it is convenient to consider the case where the fixed point is ∞ on the Riemann sphere. By scaling the coordinate, we consider a family of holomorphic maps of the form

$$f_\tau(z) = \tau z + 1 + \frac{a_{1,\tau}}{z} + \frac{a_{2,\tau}}{z^2} + \cdots$$

defined in a neighborhood of ∞. Here the parameter $\tau = 1/\lambda$ varies in a neighborhood of $\tau = 1$. For $|\tau| > 1$, let $\chi_\tau(z)$ denote the unique solution of the equation

$$\chi_\tau(f(z)) = \tau\chi_\tau(z)$$

with $\chi_\tau(\infty) = \infty$ and normalized so that $\chi(z) = z + O(1)$ in a neighborhood of ∞. We will show that, when τ tends to 1 non-tangentially within the domain $|\tau| > 1$, the sequence

$$\chi_\tau(z) - \frac{1}{\tau - 1} - a_{1,\tau}\log(\tau - 1)$$

converges to a solution to the Abel equation

$$\varphi(f_1(z)) = \varphi(z) + 1.$$

Precise statement and different formulations of the results are given in Theorems 3.3–3.6.

An alternative proof and a generalization is given recently by T. Kawahira [4]. As a related result, we note that T. C. McMullen showed the existence of quasiconformal maps giving conjugacies between f_τ and linear maps (see [7], Theorem 8.2).

§2. Preliminaries

§2.1. A family of linear maps

We begin with studying the family $\{\ell_\tau\}_\tau$ of linear maps

$$\ell_\tau(z) = \tau z + 1 \tag{1}$$

on the Riemann sphere $\widehat{\mathbb{C}}$ depending on the complex parameter τ. When $|\tau| > 1$, the map ℓ_τ has ∞ as an attracting fixed point and all points except for

$1/(1-\tau)$ converge locally uniformly to ∞ by the iterates of ℓ_τ. When $\tau = 1$, then ∞ is a parabolic fixed point and all points in $\widehat{\mathbb{C}}$ converges to ∞, though the convergence is not uniform in the neighborhood of ∞.

We will investigate the uniformity, with respect to the parameter τ, of the convergence of the iterates ℓ_τ^n, when τ tends to 1 non-tangentially from outside of the unit disk. So we will restrict the parameter τ in the closed sector

$$T_\alpha = \{\tau \in \mathbb{C} \mid \operatorname{Re}\tau - 1 \geq |\tau - 1| \cos\alpha\},$$

where α is a real number with $0 < \alpha < \pi/2$, fixed throughout this paper.

To measure the rate of convergence to ∞, we introduce the function $N : \widehat{\mathbb{C}} \times T_\alpha - \{(\infty, 1)\} \to \mathbb{R} \cup \{\infty\}$ as follows.

$$N_\tau(z) = \left| z - \frac{1}{1-\tau} \right| - \left| \frac{1}{1-\tau} \right| \qquad \text{for } (z,\tau) \in \widehat{\mathbb{C}} \times (T_\alpha - \{1\});$$
$$N_1(z) = \sup_{|\theta| \leq \alpha} \operatorname{Re}(e^{i\theta} z) \qquad \text{for } z \in \mathbb{C}.$$

We will not define $N_1(\infty)$. It is easy to see that the inequality

$$|N_\tau(z_1) - N_\tau(z_2)| \leq |z_1 - z_2| \quad z_1, z_2 \in \mathbb{C}, \tau \in T_\alpha$$

holds. In particular we have

$$N_\tau(z) \leq |z|, \quad z \in \mathbb{C}, \tau \in T_\alpha.$$

Lemma 2.1. *$N_\tau(z)$ is upper semi-continuous as a function of two variables $(z, \tau) \in \mathbb{C} \times T_\alpha - \{z = \infty\}$ and*

$$N_1(z) = \limsup_{T_\alpha \ni \tau \to 1} N_\tau(z).$$

Proof. For $r > 0$, we let $\hat{N}_{(r,\theta)}(z) = N_{1+re^{i\theta}}(z)$. Then

$$\hat{N}_{(r,\theta)}(z) = \left| z + \frac{1}{re^{i\theta}} \right| - \frac{1}{r} = \frac{1}{r}\{(1 + 2r \operatorname{Re}(e^{i\theta} z) + r^2 |z|^2)^{1/2} - 1\}.$$

This can be extended to a continuous function on $\widehat{\mathbb{C}} \times \{r \geq 0\} \times \mathbb{R}$, by defining $\hat{N}_{(0,\theta)}(z) = \operatorname{Re}(e^{i\theta} z)$. Hence

$$\limsup_{T_\alpha \ni \tau \to 1} N_\tau(z) = \sup_{|\theta| \leq \alpha} \hat{N}_{(0,\theta)}(z) = \sup_{|\theta| \leq \alpha} \operatorname{Re}(e^{i\theta} z) = N_1(z).$$

This shows the assertion. □

To have a uniform estimate of the rate of convergence of the iterats of ℓ_τ, let us first show the following:

Lemma 2.2. *For* $(z,\tau) \in \widehat{\mathbb{C}} \times T_\alpha - \{(\infty, 1)\}$*, we have*

$$N_\tau(\ell_\tau(z)) \geq |\tau| N_\tau(z) + \cos\alpha.$$

Proof. First, if $\tau \in T - \{1\}$, then (1) is rewritten as

$$\ell_\tau(z) - \frac{1}{1-\tau} = \tau\left(z - \frac{1}{1-\tau}\right).$$

Hence

$$\begin{aligned} N_\tau(\ell_\tau(z)) &= \left|\ell_\tau(z) - \frac{1}{1-\tau}\right| - \left|\frac{1}{1-\tau}\right| \\ &= |\tau|\left|z - \frac{1}{1-\tau}\right| - \left|\frac{1}{1-\tau}\right| \\ &= |\tau| N_\tau(z) + \frac{|\tau| - 1}{|1-\tau|} \\ &\geq |\tau| N_\tau(z) + \cos\alpha. \end{aligned}$$

Here we have used the fact that

$$\frac{|\tau| - 1}{|\tau - 1|} \geq \frac{\mathrm{Re}(\tau) - 1}{|\tau - 1|} \geq \cos\alpha.$$

If $\tau = 1$, then $\ell_1(z) = z + 1$, and hence

$$\mathrm{Re}(e^{i\theta}\ell_1(z)) = \mathrm{Re}(e^{i\theta}z) + \cos\theta \geq \mathrm{Re}(e^{i\theta}z) + \cos\alpha.$$

Therefore

$$N_1(\ell_1(z)) \geq N_1(z) + \cos\alpha$$

and the lemma is proved. □

Let R be a real number and define

$$\mathcal{V}_\alpha(R) = \{(z,\tau) \in \widehat{\mathbb{C}} \times T_\alpha - \{(\infty,1)\} \mid N_\tau(z) > R\}.$$

We note that $\mathcal{V}_\alpha(R)$ is not open. Slices of $\mathcal{V}_\alpha(R)$ by $\tau =$ const. are open sets given by

$$\begin{aligned} V_\tau(R) &= \{z \in \widehat{\mathbb{C}} \mid N_\tau(z) > R\} \qquad (\tau \neq 1); \\ V_1(R) &= \{z \in \mathbb{C} \mid N_1(z) > R\} = \bigcup_{|\theta| \leq \alpha} \{\mathrm{Re}(e^{i\theta}z) > R\}. \end{aligned}$$

Proposition 2.3. *We have*

$$|\ell_\tau^n(z)| \geq n\cos\alpha \qquad \textit{for } \mathcal{V}_\alpha(0) \tag{2}$$

and hence the sequence $\{\ell_\tau^n(z)\}_n$ *converges to* ∞ *uniformly* $\mathcal{V}_\alpha(0)$.

Proof. $N_\tau(z) > 0$ implies $N_\tau(\ell(z)) \geq N_\tau(z) + \cos\alpha > 0$ by Lemma 2.2. Hence, if $\tau \in T_\alpha$ and $z \in V_\tau(0)$, then $\ell_\tau^n(z) \in V_\tau(0)$ and

$$|\ell_\tau^n(z)| \geq N_\tau(\ell_\tau^n(z)) \geq N_\tau(z) + n\cos\alpha \geq n\cos\alpha,$$

for all n. This proves the assertion. □

§2.2. Solution to a difference equation

We consider the difference equation

$$h_\tau(\ell_\tau(z)) - \tau h_\tau(z) = \frac{1}{z} + C_\tau, \tag{3}$$

where $\ell_\tau(z) = \tau z + 1$ with $|\tau| > 1$ or $\tau = 1$; and C_τ is a constant depending on τ, which will be given later.

A solution to this equation is given by

$$h_\tau(z) = -\frac{1}{\tau z} + \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1}} \left\{ \frac{1}{\ell_\tau^n(0)} - \frac{1}{\ell_\tau^n(z)} \right\}. \tag{4}$$

We note that $\ell_\tau(z) = \tau^n z + \tau^{n-1} + \cdots + \tau + 1$ and $\ell_\tau(0) = \tau^{n-1} + \cdots + \tau + 1$. In the following, we will investigate some properties of this function.

First, for a τ fixed, the following properties of $h_\tau(z)$ can be easily verified: In the case $|\tau| > 1$, the function $h_\tau(z)$ is meromorphic on $\widehat{\mathbb{C}}$ except the essential singularity at $1/(1-\tau)$, and has poles at $(1-\tau^{-n})/(1-\tau)$, $(n = 0, 1, 2, \ldots)$. This function $h_\tau(z)$ is holomorphic at ∞. We write

$$H_\tau = h_\tau(\infty) = \sum_{n=1}^{\infty} \frac{1}{\tau^{n+1}\ell_\tau^n(0)}. \tag{5}$$

We can easily verify that $h_\tau(z)$, with $\tau \neq 1$, satisfies the equation (3) with the constant

$$C_\tau = (1-\tau)H_\tau. \tag{6}$$

In the case $\tau = 1$, we have $\ell^n(z) = z + n$ and

$$h_1(z) = -\frac{1}{z} + \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{z+n} \right\}.$$

This function is meromorphic on $\mathbb{C}$ and has poles at $0, -1, -2, \ldots$. As is easily verified, $h_1(z)$ satisfies the equation (3) with $C_1 = 0$.

We note that

$$h_1(z) = \frac{\Gamma'(z)}{\Gamma(z)} + \gamma$$

where $\Gamma(z)$ denotes the gamma function and γ denotes the Euler constant

$$\gamma = \lim_{n\to\infty} \Big(\sum_{k=1}^{n} \frac{1}{k} - \log n \Big).$$

Now we study the dependence of $h_\tau(z)$ on the parameter τ.

Proposition 2.4. *The function $h_\tau(z)$ is continuous on $\mathcal{V}_\alpha(0)$.*

Proof. The continuity at the points (z, τ) with $\tau \neq 1$ is clear. Using $\ell_\tau^n(z) - \ell_\tau^n(0) = \tau^n z$ and the estimate (2), we have

$$\left| \frac{1}{\tau^{n+1}} \left\{ \frac{1}{\ell_\tau^n(0)} - \frac{1}{\ell_\tau^n(z)} \right\} \right| = \left| \frac{z}{\tau \ell_\tau^n(0) \ell_\tau^n(z)} \right| \leq \left| \frac{z}{\tau} \right| \frac{1}{n^2 \cos^2 \alpha}.$$

This shows that the series (4) is locally uniformly convergent on $\mathcal{V}_\alpha(0) - \{z = \infty\}$ and hence $h_\tau(z)$ is continuous there. □

Corollary 2.5. *The constat C_τ is a continuous function of $\tau \in T_\alpha$.*

Proof. By the difference equation (3), we have $C_\tau = h_\tau(\ell_\tau(z)) - \tau h_\tau(z) - 1/z$, which is continuous on $\mathcal{V}_\alpha(0)$ by Proposition 2.4. Hence C_τ is continuous on T_α. □

Proposition 2.6. *For any $\varepsilon > 0$, there is a constant M such that*

$$|h_\tau'(z)| \leq \frac{M}{N_\tau(z)} \qquad \text{on } \mathcal{V}_\alpha(\varepsilon).$$

Proof. Differentiation of (3) with respect to z yields

$$h_\tau'(z) = \frac{1}{\tau} \sum_{n=0}^{\infty} \frac{1}{\{\ell_\tau^n(z)\}^2}.$$

Hence

$$|h_\tau'(z)| \leq \sum_{n=0}^{\infty} \frac{1}{|\ell_\tau^n(z)|^2} \leq \sum_{n=0}^{\infty} \frac{1}{(N_\tau(z) + n\cos\theta)^2}$$
$$\leq \int_0^\infty \frac{dx}{(N_r(z) + x\cos\theta)^2}.$$

Therefore $|h_\tau'(z)|$ is bounded by $M/N_\tau(z)$ with some constant M. □

§2.3. Behavior of H_τ

Now we look at the behavior of the function H_τ defined by (5), when $\tau \to 1$ within the sector T. It is clear from the expression (5) that H_τ is unbounded, while $C_\tau = (1-\tau)H_\tau$ tends to 0 by Corollary 2.5. Here we give a more precise description of its behavior.

Proposition 2.7. *We have*

$$H_\tau = -\log(\tau - 1) + \gamma - 1 + o(1)$$

as $\tau \to 1$ within the sector T_α. Here γ denotes the Euler constant.

Proof. To begin with, letting $\lambda = 1/\tau$, we have

$$\begin{aligned} H_{1/\lambda} &= \sum_{n=1}^{\infty} \frac{\lambda^{2n}}{1+\lambda+\cdots+\lambda^{n-1}} \\ &= (1-\lambda)\sum_{n=1}^{\infty}\Big(\frac{\lambda^n}{1-\lambda^n} - \lambda^n\Big) \\ &= (1-\lambda)L(\lambda) - \lambda. \end{aligned}$$

Here $L(\lambda)$ denotes the Lambert series defined by

$$L(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{1-\lambda^n}.$$

This series $L(\lambda)$ defines a holomorphic function on $|\lambda| < 1$. We want to know the behavior of this function when λ tends to 1 non-tangentially within the unit disk.

$L(\lambda)$ is developped into the power series

$$L(\lambda) = \sum_{n=1}^{\infty} d(n)\lambda^n = \lambda + 2\lambda^2 + 2\lambda^3 + 3\lambda^4 + \cdots,$$

where $d(n)$ denotes the number of divisors of n. We write

$$D(n) = d(1) + \cdots + d(n).$$

Then

$$\frac{L(\lambda)}{1-\lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n.$$

The asymptotic behavior of $D(n)$ is given by a theorem of Dirichlet (see Apostol [1], Chandrasekharan [2]) :

$$D(n) = n\log n + (2\gamma - 1)n + O(\sqrt{n}) \quad (n \to \infty).$$

From this and the fact that

$$\sum_{k=1}^{n} \frac{1}{k} - \log n = \gamma + O\Big(\frac{1}{n}\Big),$$

it follows that

$$\begin{aligned} D(n) &= n\sum_{k=1}^{n} \frac{1}{k} + (\gamma - 1)n + p_n \\ &= \sum_{k=1}^{n} \frac{n-k}{k} + \gamma n + p_n \end{aligned}$$

where $p_n = O(\sqrt{n})$ as $n \to \infty$. Therefore, noting that

$$\frac{\lambda}{(1-\lambda)^2} = \sum_{n=1}^{\infty} n\lambda^n, \quad \log(1-\lambda) = -\sum_{n=1}^{\infty} \frac{\lambda^n}{n}$$

we have

$$\frac{L(\lambda)}{1-\lambda} = \sum_{n=1}^{\infty} D(n)\lambda^n = -\frac{\lambda\log(1-\lambda)}{(1-\lambda)^2} + \frac{\gamma\lambda}{(1-\lambda)^2} + P(\lambda)$$

where $P(\lambda) = \sum_{n=1}^{\infty} p_n\lambda^n$. Since $p_n = O(\sqrt{n}) = o(n)$, we have

$$P(\lambda) = o((1-\lambda)^{-2}) \qquad \text{as } \lambda \to 1 \text{ non-tangentially.}$$

Thus we obtain

$$\begin{aligned} H_\tau &= (1-\lambda)L(\lambda) - \lambda \\ &= -\lambda\log(1-\lambda) + (\gamma - 1)\lambda + (1-\lambda)^2 P(\lambda) \\ &= -\log(\tau - 1) + \gamma - 1 + o(\tau - 1) \end{aligned}$$

and the proposition is proved. □

§3. Families of Maps with Attracting/Parabolic Fixed Points

§3.1. Domain of convergence

Let $U(R) = \{z \in \widehat{\mathbb{C}} \mid R < |z| \le \infty\}$ be a neighborhood of $\infty \in \widehat{\mathbb{C}}$ and we consider a family of holomorphic maps $f_\tau : U(R) \to \widehat{\mathbb{C}}$ of the form

$$f_\tau(z) = \tau z + 1 + A_\tau(z) \tag{7}$$

with

$$A_\tau(z) = \frac{a_{1,\tau}}{z} + \frac{a_{2,\tau}}{z^2} + \cdots .$$

We suppose that f_τ depends holomorphically on the parameter $\tau \in \Delta_\rho = \{\tau \in \mathbb{C} \mid |\tau - 1| < \rho\}$.

As in the previous section, we choose and fix α so that $0 < \alpha < \pi/2$ and let $\delta = \frac{1}{2}\cos\alpha$. By shrinking the neighbohoods $U(R)$ and Δ_ρ, we assume that there is a constant K such

$$|A_\tau(z)| < \frac{K}{|z|} < \delta \tag{8}$$

for $(z, \tau) \in U(R) \times \Delta_\rho$. Further we assume that $f_\tau(z)$ is injective in z for every $\tau \in \Delta_\rho$.

Now we have results on uniformity of convergence for $f_\tau^n(z)$, corresponding to Lemma 2.2 and Proposition 2.3 for $\ell_\tau(z)$. We set

$$T_{\alpha,\rho} = T_\alpha \cap \Delta_\rho = \{\tau \in \mathbb{C} \mid \mathrm{Re}(\tau - 1) \le |\tau - 1| \cos\alpha,\ |\tau - 1| < \rho\}.$$

Lemma 3.1. *For $(z, \tau) \in U(R) \times T_{\alpha,\rho} - \{(\infty, 1)\}$ we have*

$$N_\tau(f_\tau(z)) \ge |\tau| N_\tau(z) + \delta.$$

Proof. From $f_\tau(z) = \ell_\tau(z) + A_\tau(z)$, it follows that

$$\begin{aligned} N_\tau(f_\tau(z)) &\ge N_\tau(\ell_\tau(z)) - |A_\tau(z)| \\ &\ge |\tau| N_\tau(z) + \cos\alpha - \delta \\ &= |\tau| N_\tau(z) + \delta, \end{aligned}$$

which proves the lemma. □

Now let

$$\mathcal{V}_{\alpha,\rho}(R) = \{(z, \tau) \in \mathcal{V}_\alpha(R) \mid \tau \in T_{\alpha,\rho}\}.$$

We note that $\mathcal{V}_{\alpha,\rho}(R) \subset U(R) \times T_{\alpha,\rho}$ since $N_\tau(z) \le |z|$.

Proposition 3.2. *If $\tau \in T_{\alpha,\rho}$ and $z \in V_\tau(R)$, then $f_\tau(z) \in V_\tau(R)$. The sequence $\{f_\tau^n(z)\}_n$ converges uniformly on $\mathcal{V}_{\alpha,\rho}(R)$ to ∞.*

Proof. If $\tau \in T_{\alpha,\rho}$ and $z \in V_\tau(R)$, then $N_\tau(z) > R$. Hence $N_\tau(f_\tau(z)) \geq N_\tau(z) + \delta > R + \delta$ and $f_\tau(z) \in V_\tau(R)$. Further

$$|f_\tau^n(z)| \geq N_\tau(f_\tau^n(z)) \geq N_\tau(z) + n\delta > R + n\delta.$$

This shows the uniform convergence of $\{f_\tau^n(z)\}_n$ to ∞ on $\mathcal{V}_{\alpha,\rho}(R)$. □

§3.2. Schröder-Abel equation

We recall that C_τ is continuous on T_α and holomorphic in the interior of T_α and that $C_\tau = (1-\tau)H_\tau$ when $\tau \neq 1$. Let

$$B_\tau = 1 - a_{1,\tau}C_\tau. \tag{9}$$

The following theorem constitutes the main ingredient of this paper.

Theorem 3.3. *There exists a function $\varphi_\tau(z)$ on $\mathcal{V}_{\alpha,\rho}(R)$ with values in $\widehat{\mathbb{C}}$ satisfying the following conditions*:

(i) *$\varphi_\tau(z)$ is continuous on $\mathcal{V}_{\alpha,\rho}(R)$ and holomorphic in its interior as a mapping to $\widehat{\mathbb{C}}$.*

(ii) *For each $\tau \in T_{\alpha,\rho} - \{1\}$ fixed, the function $\varphi_\tau(z)$ is holomorphic in $V_\tau(R)$ except for a simple pole at ∞; and $\varphi_1(z)$ is holomorphic in $V_1(R)$. Further $\varphi_\tau(z)$ satisfies the functional equation*

$$\varphi_\tau(f_\tau(z)) = \tau\varphi_\tau(z) + B_\tau. \tag{10}$$

(iii) *For each $\tau \in T_{\alpha,\rho} - \{1\}$ fixed, the function $\varphi_\tau(z)$ is of the form*

$$\varphi_\tau(z) = z - a_{1,\tau}H_\tau + o(1)$$

in a neighborhood of $z = \infty$.

The proof is given in the next subsection.

This theorem implies in particular the following: Suppose that τ tends to 1 from outside of the unit disk with direction θ, i.e., $\tau = 1 + re^{i\theta}$ which fixed θ and r tending to 0. Then the domain $V_\tau(R)$ of $\varphi_\tau(z)$ converges to the half plane $\{\operatorname{Re} e^{i\theta}z > R\} \subset V_1(R)$, and $\varphi_\tau(z)$ converges to $\varphi_1(z)$ on this half plane. This remark applies also to ψ_τ given below.

To make clear the meaning of this theorem, we will give the relation between $\varphi_\tau(z)$ and the solution to Schröder equation.

Suppose $|\tau| > 1$ and consider the equation

$$\chi_\tau(f_\tau(z)) = \tau\chi_\tau(z), \tag{11}$$

which is a variant of the Schröder equation formutated for the case where the fixed point is ∞. It is classical that this equation has a unique solution $\chi_\tau(z)$ of the form $\chi_\tau(z) = z + O(1)$ in a neighborhood of ∞. By comparing the coefficients of the Laurent expansion we can see that

$$\chi_\tau(z) = z + \frac{1}{\tau - 1} + O(1/z).$$

On the other hand, we can easily verify that $\varphi(z) + B_\tau/(\tau - 1)$ satisfies the equation (11). Since $B_\tau = 1 - a_{1,\tau}C_\tau = 1 - a_{1,\tau}(1-\tau)H_\tau$ by (9) and (6), we have the following.

Theorem 3.4. *For $\tau \in T_{\alpha,\rho} - \{1\}$ we have*

$$\begin{aligned}\varphi_\tau(z) &= \chi_\tau(z) - \frac{B_\tau}{\tau - 1}\\ &= \chi_\tau(z) - \frac{1}{\tau-1} - a_{1,\tau}H_\tau.\end{aligned}$$

Our result may be stated, without referring to $\varphi_\tau(z)$, as follows.

Theorem 3.5. *When τ tends to 1 non-tangentially from outside of the unit disk, the function*

$$\chi_\tau(z) - \frac{1}{\tau - 1} + a_{1,\tau}\log(\tau - 1)$$

for $\tau \in T - \{1\}$ converges to a solution to the Abel equation for $f_1(z)$.

Here we may replace $a_{1,\tau}$ by $a_{1,1}$, since $a_{1,\tau}H_\tau$ and $-a_{1,1}\log(\tau-1)$ differ only by a continuous function on T_α, ρ.

We may normalize $\varphi_\tau(z)$ by letting $\varphi^*_\tau(z) = \varphi_\tau(z)/B_\tau$. Then $\varphi^*_\tau(z)$ satisfies the conditions of Theorem 3.3, replacing B_τ by 1. For $\tau \neq 1$, we have

$$\varphi^*_\tau(z) = \frac{\chi_\tau(z)}{B_\tau} - \frac{1}{\tau - 1}.$$

Now we give another reformulations of the result. The function φ_τ has pole on $z = \infty$. By a linear fractional transformation, we obtain a function which is holomorphic on $\mathcal{V}_{\alpha,\rho}(R)$.

Theorem 3.6. *There exists a function $\psi_\tau(z)$ on $\mathcal{V}_{\alpha,\rho}(R)$ satisfying the following conditions*:

(i) *$\psi_\tau(z)$ is continuous on $\mathcal{V}_{\alpha,\rho}(R)$ and holomorphic in its interior, as a function of two variables.*

(ii) *For each $\tau \in T_{\alpha,\rho}$ fixed, the function $\psi_\tau(z)$ is holomorphic in $V_\tau(R)$ and satisfies the functional equation*

$$\psi_\tau(f_\tau(z)) = \frac{1}{\tau}\psi_\tau(z) + 1.$$

(iii) *For each $\tau \in T_{\alpha,\rho}$ fixed, the function $\psi_\tau(z)$ is of the form*

$$\psi_\tau(z) = \frac{\tau}{\tau - 1} - \frac{\tau B_\tau}{(\tau-1)^2}\frac{1}{z} + O\Big(\frac{1}{z^2}\Big).$$

In the neighborhood of $z = \infty$,

Proof. We define

$$\psi_\tau(z) = \frac{\tau\varphi_\tau(z)}{(\tau-1)\varphi_\tau(z) + B_\tau}.$$

Then

$$\psi_\tau(z) = \frac{\tau}{\tau - 1} - \frac{\tau B_\tau}{(\tau-1)^2}\frac{1}{\chi_\tau(z)} \quad \text{when } \tau \neq 1.$$
$$\psi_1(z) = \varphi_1^*(z).$$

We can easily verify that $\psi_\tau(z)$ satisfies the required conditions. □

§3.3. Proof of Theorem 3.3

To simplify the notation, we omit the subscript τ for f_τ etc.

We rewrite the expression (7) in the form

$$f(z) = \tau z + 1 + \frac{a_{1,\tau}}{z} + A_1(z) \tag{12}$$

with $A_{1,\tau}(z) = a_2(\tau)/z^2 + \dots$. There exits some constant K_1 such that

$$|A_1(z)| \leq \frac{K_1}{|z|^2}.$$

To make clear the idea of the proof, we will first consider the case where $a_{1,\tau} = 0$ identically. Replacing z by $f^{n-1}(z)$ for in (11) and dividing by τ^n, we obtain

$$\frac{1}{\tau^n}f^n(z) = \frac{1}{\tau^{n-1}}f^{n-1}(z) + \frac{1}{\tau^n} + \frac{1}{\tau^n}A_1(f^{n-1}(z)). \tag{13}$$

We define

$$\varphi_n(z) := \frac{1}{\tau^n} f^n(z) - \sum_{k=1}^{n} \frac{1}{\tau^k} = z + \sum_{k=1}^{n} \frac{1}{\tau^k} A_1(f^{k-1}(z)).$$

Since

$$\left| \frac{1}{\tau^k} A_1(f^{n-1}(p)) \right| \leq \frac{K_1}{|f^{n-1}(z)|^2} \leq \frac{K_1}{N(f^{n-1}(z))^2} \leq \frac{K_1}{(N(z)+(n-1)\delta)^2}, \tag{14}$$

we conclude that $\varphi_n(z)$ converges uniformly as $n \to \infty$. Therefore the limit $\varphi(z) := \lim_{n\to\infty} \varphi_n(z)$ is continuous on $\mathcal{V}_{\alpha,\rho}(R)$. From $\varphi_n(f(z)) = \tau\varphi_{n+1}(z) + 1$, it follows that $\varphi(z)$ satisfies the equation (10) with $B_\tau = 1$.

Now, in the general case where $a_{1,\tau}$ does not vanish identically, we have to modify the above construction to have convergent sequence. Let us recall the function $h(z)$ satisfying the difference equation (3) in the previous section. We set

$$A_2(z) = h(f(z)) - h(\ell(z)).$$

Then

$$h(f(z)) = \tau h(z) + C_\tau + \frac{1}{z} + A_2(z).$$

Combining this with (12), we get

$$f(z) - a_1 h(f(z)) = \tau\{z - a_1 h(z)\} + B_\tau + \tilde{A}(z)$$

with $B_\tau = 1 - a_1 C_\tau$, where we have set

$$\tilde{A}(z) = A_1(z) - a_1 A_2(z).$$

In the same manner as in (13), we obtain

$$\frac{1}{\tau^n}\{f^n(z) - a_1 h(f^n(z))\} = \frac{1}{\tau^{n-1}}\{f^{n-1}(z) - a_1 h(f^{n-1}(z))\} + \frac{B_\tau}{\tau^n} + \frac{1}{\tau^n}\tilde{A}(f^{n-1}(z)).$$

We define

$$\begin{aligned}\varphi_n(z) &= \frac{1}{\tau^n}\{f^n(z) - a_1 h(f^n(z))\} - B_\tau \sum_{k=1}^{n} \frac{1}{\tau^k} \\ &= z - a_1 h(z) + \sum_{k=1}^{n} \frac{1}{\tau^k} \tilde{A}(f^{k-1}(z)).\end{aligned}$$

The sum on the right is

$$\sum_{k=1}^{n} \frac{1}{\tau^k} A_1(f^{k-1}(z)) - a_{1,\tau} \sum_{k=1}^{n} \frac{1}{\tau^k} A_2(f^{k-1}(z)).$$

When $n \to \infty$, the first sum is uniformly convergent by the estimate (14). The convergence of the second sum follows from Lemma 3.7 below. Thus $\varphi_n(z)$ converges uniformly on $\mathcal{V}_{\alpha,\rho}(R)$ as $n \to \infty$. Hence the limit $\varphi(z) = \lim_{n\to\infty} \varphi_n(z)$ is continuous on $\mathcal{V}_{\alpha,\rho}(R)$. From $\varphi_n(f(z)) = \tau\varphi_{n+1}(z) + B_\tau$ it follows that $\varphi(z)$ satisfies the equation (10).

Since $\tilde{A}(z)$ vanishes at $z = \infty$, we have $\varphi_n(z) = z - a_1 H_\tau + o(1)$ in the neighborhood of $z = \infty$. Letting $n \to \infty$ yields the assertion (iii). □

Lemma 3.7. *We have*

$$|A_2(z)| \le \frac{KM}{|z|N(z)}$$

on $(z, \tau) \in \mathcal{V}_{\alpha,\rho}(R)$.

Proof. Let z be a point with $N(z) > R$ and let $[\ell(z), f(z)]$ denote the segment joining $\ell(z)$ and $f(z)$ in $\mathbb{C}$. The length of this segment is

$$|f(z) - \ell(z)| = |A(z)| < \frac{K}{|z|} < \delta,$$

by (8). For any ζ in this segment, we have $|N(\zeta) - N(\ell(z))| \le |\zeta - \ell(z)| < \delta$. Hence $N(\zeta) > N(\ell(z)) - \delta > N(z)$. Hence, by Proposition 2.6 we have

$$|h'(\zeta)| \le \frac{M}{N(\zeta)} \le \frac{M}{N(z)}$$

on this segment. Thus we have

$$|A_2(z)| = \left| \int_{[\ell(z),f(z)]} h'(\zeta)d\zeta \right| \le \frac{KM}{|z|N(z)},$$

which proves the assertion. □

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