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On $\mathbb{Q}$-conic Bundles

By

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Abstract

A $\mathbb{Q}$-conic bundle is a proper morphism from a threefold with only terminal singularities to a normal surface such that fibers are connected and the anti-canonical divisor is relatively ample. We study the structure of $\mathbb{Q}$-conic bundles near their singular fibers. One corollary to our main results is that the base surface of every $\mathbb{Q}$-conic bundle has only Du Val singularities of type A (a positive solution of a conjecture by Iskovskikh). We obtain the complete classification of $\mathbb{Q}$-conic bundles under the additional assumption that the singular fiber is irreducible and the base surface is singular.

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§1. Introduction

In this paper we study the local structure of extremal contractions from threefolds to surfaces. Such contractions naturally appear in the birational classification of three-dimensional algebraic varieties of negative Kodaira dimension. More precisely, according to the minimal model program every algebraic projective threefold V with $\kappa(V) = -\infty$ is birationally equivalent to a $\mathbb{Q}$-factorial terminal threefold X having a K_X-negative extremal contraction to a lower dimensional variety Z. There are three cases:

a) Z is a point and then X is a $\mathbb{Q}$-Fano variety with $\rho(X) = 1$,
b) Z is a smooth curve and then X/Z is a del Pezzo fibration,
c) Z is a normal surface and then X/Z is a rational curve fibration.

We study the last case.

(1.1) Definition. By a $\mathbb{Q}$-*conic bundle* we mean a projective morphism $f\colon X \to Z$ from an (algebraic or analytic) threefold to a surface that satisfies the following properties:

(i) X is normal and has only terminal singularities,

(ii) $f_*\mathcal{O}_X = \mathcal{O}_Z$,

(iii) all fibers are one-dimensional,

(iv) $-K_X$ is f-ample.

For $f\colon X \to Z$ as above and for a point $o \in Z$, we call the *analytic* germ $(X, f^{-1}(o)_{\mathrm{red}})$ a $\mathbb{Q}$-*conic bundle germ*.

The easiest example of $\mathbb{Q}$-conic bundles is a *standard Gorenstein conic bundle*: Z is smooth and X is embedded in the projectivization $\mathbb{P}_Z(\mathcal{E})$ of a rank 3 vector bundle so that the fibers X_z, $z \in Z$ are conics in $\mathbb{P}_Z(\mathcal{E})_z$. More complicated examples can be constructed as quotients:

(1.1.1) Example-Definition (toroidal example). Consider the following action of $\boldsymbol{\mu}_m$ on $\mathbb{P}^1_x \times \mathbb{C}^2_{u,v}$:

$$(x; u, v) \longmapsto (\varepsilon x; \varepsilon^a u, \varepsilon^b v),$$

where ε is a primitive m-th root of unity and $\gcd(m,a)=\gcd(m,b)=1$. Let $X := \mathbb{P}^1 \times \mathbb{C}^2/\boldsymbol{\mu}_m$, $Z := \mathbb{C}^2/\boldsymbol{\mu}_m$ and let $f\colon X \to Z$ be the natural projection. Since $\boldsymbol{\mu}_m$ acts freely in codimension one, $-K_X$ is f-ample. Two fixed points on $\mathbb{P}^1 \times \mathbb{C}^2$ gives two cyclic quotient singularities of types $\frac{1}{m}(1,a,b)$ and $\frac{1}{m}(-1,a,b)$ on X. These points are terminal if and only if $a+b \equiv 0 \bmod m$. In this case, f is a $\mathbb{Q}$-conic bundle and the base surface Z has a Du Val singularity of type A_{m-1}. We say that a $\mathbb{Q}$-conic bundle germ is *toroidal* if it is biholomorphic to $f\colon (X, f^{-1}(0)_{\mathrm{red}}) \to (Z,0)$ above (with $a+b \equiv 0 \bmod m$).

Our first main result is a complete classification of $\mathbb{Q}$-conic bundle germs with irreducible central fiber under the assumption that the base surface is singular:

(1.2) Theorem. *Let $f\colon (X,C) \to (Z,o)$ be a $\mathbb{Q}$-conic bundle germ, where C is irreducible. Assume that (Z,o) is singular. Then one of the following holds:*

Cases where X is locally primitive.

(1.2.1) *(X,C) is toroidal.*

(1.2.2) *(X,C) is biholomorphic to the quotient of the smooth $\mathbb{Q}$-conic bundle*

$$X' = \{y_1^2 + uy_2^2 + vy_3^2 = 0\} \subset \mathbb{P}^2_{y_1,y_2,y_3} \times \mathbb{C}^2_{u,v} \longrightarrow \mathbb{C}^2_{u,v}.$$

by $\boldsymbol{\mu}_m$-action:

$$(y_1,y_2,y_3,u,v) \longmapsto (\varepsilon^a y_1, \varepsilon^{-1} y_2, y_3, \varepsilon u, \varepsilon^{-1} v).$$

Here $m = 2a+1$ is odd and ε is a primitive m-th root of unity. The singular locus of X consists of two cyclic quotient singularities of types $\frac{1}{m}(a,-1,1)$ and $\frac{1}{m}(a+1,1,-1)$. The base surface $\mathbb{C}^2/\boldsymbol{\mu}_m$ has a singularity of type A_{m-1}.

Cases where X is not locally primitive. *Let $P \in X$ be the imprimitive point and let m, s and $\bar{m}$ be its index, splitting degree and subindex, respectively. In this case, P is the only non-Gorenstein point and X has at most one Gorenstein singular point. We refer to* (5.3.1) *for the definition of types* (IA$^\vee$) – (IE$^\vee$).

(1.2.3) *(X,C) is of type* (IE$^\vee$) *at P, $s=4$, $\bar{m}=2$, (Z,o) is Du Val of type A_3, X has a cyclic quotient singularity P of type $\frac{1}{8}(5,1,3)$ and has no other singular points. Furthermore, (X,C) is the quotient of the index-two $\mathbb{Q}$-conic bundle germ given by the following two equations in $\mathbb{P}(1,1,1,2)_{y_1,\dots,y_4} \times \mathbb{C}^2_{u,v}$*

$$\begin{aligned} y_1^2 - y_2^2 &= u\psi_1(y_1,\dots,y_4;u,v) + v\psi_2(y_1,\dots,y_4;u,v), \\ y_1y_2 - y_3^2 &= u\psi_3(y_1,\dots,y_4;u,v) + v\psi_4(y_1,\dots,y_4;u,v) \end{aligned}$$

by $\boldsymbol{\mu}_4$-action:

$$y_1 \mapsto -\mathrm{i}\, y_1, \quad y_2 \mapsto \mathrm{i}\, y_2, \quad y_3 \mapsto -y_3, \quad y_4 \mapsto \mathrm{i}\, y_4, \quad u \mapsto \mathrm{i}\, u, \quad v \mapsto -\mathrm{i}\, v,$$

(*see Example* (7.7.1)).

(1.2.4) *(X, C) is of type* (ID$^\vee$) *at P, $s = 2$, $\bar{m} = 1$, (Z, o) is Du Val of type A_1, (X, C) is a quotient of a Gorenstein conic bundle given by the following equation in $\mathbb{P}^2_{y_1,y_2,y_3} \times \mathbb{C}^2_{u,v}$*

$$y_1^2 + y_2^2 + \psi(u, v) y_3^2 = 0, \qquad \psi(u, v) \in \mathbb{C}\{u^2,\, v^2,\, uv\},$$

by $\boldsymbol{\mu}_2$-action:

$$u \mapsto -u, \quad v \mapsto -v, \quad y_1 \mapsto -y_1, \quad y_2 \mapsto y_2, \quad y_3 \mapsto y_3.$$

Here $\psi(u, v)$ has no multiple factors. In this case, (X, P) is the only singular point and it is of type $cA/2$ or $cAx/2$.

(1.2.5) *(X, C) is of type* (IA$^\vee$) *at P, $\bar{m} = 2$, $s = 2$, (Z, o) is Du Val of type A_1, (X, P) is a cyclic quotient singularity of type $\frac{1}{4}(1, 1, 3)$, and (X, C) is the quotient of the index-two $\mathbb{Q}$-conic bundle germ given by the following two equations in $\mathbb{P}(1, 1, 1, 2)_{y_1,\ldots,y_4} \times \mathbb{C}^2_{u,v}$*

$$\begin{aligned} y_1^2 - y_2^2 &= u\psi_1(y_1, \ldots, y_4; u, v) + v\psi_2(y_1, \ldots, y_4; u, v), \\ y_3^2 \qquad &= u\psi_3(y_1, \ldots, y_4; u, v) + v\psi_4(y_1, \ldots, y_4; u, v) \end{aligned}$$

by $\boldsymbol{\mu}_2$-action:

$$y_1 \mapsto y_1, \quad y_2 \mapsto -y_2, \quad y_3 \mapsto y_3, \quad y_4 \mapsto -y_4, \quad u \mapsto -u, \quad v \mapsto -v$$

(*see Example* (7.7.2)).

(1.2.6) *(X, C) is of type* (II$^\vee$) *at P, $\bar{m} = 2$, $s = 2$, (Z, o) is Du Val of type A_1, (X, P) is a singularity of type $cAx/4$, and (X, C) is the quotient of the same form as in* (1.2.5) (*see Example* (7.7.3)).

All the cases (1.2.1) – (1.2.6) *occur.*

By running MMP over the base Z we immediately obtain the following fact which was conjectured by Iskovskikh:

(1.2.7) Theorem. *Let $f \colon X \to Z$ be a $\mathbb{Q}$-conic bundle* (*possibly with reducible fibers*). *Then Z has only Du Val singularities of type A.*

We also note that the singularity (Z, o) is unbounded only in locally primitive cases (1.2.1) and (1.2.2). In all other cases (Z, o) is either of type A_1 or A_3. Theorem (1.2.7) has important applications to rationality problem of conic bundles [Isk96].

Note that the condition that X has only terminal singularities is essential in Theorem (1.2.7): put in Example (1.1.1) $a = 1$ and $b = -2$ (m is odd). We get an extremal contraction having two singular points which are canonical Gorenstein of type $\frac{1}{m}(1, 1, -2)$ and terminal of type $\frac{1}{m}(-1, 1, -2)$. The base surface has a singularity of type $\frac{1}{m}(1, -2)$ which is not Du Val.

(1.2.8) Corollary. *If in notation of* (1.2) *the base* (Z, o) *is not of type* A_1*, then* X *has only cyclic quotient singularities.*

In the case of smooth base our results are not so strong:

(1.3) Theorem. *Let* $(X, C \simeq \mathbb{P}^1)$ *be a* $\mathbb{Q}$*-conic bundle germ over a smooth base* (Z, o)*. Then* (X, C) *is locally primitive and the configuration of singular points is one of the following* (*notation* (IA) – (III) *are explained in* (5.2.1)):

(1.3.1) $\emptyset$, (III), (III)+(III) (X *is Gorenstein*).

(1.3.2) (IA), (IA)+(III), (IA)+(III)+(III).

(1.3.3) (IIA), (IIA)+(III).

(1.3.4) (IC), (IIB).

(1.3.5) (IA)+(IA) *of indices* 2 *and odd* $m \geq 3$.

(1.3.6) (IA)+(IA)+(III) *of indices* 2, *odd* $m \geq 3$ *and* 1.

In contrast with Theorem (1.2) we can say nothing about the existence of $\mathbb{Q}$-conic bundles as in (1.3.3)–(1.3.6). There are examples of index-two $\mathbb{Q}$-conic bundles as in (1.3.2) (see [Pro97a, §3] and (12.1)). One can also easily construct examples of Gorenstein standard conic bundles of type (1.3.1).

(1.3.7) Proposition (Reid's conjecture about general elephant). *Let* $(X, C \simeq \mathbb{P}^1)$ *be a* $\mathbb{Q}$*-conic bundle germ. Then, except possibly for cases* (1.3.4), (1.3.5), *and* (1.3.6), *a general member of* $|-K_X|$ *has only Du Val singularities. In these exceptional cases a general member of* $D \in |-2K_X|$ *does not contain* C *and the log divisor* $K_X + \frac{1}{2}D$ *is log terminal.*

Proposition (1.3.7) follows from Remarks (7.6.1), (8.6.1) and Theorem (10.10).

(1.4) Comments on the approach. Contractions similar to $\mathbb{Q}$-conic bundles were considered in [Mor88]. In fact, [Mor88] deals with birational contractions of threefolds $f\colon X \to (Z,o)$ such that X has only terminal singularities, $-K_X$ is f-ample, and $C := f^{-1}(o)_{\mathrm{red}}$ is a curve. In this case, we have vanishings $R^1f_*\mathcal{O}_X = 0$ and $R^1f_*\omega_X = 0$. (cf. (2.3)). Though the former vanishing was used all over the places in [Mor88], the latter vanishing $R^1f_*\omega_X = 0$ was used only occasionally. It is easy to find the places where the corollaries of $R^1f_*\omega_X = 0$ were used. In this paper we follow the arguments of [Mor88] paying special attention to those corollaries of $R^1f_*\omega_X = 0$ and furthermore give comments to modify the arguments when the corollaries are used.

Though fewer vanishing conditions are available, we have new tools Lemma (2.8) and Theorem (4.4) for $\mathbb{Q}$-conic bundles. These results together with [Mor88] form the basis of our approach.

§2. Preliminaries

(2.2) Let $f\colon (X,C) \to (Z,o)$ be a $\mathbb{Q}$-conic bundle germ. The following is an immediate consequence of the Kawamata-Viehweg vanishing theorem.

(2.3) Theorem. *$R^if_*\mathcal{O}_X = 0$ for $i > 0$.*

(2.3.1) Corollary (cf. [Mor88, Remark 1.2.1, Cor. 1.3]).

(i) *If J be an ideal such that* $\operatorname{Supp}\mathcal{O}_X/J \subset C$*, then* $H^1(\mathcal{O}_X/J) = 0$.

(ii) $p_a(C) = 0$ *and C is a union of smooth rational curves.*

(iii) $\operatorname{Pic} X \simeq H^2(C,\mathbb{Z}) \simeq \mathbb{Z}^\rho$*, where ρ is the number of irreducible components of C.*

(2.3.2) Remark. If C is reducible, then $\rho(X/Z) > 1$ and for every closed curve $C' \subsetneq C$ the germ (X,C') is an extremal neighborhood (isolated or divisorial). These were classified in [Mor88] and [KM92] under the condition that C' is irreducible.

(2.3.3) Remark. In general, we do not assume that X is $\mathbb{Q}$-factorial (i.e., a Weil divisor on X is not necessarily $\mathbb{Q}$-Cartier). In fact, the following are equivalent

(i) X is $\mathbb{Q}$-factorial and $\rho(X/Z) = 1$,

(ii) the preimage of an arbitrary irreducible curve $\Gamma \subset Z$ is also irreducible.

Indeed, the implication (i) $\Rightarrow$ (ii) is obvious. To show (ii) $\Rightarrow$ (i), consider a $\mathbb{Q}$-factorialization $Y \to X$ [Kaw88] and run the MMP over (Z, o). If (i) does not hold, $\rho(Y/Z) > 1$. On the last step of the MMP we get a divisorial contraction $Y_{n-1} \to Y_n$ over (Z, o). Let E be the corresponding exceptional divisor and let Γ be its image on Z. Then Γ is an irreducible curve and $f^{-1}(\Gamma)$ has two components.

(2.4) We need the following easy construction which is to be used throughout the paper. First, we claim that (Z, o) is a quotient singularity. Indeed, the general hyperplane section $H \subset X$ is smooth and the restriction $f_H \colon H \to Z$ is a finite morphism. Thus (Z, o) is a log terminal singularity [KM98, Prop. 5.20]. Therefore, (Z, o) is a quotient of a smooth germ (Z', o') by a finite group G which acts freely outside of o' [Kaw88, Th. 9.6]. Consider the base change

(2.4.1)
$$\begin{array}{ccc} (X', C') & \xrightarrow{g} & (X, C) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (Z', o') & \xrightarrow{h} & (Z, o) \end{array}$$

where X' is the normalization of $X \times_Z Z'$ and $C' := f'^{-1}(C)_{\mathrm{red}}$. The group G naturally acts on X' so that $X = X'/G$. Since X has only isolated singularities, g is étale in codimension 2. Moreover, $K_{X'} = g^* K_X$ and singularities of X' are terminal. In particular, $f' \colon (X', C') \to (Z', o')$ is a $\mathbb{Q}$-conic bundle germ.

(2.4.2) Corollary ([Cut88]). *Let $f \colon X \to Z$ be a $\mathbb{Q}$-conic bundle. If X is Gorenstein* (*and terminal*)*, then Z is smooth and there is a vector bundle $\mathcal{E}$ of rank 3 on Z and an embeddings $X \hookrightarrow \mathbb{P}(\mathcal{E})$ such that every scheme fiber X_z, $z \in Z$ is a conic in $\mathbb{P}(\mathcal{E})_z$.*

Sketch of the proof. The question is local, so we assume that $f \colon (X, C) \to (Z, o)$ is a $\mathbb{Q}$-conic bundle germ. If (Z, o) is smooth, the assertion can be proved in the standard way: f is flat because X is Cohen-Macaulay and we can put $\mathcal{E} = f_* \mathcal{O}_X(-K_X)$ (see, e.g., [Cut88]). Assume that (Z, o) is singular. Consider the base change (2.4.1). Then (Z', o') is smooth and $G \neq \{1\}$. Since X is Gorenstein terminal, the action of G on X' and C' is free. On the other hand, X' is also Gorenstein. By the above arguments f' is a standard Gorenstein conic bundle. In particular, the central fiber $C' := f^{-1}(o')_{\mathrm{red}}$ is a conic. If C' is reducible, then the singular point $P' \in C'$ is G-invariant, a contradiction. Hence, $C' \simeq \mathbb{P}^1$. This contradicts the fact that the action of G on C' is free. $\square$

(2.5) Definition ([Mor88, (0.4.16), (1.7)]). Let (X, P) be a terminal 3-dimensional singularity of index m and let $C \subset X$ be a smooth curve passing through P. We say that C is (locally) *primitive* at P if the natural map

$$\varrho\colon \mathbb{Z} \simeq \pi_1(C \setminus \{P\}) \to \pi_1(X \setminus \{P\}) \simeq \mathbb{Z}/m\mathbb{Z}$$

is surjective and *imprimitive* at P otherwise. The order s of $\operatorname{Coker} \varrho$ is called the *splitting degree* and the number $\bar{m} = m/s$ is called the *subindex* of $P \in C$.

It is easy to see that the splitting degree coincides with the number of irreducible components of the preimage $C^\sharp$ of C under the index-one cover $X^\sharp \to X$ near P. If P is primitive, we put $s = 1$ and $\bar{m} = m$.

(2.6) From now on we assume that $f\colon (X, C) \to (Z, o)$ is a $\mathbb{Q}$-conic bundle germ with $C \simeq \mathbb{P}^1$. There are two cases (cf. [Mor88, (1.12)]):

(2.6.1) Case: C' is irreducible.

(2.6.2) Case: $C' = \cup_{i=1}^s C'_i$, where $s > 1$ and $C'_i \simeq \mathbb{P}^1$. In this case, G acts on $\{C'_i\}$ transitively.

(2.6.3) Claim. *In the case* (2.6.2)*, all the irreducible components C'_i pass through one point P' and do not intersect each other elsewhere.*

Proof. Since G acts on $\{C'_i\}$ transitively and $p_a(C') = 0$, each component C'_i meets the closure of $C' - C'_i$ at one point. □

(2.6.4) Proposition (cf. [Mor88, 1.11-1.13]). *Notation as in* (2.4) *and* (2.6.3)*.*

(i) *In the case* (2.6.2)*, C is imprimitive at $g(P')$. Conversely, if C is imprimitive at some point P, then f is such as in* (2.6.2) *and $P = g(P')$. Moreover, the splitting degree s coincides with the number of irreducible components of C'.*

(ii) (X, C) *has at most one imprimitive point.*

(2.7) Proposition ([Pro97a, Lemma 1.10]). *(Z, o) is a cyclic quotient singularity.*

Proof. It is sufficient to show that G is a cyclic group. In the case where X is locally primitive, G effectively acts on $C' \simeq \mathbb{P}^1$ and on the tangent space $T_{Z',o'} \simeq \mathbb{C}^2$. This gives us two embeddings: $G \subset PGL(2, \mathbb{C})$ and $G \subset GL(2, \mathbb{C})$. Assume that G is not cyclic. By the classification of finite subgroups in $PGL(2, \mathbb{C})$ G is either $\mathfrak{A}_5$, $\mathfrak{S}_4$, $\mathfrak{A}_4$, or the dihedral group $\mathfrak{D}_n$ of order

$2n$ (see, e.g., [Spr77]). In all cases there are at least two different elements of order two in G. But then at least one of them is a reflection in $G \subset GL(2, \mathbb{C})$, a contradiction.

In the case where f is not locally primitive, by Claim (2.6.3), G has a fixed point $P' \in X'$. Let $P = g(P')$ and let $U \ni P$ be a small neighborhood. There is a surjection $\pi_1(U \setminus \{P\}) \twoheadrightarrow G$. On the other hand, $\pi_1(U \setminus \{P\})$ is cyclic [Kaw88, Lemma 5.1]. □

Thus we may assume that $G = \boldsymbol{\mu}_d$ and $Z \simeq \mathbb{C}^2/\boldsymbol{\mu}_d$, where the action of $\boldsymbol{\mu}_d$ on $\mathbb{C}^2 \simeq Z'$ is free outside of 0. We call this d the *topological index* of $f\colon (X, C) \to (Z, o)$.

Let $\operatorname{Cl}^{\mathrm{sc}} X$ be the subgroup of the divisor class group $\operatorname{Cl} X$ consisting of Weil divisor classes which are $\mathbb{Q}$-Cartier.

(2.7.1) Corollary. *$\pi_1(X \setminus \operatorname{Sing} X) \simeq \boldsymbol{\mu}_d$ and $\operatorname{Cl}^{\mathrm{sc}} X \simeq \mathbb{Z} \oplus \mathbb{Z}_d$, where $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$ and d is the topological index of f.*

(2.7.2) Corollary. *In the above notation, let $P_1, \ldots, P_l$ be all the non-Gorenstein points and let $m_1, \ldots, m_l$ be their indices (the case $l = 1$ is not excluded). Assume that $P_2, \ldots, P_l$ are primitive. Let s_1 and $\bar{m}_1$ be the splitting degree and the subindex of P_1.*

(i) *For each prime p the number of the m_i's divisible by p is ≤ 2.*

(ii) *There is a $\mathbb{Q}$-Cartier Weil divisor D on X such that $D \cdot C = d/m_1 \cdots m_l$. Moreover, D generates $\operatorname{Cl}^{\mathrm{sc}} X/\operatorname{Torsion} = \operatorname{Cl}^{\mathrm{sc}} X/\!\equiv$.*

(iii) $\displaystyle\prod_{i=1}^{l} m_i = d \cdot \operatorname{lcm}(\bar{m}_1, m_2, \ldots, m_l)$.

Proof. Let H be an ample generator of $\operatorname{Pic} X$ so that $H \cdot C = 1$. Clearly, the following sequence

$$0 \longrightarrow \operatorname{Pic} X \longrightarrow \operatorname{Cl}^{\mathrm{sc}} X \xrightarrow{\varsigma} \oplus_i \operatorname{Cl}^{\mathrm{sc}}(X, P_i) \longrightarrow 0 \qquad (2.7.3)$$

is exact. Here $\operatorname{Cl}^{\mathrm{sc}}(X, P_i) \simeq \mathbb{Z}_{m_i}$ by [Kaw88, Lemma 5.1]. Then (i) immediately follows by (2.7.1).

Let us prove (ii). We have $\operatorname{Cl}^{\mathrm{sc}} X/\mathbb{Z}_d \simeq \mathbb{Z}$ by (2.7.1) and the order of $(\operatorname{Cl}^{\mathrm{sc}} X/\mathbb{Z}_d)/\operatorname{Pic} X$ is $\frac{1}{d}\prod m_i$ by (2.7.3). Let D be an ample Weil divisor generating $\operatorname{Cl}^{\mathrm{sc}} X/\mathbb{Z}_d$. Since $H \cdot C = 1$, We have $\frac{1}{d}\prod m_i D \cdot C = H \cdot C = 1$. This proves (ii).

Finally, by [Mor88, 1.9, 1.7] the $\mathbb{Z}$-module $\operatorname{Cl}^{\mathrm{sc}} X$ is generated by H and some ample Weil divisors $D_1, \ldots, D_l$ with relations $m_i D_i - n_i H \sim 0$, where

$n_i = m_i D_i \cdot C$, $\gcd(m_1, n_1) = s_1$, and $\gcd(m_i, n_i) = 1$ for $i = 2, \dots, l$. Now (iii) can be proved by considering the p-prime component of $\operatorname{Cl}^{\mathrm{sc}} X$ for each prime p. □

(2.7.4) Corollary. *In the locally primitive case, $\boldsymbol{\mu}_d$ has exactly two fixed points on $(X', C' \simeq \mathbb{P}^1)$. Therefore, there are two points on (X, C) whose indices are divisible by d. Conversely, if there are two primitive points on (X, C) whose indices divisible by r, then r divides d.*

(2.7.5) Corollary. *In the case (X, C) is imprimitive at $P = g(P')$, the splitting degree s (> 1) divides d, and let $r := d/s$. Put $X^\flat := X'/\boldsymbol{\mu}_r$, $Z^\flat := Z'/\boldsymbol{\mu}_r$, and $C^\flat := C'/\boldsymbol{\mu}_r$. We have the following decomposition:*

(2.7.6)
$$\begin{array}{ccccc}
(X', C') & \xrightarrow{g''} & (X^\flat, C^\flat) & \xrightarrow{g^\flat} & (X, C) \\
{\scriptstyle f'}\downarrow & & {\scriptstyle f^\flat}\downarrow & & {\scriptstyle f}\downarrow \\
(Z', o') & \xrightarrow{h'} & (Z^\flat, o^\flat) & \xrightarrow{h^\flat} & (Z, o)
\end{array}$$

and the following hold:

(i) *The group $\boldsymbol{\mu}_r$ does not permute components of C', so $C^\flat$ has exactly s irreducible components $C_i^\flat$ passing through one point $P^\flat = g''(P')$. The group $\boldsymbol{\mu}_s = \boldsymbol{\mu}_d/\boldsymbol{\mu}_r$ naturally acts on $X^\flat$ so that $X = X^\flat/\boldsymbol{\mu}_r$.*

(ii) *If $d > s$, then $\boldsymbol{\mu}_r$ has two fixed points on each component $C_i' \subset C'$, P' and $Q_i' \neq P'$.*

(iii) *$(X^\flat, C_i^\flat)$ is a locally primitive extremal neighborhood* (2.3.2)*, $X^\flat \to X$ is étale outside $P^\flat$ and $C_i^\flat \to C$ is an isomorphism.*

The base change $g^\flat$ as in (2.7.6) is called the *splitting cover* [Mor88, 1.12.1].

Proof. Let $G \subset \boldsymbol{\mu}_d$ be the stabilizer of some component $C_i' \subset C'$. Then $G = \boldsymbol{\mu}_r$ and $X^\flat := X'/G$ satisfies the desired properties. □

(2.8) Lemma. *Let (X, C) be a $\mathbb{Q}$-conic bundle germ with $C \simeq \mathbb{P}^1$. Let d be the topological index of (X, C) and let $m_1, \dots, m_r$ be indices of all the non-Gorenstein points. Assume that f is not toroidal. Then*

$$-K_X \cdot C = d/m_1 \cdots m_r.$$

Proof. Take D as in (ii) of Corollary (2.7.2). Then $-K_X \equiv rD$ for some $r \in \mathbb{Z}_{>0}$. We claim that $r = 1$. Indeed, for the general fiber L we have

$2 = -K_X \cdot L = rD \cdot L$. Since $D \cdot L$ is an integer, $r = 1$ or 2. If $r = 2$, then $D \cdot L = 1$, i.e., D is f-ample with $\deg = 1$ on the general fiber. Apply construction (2.4.1). Then $D' := f'^{*}D$ satisfies the same property: it is f'-ample with $\deg = 1$ on the general fiber. Since $X' \setminus f'^{-1}(o') \to Z' \setminus \{o'\}$ is a standard conic bundle (see (2.4.2)), this implies that all the fibers over $Z' \setminus \{o'\}$ are smooth rational curves. In particular, the morphism f' is smooth outside of C'. We claim that f' is smooth everywhere. Denote $\mathcal{F} := \mathcal{O}_{X'}(D')$. Then locally near a singular point $P' \in X'$, $\mathcal{F}$ is a direct summand of $\pi_*\mathcal{F}^\sharp$, where $\pi\colon (X^\sharp, P^\sharp) \to (X', P')$ is the index-one cover and $\mathcal{F}^\sharp$ is the lifting of $\mathcal{F}$. Since $\mathcal{F}^\sharp$ is Cohen-Macaulay and Z' is smooth, $\mathcal{F}$ is flat over Z'. Then by the base change theorem ([Mum66, Lect. 7, (iii), p. 51]) $f'_*\mathcal{F}$ is locally free. Put $\hat{X} := \mathbb{P}(f'_*\mathcal{F})$ with natural projection $\hat{f}\colon \hat{X} \to Z'$. We have a bimeromorphic map $\hat{X} \dashrightarrow X'$ over Z' that indices an isomorphism $(\hat{X} \setminus \hat{C}) \simeq (X' \setminus C')$, where $\hat{C} = \hat{f}^{-1}(o)$. Since f', $\hat{f}$ are projective and $\rho(X'/Z') = \rho(\hat{X}/Z') = 1$, we have $\hat{X} \simeq X'$. But then X' is smooth and so is the morphism f'. This proves our claim.

Thus we may assume $X' \simeq Z' \times \mathbb{P}^1$. Recall that X is the quotient of X' by $\boldsymbol{\mu}_d$. By [Pro97a, §2] the action of $\boldsymbol{\mu}_d$ is as in (1.1.1) and f is toroidal, a contradiction to our assumption. Therefore, $r = 1$ and $-K_X \cdot C = D \cdot C$. This proves our equality. □

(2.8.1) Corollary. *If X has a unique non-Gorenstein point which is imprimitive of splitting degree s and subindex $\bar{m}$, then $2\bar{m} \equiv 0 \bmod s$.*

Proof. Let $f'^{-1}(o')$ be the scheme fiber. Then $f'^{-1}(o') \equiv rC'$ for some $r \in \mathbb{Z}_{>0}$. Thus $2 = -K_{X'} \cdot f'^{-1}(o') = -rK_{X'} \cdot C' = -rsK_X \cdot C = rs/\bar{m}$. This proves our statement. □

The following fact will be used freely.

(2.9) Proposition ([Pro97a, Th. 2.4]). *In notation of* (2.4.1) *assume that X' is Gorenstein* (*we do not assume that C is irreducible*). *Assume further that $d > 1$. Then (X, C) is in one of the cases* (1.2.1), (1.2.2), (1.2.4).

Sketch of the proof. By (2.4.2) there is a $\boldsymbol{\mu}_d$-equivariant embedding $X' \hookrightarrow \mathbb{P}^2 \times Z'$ over Z'. Then one can choose a suitable coordinate system in $\mathbb{P}^2$ and $Z' \simeq \mathbb{C}^2$. □

§3. Numerical Invariants i_P, w_P and w_P^*

For convenience of the reader we recall some basic notation of [Mor88].

(3.1) Let X be an analytic threefold with terminal singularities and let $C \subset X$ be a reduced curve. Let $I_C \subset \mathcal{O}_X$ be the ideal sheaf of C and let $I_C^{(n)}$ be the symbolic nth power of I_C, that is, the saturation of I_C^n in $\mathcal{O}_X$. Put $\operatorname{gr}_C^n \mathcal{O} := I_C^{(n)}/I_C^{(n+1)}$. Further, let $F^n\omega_X$ be the saturation of $I_C^n\omega_X$ in ω_X and let $\operatorname{gr}_C^n \omega := F^n\omega_X/F^{n+1}\omega_X$. Let m be the index of K_X. There are natural homomorphisms

$$\begin{aligned}
\alpha_1 &: \textstyle\bigwedge^2 \operatorname{gr}_C^1 \mathcal{O} &&\longrightarrow \mathcal{H}om_{\mathcal{O}_C}(\Omega_C^1, \operatorname{gr}_C^0 \omega),\\
\alpha_n &: S^n \operatorname{gr}_C^1 \mathcal{O} &&\longrightarrow \operatorname{gr}_C^n \mathcal{O}, \quad n \ge 2,\\
\beta_0 &: (\operatorname{gr}_C^0 \omega)^{\otimes m} &&\longrightarrow (\omega_X^{\otimes m})^{**} \otimes \mathcal{O}_C,\\
\beta_n &: \operatorname{gr}_C^0 \omega \otimes S^n \operatorname{gr}_C^1 \mathcal{O} &&\longrightarrow \operatorname{gr}_C^n \omega, \quad n \ge 1,
\end{aligned}$$

where M^* for an $\mathcal{O}_X$-module M denotes its dual, $\mathcal{H}om\,\mathcal{O}_X(M, \mathcal{O}_X)$. Denote

$$i_P(n) := \operatorname{len}_P \operatorname{Coker} \alpha_n, \qquad w_P(0) := \operatorname{len}_P \operatorname{Coker} \beta_0/m,$$

$$w_P(n) := \operatorname{len}_P \operatorname{Coker} \beta_n, \quad w_P^*(n) := \binom{n+1}{2} i_P(1) - w_P(n), \quad n \ge 1.$$

Assume that $C \simeq \mathbb{P}^1$. Then we have by [Mor88, 2.3.1]

(3.1.1) $$-\deg \operatorname{gr}_C^0 \omega = -K_X \cdot C + \sum_P w_P(0)$$

(3.1.2) $$2 + \deg \operatorname{gr}_C^0 \omega - \deg \operatorname{gr}_C^1 \mathcal{O} = \sum_P i_P(1).$$

(3.1.3) $$\deg \operatorname{gr}_C^n \mathcal{O} = \frac{1}{2} n(n+1) \deg \operatorname{gr}_C^1 \mathcal{O} + \sum_P i_P(n), \qquad n \ge 2,$$

and therefore the following corollaries to $\operatorname{rk} \operatorname{gr}_C^i \mathcal{O} = i + 1$ and $R^1 f_* \mathcal{O}_X = 0$:

(3.1.4) $$\sum_{i=1}^{n} (\deg \operatorname{gr}_C^i \mathcal{O} + i + 1) \ge 0, \qquad n \ge 1,$$

(3.1.5) $$4 \ge -\deg \operatorname{gr}_C^0 \omega + \sum_P i_P(1) = -K_X \cdot C + \sum_P w_P(0) + \sum_P i_P(1).$$

(3.1.6) Remark. In the case of extremal neighborhoods by the Grauert-Riemenshneider vanishing one has $\operatorname{gr}_C^0 \omega = \mathcal{O}_C(-1)$ (see [Mor88, 2.3]). This is no longer true for $\mathbb{Q}$-conic bundles: in Example (1.1.1) easy computations show $\deg \operatorname{gr}^0_C \omega = -2$ (see (3.1.1)). Similarly, in (1.2.4) we also have $\deg \operatorname{gr}^0_C \omega = -2$. We will show below that these two examples are the only exceptions (see Corollaries (4.4.3) and (7.2.2)).

(3.1.7) Lemma. *If $\operatorname{gr}^0_C \omega = \mathcal{O}(-1)$, then*

$$\deg \operatorname{gr}^n_C \omega = \frac{1}{2}(n+1)(n-2) - \sum_P w^*_P(n), \quad n \geq 1. \tag{3.1.8}$$

If furthermore $H^1(\omega_X/F^{n+1}\omega_X) = 0$, then

$$\sum_{i=1}^{n} \bigl(\deg \operatorname{gr}^i_C \omega + i + 1\bigr) \geq 0, \qquad n \geq 1. \tag{3.1.9}$$

Proof. Follows by the exact sequences

$$0 \longrightarrow \operatorname{gr}^i_C \omega \longrightarrow \omega_X/F^{i+1}\omega_X \longrightarrow \omega_X/F^i\omega_X \longrightarrow 0$$

(see [Mor88, 2.3]). □

(3.1.10) Lemma ([Mor88, 2.15]). *If (X, P) is singular, then $i_P(1) \geq 1$.*

Proof. The proof of [Mor88, 2.15] applies because it uses only local computations near P that are not based on $R^1 f_* \omega_X$. □

(3.1.11) Corollary. *A $\mathbb{Q}$-conic bundle germ $(X, C \simeq \mathbb{P}^1)$ has at most three singular points.*

§4. Sheaves $\operatorname{gr}^n_C \omega$

(4.1) Lemma. *Let $f: X \to Z$ be a $\mathbb{Q}$-conic bundle. Assume that the base surface Z is smooth. Then there is a canonical isomorphism $R^1 f_* \omega_X \simeq \omega_Z$.*

Proof. Let $g: W \to X$ be a resolution. By [Kol86, Prop. 7.6] we have $R^1(f \circ g)_* \omega_W = \omega_Z$. Since X has only terminal singularities, $g_* \omega_W = \omega_X$ and by the Grauert-Riemenshneider vanishing, $R^i g_* \omega_W = 0$ for $i > 0$. Then the Leray spectral sequence gives us $R^1 f_* \omega_X = R^1(f \circ g)_* \omega_W = \omega_Z$. □

For convenience of the reader we recall basic definitions [Mor88, 8.8].

(4.2) Let (X, P) be three-dimensional terminal singularity of index m and let $\pi\colon (X^\sharp, P^\sharp) \to (X, P)$ be the index-one cover. Let $\mathcal{L}$ be a coherent sheaf on X without submodules of finite length > 0. An *ℓ-structure* of $\mathcal{L}$ at P is a coherent sheaf $\mathcal{L}^\sharp$ on $X^\sharp$ without submodules of finite length > 0 with $\boldsymbol{\mu}_m$-action endowed with an isomorphism $(\mathcal{L}^\sharp)^{\boldsymbol{\mu}_m} \simeq \mathcal{L}$. An *$\ell$-basis of* $\mathcal{L}$ *at* P is a collection of $\boldsymbol{\mu}_m$-semi-invariants $s_1^\sharp, \dots, s_r^\sharp \in \mathcal{L}^\sharp$ generating $\mathcal{L}^\sharp$ as an $\mathcal{O}_{X^\sharp}$-module at $P^\sharp$. Let Y be a closed subscheme of X such that $P \notin \operatorname{Ass} \mathcal{O}_Y$ and let $Y^\sharp \subset X^\sharp$ be the canonical lifting. Note that $\mathcal{L}$ is an $\mathcal{O}_Y$-module if and only if $\mathcal{L}^\sharp$ is an $\mathcal{O}_{Y^\sharp}$-module. We say that $\mathcal{L}$ is *ℓ-free* $\mathcal{O}_Y$-module at P if $\mathcal{L}^\sharp$ is a free $\mathcal{O}_{Y^\sharp}$-module at $P^\sharp$. If $\mathcal{L}$ is ℓ-free $\mathcal{O}_Y$-module at P, then an ℓ-basis of $\mathcal{L}$ at P is said to be *ℓ-free* if it is a free $\mathcal{O}_{Y^\sharp}$-basis.

Let $\mathcal{L}$ and $\mathcal{M}$ be $\mathcal{O}_Y$-modules at P with ℓ-structures $\mathcal{L} \subset \mathcal{L}^\sharp$ and $\mathcal{M} \subset \mathcal{M}^\sharp$. Define the following operations:

- $\mathcal{L} \mathbin{\tilde{\oplus}} \mathcal{M} \subset (\mathcal{L} \oplus \mathcal{M})^\sharp$ is an $\mathcal{O}_Y$-module at P with ℓ-structure
$$(\mathcal{L} \mathbin{\tilde{\oplus}} \mathcal{M})^\sharp = \mathcal{L}^\sharp \oplus \mathcal{M}^\sharp.$$
- $\mathcal{L} \mathbin{\tilde{\otimes}} \mathcal{M} \subset (\mathcal{L} \otimes \mathcal{M})^\sharp$ is an $\mathcal{O}_Y$-module at P with ℓ-structure
$$(\mathcal{L} \mathbin{\tilde{\otimes}} \mathcal{M})^\sharp = (\mathcal{L}^\sharp \otimes_{\mathcal{O}_{X^\sharp}} \mathcal{M}^\sharp) / \operatorname{Sat}_{\mathcal{L}^\sharp \otimes \mathcal{M}^\sharp}(0),$$
where $\operatorname{Sat}_{\mathcal{F}_1} \mathcal{F}_2$ is the saturation of $\mathcal{F}_2$ in $\mathcal{F}_1$.

These operations satisfy standard properties (see [Mor88, 8.8.4]). If X is an analytic threefold with terminal singularities and Y is a closed subscheme of X, then the above local definitions of $\tilde{\oplus}$ and $\tilde{\otimes}$ patch with corresponding operations on $X \setminus \operatorname{Sing} X$. Therefore, they give well-defined operations of global $\mathcal{O}_Y$-modules.

(4.3) Let $f\colon (X, C) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ (we do not assume that C is irreducible).

(4.4) Theorem. *Assume that (Z, o) is smooth. Let $J \subset \mathcal{O}_X$ be an ideal such that* $\operatorname{Supp} \mathcal{O}_X/J \subset C$ *and $\mathcal{O}_X/J$ has no embedded components. Assume that $H^1(\omega_X \mathbin{\tilde{\otimes}} \mathcal{O}_X/J) \neq 0$. Then* $\operatorname{Spec}_X \mathcal{O}_X/J \supset f^{-1}(o)$*, where $f^{-1}(o)$ is the scheme fiber (in other words, $J \subset \mathfrak{m}_{Z,o}\mathcal{O}_X$, where $\mathfrak{m}_{Z,o} \subset \mathcal{O}_Z$ is the maximal ideal of o).*

Proof. First we assume that $\operatorname{Spec}_X \mathcal{O}_X/J \subsetneq f^{-1}(o)$. Denote $\Gamma := f^{-1}(o)$ and $V := \operatorname{Spec} \mathcal{O}_X/J$. Then $\omega_\Gamma \simeq \omega_X \otimes \mathcal{O}_\Gamma$, and so $\omega_X \mathbin{\tilde{\otimes}} \mathcal{O}_V \simeq \omega_\Gamma \mathbin{\tilde{\otimes}} \mathcal{O}_V$ in this case.

Let $\mathcal{I}_V$ be the ideal sheaf of V in Γ where we note $\mathcal{I}_V \neq 0$ by $V \subsetneq f^{-1}(o)$, and let $\mathcal{I}_D$ be an associated prime of $\mathcal{I}_V$ (i.e. $\mathcal{I}_D \in \operatorname{Ass}(\mathcal{I}_V)$), and let $D \subset C$ be the corresponding irreducible component. By the Serre duality, we have

$$\omega_D = \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \omega_\Gamma) = \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma) \tilde{\otimes} \omega_\Gamma.$$

Hence $\mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma)$ is a torsion-free $\mathcal{O}_D$-module of rank 1. We also see $\mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{I}_V) \neq 0$ by $\mathcal{I}_D \in \operatorname{Ass}(\mathcal{I}_V)$. Thus the cokernel of the inclusion

$$0 \neq \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{I}_V) \hookrightarrow \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma).$$

is of finite length and is a submodule of $\mathcal{O}_V = \mathcal{O}_\Gamma/\mathcal{I}_V$. Since $\mathcal{O}_V = \mathcal{O}_\Gamma/\mathcal{I}_V$ has no embedded primes, we have $\mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{I}_V) = \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma)$ and

$$\mathcal{I}_V = \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_\Gamma, \mathcal{I}_V) \supset \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma).$$

Considering the trace map one can see that $\mathbb{C} \simeq H^1(\omega_D) \to H^1(\omega_\Gamma)$ is an injection (and moreover $H^1(\omega_D) \simeq H^1(\omega_\Gamma) \simeq \mathbb{C}$). Since ω_Γ is ℓ-invertible, the composition map

$$\upsilon \colon H^1(\omega_D) \xrightarrow{\sim} H^1(\omega_\Gamma) \to H^1(\omega_\Gamma \tilde{\otimes} \left(\mathcal{O}_\Gamma/ \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma)\right))$$

is zero.

On the other hand, $\omega_\Gamma \to \omega_X \tilde{\otimes} \mathcal{O}_V$ has the following decomposition

$$\omega_\Gamma \to \omega_\Gamma \tilde{\otimes} \left(\mathcal{O}_\Gamma/ \mathcal{H}om_{\mathcal{O}_\Gamma}(\mathcal{O}_D, \mathcal{O}_\Gamma)\right) \to \omega_\Gamma \tilde{\otimes} \mathcal{O}_\Gamma/\mathcal{I}_V \simeq \omega_X \tilde{\otimes} \mathcal{O}_V,$$

and the induced surjective map

$$H^1(\omega_\Gamma) \longrightarrow H^1(\omega_\Gamma \tilde{\otimes} \mathcal{O}_V) \neq 0$$

factors through υ which is zero, a contradiction.

This proves that $\operatorname{Spec}_X \mathcal{O}_X/J = f^{-1}(o)$ if $\operatorname{Spec}_X \mathcal{O}_X/J \subset f^{-1}(o)$.

Now we treat the general case. By Nakayama's lemma $H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/J) \otimes_{\mathcal{O}_Z} \mathcal{O}_Z/\mathfrak{m}_{Z,o} \neq 0$. Since H^1 is right exact for $\mathcal{O}_X$-sheaves, we see that

$$H^1((\omega_X \tilde{\otimes} \mathcal{O}_X/J) \otimes \mathcal{O}_X/\mathfrak{m}_{Z,o}\mathcal{O}_X) \simeq H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/J) \otimes_{\mathcal{O}_Z} \mathcal{O}_Z/\mathfrak{m}_{Z,o} \neq 0.$$

Let us consider the homomorphism

$$(\omega_X \tilde{\otimes} \mathcal{O}_X/J) \otimes \mathcal{O}_X/\mathfrak{m}_{Z,o}\mathcal{O}_X \to \omega_X \tilde{\otimes} \mathcal{O}_X/J^s,$$

where J^s is the saturation of $J + \mathfrak{m}_{Z,o}\mathcal{O}_X$ in $\mathcal{O}_X$. It is surjective and its kernel is supported at a finite number of points. Thus

$$H^1\big(\omega_X \tilde{\otimes} \mathcal{O}_X/J^s\big) \simeq H^1\big((\omega_X \tilde{\otimes} \mathcal{O}_X/J) \otimes \mathcal{O}_X/\mathfrak{m}_{Z,o}\mathcal{O}_X\big) \neq 0$$

and $J^s \supset \mathfrak{m}_{Z,o}\mathcal{O}_X$. By the special case treated above we have $J + \mathfrak{m}_{Z,o}\mathcal{O}_X \subset J^s = \mathfrak{m}_{Z,o}\mathcal{O}_X$, i.e., $J \subset \mathfrak{m}_{Z,o}\mathcal{O}_X$.

□

(4.4.1) Corollary. *Assume that (Z,o) is smooth. If $H^1(\mathrm{gr}^0_C\,\omega) \neq 0$, then $C = f^{-1}(o)$.*

Proof. Apply Theorem (4.4) with $J = I_C$. □

(4.4.2) Lemma ([Kol99, Prop. 4.2]). *If X is not Gorenstein, then X has index > 1 at all singular points of C.*

Proof. If C has at least three irreducible components, the assertion follows by Remark (2.3.2) and [Mor88, Cor. 1.15]. Thus we assume that $C = C_1 \cup C_2$ and X is Gorenstein at $P \in C_1 \cap C_2$. First we consider the case when (Z,o) is smooth. By our assumption $\mathrm{gr}^0_C\,\omega = \omega_X \otimes \mathcal{O}_C$ is invertible at P. Consider the injection $\varphi\colon \mathrm{gr}^0_C\,\omega \hookrightarrow \mathrm{gr}^0_{C_1}\,\omega \oplus \mathrm{gr}^0_{C_2}\,\omega$. Recall that (X, C_i) is an extremal neighborhood by Remark (2.3.2). Then by [Mor88, Prop. 1.14] $\mathrm{gr}^0_{C_i}\,\omega = \mathcal{O}_{C_i}(-1)$, so $H^0(\mathrm{Coker}\,\varphi) = H^1(\mathrm{gr}^0_C\,\omega)$. On the other hand, $\mathrm{Coker}\,\varphi$ is a sheaf of finite length supported at P. Since $\mathrm{gr}^0_C\,\omega$ is invertible, $\mathrm{Coker}\,\varphi$ is non-trivial. So, $H^1(\mathrm{gr}^0_C\,\omega) \neq 0$ and by Corollary (4.4.1) $C_1 \cup C_2 = f^{-1}(o)$. Thus X is smooth outside of $\mathrm{Sing}\,C$. Since P is the only singular point of C by Corollary (2.3.1), we are done.

Now we assume that (Z,o) is singular. Consider the base change (2.4.1). Since X is Gorenstein terminal at P, so is X' at all the points $P'_i \in g^{-1}(P)$. Moreover, g is étale over P. Hence, the central curve C' is singular at P'_i. By the above, X' is Gorenstein and by Corollary (2.4.2) $f'\colon X' \to Z'$ is a standard Gorenstein conic bundle. In particular, C' is a plane conic. Since the set $g^{-1}(P)$ is contained in the singular locus of C', it consists of one point, a contradiction. □

(4.4.3) Corollary (cf. [Mor88, Prop. 1.14]). *Assume that C is irreducible. If $\mathrm{gr}^0_C\,\omega \not\simeq \mathcal{O}_C(-1)$, then in notation of (2.4.1) we have $f'^{-1}(o') = C'$. If furthermore (X,C) is locally primitive, then it is toroidal.*

Proof. Let m be the index of X. Since there is an injection $(\mathrm{gr}^0_C\,\omega)^{\otimes m} \hookrightarrow \mathcal{O}_C(mK_X)$, $\deg \mathrm{gr}^0_C\,\omega < 0$. Since $\mathrm{gr}^0_C\,\omega \not\simeq \mathcal{O}_C(-1)$, $H^1(\mathrm{gr}^0_C\,\omega) \neq 0$. In notation of (2.4.1) we have $H^1(\mathrm{gr}^0_{C'}\,\omega) \neq 0$ (because $H^1(\mathrm{gr}^0_C\,\omega) = H^1(\mathrm{gr}^0_{C'}\,\omega)^{\boldsymbol{\mu}_d}$).

By Corollary (4.4.1) $C' = f'^{-1}(o')$. If f is locally primitive, C' is irreducible (see (2.6.1)). So $C' \simeq \mathbb{P}^1$ and X' is smooth. Up to analytic isomorphism we may assume that $X' \simeq Z' \times \mathbb{P}^1$. Then in some coordinate system the action of $\boldsymbol{\mu}_d$ on X' is as in (1.1.1) (see [Pro97a, §2]), so f is toroidal. □

(4.4.4) Remark. In notation of Theorem (4.4) assume that the map $H^0(I_C) \to H^0(I_C/J)$ is zero. Then $\operatorname{Spec}_X \mathcal{O}_X/J \subset f^{-1}(o)$. Therefore, the nonvanishing $H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/J) \neq 0$ implies $\operatorname{Spec}_X \mathcal{O}_X/J = f^{-1}(o)$.

(4.4.5) Corollary. *Notation as in* (4.3). *Assume that* (Z, o) *is smooth. If the map* $H^0(I_C) \to H^0(\operatorname{gr}^1_C \mathcal{O})$ *is zero, then* $H^1(\operatorname{gr}^1_C \omega) = 0$.

Proof. Assume that $H^1(\operatorname{gr}^1_C \omega) \neq 0$. In notation of Theorem (4.4), put $J = I_C^{(2)}$ and $V := \operatorname{Spec}_X \mathcal{O}_X/I_C^{(2)}$. From the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{gr}^1_C \mathcal{O} \tilde{\otimes} \omega_X & \longrightarrow & \mathcal{O}_X/I_C^{(2)} \tilde{\otimes} \omega_X & \longrightarrow & \mathcal{O}_C \tilde{\otimes} \omega_X & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ & & \operatorname{gr}^1_C \omega & & \mathcal{O}_V \tilde{\otimes} \omega_X & & \operatorname{gr}^0_C \omega & & \end{array}$$

and $\deg \operatorname{gr}^0_{C_i} \omega < 0$ for each i we get $H^1(\mathcal{O}_V \tilde{\otimes} \omega_X) \neq 0$. Then by Theorem (4.4) and Remark (4.4.4) $V = f^{-1}(o)$. Let $P \in C$ be a general point. Then in a suitable coordinate system (x, y, z) near P we may assume that C is the z-axis. So, $I_C = (x, y)$ and $I_C^{(2)} = (x^2, xy, y^2)$. But then V is not a local complete intersection near P, a contradiction. □

(4.4.6) Corollary. *Assume that* (Z, o) *is smooth and* C *is irreducible. If* $\sum_P i_P(1) \geq 3$, *then* $\sum w^*_P(1) \leq 1$.

Proof. By (3.1.5) $\operatorname{gr}^0_C \omega = \mathcal{O}(-1)$. Further, by (3.1.2) $\deg \operatorname{gr}^1_C \mathcal{O} \leq -2$. Hence, $H^0(\operatorname{gr}^1_C \mathcal{O}) = 0$ (cf. [Mor88, Remark 2.3.4]). Now the desired inequality follows by Corollary (4.4.5) and (3.1.8). □

§5. Preliminary Classification of Singular Points

(5.1) Notation. Let $f\colon (X, C \simeq \mathbb{P}^1) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ. Let $P \in C$ be a point of index $m \geq 1$. Let s and $\bar{m}$ be the splitting degree and subindex, respectively. Thus $m = s\bar{m}$. Consider the canonical $\boldsymbol{\mu}_m$-cover $\pi\colon (X^\sharp, P^\sharp) \to (X, P)$ and let $C^\sharp := \pi^{-1}(C)$. Take normalized ℓ-coordinates $(x_1, \dots, x_4)$ and t and let ϕ be an ℓ-equation of $X \supset C \ni P$ (see [Mor88, 2.6]). Put $a_i = \operatorname{ord} x_i$.

Note that $a_i < \infty$ and $\operatorname{wt} x_i \equiv a_i \bmod \bar{m}$. If $m = 1$, then $X = X^\sharp$. In this case, P is said to be *of type* (III).

(5.2) Primitive point. Consider the case when P is primitive and $m > 1$. Then $s = 1$ and $\bar{m} = m$. In this case, the classification coincides with that in [Mor88] as shown next:

(5.2.1) Proposition (cf. [Mor88, Prop. 4.2]). *Let P and m be as above. Modulo permutations of x_i's, the semigroup* $\operatorname{ord} C^\sharp$ *is generated by a_1 and a_2. Moreover, exactly one of the following holds:*

(IA) $a_1 + a_3 \equiv 0 \bmod m$, $a_4 = m$, $m \in \mathbb{Z}_{>0}a_1 + \mathbb{Z}_{>0}a_2$, *where we may still permute x_1 and x_3 if $a_2 = 1$,*

(IB) $a_1 + a_3 \equiv 0 \bmod m$, $a_2 = m$, $2 \le a_1$,

(IC) $a_1 + a_2 = a_3 = m$, $a_4 \not\equiv a_1, a_2 \bmod m$, $2 \le a_1 < a_2$,

(IIA) $m = 4$, *P is of type $cAx/4$, and* $\operatorname{ord} x = (1, 1, 3, 2)$,

(IIB) $m = 4$, *P is of type $cAx/4$, and* $\operatorname{ord} x = (3, 2, 5, 5)$.

Proof. If X has an imprimitive point ($\neq Q$), then P is as classified in [Mor88, Prop.4.2] by (2.7.5), (iii). So we can assume that X has no imprimitive points. If (X, C) is toroidal, then at both singular points $\operatorname{ord} x = (1, a, m-1, m)$. So, these points are of type (IA). Taking Corollary (4.4.3) into account, we may assume that $\operatorname{gr}^0_C \omega = \mathcal{O}_C(-1)$. By (3.1.1) and (3.1.5) we have $w_P(0) < 1$ and $i_P(1) \le 3$. We claim that $C^\sharp$ is planar, i.e., $\operatorname{ord} C^\sharp$ is generated by two elements. Indeed, in the contrary case by [Mor88, Lemma 3.4] we have $i_P(1) = 3$. Hence P is the only singular point (see (3.1.5)) and (Z, o) is smooth (Corollary (2.7.4)). By Corollary (4.4.6) $w^*_P(1) \le 1$. In this case, arguments of [Mor88, 3.5] work. This shows that $C^\sharp$ is planar. Now we can apply [Mor88, Proof of 4.2] and obtain the above classification. □

(5.3) Imprimitive point. Now assume that P is imprimitive. Then in diagram (2.4.1) the central fiber C' has exactly s (> 1) irreducible components. Note that the classification is different from that in [Mor88] only in the case $C' = f'^{-1}(o')$.

(5.3.1) Proposition (cf. [Mor88, Prop. 4.2]). *Let P, $C^\sharp$, and s be as above. Modulo permutations of x_i's and changes of ℓ-characters, the semigroup* $\operatorname{ow} C^\sharp$ *is generated by* $\operatorname{ow} x_1$ *and* $\operatorname{ow} x_2$ *except for the case* (IE$^\vee$) *below. Moreover, exactly one of the following holds:*

(IA$^\vee$) $\bar{m} > 1$, $\operatorname{wt} x_1 + \operatorname{wt} x_3 \equiv 0 \bmod m$, $\operatorname{ow} x_4 = (\bar{m}, 0)$, $\operatorname{ow} C^\sharp$ *is generated by* $\operatorname{ow} x_1$ *and* $\operatorname{ow} x_2$, *and* $w_P(0) \ge 1/2$.

(IC$^\vee$) *$s = 2$, $\bar{m}$ is an even integer ≥ 4, and*

$$\begin{array}{ccccc} & x_1 & x_2 & x_3 & x_4 \\ \text{wt} & 1 & -1 & 0 & \bar{m}+1 \bmod m \\ \text{ord} & 1 & \bar{m}-1 & \bar{m} & \bar{m}+1 \end{array}$$

(II$^\vee$) *$\bar{m} = s = 2$, P is of type $cAx/4$, and*

$$\begin{array}{ccccc} & x_1 & x_2 & x_3 & x_4 \\ \text{wt} & 1 & 3 & 3 & 2 \bmod 4 \\ \text{ord} & 1 & 1 & 1 & 2 \end{array}$$

(ID$^\vee$) *$\bar{m} = 1$, $s = 2$, P is of type $cA/2$ or $cAx/2$, and*

$$\begin{array}{ccccc} & x_1 & x_2 & x_3 & x_4 \\ \text{wt} & 1 & 1 & 1 & 0 \bmod 2 \\ \text{ord} & 1 & 1 & 1 & 1 \end{array}$$

(IE$^\vee$) *$\bar{m} = 2$, $s = 4$, P is of type $cA/8$, and*

$$\begin{array}{ccccc} & x_1 & x_2 & x_3 & x_4 \\ \text{wt} & 5 & 1 & 3 & 0 \bmod 8 \\ \text{ord} & 1 & 1 & 1 & 2 \end{array}$$

Moreover, we are in the case (ID$^\vee$) *or* (IE$^\vee$) *if only if $C' = f'^{-1}(o')$. In this case, P is the only non-Gorenstein point.*

(5.3.2) Remark. It is easy to show that in cases (IC$^\vee$) and (IE$^\vee$) the point (X, P) is a cyclic quotient singularity (cf. [Mor88, Lemma 4.4]).

Proof. First assume that $C' \neq f'^{-1}(o')$. By Corollary (4.4.1) we have $H^1(\mathrm{gr}^0_{C'}\omega) = 0$. Therefore,

$$H^1(\mathrm{gr}^0_C\,\omega) = H^1(\mathrm{gr}^0_{C'}\,\omega)^{\boldsymbol{\mu}_d} = 0.$$

This implies $\mathrm{gr}^0_C\,\omega = \mathcal{O}_C(-1)$ and $w_P(0) < 1$. In particular, $\bar{m} > 1$ (see [Mor88, Cor. 2.10]). Let $g^\flat\colon (X^\flat, C^\flat) \to (X, C)$ be the splitting cover. Consider the exact sequence

$$0 \longrightarrow \mathrm{gr}^0_{C^\flat}\,\omega \xrightarrow{\varphi} \bigoplus_{i=1}^{s} \mathrm{gr}^0_{C^\flat_i}\,\omega \longrightarrow \operatorname{Coker}\varphi \longrightarrow 0. \tag{5.3.3}$$

Note that $\mathrm{gr}^0_{C_i^\flat}\,\omega = \mathcal{O}(-1)$ (see [Mor88, 2.3.2]). Hence,

$$H^0(\operatorname{Coker}\varphi) = H^1(\mathrm{gr}^0_{C^\flat}\,\omega) = H^1(\mathrm{gr}^0_{C'}\,\omega)^{\boldsymbol{\mu}_{d/s}} = 0.$$

Since the support of $\operatorname{Coker}\varphi$ is zero-dimensional, φ is an isomorphism. Therefore, the classification [Mor88, 4.2] holds for (X,P) in this case (see [Mor88, 3.6-3.8]).

Now we consider the case where $C' = f'^{-1}(o')$. If $\bar{m} = 1$, then by Lemma (4.4.2) the splitting cover $X^\flat$ is Gorenstein and $-K_{X^\flat}\cdot C_i^\flat$ is an integer for any component $C_i^\flat \subset C^\flat$. Hence, $2 = -K_{X^\flat}\cdot C^\flat = -sK_{X^\flat}\cdot C_i^\flat$. This implies $s=2$. We get the case (ID$^\vee$). Furthermore by Proposition (2.9) we are in the case (1.2.4) and hence P is the only non-Gorenstein point. From now on we assume that $\bar{m} > 1$.

(5.3.4) We claim that P is the only non-Gorenstein point. Indeed, assume first that there are at least two non-Gorenstein points other than P on C. Then on the splitting cover $X^\flat$ any irreducible component $C_i^\flat$ of $C^\flat$ contains at least three non-Gorenstein points by $\bar{m} > 1$ (2.7.5), (iii). Since the extremal neighborhood $(X^\flat, C_i^\flat)$ (2.7.5), (iii) can have at most two non-Gorenstein points [Mor88, Thm. 6.2], this is impossible. So we assume that (X,C) contains exactly two non-Gorenstein points, P and Q. Let n be the index of Q. Clearly, $-K_{X'}\cdot C' = 2$. On the other hand, by Lemma (2.8) $-K_{X'}\cdot C' = -sK_X\cdot C = sd/mn$. Let $r = \gcd(\bar{m}, n)$. Then $d = rs$ and $s = 2n\bar{m}/r$. Since $\bar{m} > 1$, we have $s = 2s_1$, where $s_1 = n\bar{m}/r > 1$. Consider the quotient X'' of $X^\flat$ by $\boldsymbol{\mu}_{s_1} \subset \boldsymbol{\mu}_s$. We get extremal neighborhoods (X'', C_i'') with two non-Gorenstein points: imprimitive of index $\bar{m}s_1$ and primitive of index n. By [Mor88, Th. 6.7, 9.4] this is impossible. Thus the claim is proved. In particular, $X^\flat = X'$.

As above we have $-K_{X'}\cdot C' = 2 = s/\bar{m}$. Hence, $s = 2\bar{m}$. In particular, $m = 2\bar{m}^2 \neq 4$ and P is not of type $cAx/4$. Up to permutation of x_i's we may assume that $\mathrm{wt}\,x_4 \equiv \mathrm{wt}\,x_1x_3 \equiv \mathrm{wt}\,\phi \equiv 0 \bmod m$. Since $-K_X\cdot C = 1/\bar{m}$ and P is the only non-Gorenstein point, $\mathrm{ord}(x_1\cdots x_4/\phi) \equiv -\bar{m}K_X\cdot C \equiv 1 \bmod \bar{m}$ (see [Mor88, Corollary 2.10]). So, $a_2 \equiv 1 \bmod \bar{m}$.

Consider the map $\varphi\colon \mathrm{gr}^0_{C'}\,\omega \to \oplus\,\mathrm{gr}^0_{C_i'}\,\omega$ (see (5.3.3)) and the induced map

$$\Phi\colon \mathrm{gr}^0_{C'}\,\omega = (\mathcal{O}_{C^\sharp}\bar{\omega})^{\boldsymbol{\mu}_{\bar{m}}} \to \bigoplus_{i=1}^{s}\mathrm{gr}^0_{C_i'}\,\omega\otimes\mathbb{C}(P'), \tag{5.3.5}$$

where $\bar{\omega}$ is a semi-invariant generator of $\omega_{X^\sharp}$ at $P^\sharp$. For example we can take

$$\bar{\omega} = \frac{dx_2\wedge dx_3\wedge dx_4}{\partial\phi/\partial x_1}.$$

Since $\bar{m}w_{P'_{(i)}}(0) = \bar{m} - 1$, we see that $\Phi(\nu\bar{\omega}) = 0$ for $\boldsymbol{\mu}_{\bar{m}}$-semi-invariants ν with $\operatorname{ord}\nu \geq \bar{m}$.

(5.3.6) Lemma. *If $C' = f'^{-1}(o')$, then*

$$H^1(\operatorname{gr}^0_{C'}\omega) = H^0(\operatorname{Coker}\varphi) = H^0(\operatorname{Coker}\Phi) = \mathbb{C}.$$

Proof. Since $H^j(\operatorname{gr}^0_{C'_i}\omega) = 0$, we have

$$\begin{aligned} H^0(\operatorname{Coker}\varphi) &\simeq H^1(\operatorname{gr}^0_{C'}\omega) \simeq H^1(\omega_{X'} \otimes \mathcal{O}_{f'^{-1}(o')}) \\ &\simeq (R^1 f'_*\omega_{X'}) \otimes \mathbb{C}(o') \simeq \omega_{Z'} \otimes \mathbb{C}(o') \simeq \mathbb{C}. \end{aligned}$$

(We used the base change theorem and Lemma (4.1).) □

There are two cases.

(5.3.7) Case $a_2 \geq \bar{m}$. Clearly, $a_1 + a_3 \geq \bar{m}$. Thus we may assume that $a_1 \leq a_3 \geq \bar{m}/2$. In this case, Φ factors through

$$\Big(\mathcal{O}_{C^\sharp,P^\sharp}/(x_1^{\bar{m}}, x_1x_3, x_3^2, x_2, x_4)\cdot\bar{\omega}\Big)^{\boldsymbol{\mu}_{\bar{m}}} \simeq \mathbb{C}(P')x_1^{\lambda}\cdot\bar{\omega} \oplus \big(\mathbb{C}(P')x_3\cdot\bar{\omega}\big)^{\boldsymbol{\mu}_{\bar{m}}}$$

for a unique $0 < \lambda < \bar{m}$ such that $\lambda a_1 + \operatorname{wt}\bar{\omega} \equiv 0 \bmod \bar{m}$. Since $\dim\operatorname{Coker}\Phi \leq 1$, by (5.3.5) we have $2\bar{m} = s \leq 2 + 1 = 3$, a contradiction.

(5.3.8) Case $a_2 = 1$. As above, Φ factors through

$$R := \big(\mathcal{O}_{C^\sharp,P^\sharp}/(x_1^{\bar{m}}, x_1x_3, x_3^2, x_2^{\bar{m}}, x_4)\cdot\bar{\omega}\big)^{\boldsymbol{\mu}_{\bar{m}}}.$$

This R is generated by the images of

$$x_1^i x_2^{\bar{m}-1-a_1 i}, \quad x_3^j x_2^{\bar{m}-1-a_3 j}, \qquad 0 \leq i \leq (\bar{m}-1)/a_1,\ 1 \leq j \leq (\bar{m}-1)/a_3.$$

Therefore,

$$2\bar{m} = s \leq \dim R + 1 \leq \frac{\bar{m}-1}{a_1} + 1 + \frac{\bar{m}-1}{a_3} + 1 \leq \frac{\bar{m}-1}{1} + \frac{\bar{m}-1}{1} + 2 = 2\bar{m}.$$

This immediately implies $a_1 = a_3 = 1$. Since $a_1 + a_3 \equiv 0 \bmod \bar{m}$, we have $\bar{m} = 2$, $s = 4$, and $m = 8$. Changing ℓ-characters [Mor88, 2.5] and permuting x_1 and x_3, we may assume that $\operatorname{wt} x_2 \equiv 1 \bmod 8$ and $\operatorname{wt} x_1 \equiv 1$ or $5 \bmod 8$. If $\operatorname{wt} x_1 \equiv 1 \bmod m$, then $\operatorname{ow} C^\sharp$ is generated by $\operatorname{ow} x_1$ and $\operatorname{ow} x_3$. In particular, x_1/x_2 is constant on $C^\sharp$. This means that R is generated by x_1 and x_3. Hence, by (5.3.5) $4 = s \leq \dim R + 1 \leq 3$, a contradiction. Therefore, we have the case (IE$^\vee$).

□

§6. Deformations of $\mathbb{Q}$-conic Bundles

We recall the following

(6.1) Proposition ([Mor88, 1b.8.2]). *Let (X, C) a the $\mathbb{Q}$-conic bundle germ and let $P \in C$. Then every deformation of germs $(X, P) \supset (C, P)$ can be extended to a deformation of (X, C) so that the deformation is trivial outside some small neighborhood of P.*

Proof (*cf.* [KM92, 11.4.2]). Let $P_i \in X$ be singular points. Consider the natural morphism

$$\Psi\colon \operatorname{Def} X \longrightarrow \prod \operatorname{Def}(X, P_i).$$

It is sufficient to show that Ψ is smooth (in particular, surjective). The obstruction to globalize a deformation in $\prod \operatorname{Def}(X, P_i)$ lies in $R^2 f_* T_X$. Since f has only one-dimensional fibers, $R^2 f_* T_X = 0$. □

(6.2) Proposition. *Let $f\colon (X, C \simeq \mathbb{P}^1) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ. Let (X_t, C_t), $t \in \mathfrak{T} \ni 0$ be an one-parameter deformation as in Proposition* (6.1) *and let $\mathfrak{X} \to \mathfrak{T}$ be the corresponding family so that $\mathfrak{X}_0 = X$. There exists a contraction $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Z}$ over $\mathfrak{T} \ni 0$ such that $\mathfrak{Z}_0 = Z$ and for all $t \in \mathfrak{T} \ni 0$, $\mathfrak{f}_t\colon \mathfrak{X}_t \to \mathfrak{Z}_t$ is a $\mathbb{Q}$-conic bundle germ.*

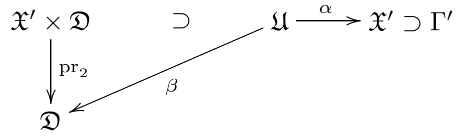
Proof. Consider the base change (2.4.1). Let $\Gamma' = f'^{-1}(o')$ be the scheme fiber (so that $\Gamma'_{\mathrm{red}} = C'$) and let $g^{-1}(P_i) = \{P'_{i,1}, \ldots, P_{i,s_i}\}$. We claim that arbitrary deformation in $\operatorname{Def}(X, P_i)$ determines a $\boldsymbol{\mu}_d$-equivariant deformation in $\prod_j \operatorname{Def}(X', P'_{i,j})$. Indeed, the total space $\mathfrak{Q}$ of a deformation of a terminal singularity (X, P) is $\mathbb{Q}$-Gorenstein (see [Ste88, §6]) and index-one cover of $\mathfrak{Q}$ is the total deformation space of the index-one cover $(X^\sharp, P^\sharp)$ of (X, P). Therefore every deformation of a terminal singularity of index m is induced by some $\boldsymbol{\mu}_m$-equivariant deformation of its index-one cover. This proves our claim. This implies that a deformation in $\operatorname{Def}(X, P_i)$ determines a deformation of X' which must be $\boldsymbol{\mu}_d$-equivariant. Therefore, the cover $X' \to X$ induces a cover $\mathfrak{X}' \to \mathfrak{X}$ so that $\mathfrak{X} = \mathfrak{X}'/\boldsymbol{\mu}_d$.

Since Γ' is a complete intersection in $\mathfrak{X}'$, the conormal sheaf $\mathcal{N}^*_{\Gamma'/\mathfrak{X}'}$ is locally free. We have the exact sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{N}^*_{X'/\mathfrak{X}'}|_{\Gamma'} & \longrightarrow & \mathcal{N}^*_{\Gamma'/\mathfrak{X}'} & \longrightarrow & \mathcal{N}^*_{\Gamma'/X'} & \longrightarrow & 0 \\
 & & \| & & & & \| & & \\
 & & \mathcal{O}_{\Gamma'} & & & & \mathcal{O}_{\Gamma'}^{\oplus 2} & &
\end{array}
\tag{6.2.1}
$$

Since $\operatorname{Ext}^1(\mathcal{O}_{\Gamma'}^{\oplus 2}, \mathcal{O}_{\Gamma'}) = H^1(\mathcal{O}_{\Gamma'}^{\oplus 2}) = 0$, the sequence splits.

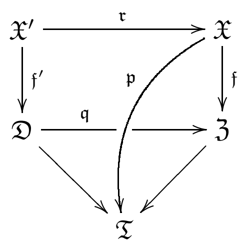
Therefore the germ $\mathfrak{D}$ of the Douady space of $\mathfrak{X}'$ at $[\Gamma']$ is smooth, where $[\Gamma']$ is the point representing Γ'. Let $\mathfrak{U} \to \mathfrak{D}$ be the corresponding universal family. There is a natural embedding $\mathfrak{U} \subset \mathfrak{X}' \times \mathfrak{D}$ such that $\mathfrak{U} \to \mathfrak{D}$ is induced by the projection $\mathfrak{X}' \times \mathfrak{D} \to \mathfrak{D}$. Thus we have the following diagram:



The natural embedding $\Gamma' = \Gamma' \times [\Gamma'] \subset \mathfrak{U}$ induces an isomorphism $\alpha|_{\Gamma'}\colon \Gamma' \to \Gamma'$. Further, $\Gamma' = \Gamma' \times [\Gamma'] \subset \mathfrak{U}$ is a fiber of β, so $\mathcal{N}^*_{\Gamma'/\mathfrak{U}}$ is locally free and isomorphic to $\mathcal{O}_{\Gamma'}^{\oplus 3}$. Hence,

$$d\alpha\colon \mathcal{N}^*_{\Gamma'/\mathfrak{X}'} \to \mathcal{N}^*_{\Gamma'/\mathfrak{U}}.$$

is an isomorphism. Shrinking $\mathfrak{U} \supset \Gamma'$ and $\mathfrak{X}' \supset \Gamma'$ we may assume that α is an isomorphism. This induces a $\boldsymbol{\mu}_d$-equivariant contraction morphism $\Phi = \alpha^{-1}\beta\colon \mathfrak{X}' \to \mathfrak{D}$ such that $\Phi(\Gamma')$ is a point. Put $\mathfrak{Z} := \mathfrak{D}/\boldsymbol{\mu}_d$. Since the morphism $\mathfrak{p}\colon \mathfrak{X} \to \mathfrak{T}$ maps X to 0, by shrinking $\mathfrak{X}$ we may assume that $\mathfrak{p}$ is constant on fibers of $\mathfrak{f}$. Then $\mathfrak{p}$ defines $\mathfrak{Z} \to \mathfrak{T}$. We obtain the following diagram



We have

$$\mathfrak{f}_*\mathcal{O}_{\mathfrak{X}} = \mathfrak{f}_*(\mathfrak{r}_*\mathcal{O}_{\mathfrak{X}'})^{\boldsymbol{\mu}_d} = (\mathfrak{f}_*\mathfrak{r}_*\mathcal{O}_{\mathfrak{X}'})^{\boldsymbol{\mu}_d} = (\mathfrak{q}_*\mathfrak{f}'_*\mathcal{O}_{\mathfrak{X}'})^{\boldsymbol{\mu}_d} = (\mathfrak{q}_*\mathcal{O}_{\mathfrak{D}})^{\boldsymbol{\mu}_d} = \mathcal{O}_{\mathfrak{Z}}.$$

Therefore $\mathfrak{f}$ has connected fibers. Clearly, $\mathfrak{X}$ is $\mathbb{Q}$-Gorenstein and $-K_{\mathfrak{X}}$ is $\mathfrak{f}$-ample. □

(6.2.2) Remark. In general, it is not true that $f_t^{-1}(o_t)_{\mathrm{red}} = C_t$. It is possible that $f_t^{-1}(o_t)$ is reducible and C_t is one of its components. In this case, (X_t, C_t) is an extremal neighborhood by Remark (2.3.2).

§7. The Case where X is not Locally Primitive

In this section we consider the case where X is not locally primitive. We classify configurations of singular points and prove Theorem (1.2) except for one case when a $\mathbb{Q}$-conic bundle germ has two non-Gorenstein points. Some weaker results were obtained in [Pro97b].

(7.1) Notation. Let $f\colon (X,C) \to (Z,o)$ be a $\mathbb{Q}$-conic bundle germ, where C is irreducible. Assume that (X,C) contains an imprimitive point P. Let m, $\bar{m}$ and s be the index, the subindex and the splitting degree of P, respectively.

(7.1.1) First we note that if $\bar{m}=1$, then by Lemma (4.4.2) X' is Gorenstein and we have the case (1.2.4) by (2.9). From now on we assume that $\bar{m}>1$ (in particular, P is not of type (ID$^\vee$)).

(7.1.2) Lemma (cf. [Mor88, Th. 6.1 (ii)]). *In notation of* (7.1)*, assume that* X *has a singular point* $Q \neq P$*. Then* $\operatorname{Sing} X = \{P, Q\}$ *and* Q *is of type* (IA) *or* (III)*. If* Q *is of type* (IA)*, then* $\operatorname{siz}_Q = 1$*.*

Recall [Mor88, 4.5] that a point $P \in X$ is said to be *ordinary* iff (X,P) is either an ordinary double point or a cyclic quotient singularity.

Proof. Assume that (X,C) has two more singular points Q and R. Then by Proposition (5.3.1), P is of type (IA$^\vee$), (IC$^\vee$), or (II$^\vee$). By Proposition (2.6.4) both Q and R are primitive. Replace (X,C) with L-deformation [Mor88, Prop.-Def. 4.7] so that P, Q, R are ordinary (cf. [Mor88, Rem. 4.5.1]). If this new (X,C) is an extremal neighborhood, the assertion follows by [Mor88, Th. 6.1 (ii)]. Thus we may assume that (X,C) is a $\mathbb{Q}$-conic bundle germ. Consider the cover $g\colon (X',C') \to (X,C)$ from (2.4.1). By Lemma (4.4.2) we may assume that $P' \in X'$ is not Gorenstein (otherwise (X',C') is a standard Gorenstein conic bundle germ and then (X,C) is as in (1.2.4), see Proposition (2.9)). Let $C'_i \subset C'$ be any irreducible component. By (2.7.5) (iii), (X',C'_i) is an extremal neighborhood with at least three singular points. Then by [Mor88, (2.3.2)] $\deg \operatorname{gr}^1_{C'_i}\mathcal{O} \le -2$. Hence, $H^0(\operatorname{gr}^1_{C'_i}\mathcal{O}) = 0$ [Mor88, Remark 2.3.4]. This implies that $H^0(\operatorname{gr}^1_{C'}\mathcal{O}) \subset \bigoplus_i H^0(\operatorname{gr}^1_{C'_i}\mathcal{O}) = 0$. Therefore, $H^0(I^{(2)}_{C'}) = H^0(I_{C'})$ and $H^0(\mathcal{O}_{X'}/I^{(2)}_{C'}) = \mathbb{C}$. By Corollary (4.4.5) we have $H^1(\operatorname{gr}^1_{C'}\omega) = 0$. Therefore, $H^1(\operatorname{gr}^1_C\omega) = H^1(\operatorname{gr}^1_{C'}\omega)^{\boldsymbol{\mu}_d} = 0$. With this extra condition, the proof of [Mor88, Th. 6.1 (i)] (resp. [Mor88, Th. 6.1 (ii)]) works if P is of type (IC$^\vee$) (resp. (IA$^\vee$)). Thus $\operatorname{Sing} X = \{P, Q\}$.

Now assume that Q is not of type (III). Consider the splitting cover $g^\flat$ from (2.7.6). Each $(X^\flat, C^\flat_i)$ is an extremal neighborhood having two non-Gorenstein points: $P^\flat$ and $Q_i^\flat$. By [Mor88, Th. 6.7, Th. 9.4] $Q_i^\flat$ is of type (IA) with siz $= 1$ and so is Q. □

(7.2) Proposition. *If $C' = f'^{-1}(o')$ (and $\bar{m} > 1$) or, equivalently if we have a point of type* (IE$^\vee$)*, then f is as in* (1.2.3).

Proof. We note that $C' = f'^{-1}(o')$ (and $\bar{m} > 1$) if and only if we are in the case (IE$^\vee$) by Proposition (5.3.1). In some (non-normalized) coordinate system, $C^\sharp \subset \mathbb{C}^4_{y_1,y_2,y_3,y_4}$ is a complete intersection given by

$$y_1^2 - y_2^2 = y_1y_2 - y_3^2 = y_4 = 0, \tag{7.2.1}$$

where $\mathrm{wt}(y) \equiv (5,1,3,0) \bmod 8$. Thus we may fix an embedding $C^\sharp \subset \mathbb{C}^3_{y_1,y_2,y_3}$ and $X^\sharp \subset \mathbb{C}^4_{y_1,\dots,y_4}$. Let (u,v) be $\boldsymbol{\mu}_8$-semi-invariant coordinates in $Z' = \mathbb{C}^2$. Since $C' = f'^{-1}(o')$, we may regard u, v as $\boldsymbol{\mu}_8$-semi-invariant generators of the ideal of $C^\sharp$ in $X^\sharp$. Therefore the ideal of $C^\sharp$ in $\mathbb{C}^4_{y_1,\dots,y_4}$ has two systems of semi-invariant generators:

$$y_1^2 - y_2^2,\ y_1y_2 - y_3^2,\ y_4 \quad \text{and} \quad u,\ v,\ \phi.$$

Up to permutation of u and v we may assume that

$$\mathrm{wt}\, u \equiv \mathrm{wt}\, y_1^2 \equiv 2, \quad \mathrm{wt}\, v \equiv \mathrm{wt}\, y_1y_2 \equiv -2 \bmod 8,$$

$$\phi = (\mathrm{unit})y_4 + (y_1^2 - y_2^2)\phi_1 + (y_1y_2 - y_3^2)\phi_2$$

because $\mathrm{wt}\, y_1^2,\ \mathrm{wt}\, y_3^2 \not\equiv \mathrm{wt}\, y_4 \bmod 8$. In particular, $X^\sharp$ is smooth and (X,P) is a cyclic quotient of type $\frac{1}{8}(5,1,3)$. Hence (X',P') is a singularity of type $\frac{1}{2}(1,1,1)$ and coordinates y_1, y_2, y_3 can be regarded as sections of $|-K_{X'}|$ on X'. Note that the linear system $|-K_{X'}|$ has a unique base point P' and $|-2K_{X'}|$ is base point free. Let z be a section of $|-2K_{X'}|$. Then y_1, y_2, y_3, z define a map $\vartheta\colon X' \dashrightarrow \mathbb{P} \times \mathbb{C}^2$, where $\mathbb{P} := \mathrm{Proj}\, \mathbb{C}[y_1,y_2,y_3,y_4] = \mathbb{P}(1,1,1,2)$. Since this ϑ is regular on each component of C' and on the tangent space to C' at P', it is an embedding. Therefore, X' can be naturally embedded into $\mathbb{P} \times \mathbb{C}^2$ and by (7.2.1) the defining equations are of the form

$$\begin{aligned} y_1^2 - y_2^2 &= u\psi_1 + v\psi_2, \\ y_1y_2 - y_3^2 &= u\psi_3 + v\psi_4, \end{aligned}$$

where $\psi_i = \psi_i(y_1,y_2,y_3,z,u,v)$. This proves our proposition. □

(7.2.2) Corollary. *In the notation of* (7.1) *the following are equivalent*: (i) $\mathrm{gr}^0_C\,\omega \not\simeq \mathcal{O}(-1)$, (ii) *we are in the case* (1.2.4)*, and* (iii) *P is a point of type* (ID$^\vee$).

Proof. Assume that $\mathrm{gr}^0_C\,\omega \not\simeq \mathcal{O}(-1)$. Then by (4.4.3) we see that $f'^{-1}(o') = C'$. If $\bar{m} > 1$, then we are in the case (1.2.3) by Proposition (7.2), in which case we have $w_P(0) = 1/2$ and hence $\mathrm{gr}^0_C\,\omega = \mathcal{O}(-1)$ by (3.1.1) and Lemma (2.8). Hence $\bar{m} = 1$, then we are in the case (1.2.4) as explained in (7.1.1). Thus (i) implies (ii) and (iii).

If P is of type (ID$^\vee$), then $\bar{m} = 1$ and again by (7.1.1) we are in the case (1.2.4). Then by [Mor88, (2.10)] we have $w_P(0) \ge 1$ and so $\deg \mathrm{gr}^0_C\,\omega < -1$ (see (3.1.1)). □

(7.3) Proposition. *In notation of* (7.1) (X, C) *has no type* (IC$^\vee$) *points.*

Proof. Assume that $P \in (X, C)$ is a type (IC$^\vee$) point. Recall that $s = 2$ and $\bar{m}$ is even ≥ 4 in this case. Then (X, C) has at most one more (primitive) singular point. Applying L-deformation we may assume one of the following:

(7.3.1) P is the only singular point of X, or

(7.3.2) (X, C) has one more ordinary singular point Q of index $n > 1$.

By [Mor88, Th. 6.1 (i)] this new (X, C) is a conic bundle germ. Following the proof of [Mor88, (i) Th. 6.1] we get $H^1(\mathcal{O}_X/I^{(2)}_C \tilde{\otimes} \omega_X) \ne 0$. Hence, $H^1(\mathcal{O}_{X'}/I^{(2)}_{C'} \tilde{\otimes} \omega_{X'}) \ne 0$.

(7.3.3) Let $V := \mathrm{Spec}\,\mathcal{O}_{X'}/I^{(2)}_{C'}$. By Theorem (4.4) we have $f'^{-1}(o') \subset V$. Moreover, as in the proof of (4.4.5) one can see that V is not a local complete intersection at the general point. Therefore, $f'^{-1}(o') \ne V$. Since $V \equiv 3C'$ (as a cycle), we have

$$2 = -K_{X'} \cdot f'^{-1}(o') < -K_{X'} \cdot V = -3K_{X'} \cdot C'.$$

Taking account of $-K_{X'} \cdot C' = d^2/mn$ (see (2.8)) we get $2nm < 3d^2$, where we put $n = 1$ in the case (7.3.1). Recall that $s = 2$. Write $m = s\bar{m} = 2\bar{m}$, $n = rn'$ and $\bar{m} = r\bar{m}'$, where $r = \gcd(\bar{m}, n)$. Then $d = 2r$ (see Corollary (2.7.2) (iii)) and $n'\bar{m}' < 3$. Note that $\bar{m}'$ is the index of P'. If $\bar{m}' = 1$, then X' is Gorenstein by (4.4.2) and (X, P) cannot be of type (IC$^\vee$) by Proposition (2.9). So, $\bar{m}' = 2$, $n' = 1$, and $\bar{m} = 2n$. In particular, $n > 1$ and the case (7.3.1) is impossible.

Note that (X, P) is a cyclic quotient singularity by [Mor88, Lemma 4.4]. Thus we may assume that $X^\sharp = \mathbb{C}^3_{x_1,x_2,x_4}$ and $C^\sharp$ is given by the equations $x_4 = x_2^2 - x_1^{m-2} = 0$. Thus C' near P' is isomorphic to $\{x_4 = x_2^2 - x_1^{m-2} = 0\}/\boldsymbol{\mu}_2(1,1,1)$. Putting $w_1 = x_1^2$, $w_2 = x_2^2$, $w_3 = x_1x_2$ we get that near P' the curve $C' \subset \mathbb{C}^3_{w_1,w_2,w_3}$ can be given by two equations $w_2 = w_1^{\bar{m}-1}$ and

$w_1w_2 = w_3^2$. Eliminating w_2 we obtain $C' := \{w_3^2 = w_1^{2n}\}$. It is easy to see that C' has an ordinary double point at the origin only if $n = 1$. This contradicts $p_a(C') = 0$. □

The following lemma was proved in [Pro97b, §3]. However it was implicitly assumed in the proof that X is $\mathbb{Q}$-factorial. Below is a corrected version.

(7.3.4) Lemma. *In notation of* (7.1) *assume that X is $\mathbb{Q}$-factorial. Then $s = 2^k$. If furthermore X has two non-Gorenstein points, then $s = 2$.*

Proof. Write $d = sr$ and $s = 2^k q$, where q is odd. We will derive a contradiction assuming $q > 1$. Consider the quotient $X^\dagger/Z^\dagger$ of X'/Z' from (2.4.1) by $\boldsymbol{\mu}_{2^k r} \subset \boldsymbol{\mu}_d$:

$$\begin{array}{ccc} X^\dagger & \xrightarrow{g^\dagger} & X \\ {\scriptstyle f^\dagger}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ Z^\dagger & \xrightarrow{h^\dagger} & Z \end{array}$$

where $h^\dagger\colon Z^\dagger \to Z$ is a $\boldsymbol{\mu}_q$-cover. Then $C^\dagger := {g^\dagger}^{-1}(C)$ has q irreducible components because $X^\dagger$ is a $\boldsymbol{\mu}_{2^k}$-quotient of the splitting cover (2.7.5) and therefore, $\rho(X^\dagger/Z^\dagger) = q$ by Corollary (2.3.1). There is a curve $V \subset Z^\dagger$ such that $f^{\dagger-1}(V)$ has exactly two components, say E_1 and $E_{1'}$. For a general point $z \in V$ the preimage ${f^\dagger}^{-1}(z)$ is a reducible conic, so ${f^\dagger}^{-1}(z) = \ell_1 + \ell_2$. Consider the orbit $\{E_1, E_2, \dots, E_t\}$ of E_1 under the action of $\boldsymbol{\mu}_q$. Obviously, every $f^\dagger(E_i)$ is a curve on $Z^\dagger$. Further, $\sum_{i=1}^t E_i \sim_{\mathbb{Q}} g^{\dagger *}M$, where M is a Weil $\mathbb{Q}$-Cartier divisor on X. On the other hand, $\rho(X/Z) = 1$ and M is f-vertical. Hence, $M \sim_{\mathbb{Q}} 0$ and $\sum E_i \sim_{\mathbb{Q}} 0$. We can choose components $\ell_1, \ell_2 \subset {f^\dagger}^{-1}(z)$, $z \in V$ so that $\ell_1 \cdot E_1 < 0$ and $\ell_1 \cdot E_{1'} > 0$. This gives us $\ell_1 \cdot E_i > 0$ for some $E_i \in \{E_1, E_2, \dots, E_t\}$. Then $E_{1'} = E_i$, i.e., there exists $\sigma \in \boldsymbol{\mu}_q$ such that $\sigma(E_1) = E_{1'}$. From the symmetry we get that the orbit $\{E_1, E_2, \dots, E_t\}$ may be divided into pairs of divisors E_j, $E_{j'}$ such that $f^\dagger(E_j) = f^\dagger(E_{j'})$ is a curve. Thus both t and q are even. This proves the first statement.

Now assume that X has two non-Gorenstein points and let $s = 2^k$, $k \geq 2$. Consider the quotient (X'', C'') of (X', C') by $\boldsymbol{\mu}_{2^{k-1}}$. Then the central fiber C'' is reducible and every germ (X'', C_i'') is an extremal neighborhood having two non-Gorenstein points: imprimitive and primitive. By the classification [Mor88, Th. 6.7, 9.3] this is impossible. □

(7.4) Proposition (cf. [Mor88, Th. 6.1 (iii)]). *Notation as in* (7.1). *Assume that (X, C) has one more non-Gorenstein point Q. Then P is of type* (IA$^\vee$), $\mathrm{siz}_P = 1$, *and $w_P(0) \geq 2/3$.*

(7.4.1) Corollary. *In the above notation we have $w_Q(0) < 1/3$. In particular, the index of Q is ≥ 4.*

Proof. (7.4.1) immediately follows from (7.4) and (7.2.2). We assume that $\operatorname{siz}_P \geq 2$ or $w_P(0) < 2/3$ and we will derive a contradiction. Let n be the index of Q. By Lemma (7.1.2) $\operatorname{Sing} X = \{P, Q\}$ and Q is of type (IA) and by Propositions (7.2), (7.3) and Corollary (7.2.2), P is of type ($\text{IA}^\vee$) or ($\text{II}^\vee$). Replacing (X, C) with L-deformation, we may assume that X has only ordinary points (in particular, P is of type ($\text{IA}^\vee$)). If this new (X, C) is an extremal neighborhood, the assertion follows by [Mor88, Th. 6.1 (iii)]. Thus we may assume that (X, C) is again a $\mathbb{Q}$-conic bundle germ.

If $H^1(\mathcal{O}_X/I_C^{(2)} \tilde{\otimes} \omega_X) = 0$, then following the proof of [Mor88, (iii) Th. 6.1] we derive a contradiction. Hence, in notation of (2.4.1) we have $H^1(\mathcal{O}_{X'}/I_{C'}^{(2)} \tilde{\otimes} \omega_{X'}) \neq 0$. Let $V := \operatorname{Spec} \mathcal{O}_{X'}/I_{C'}^{(2)}$. As in (7.3.3) by Theorem (4.4) $f'^{-1}(o') \subset V$, and so

$$2 = -K_{X'} \cdot f'^{-1}(o') < -K_{X'} \cdot V = -3K_{X'} \cdot C'.$$

Taking account of $-K_{X'} \cdot C' = d^2/mn$ (see (2.8)) we obtain

$$2mn < 3d^2.$$

Since X has only ordinary points of index > 1, X is $\mathbb{Q}$-factorial. By Lemma (7.3.4) $s = 2$. Write $m = s\bar{m} = 2\bar{m}$, $n = rn'$ and $\bar{m} = r\bar{m}'$, where $r = \gcd(\bar{m}, n)$. By Corollary (2.7.2) (iii) we have $d = sr = 2r$. Then $n'\bar{m}' < 3$. Since $\bar{m}'$ is the index of P', we may assume that $\bar{m}' > 1$ (otherwise by Lemma (4.4.2) and Proposition (2.9) we have the case (1.2.4)). Therefore, $\bar{m}' = 2$, $n' = 1$, and $\bar{m} = 2n$.

We may assume that $X^\sharp = \mathbb{C}^3_{x_1,x_2,x_3}$ at $P^\sharp$, the curve $C^\sharp$ is given by the equations $x_3 = x_2^{a_1 s} - x_1^{a_2 s} = 0$, and $(X', P') = \mathbb{C}^3_{x_1,x_2,x_3}/\boldsymbol{\mu}_2(1,1,1)$. Putting $w_1 = x_1^2$, $w_2 = x_2^2$, $w_3 = x_1 x_2$ we get that near P' the curve $C' \subset \{x_3 = 0\}/\boldsymbol{\mu}_2 = \mathbb{C}^2_{x_1,x_2}(1,1)/\boldsymbol{\mu}_2$ can be given by two equations: $w_2^{a_1} = w_1^{a_2}$ and $w_1 w_2 = w_3^2$. We claim that $a_1 = a_2 = 1$. Indeed, assume for example that $a_1 > 1$. Since $p_a(C') = 0$, C' has an ordinary double point at the origin. Hence, $a_2 = 1$. Eliminating w_1 we get the following equation for C': $w_2^{a_1+1} = w_3^2$. Again the origin is an ordinary double point only if $a_1 = 1$, a contradiction. Thus, $a_1 = a_2 = 1$ and $w_P(0) = 1 - 1/\bar{m}$ by [Mor88, Th. 4.9]. This gives

$$w_Q(0) = 1 - w_P(0) + K_X \cdot C = 1/\bar{m} - 1/\bar{m} = 0,$$

a contradiction. Hence we have $\operatorname{siz}_P = 1$ and $w_P(0) \geq 2/3$. If P is of type ($\text{II}^\vee$), then $w_P(0) = 1/2$ (see [Mor88, Th. 4.9]). So, P is of type ($\text{IA}^\vee$). □

(7.5) Proposition. *In notation of* (7.1) *assume that* P *is of type* (IA$^\vee$). *Then* P *is of index* 4*, splitting degree* 2 *and subindex* 2*. Moreover,* (X,P) *is a cyclic quotient and*

$$\begin{array}{rcccc} & x_1 & x_2 & x_3 & x_4 \\ \mathrm{wt} & 1 & -1 & -1 & 0 \ \mathrm{mod}\, 4 \\ \mathrm{ord} & 1 & 1 & 1 & 2 \end{array}$$

Proof. By (10.7.2) below P is the only non-Gorenstein point on X. * Since $-K_X \cdot C = 1/\bar{m}$, $w_P(0) = 1 - 1/\bar{m}$. Hence we have $a_2 = 1$ by [Mor88, Th. 4.9.(i)]. The general member $F \in |-K_{(X,P)}|$ has only Du Val singularity of type A_{mk-1}, $k \in \mathbb{Z}_{>0}$. It is easy to see that $F^\sharp$ is given by $x_2 = 0$, so $F \cdot C = 1/\bar{m}$. Hence, $K_X + F$ is a numerically trivial Cartier divisor. Since $\operatorname{Pic} X \simeq H^2(X,\mathbb{Z}) \simeq \mathbb{Z}$, $K_X + F \sim 0$. Thus, the general member $F \in |-K_X|$ does not contain C and has only Du Val singularity of type A_{mk-1}. Consider the double cover $f|_F \colon (F,P) \to (Z,o)$. Diagram (2.4.1) induces the following

$$\begin{array}{ccc} (F',P') & \xrightarrow{g_{F'}} & (F,P) \\ {\scriptstyle f'_{F'}}\downarrow & & {\scriptstyle f_F}\downarrow \\ (Z',o') & \xrightarrow{h} & (Z,o) \end{array}$$

where $F' \in |-K_{X'}|$, $P' = g^{-1}(P)$, $F' \cap C' = \{P'\}$, and $g_{F'}$ is étale outside of P'. Since $Z' \to Z$ is of degree s, s divides mk and (F',P') is of type A_{n-1}, where $n = mk/s = \bar{m}k$. We see $n = 2$ because otherwise we have a contradiction by Lemma (7.5.1) below. Thus $n = 2$ and $\bar{m} = 2$ (recall that $\bar{m} > 1$ by the assumption of (IA$^\vee$) (5.3.1)). In this case, by Corollary (2.8.1) $s = 4$ or 2. If $s = 4$, then $-K_{X'} \cdot C' = s/\bar{m} = 2$ (see Lemma (2.8)). Hence, $C' = f'^{-1}(o')$ and we have the case (1.2.3) by Proposition (7.2). But then P is not of type (IA$^\vee$), a contradiction. Hence $s = 2$ and the rest is easy. □

(7.5.1) Lemma. *Let* (S,Q) *be a Du Val singularity of type* A_{n-1}, $n \geq 3$ *and let* $\pi \colon (S,Q) \to (\mathbb{C}^2,0)$ *be a double cover. Assume that* $\boldsymbol{\mu}_d$ *acts on* (S,Q) *and* $(\mathbb{C}^2,0)$ *freely in codimension one and so that* π *is* $\boldsymbol{\mu}_d$*-equivariant. Then the quotient* $(S,Q)/\boldsymbol{\mu}_d$ *cannot be Du Val of type* A.

Proof. Let $R \subset \mathbb{C}^2$ be the branch divisor of π. Since (S,Q) is of type A, the equation of R must contain a quadratic term. Hence, in some $\boldsymbol{\mu}_d$-semi-invariant coordinates u, v in $\mathbb{C}^2$, the curve R can be given by $u^2 + v^n = 0$. In

*In (10.1) – (10.7.2), no results in (7.5) – (9.4.2) are used when P is imprimitive. Thus the back reference (10.7.2) here does not cause any trouble.

this case, there is a $\boldsymbol{\mu}_d$-equivariant embedding $(S, Q) \hookrightarrow (\mathbb{C}^3_{u,v,w}, 0)$ such that S is given by $w^2 = u^2 + v^n$ and w is a semi-invariant. Assume that $S/\boldsymbol{\mu}_d$ is Du Val. Since $K_{S/\boldsymbol{\mu}_d}$ is Cartier, we have $\operatorname{wt}(uvw) = \operatorname{wt} w^2 = \operatorname{wt} u^2 = \operatorname{wt} v^n$. This implies $\operatorname{wt} w = \operatorname{wt}(uv)$ and $\operatorname{wt} v^2 = 0$. Since the action of $\boldsymbol{\mu}_d$ on $\mathbb{C}^2$ is free in codimension one, $d = 2$ and n is even. So, $n = 2l$ for some $l \geq 2$. Hence, $S/\boldsymbol{\mu}_d$ is a quotient of $\{u^2 + v^{2l} = w^2\}$ by $\boldsymbol{\mu}_2(1, 1, 0)$. But if $l \geq 2$, this quotient is not of type A, a contradiction. □

(7.6) Proposition. *In notation of* (7.1) *assume that P is of type* (IA$^\vee$) (*resp.* (II$^\vee$)). *Then $f\colon (X, C) \to (Z, o)$ is as in* (1.2.5) (*resp.* (1.2.6)).

Proof. In the case (II$^\vee$) X has no other non-Gorenstein points by (7.4). Then applying (2.4.1) we will see that X/Z is the quotient of an index-two $\mathbb{Q}$-conic bundle $f'\colon (X', C') \to (Z', o')$ by $\boldsymbol{\mu}_2$. The components of the central curve C' are permuted, so C' has two components of the same multiplicity. Hence X'/Z' is in the case (12.1.6). The action on X' is described in (12.1.12). □

(7.6.1) Remark. We have treated all the types (IA$^\vee$)–(II$^\vee$) of imprimitive points. Finally we note that the existence of a good anicanonical divisor (Proposition (1.3.7)) in the imprimitive cases (1.2.3)–(1.2.6) can be shown exactly as in [Mor88, 7.3] (see also [Pro97b]).

(7.7) Examples. Below we propose explicit examples of $\mathbb{Q}$-conic bundles as in (1.2.3), (1.2.5) and (1.2.6).

(7.7.1) Example. Under the notation of (1.2.3) consider the subvariety X' defined by

$$\begin{cases} y_1^2 - y_2^2 = uy_4 \\ y_1 y_2 - y_3^2 = vy_4. \end{cases}$$

The projection $f'\colon X' \to \mathbb{C}^2$ is a $\mathbb{Q}$-conic bundle of index 2 (see (12.1.3)). Then $X'/\boldsymbol{\mu}_4 \to \mathbb{C}^2/\boldsymbol{\mu}_4$ is a $\mathbb{Q}$-conic bundle with an imprimitive point as in (1.2.3). The singular point is unique and is of type (IE$^\vee$).

(7.7.2) Example. Let $X' \subset \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2$ be the subvariety given by the equations

$$\begin{cases} y_1^2 - y_2^2 = uy_4 \\ y_3^2 = vy_4 + u^2 y_2^2 + \lambda u y_1 y_2, \quad \lambda \in \mathbb{C}. \end{cases}$$

Consider the action of $\boldsymbol{\mu}_2$ on X':

$$y_1 \mapsto y_1, \quad y_2 \mapsto -y_2, \quad y_3 \mapsto y_3, \quad y_4 \mapsto -y_4, \quad u \mapsto -u, \quad v \mapsto -v.$$

Then $X := X'/\boldsymbol{\mu}_2 \to \mathbb{C}^2/\boldsymbol{\mu}_2$ is a $\mathbb{Q}$-conic bundle with an imprimitive point as in (1.2.5). It has a singularity of type (IA$^\vee$) which is the cyclic quotient $\frac{1}{4}(1,-1,1)$. If $\lambda = 0$, then X also has a (Gorenstein) ordinary double point.

(7.7.3) Example. Let $X' \subset \mathbb{P}(1,1,1,2) \times \mathbb{C}^2$ be the subvariety given by the equations

$$\begin{cases} y_1^2 - y_2^2 = u^3 y_4 + v y_4 \\ y_3^2 \qquad\quad = v y_4 + u^2 y_2^2 + \lambda u y_1 y_2, \quad \lambda \in \mathbb{C}. \end{cases}$$

Define the action of $\boldsymbol{\mu}_2$ on X' as in (7.7.2). Then $X := X'/\boldsymbol{\mu}_2 \to \mathbb{C}^2/\boldsymbol{\mu}_2$ is a $\mathbb{Q}$-conic bundle with an imprimitive point as in (1.2.6). The non-Gorenstein point is of type (II$^\vee$). It is the only singular point if $\lambda \neq 0$. If $\lambda = 0$, then X has one more singular point which is of type (III).

In case of a $\mathbb{Q}$-conic bundle with an imprimitive point, the proofs of Theorems (1.2) and (1.2.7) are completed here modulo the arguments in (10.1) – (10.7.2).

§8. The Case where X is Locally Primitive. Possible Singularities.

In this section we consider locally primitive $\mathbb{Q}$-conic bundles. The main result is summarized in Theorem (8.6).

(8.1) Notation. Let $f: (X, C \simeq \mathbb{P}^1) \to (Z, o)$ be a locally primitive $\mathbb{Q}$-conic bundle germ. Let $P \in (X, C)$ be a (primitive) non-Gorenstein singular point and let $m \geq 2$ be its index. We may assume that $\operatorname{gr}_C^0 \omega \simeq \mathcal{O}_C(-1)$ (see (4.4.3)).

(8.1.1) Lemma. *There are at most* 3 *singular points of X on C.*

Proof. By $\deg \operatorname{gr}_C^0 \omega < 0$ and (3.1.5), we have $\sum_Q i_Q(1) \leq 3$, and the proof of [Mor88, 6.2(i)] works. □

(8.1.2) Lemma. *If P is a point of type* (IB) *or* (IC)*, then the base (Z, o) is smooth. In the case* (IIB)*, (Z, o) is either smooth or Du Val of type A_1.*

Proof. Assume that (Z, o) is singular and consider the base change as in (2.4.1). Let $P' \in g^{-1}(P)$ and let m' be the index of (X', P'). We note that $(X^\sharp, P^\sharp)$ is also the index-one cover of (X', P'). Clearly, m' divides m.

We claim that $m' < m$. Suppose $m' = m$. Then the Galois cover $X' \to X$ is étale at P' and X' has at least two points of the same index m on C'. This means that Z' is singular by (2.7.2), a contradiction. Thus $m' < m$ as claimed.

Since C' is smooth, $m' \in \operatorname{ord} C^\sharp$. This is not possible in cases (IB) and (IC) (because modulo renumbering of a_i's $a_4 = m$, $a_1 + a_2 \geq m$, $\gcd(a_i, m') = 1$, and $a_i > 1$ for $i = 1, 2, 3$). In the case (IIB) the only possibility is $m' = 2$ and then the topological index of f is 2. □

(8.2) Proposition. *Assume that $P \in (X, C)$ is a type* (IC) *point. Let m be its index. Then (X, C) has no other singular points. Moreover, $i_P(1) = 2$, $w_P(0) = 1 - 1/m$, $a_1 = 2$, and $a_4 = m + 1$.*

Proof. Assume that (X, C) has one more singular point Q of index $n \geq 1$. By Lemma (8.1.2) the base surface Z is smooth. Since $i_P(1) \geq 2$ [Mor88, Prop. 5.5], we have $i_P(1) = 2$ and $i_Q(1) = 1$. We may assume that Q is ordinary of type (IA) or (III) by L-deformation. Further, by Corollary (4.4.6) $w^*_P(1) + w^*_Q(1) \leq 1$.

If $w_P(0) \neq 1 - 1/m$, all the arguments of [Mor88, 6.5.2] can be applied and we derive a contradiction. Assume that $w_P(0) = 1 - 1/m$. We follow the arguments of [Mor88, 6.5.3]. Since P is of type (IC), $m \geq 5$, so $w_Q(0) = 1 - w_P(0) - 1/nm = 1/m - 1/nm < 1/5$. Let Q be of type (IA) (resp. (III)). Then, for $1 \leq d \leq 4$, by [Mor88, 5.1] (resp. [Mor88, 4.9]), one has $w^*_Q(d) = d(d+1)/2$ (resp. $w^*_Q(d) = \lfloor (d+1)^2/4 \rfloor$) for $d \leq 4$. On the other hand, by [Mor88, 5.5] $w^*_P(1) = 1 - \delta_{m,5}$. Therefore, $m = 5$. Further, by [Mor88, 5.5 (v)] $w^*_P(2) = 0$ and $w^*_P(3) = 4$. Thus,

$$\textstyle\sum_{d=1}^{3}(1 + d + \deg \operatorname{gr}^d_C \omega) = \sum_{d=1}^{3} \frac{d(d+1)}{2} - 4 - \sum_{d=1}^{3} \frac{d(d+1)}{2} = -4 < 0$$
$$\left(\text{resp.} \quad \textstyle\sum_{d=1}^{3} \frac{d(d+1)}{2} - 4 - \sum_{d=1}^{3} \left\lfloor \frac{(d+1)^2}{4} \right\rfloor = -1 < 0\right).$$

Therefore, $H^1(\mathcal{O}_X / I^{(4)}_C \tilde{\otimes} \omega_X) \neq 0$. Let $V := \operatorname{Spec} \mathcal{O}_X / I^{(4)}_C$. Then by Theorem (4.4) $V \supset f^{-1}(o)$. Moreover, $V \neq f^{-1}(o)$ because V is not a local complete intersection inside X. Hence,

$$2 = -K_X \cdot f^{-1}(o) < -K_X \cdot V \leq -10K_X \cdot C = 10/nm.$$

This gives $m \leq nm < 5$, a contradiction. Hence P is the only singular point.

By (2.8) and (3.1.1) $w_P(0) = 1 - 1/m$. Hence, $a_4 \equiv 1 \bmod m$ ([Mor88, Th. 4.9]). Now assume that $i_P(1) = 3$. By [Mor88, Prop. 5.5] $w^*_P(1) \geq 2$. This contradicts (4.4.6). Thus, $a_1 = i_P(1) = 2$ ([Mor88, Prop. 5.5]) and $m + 1 \in \operatorname{ord} C^\sharp$. By normalizedness $a_4 = m + 1$ (see (5.1)). □

(8.3) Proposition. (X, C) *has no type* (IB) *points.*

Proof. By [Mor88, Prop. 4.7] we can deform (X, C) to $(X_\lambda, C_\lambda \simeq \mathbb{P}^1)$, where X_λ has at least two non-Gorenstein points of the same index m. If (X_λ, C_λ) is an extremal neighborhood, the assertion follows as in the proof of [Mor88, Th. 6.3]. Otherwise (X_λ, C_λ) is a $\mathbb{Q}$-conic bundle germ over a singular base (Z_λ, o_λ) by (2.7.2) (iii). But Z is smooth by Lemma (8.1.2) and Z_λ is a deformation of Z, a contradiction. □

(8.4) Proposition. *If (X, C) has a point P of type* (IIB)*, then P is the only singular point and the base surface is smooth.*

Proof. Assume that (X, C) has a singular point $Q \neq P$ of index $m \geq 1$. By [Mor88, Prop. 4.7] we can deform (X, C) to $(X_\lambda, C_\lambda \simeq \mathbb{P}^1)$, where X_λ has three singular points P_λ, P'_λ and Q_λ of indices 2, 4 and m. If (X_λ, C_λ) is an extremal neighborhood, the assertion follows by [Mor88, Th. 6.2]. Assume that (X_λ, C_λ) is a $\mathbb{Q}$-conic bundle germ over (Z_λ, o_λ). Since the indices of P_λ and P'_λ are not coprime, the base surface (Z_λ, o_λ) is singular by (2.7.2) (iii). So is the base surface (Z, o) of (X, C). By Lemma (8.1.2) (Z, o) is of type A_1. This implies that m is even and $\operatorname{Cl}^{\mathrm{sc}} X \simeq \mathbb{Z} \oplus \mathbb{Z}_2$ (see Corollary (2.7.1)). Then (X_λ, C_λ) contains three non-Gorenstein points of even indices. But in this case, the map ς in (2.7.3) cannot be surjective, a contradiction. □

(8.5) Proposition (cf. [Mor88, Th. 6.6]). *Let $P \in X$ be a type* (IA) *point of index m. Then $\operatorname{siz}_P = 1$. If moreover P is the only non-Gorenstein point on X, then (Z, o) is smooth, $w_P(0) = 1 - 1/m$ and $a_2 = 1$.*

Proof. Assume that $\operatorname{siz}_P \geq 2$ and let m be the index of P.

(8.5.1) First we consider the case when (Z, o) is smooth. We claim that P is the only singular point of X. Let $Q \in X$ be a singular point of index $n \geq 1$. To derive a contradiction we note that the proof of [Mor88, Th. 6.6] works whenever

$$H^1(\omega_X / F^2\omega_X) = H^1(\omega_X / F^3\omega_X) = 0.$$

Assume that one of the above vanishings does not hold.

Let $V := \operatorname{Spec} \mathcal{O}_X / I_C^{(j)}$, $j = 2$ or 3. By Theorem (4.4) (cf. (7.3.3)) we have $f^{-1}(o) \subset V$. Hence,

$$2 = -K_X \cdot f^{-1}(o) < -K_X \cdot V = -6K_X \cdot C.$$

Taking account of $-K_X \cdot C = 1/mn$ (see (2.8)) we obtain $mn < 3$. So, $m = 2$ and $n = 1$. On the other hand, a point of type (IA) and index two has

$\operatorname{ord} x = (1, 1, 1, 2)$. Such a point is of size 1, a contradiction. Thus P is the only singular point of X. By (2.8) and (3.1.1) $w_P(0) = 1 - 1/m$. Therefore, $a_2 \equiv 1 \bmod m$. On the other hand, $a_2 < m$ by definition of (IA) point. Hence, $a_2 = 1$, $m - a_1 a_2 = a_3 \in \operatorname{ord} C^\sharp$, and $\operatorname{siz}_P = 1$, a contradiction.

(8.5.2) Now we consider the case when (Z, o) is singular. So the topological index of f is $d > 1$. Put $m' := m/d$. By definition of size we have

(8.5.3) $$2 \le U_P(a_1 a_2) := \min\{k \mid km'd - a_1 a_2 \in \operatorname{ord} C^\sharp\}.$$

Consider the base change (2.4.1). Note that $P' = g^{-1}(P)$ is also a point of type (IA) of index m' having the same index-one cover as that of P. At P' we have

$$U_{P'}(a_1 a_2) = \min\{l \mid lm' - a_1 a_2 \in \operatorname{ord} C^\sharp\}.$$

Write $l := U_{P'}(a_1 a_2)$ as $l = qd - r$, where $0 \le r < d$. Then

$$qm'd - a_1 a_2 = lm' - a_1 a_2 + rm' \in \operatorname{ord} C^\sharp.$$

(We used the fact that $m' \in \mathbb{Z}_{>0} a_1 + \mathbb{Z}_{>0} a_2$, see (5.2.1).) It is easy to see now from (8.5.3) that $U_{P'}(a_1 a_2) = l > qd - d \ge d(U_P(a_1 a_2) - 1) \ge 2$. This contradicts the case (8.5.1) above.

(8.5.4) Finally assume that P is the only non-Gorenstein point on X. Then (Z, o) is smooth by Corollary (2.7.4). Hence by (2.8) and (3.1.1) we have $-K_X \cdot C = 1/m$ and $w_P(0) = 1 - 1/m$. Thus $a_2 = 1$, see [Mor88, Th. 4.9].

□

Summarizing the results of this section we obtain

(8.6) Theorem. *Let $f\colon (X, C \simeq \mathbb{P}^1) \to (Z, o)$ be a locally primitive $\mathbb{Q}$-conic bundle germ. Assume that X is not Gorenstein. Then the configuration of singular points is one of the following:*

(i) *type* (IC) *point P of size* 1 *and index m with $i_P(1) = 2$, $w_P(0) = 1 - 1/m$, $a_1 = 2$, and $a_4 = m + 1$;*

(ii) *type* (IIB) *point;*

(iii) *type* (IA) *point P of size* 1 *and index m with $w_P(0) = 1 - 1/m$ and $a_2 = 1$, and possibly at most two more type* (III) *points;*

(iv) *type* (IIA) *point P, and possibly at most two more type* (III) *points;*

(v) *two non-Gorenstein points which are of types* (IA) *or* (IIA), *and possibly at most one more type* (III) *point*;

(vi) *three non-Gorenstein singular points and no other singularities* (*cf.* (8.1.1)).

(8.6.1) Remark. The existence of a good member of $|-K_X|$ or $|-2K_X|$ in the cases (i)–(iv) can be shown as in [Mor88, 7.3]. The cases (v) and (vi) will be studied in the following sections.

§9. The Case of Three Singular Points

In this section we consider $\mathbb{Q}$-conic bundles with exactly three singular points (cf. (8.1.1)). The main result is the following

(9.1) Theorem (cf. [Mor88, Th. 6.2]). *Let* $(X, C \simeq \mathbb{P}^1)$ *be a* $\mathbb{Q}$*-conic bundle germ with three singular points. Up to permutations the configuration of singular points is one of the following*:

(i) (IA), (III), (III) (*cf.* (8.6) (iii));

(ii) (IA), (IA), (III). *In this case, the indices are* 2, *odd* ≥ 3, *and* 1.

In both cases (Z, o) *is smooth.*

(9.2) Notation. To the end of this section we assume that $f\colon (X, C \simeq \mathbb{P}^1) \to (Z, o)$ is a $\mathbb{Q}$-conic bundle germ with three singular points P, Q, and R. Let k, m, n be the indices of P, Q, and R, respectively.

(9.2.1) By (3.1.5) $i_P(1) = i_Q(1) = i_R(1) = 1$ and by Lemma (7.1.2) all these points are primitive. By Propositions (8.2), (8.3), and (8.4) P, Q, R are of types (IA), (IIA), or (III). We may assume that $\operatorname{gr}^0_C \omega \simeq \mathcal{O}_C(-1)$ (see (4.4.3)).

(9.2.2) Lemma. *A* $\mathbb{Q}$*-conic bundle germ* $(X, C \simeq \mathbb{P}^1)$ *cannot have three Gorenstein singular points.*

Proof. Indeed, assume that (X, C) has three Gorenstein singular points P_1, P_2, P_3. In this case, (Z, o) is smooth and (X, C) has no other singular points. Applying L-deformation we may assume that P_i are ordinary. Then by [Mor88, Th. 4.9] $w^*_{P_i}(1) = 1$. This contradicts Corollary (4.4.6). □

(9.2.3) Lemma (cf. [Mor88, 0.4.13.3, Th. 6.2]). *A* $\mathbb{Q}$*-conic bundle germ* $(X, C \simeq \mathbb{P}^1)$ *has at most two non-Gorenstein points.*

Proof (*following* [Mor88, 0.4.13.3]). Assume that P, Q, $R \in X$ are singular points of indices k, m and $n > 1$. By (7.1.1) we may assume that the subindex > 1 for imprimitive points and hence that (X, C) is locally primitive (cf. (5.3.4)). By L-deformation at P, Q and R, and by [Mor88, Th. (6.2)] we may assume that P, Q and R are cyclic quotient singularities. Using Van Kampen's theorem it is easy to compute the fundamental group of $X \setminus \{P, Q, R\}$:

$$\pi_1(X \setminus \{P, Q, R\}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle / \{\sigma_1^k = \sigma_2^m = \sigma_3^n = \sigma_1\sigma_2\sigma_3 = 1\}.$$

The target group has a finite quotient group G in which the images of σ_1, σ_2, σ_3 are exactly of order k, m and n, respectively (see, e.g., [Feu71]). The above quotient defines a finite Galois cover $g\colon (X', C') \to (X, C)$. By taking Stein factorization we obtain a $\mathbb{Q}$-conic bundle $f'\colon (X', C') \to (Z', o')$ with irreducible central fiber C'. By Corollary (2.7.1) G is cyclic. This contradicts Corollary (2.7.4). □

(9.2.4) Remark. One can check also that the arguments of [Mor88, (6.2.2)] work in this case without any changes.

(9.3) Proposition. *In notation* (9.2) (X, C) *cannot have three singular points of types* (IIA)*,* (III)*, and* (III).

Proof. Assume that X contains a type (IIA) point P and two type (III) points Q and R. The base (Z, o) is smooth by Corollary (2.7.4). Applying L-deformations at Q and R and L' deformation at P (see [Mor88, 4.12.2]) we may assume that Q, R are ordinary and $(X, P) \simeq \{y_1y_2 + y_3^2 + y_4^3 = 0\}/\boldsymbol{\mu}_4(1, 1, 3, 2)$, where $C^\sharp$ is the y_1-axis. This new (X, C) is again a $\mathbb{Q}$-conic bundle germ by [Mor88, 6.2].

We claim that $H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/I_C^{(3)}) = 0$. Indeed, otherwise we can apply Theorem (4.4) to $V := \operatorname{Spec}_X \mathcal{O}_X/I_C^{(3)}$ (cf. (7.3.3)):

$$2 = -K_X \cdot f^{-1}(o) < -K_X \cdot V = -6K_X \cdot C.$$

Taking account of $-K_X \cdot C = 1/4$ (see (2.8)) we obtain a contradiction. Therefore, $H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/I_C^{(3)}) = 0$. This implies

$$\deg \operatorname{gr}_C^1 \omega + 2 + \deg \operatorname{gr}_C^2 \omega + 3 \geq 0 \qquad (9.3.1)$$

(see (3.1.9)).

By [Mor88, 4.9] $w_Q^*(1) = w_R^*(1) = 1$ and $w_Q^*(2) = w_R^*(2) = 2$. By Lemma (9.3.2) below and using (3.1.8) we obtain

$$\deg \operatorname{gr}_C^1 \omega = -2, \qquad \deg \operatorname{gr}_C^2 \omega = -4.$$

This contradicts (9.3.1). □

(9.3.2) Lemma. *Let (X, P) be a $cAx/4$-singularity of the form $\{y_1y_2 + y_3^2 + y_4^3 = 0\}/\boldsymbol{\mu}_4(1, 1, 3, 2)$ and let $C = (y_1\text{-axis})/\boldsymbol{\mu}_4$. Then*

$$i_P(1) = 1,\ w_P(1) = 2,\ w_P^*(1) = -1,\ i_P(2) = 1,\ w_P(2) = 3,\ w_P^*(2) = 0.$$

Proof. Let $I_{C^\sharp}$ be the ideal of $C^\sharp$ in $X^\sharp$. Since $I_{C^\sharp} = (y_2, y_3, y_4)$, we have $I_{C^\sharp}^{(2)} = (y_2) + I_{C^\sharp}^2$ and $I_{C^\sharp}^{(3)} = y_2 I_{C^\sharp} + I_{C^\sharp}^3$. Let $\bar{\omega}$ be a semi-invariant generator of $\omega_{X^\sharp}$. For example we can take

$$\bar{\omega} = \frac{dy_1 \wedge dy_3 \wedge dy_4}{y_1}.$$

Obviously, $\operatorname{wt} \bar{\omega} \equiv 1 \bmod 4$. By definitions of $\operatorname{gr}_C^i \mathcal{O}$ and $\operatorname{gr}_C^i \omega$ we get

$$\operatorname{gr}_C^1 \mathcal{O} = y_3y_1 \cdot \mathcal{O}_C \oplus y_4y_1^2 \cdot \mathcal{O}_C,$$

$$S^2 \operatorname{gr}_C^1 \mathcal{O} = y_3^2y_1^2 \cdot \mathcal{O}_C \oplus y_3y_4y_1^3 \cdot \mathcal{O}_C \oplus y_4^2y_1^4 \cdot \mathcal{O}_C,$$

$$\operatorname{gr}_C^2 \mathcal{O} = y_2y_1^3 \cdot \mathcal{O}_C \oplus y_3y_4y_1^3 \cdot \mathcal{O}_C \oplus y_4^2 \cdot \mathcal{O}_C,$$

$$\operatorname{gr}_C^0 \omega = y_1^3\bar{\omega} \cdot \mathcal{O}_C,$$

$$\operatorname{gr}_C^2 \omega = y_2y_1^2\bar{\omega} \cdot \mathcal{O}_C \oplus y_3y_4y_1^2\bar{\omega} \cdot \mathcal{O}_C \oplus y_4^2y_1^3\bar{\omega} \cdot \mathcal{O}_C,$$

where we note that $y_3^2 = -y_1y_2$ in $\operatorname{gr}_C^2 \mathcal{O}$. By definitions of i_P and w_P (see (3.1)), $i_P(2) = \operatorname{len} \operatorname{Coker} \alpha_2 = 1$ and $w_P(2) = \operatorname{len} \operatorname{Coker} \beta_2 = 3$. Hence, $w_P^*(2) = 0$. Computations for $i_P(1)$, $w_P(1)$ and $w_P^*(1)$ are similar (for $i_P(1)$, see [Mor88, 2.16]). □

(9.4) Proposition. *Let $(X, C \simeq \mathbb{P}^1)$ be a $\mathbb{Q}$-conic bundle germ having two non-Gorenstein points P and Q and one Gorenstein singular point R. Then the indices of non-Gorenstein points are* 2 *and odd* ≥ 3*. In particular, both P and Q are of type* (IA) (*cf.* (9.2.1)).

Proof. We use notation of (9.2). Assume $k \geq 2$, $m \geq 2$, $n = 1$ by hypothesis. Up to permutation we also may assume that $k \leq m$. Apply L-deformation so that P, Q, R become ordinary. In particular, P and Q are of type (IA) points. If this new (X, C) is an extremal neighborhood, the fact follows by [Mor88, Th. 6.2]. Thus we may assume that (X, C) is a $\mathbb{Q}$-conic bundle germ. By (2.8) and (2.7.2) (iii) we have $-K_X \cdot C = d/km$ and

$-K_{X'}\cdot C' = d^2/km$, where $d = \gcd(k, m)$ and (X', C') is as in (2.4.1). If $k = m$, then X' is Gorenstein and X cannot have three singular points by (2.9). Thus, $d \le k < m$. If $H^1(\omega_{X'} \tilde{\otimes} \mathcal{O}_{X'}/I_{C'}^{(2)}) \ne 0$, then as in (7.3.3) by Theorem (4.4) we have $f'^{-1}(o') \subset \operatorname{Spec} \mathcal{O}_{X'}/I_{C'}^{(2)}$, and so

$$2 = -K_{X'} \cdot f'^{-1}(o') < -3K_{X'} \cdot C'.$$

We get $3d^2 > 2km$ and $d = k = m$, a contradiction. Therefore,

$$H^1(\omega_{X'} \tilde{\otimes} \mathcal{O}_{X'}/I_{C'}^{(2)}) = 0, \qquad H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/I_C^{(2)}) = 0. \tag{9.4.1}$$

If $H^1(\operatorname{gr}_C^2 \omega) = 0$, the arguments of [Mor88, 6.2.3] apply and we are done. So we assume $H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/I_C^{(3)}) \ne 0$ and

$$H^1(\omega_{X'} \tilde{\otimes} \mathcal{O}_{X'}/I_{C'}^{(3)}) \ne 0.$$

Again apply Theorem (4.4) to (X', C') with $I_{C'}^{(3)}$. We obtain $4 \le km < 3d^2$. Thus $d > 1$ and $X \ne X'$. Note that in diagram (2.4.1) the preimage $g^{-1}(R)$ consists of d Gorenstein points. By Lemma (9.2.2) $d = 2$. Hence, $k = 2$ and $m = 4$. Clearly, $w_P(0) = 1/2$. By Lemma (2.8) and (3.1.1) $w_Q(0) = 1 - 1/2 - 1/4 = 1/4$. Therefore, near Q we have $\operatorname{ord} x_2 = 3$, so $\operatorname{ord} x = (1, 3, 3)$. Further, by [Mor88, 5.1 (ii), 4.9 (ii), 5.3], $w_Q^*(2) = 3$, $w_R^*(2) = 2$, and $w_P^*(2) = 0$. By (3.1.8) $\deg \operatorname{gr}_C^2 \omega = -5$. In particular, $h^1(\operatorname{gr}_C^2 \omega) \ge 2$ and $h^1(\operatorname{gr}_{C'}^2 \omega') \ge 2$.

Now we claim $H^0(X', \operatorname{gr}_{C'}^1 \omega') = 0$. Note that (X', C') has three singular points: Q' of index 2 and two (III) points R', R''. By [Mor88, 4.9, 5.3] we have $w_{P'}^*(1) = -1$ and $w_{R'}^*(1) = w_{R''}^*(1) = 1$. Thus, $\deg \operatorname{gr}_{C'}^1 \omega' = -2$ (see (3.1.8)). Since by (9.4.1) $H^1(\operatorname{gr}_{C'}^1 \omega') = 0$, we have $\operatorname{gr}_{C'}^1 \omega' \simeq \mathcal{O}_{C'}(-1) \oplus \mathcal{O}_{C'}(-1)$ and $H^0(\operatorname{gr}_{C'}^1 \omega') = 0$.

Finally we apply Proposition (9.4.2) below for (X', C') with $a = 2$ and derive a contradiction. □

(9.4.2) Proposition. *Let (X, C) be a $\mathbb{Q}$-conic bundle germ with smooth (Z, o) (C may be reducible). Assume that there exists a positive integer a such that $H^j(X, \operatorname{gr}_C^i \omega) = 0$ for all j and all $i < a$ and such that $H^1(X, \operatorname{gr}_C^a \omega) \ne 0$. Then*

$$H^j(X, \omega_X \tilde{\otimes} I_C^{(i)}) = \begin{cases} 0 & (j = 0) \\ \omega_Z & (j = 1,\ i \le a), \end{cases} \tag{9.4.3}$$

and $H^1(X, \operatorname{gr}_C^a \omega) \simeq \mathbb{C}$.

Proof. We note that the first assertion follows from $H^0(X, \omega_X) = 0$ when $j = 0$ and from Lemma (4.1) when $j = 1$ and $i = 0$. Consider the natural exact sequence

$$0 \to \omega_X \tilde{\otimes} I_C^{(i+1)} \to \omega_X \tilde{\otimes} I_C^{(i)} \to \mathrm{gr}_C^i \omega \to 0.$$

If (9.4.3) is proved for an i ($< a$), we have $H^1(X, \omega_X \tilde{\otimes} I_C^{(i+1)}) \simeq H^1(X, \omega_X \tilde{\otimes} I_C^{(i)})$, which proves the assertion for $i+1$. If we set $i = a$, we have a surjection $\mathcal{O}_Z \simeq \omega_Z \to H^1(X, \mathrm{gr}_C^a \omega)$ which kills $\mathfrak{m}_{Z,o}$. Thus $H^1(X, \mathrm{gr}_C^a \omega) \simeq \mathbb{C}$. □

Proof of Theorem (9.1). By Lemmas (9.2.2) and (9.2.3) X has one or two non-Gorenstein points and by (9.2.1) these points are of types (IA), (IIA), or (III). The case (IIA)+(III)+(III) is disproved in Proposition (9.3) and the cases (IA)+(IIA)+(III) and (IIA)+(IIA)+(III) are disproved in Proposition (9.4). □

Finally we note that the existence of a good member of $|-K_X|$ or $|-2K_X|$ in the cases (i) and (ii) of Theorem (9.1) can be shown as in [Mor88, 7.3].

§10. Two Non-Gorenstein Points Case: General (Bi)Elephants

In this section we consider ℚ-conic bundles with two non-Gorenstein points and no other singularities. The main result of this section is Theorem (10.10).

(10.1) Notation. Let $(X, C \simeq \mathbb{P}^1)$ be a ℚ-conic bundle germ having two singular points P, P' of indices m, $m' \geq 2$. We assume that (X, C) is not toroidal because in the toroidal case the existence of a good divisor in $|-K_X|$ is an easy exercise (see (i) of (10.10)). Since (X, C) has at most one imprimitive point, we will assume that P' is primitive. Let s and $\bar{m}$ be the splitting degree and the subindex of P. Recall that $m = s\bar{m}$, that $m \geq 4$ by (7.1.1) if P is imprimitive, and that $s = 1$ if P is primitive. Let I_C be the sheaf of ideals defining C in $\mathcal{O}_X$. Let $\pi^\sharp \colon (X^\sharp, P^\sharp) \to (X, P)$ (resp. $\pi^\flat \colon (X^\flat, P'^\flat) \to (X, P')$) be the index-one cover and $C^\sharp = \pi^{\sharp -1}(C)_{\mathrm{red}}$ (resp. $C^\flat = \pi^{\flat -1}(C)_{\mathrm{red}}$). Let $I_C^\sharp$ (resp. $I_C^\flat$) be the canonical lifting of I_C at P (resp. P'). Take normalized ℓ-coordinates $(x_1, \ldots, x_4)$ (resp. $(x'_1, \ldots, x'_4)$) at P (resp. P') such that $a_i = \mathrm{ord}\, x_i$ (resp. $a'_i = \mathrm{ord}\, x'_i$).

By (4.4.3) and (7.2.2) we have $\mathrm{gr}_C^0 \omega \simeq \mathcal{O}_C(-1)$. We note that if $m = 2$ then P is primitive as seen above and we can reduce $m = 2$ to the case $m' = 2$ by switching P and P'.

Thus we distinguish the following three cases:

(10.1.1) $m' = 2$ and m is odd,

(10.1.2) $m, m' \geq 3$, and

(10.1.3) $m' = 2$, $m = 2n$, $n \in \mathbb{Z}_{>0}$.

(10.2) The case (10.1.1) is easy. Indeed, by (7.1.1) and Corollary (7.4.1) (X, C) is primitive. Hence the base (Z, o) is smooth by Corollary (2.7.4) and both non-Gorenstein points are of type (IA) by Theorem (8.6) (v). We get the case (1.3.5). The existence of a good member in $|-2K_X|$ (Proposition (1.3.7)) can be shown exactly as in [Mor88, 7.3]. From now on we consider cases (10.1.2) and (10.1.3).

(10.3) First, we treat the case (10.1.3) till the end of (10.4).

(10.3.1) Lemma. *In the case* (10.1.3)*, (X, C) is locally primitive, n is even, and we have*

$$a_1 = 1, \quad a_2 = n + 1, \quad a_3 = 2n - 1, \quad a_4 = 2n.$$

In particular, (X, P) is of type (IA)*.*

Proof. By (7.4.1) P is primitive. Hence by (2.8) and (2.7.2), (iii) we have $-K_X \cdot C = \frac{1}{2n}$. Further, (4.4.3) implies $\mathrm{gr}^0_C\,\omega \simeq \mathcal{O}_C(-1)$. Thus by (3.1.1) we have

$$w_P(0) = 1 - w_{P'}(0) + K_X \cdot C = \frac{1}{2} - \frac{1}{2n}.$$

Hence, by Proposition (8.6) and [Mor88, 4.9 (i)] the point P is of type (IA). In this case, $w_P(0) = 1 - a_2/2n$ (see [Mor88, 4.9 (i)]). This gives us $a_2 = n + 1$. Since $\gcd(2n, a_2) = 1$, n is even. Finally, by Proposition (8.5) $\mathrm{siz}_P = 1$. Therefore, $a_1 a_2 < 2n$ and $a_1 = 1$. The rest is obvious. □

(10.3.2) Lemma (cf. [KM92, 2.13.1]). *In the case* (10.1.3)*, we can write*

$$(X, P) = (y_1, y_2, y_3, y_4; \phi)/\boldsymbol{\mu}_m(1, a_2, -1, 0; 0) \supset (C, P) = y_1\text{-}axis/\boldsymbol{\mu}_m,$$

$$(X, P') = (y'_1, y'_2, y'_3, y'_4; \phi')/\boldsymbol{\mu}_2(1, 1, 1, 0; 0) \supset (C, P') = y'_1\text{-}axis/\boldsymbol{\mu}_2,$$

where $\phi \equiv y_1 y_3 \bmod (y_2, y_3)^2 + (y_4)$.

Proof. We only need to prove the last equality, which follows from the fact that (X, P) is a point of type cA/m. □

Denote $\ell(P) := \mathrm{len}_P\, I^{\sharp(2)}/I^{\sharp 2}$, where $I^\sharp$ is the ideal defining $C^\sharp$ in $X^\sharp$ near $P^\sharp$.

(10.3.3) Lemma (cf. [KM92, 2.13.2]). *In the case* (10.1.3) *we have $\ell(P) \le 1$ and $i_P(1) = 1$.*

Proof. Follows by (10.3.2) and [Mor88, 2.16]. □

(10.3.4) Lemma (cf. [KM92, 2.13.3]). *In the case* (10.1.3) *we have* $\ell(P') \le 1$ *and* $i_{P'}(1) = 1$.

Proof. Assume that $r := \ell(P') \ge 2$. Then by [Mor88, 2.16] the equation of $X^\flat$ near $P'^\flat$ has the following form: $\phi' \equiv y_1'^r y_i' \bmod (y_2', y_3', y_4')^2$, where $i = 3$ (resp. 4) if r is odd (resp. even). Consider the following L-deformation $\phi'_\lambda = \phi' + \lambda y_1'^{r-2} y_i'$. Then (X_λ, C_λ) has three singular points of indices $m = 2n$, 2, and 1. This is impossible by [Mor88, 6.2] and (9.1). Therefore, $r \le 1$. The last statement follows by [Mor88, 2.16 (ii)]. □

(10.3.5) Corollary. $\operatorname{gr}^1_C \mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$ *in the case* (10.1.3).

Proof. Follows by (3.1.2) because $H^1(\operatorname{gr}^1_C \mathcal{O}) = 0$. □

(10.3.6) Let $\mathcal{L} \subset \operatorname{gr}^1_C \mathcal{O}$ be a (unique) subsheaf such that $\mathcal{L} \simeq \mathcal{O}$. Note that $\mathcal{L}$ is an ℓ-invertible $\mathcal{O}_C$-module. Let u_1 (resp. u_1') be an ℓ-free ℓ-basis at P (resp. P'). By [Mor88, Cor. 9.1.7] there is a subbundle $\mathcal{M} \simeq \mathcal{O}(-1)$ of $\operatorname{gr}^1_C \mathcal{O}$ such that $\operatorname{gr}^1_C \mathcal{O} = \mathcal{L} \oplus \mathcal{M}$ is an ℓ-splitting. Let u_2 (resp. u_2') be an ℓ-free ℓ-basis of $\mathcal{M}$ at P (resp. P').

(10.3.7) Lemma (cf. [KM92, 2.13.8]). $ql \deg(\mathcal{M}, P') = 1$.

Proof. Since $ql \deg(\mathcal{M}, P') < m' = 2$, it is sufficient only to disprove the case $ql \deg(\mathcal{M}, P') = 0$. Assume that $ql \deg(\mathcal{M}, P') = 0$. Since y_2, y_3, y_4 form an ℓ-basis of $\operatorname{gr}^1_C \mathcal{O}$ at P, we have $\mathcal{M} \simeq (-1 + iP^\sharp)$, where $i = 0$, 1, or $m - a_2$ $(= n - 1)$. Recall that $\operatorname{gr}^0_C \omega \simeq (-1 + (m - a_2)P^\sharp + P'^\flat)$. Therefore,

$$\operatorname{gr}^1_C \omega \simeq \operatorname{gr}^1_C \mathcal{O} \mathbin{\tilde\otimes} \operatorname{gr}^0_C \omega \simeq \mathcal{L} \mathbin{\tilde\otimes} \operatorname{gr}^0_C \omega \mathbin{\tilde\oplus} (-2 + (m - a_2 + i)P^\sharp + P'^\flat).$$

The last expression is normalized (because $m - a_2 + i = n - 1 + i < m = 2n$). Hence, $H^1(\operatorname{gr}^1_C \omega) \ne 0$. Put $V := \operatorname{Spec} \mathcal{O}_X / I_C^{(2)}$ and $V' := \operatorname{Spec} \mathcal{O}_{X'} / I_{C'}^{(2)}$ (notation of (2.4.1)). As in the proof of Corollary (4.4.5) we get $H^1(\mathcal{O}_V \mathbin{\tilde\otimes} \omega_X) \ne 0$ and therefore $H^1(\mathcal{O}_{V'} \mathbin{\tilde\otimes} \omega_{X'}) \ne 0$ (notation of (2.4.1)). By Theorem (4.4) $V' \supset f'^{-1}(o')$. In particular,

$$2 = -K_{X'} \cdot f'^{-1}(o') \le -K_{X'} \cdot V' = -3K_{X'} \cdot C' = 3/n.$$

This implies $n = 1$, a contradiction. □

(10.3.8) Lemma (cf. [KM92, 2.13.10]). $ql \deg(\mathcal{M}, P) = m - a_2$.

Proof. First we note that the arguments of [KM92, 2.13.10.1-2] apply to our case and show in particular that if $ql\deg(\mathcal{M},P) \neq m - a_2$ and if $H^1(\omega_X/F^4(\omega,J)) = 0$, then m is odd while m is even in our case. Hence it is enough to derive a contradiction assuming that $ql\deg(\mathcal{M},P) \neq m-a_2$ and $H^1(\omega_X/F^4(\omega,J)) \neq 0$.

Let J be the C-laminal ideal of width 2 such that $J/I_C^{(2)} = \mathcal{L}$. Then

$$0 \neq H^1(\omega_X/F^4(\omega,J)) = H^1(\omega_X/J^2\omega_X) = H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/J^{(2)}).$$

As in the proof of Lemma (10.3.7) put $V := \operatorname{Spec}_{X'} \mathcal{O}_{X'}/J'^{(2)}$, where J' is the pull-back of J on X' (we use notation of (2.4.1)). Recall that $I_C \supset J \supset I_C^2$. Thus $I'_C \supset J' \supset I'^2_C$, where I'_C is the ideal sheaf of C'. Since $H^1(\omega_X \tilde{\otimes} \mathcal{O}_X/J^{(2)}) \neq 0$, we have $H^1(\omega_{X'} \tilde{\otimes} \mathcal{O}_{X'}/J'^{(2)}) \neq 0$. By Theorem (4.4) $V \supset f'^{-1}(o')$. Let $Q' \in C'$ be a general point. Then in a suitable coordinate system (x,y,z) near Q' we may assume that C' is the z-axis. So, $I'_C = (x,y)$ and $I'^{(2)}_C = (x^2,xy,y^2)$. Since $J'/I^{(2)}_{C'}$ is of rank 1, by changing coordinates x,y we may assume that $J' = (x,y^2)$ near Q'. Then $J'^{(2)} = (x^2,xy^2,y^4)$ and $V \equiv lC'$, where $l = \operatorname{len}_0 \mathbb{C}[x,y]/(x^2,xy^2,y^4) = 6$. Similarly to the proof of Lemma (10.3.7) we have

$$2 = -K_{X'} \cdot f'^{-1}(o') \leq -K_{X'} \cdot V = -6K_{X'} \cdot C' = 6/n.$$

Since n is even ≥ 2, we get only one possibility $n = 2$.

As in the proof of Lemma (10.3.7) we see that $ql\deg(M,P) = 0$ (because $m-a_2 = 1$). Then by (10.3.7) $\mathcal{M} \simeq (-1+P'^{\flat})$. Now we consider the base change (2.4.1). Here g is a double cover and X' is of index two. Set $P^{\natural} := g^{-1}(P)$, the unique non-Gorenstein point of X'. (Note that in our notation $P' \in X$, which is different from the notation of (2.4)–(2.7))). Note that the index-one covers of (X,P) and $(X',P^{\natural})$ coincide. Let $\mathcal{M}'$ be the ℓ-invertible sheaf on X', the pull-back of $\mathcal{M}$ to X'. We have $\mathcal{M}' \simeq (-1+0P^{\sharp})$ and $\operatorname{gr}^0_{C'}\omega_{X'} \simeq (-1+P^{\sharp})$, where $C' := g^{-1}(C)_{\mathrm{red}}$. Hence, $H^1(X',\omega_{X'} \tilde{\otimes} \mathcal{M}') \neq 0$. Let J' be the ideal on X' lifting J. Then taking account of the exact sequence

$$0 \longrightarrow I_{C'}/J'(=\mathcal{M}') \longrightarrow \mathcal{O}_{X'}/J' \longrightarrow \mathcal{O}_{C'} \longrightarrow 0$$

and isomorphisms $\omega_{X'} \tilde{\otimes} \mathcal{O}_{C'} \simeq \operatorname{gr}^0_{C'}\omega \simeq \mathcal{O}_{C'}(-1)$, we get $H^1(X',\omega_{X'} \tilde{\otimes} \mathcal{O}_{X'}/J') \simeq H^1(X',\omega_{X'} \tilde{\otimes} \mathcal{M}') \neq 0$. Hence by Theorem (4.4) we have $f'^{-1}(o') \subset \operatorname{Spec}\mathcal{O}_{X'}/J'$ which means $2 \leq 2/2 = 1$, a contradiction. □

(10.3.9) Remark. In the above notation the case $n = 2$ can be disproved also by considering possible actions of involutions on index two $\mathbb{Q}$-conic bundles $f'\colon X' \to Z'$ (see (12.1.7) and (12.1.8)).

Thus we have proved the following

(10.4) Proposition. *In the case* (10.1.3) *there is an ℓ-isomorphism* $\mathcal{M} \simeq \operatorname{gr}^0_C \omega$.

(10.5) Lemma. *Up to permutations we may assume that P' is of type* (IA) *and P is of type* (IA$^\vee$), (IA), *or* (IIA). *Moreover,* $\operatorname{siz}_P = \operatorname{siz}_{P'} = 1$.

Proof. If (X, C) is not locally primitive, the assertion follows by (7.1.2) and (7.4). We assume that (X, C) is locally primitive. By Propositions (8.2), (8.3), and (8.4) points P and P' are of types (IA) or (IIA). If both P and P' are of type (IIA), then $w_P(0) + w_{P'}(0) = 3/2 > 1$ (see [Mor88, 4.9 (i)]). This contradicts (3.1.1). Thus we may assume that P' is of type (IA) modulo permutation of P and P'. To prove the last statement consider L-deformation $(X_\lambda, C_\lambda \ni P_\lambda, P'_\lambda)$ of $(X, C \ni P, P')$ so that P_λ, P'_λ are ordinary points. In particular, they are of type (IA). By [Mor88, 4.7] $\operatorname{siz}_{P_\lambda} = \operatorname{siz}_P$ and $\operatorname{siz}_{P'_\lambda} = \operatorname{siz}_{P'}$. If (X_λ, C_λ) is a $\mathbb{Q}$-conic bundle germ, the assertion follows by Proposition (8.5). Otherwise we can apply [Mor88, Th. 6.6]. □

Temporarily we consider the following situation.

(10.6) Notation. Assume that P and P' are ordinary. Then $i_P(1) = \operatorname{siz}_P = 1$ and $i_{P'}(1) = \operatorname{siz}_{P'} = 1$. Hence, $\operatorname{gr}^1_C \mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$ (because $H^1(\operatorname{gr}^1_C \mathcal{O}) = 0$). Let $\mathcal{L} \subset \operatorname{gr}^1_C \mathcal{O}$ be a (unique) subsheaf such that $\mathcal{L} \simeq \mathcal{O}$. Note that $\mathcal{L}$ is an ℓ-invertible $\mathcal{O}_C$-module. Let u_1 (resp. u'_1) be an ℓ-free ℓ-basis at P (resp. P').

(10.6.1) Theorem (cf. [Mor88, 9.3]). *Notation as in* (10.6). *Then both* $(C^\sharp, P^\sharp)$ *and* $(C^\flat, P'^\flat)$ *are smooth,* $\operatorname{wt} u_1 \equiv -1 \bmod m$, $\operatorname{wt} u'_1 \equiv -1 \bmod m'$ *and furthermore we may assume that* $a_1 = a'_1 = 1$.

The proof follows [Mor88, 9.3]. We will treat (10.6.1) in four cases (10.6.2)–(10.6.5) below.

(10.6.2) Case: P is of type (IA$^\vee$).

If both P and P' are primitive, then in a suitable coordinate system near $P^\sharp$ the ideal $I^\sharp_C$ is generated by $x_1^{a_2} - x_2^{a_1}$ and x_3 (because P is ordinary). Hence, either $\operatorname{wt} u_1 \equiv a_1 a_2$ or $\operatorname{wt} u_1 \equiv -a_1 \bmod m$ holds and the corresponding assertion also holds for P'. Modulo permutation of P and P' and if $a_2 = 1$ (resp. $a'_2 = 1$) modulo further permutation of a_1 and a_3 (resp. a'_1 and a'_3) there are three cases:

(10.6.3) Case: $\operatorname{wt} u_1 \equiv a_1 a_2$, $a_2 \not\equiv \pm 1 \bmod m$, $\operatorname{wt} u'_1 \equiv a'_1 a'_2$, $a'_2 \not\equiv \pm 1 \bmod m'$.

(10.6.4) Case: $\operatorname{wt} u_1 \equiv a_1 a_2$, $a_2 \not\equiv \pm 1 \bmod m$, $\operatorname{wt} u_1' \equiv a_3' \bmod m'$.

(10.6.5) Case: $\operatorname{wt} u_1 \equiv a_3 \bmod m$, $\operatorname{wt} u_1' \equiv a_3' \bmod m'$.

We will show that only the case (10.6.5) is possible and $a_1 = a_1' = 1$. By [Mor88, Cor. 9.1.7] there is a subbundle $\mathcal{M} \simeq \mathcal{O}(-1)$ of $\operatorname{gr}^1_C \mathcal{O}$ such that $\operatorname{gr}^1_C \mathcal{O} = \mathcal{L} \oplus \mathcal{M}$ is an ℓ-splitting. Let u_2 (resp. u_2') be an ℓ-free ℓ-basis of $\mathcal{M}$ at P (resp. P').

(10.6.6) Let J be the C-laminal ideal of width 2 such that $J/I_C^{(2)} = \mathcal{L}$, and our symbols are compatible with those in [Mor88, 9.3.2].

Note that $w_P(0) = 1 - a_2/\bar{m}$ and $w_{P'}(0) = 1 - a_2'/m'$. Then by (3.1.1) and $(K_X \cdot C) < 0$ we have

$$1 < \frac{a_2}{\bar{m}} + \frac{a_2'}{m'} \quad ([\text{Mor88}, 9.3.4]). \tag{10.6.7}$$

Using Lemma (2.8) that holds only for $\mathbb{Q}$-conic bundles, we have

$$1 + \frac{d}{mm'} = \frac{a_2}{\bar{m}} + \frac{a_2'}{m'}, \tag{10.6.8}$$

where $d = mm'/\operatorname{lcm}(\bar{m}, m') = s\gcd(\bar{m}, m')$ by (2.7.2), (iii).

(10.6.9) Disproof of the case (10.6.2). This case corresponds to [Mor88, 9.3.ipr], and is disproved by the same argument as [Mor88, 9.3.5].

Hence we note that $\bar{m} = m$ below till the end of the proof of (10.6.1).

(10.6.10) Disproof of the case (10.6.3). This case corresponds to [Mor88, 9.3.a], and is disproved by the same argument as [Mor88, 9.3.6].

(10.6.11) Disproof of the case (10.6.4). This case corresponds to [Mor88, 9.3.b], and is disproved by the same argument as [Mor88, 9.3.7].

(10.6.12) Treatment of the case (10.6.5). This case corresponds to [Mor88, 9.3c], and the arguments of [Mor88, 9.3.8] work except for [Mor88, 9.3.8.6], which we prove below using (10.6.8).

We have $\operatorname{wt} u_1 \equiv a_3 \bmod m$, $\operatorname{wt} u_1' \equiv a_3' \bmod m'$. We will prove that $a_1 = a_1' = 1$. By symmetry we may assume that $a_2'/m' > 1/2$ (see (10.6.7)). Since $\operatorname{siz}_{P'} = 1$, $m' \geq a_1' a_2'$. This gives us $a_1' = 1$. We will prove $a_1 = 1$. Assume that $a_1 \geq 2$. Then computations [Mor88, 9.3.8.3–9.3.8.4] apply and give us $a_1 = 2$, $a_2 = 1$. In particular, $C^\sharp$ is smooth over P and P'. Further by [Mor88, 9.3.8.5] we have

$$\begin{aligned} ql_C(\mathcal{L}) &= 2P^\sharp + P'^\sharp, \\ ql_C(\mathcal{M}) &= -1 + (m-2)P^\sharp + (m' - a_2')P'^\sharp, \end{aligned} \tag{10.6.13}$$

and $ql_C(\mathrm{gr}^0_C\,\omega) = -1 + (m-1)P^\sharp + (m' - a'_2)P'^\sharp$.

(10.6.14) Claim (cf. [Mor88, 9.3.8.6]). $m \geq 5$.

Proof. Assume that $m < 5$. Since $a_1 = 2$ and $\gcd(m, a_1) = 1$, we have $m = 3$. Then by (10.6.8)

$$2m' + d = 3a'_2, \quad d = \gcd(3, m').$$

Now one can see that computations of [Mor88, 9.3.8.6] apply and give us $\chi(F^1(\omega, J)/F^4(\omega, J)) < 0$. From the exact sequence

$$0 \longrightarrow F^1(\omega, J)/F^4(\omega, J) \longrightarrow \omega_X/F^4(\omega, J) \longrightarrow \mathrm{gr}^0_C\,\omega \longrightarrow 0$$

we get

$$\chi(\omega_X/F^4(\omega, J)) = \chi(F^1(\omega, J)/F^4(\omega, J)) + \chi(\omega_X/F^1(\omega, J)) < 0.$$

(we note that $F^1(\omega, J) = \mathrm{Sat}_{\omega_X}(I_C\omega_X)$ and $\omega_X/F^1(\omega, J) = \mathrm{gr}^0_C\,\omega$). In particular, we have $H^1(\omega_X/F^4(\omega, J)) \neq 0$. Recall that $I_C \supset J \supset I_C^{(2)}$ and $F^4(\omega, J) = \mathrm{Sat}_{\omega_X}(J^2\omega_X)$. Assume that (Z, o) is smooth, i.e., $3 \nmid m'$. Then by Theorem (4.4) $f^{-1}(o) \subset \mathrm{Spec}\,\mathcal{O}_X/J^{(2)} \subset \mathrm{Spec}\,\mathcal{O}_X/I_C^{(4)}$. We get a contradiction (cf. (7.3.3)):

$$2 = -K_X \cdot f^{-1}(o) < -10K_X \cdot C = 10/(3m'), \quad m' = 1.$$

Now assume that $m' = 3m''$, $m'' \geq 2$. Take a Weil divisor ξ such that $ql_C\xi = P^\sharp - m''P'^\sharp$. Then ξ is a 3-torsion in $\mathrm{Cl}^{\mathrm{sc}}\,X$. Taking (10.6.13) into account we obtain

$$\begin{aligned} ql_C(\mathcal{M}\,\tilde{\otimes}\,\xi) &= -1 + (m-1)P^\sharp + (2m'' - a'_2)P'^\sharp = \\ &= -2 + (m-1)P^\sharp + (5m'' - a'_2)P'^\sharp. \end{aligned}$$

Since $a'_2 = 2m'' + 1$, $5m'' - a'_2 = 3m'' - 1$. So, the last expression is normalized. By [Mor88, 8.9.1 (iii)] $\deg_C \mathcal{M}\tilde{\otimes}\xi = -2$. Note that $\mathrm{gr}^1_C\,\mathcal{O}\tilde{\otimes}\xi = (\mathcal{L}\tilde{\otimes}\xi)\tilde{\oplus}(\mathcal{M}\tilde{\otimes}\xi)$. Hence, $H^1(\mathrm{gr}^1_C\,\mathcal{O}\,\tilde{\otimes}\,\xi) \neq 0$. This is a contradiction, and $m \geq 5$ as proved. □

The remainder of the proof is the same as [Mor88, 9.3.8.7]. Thus Theorem (10.6.1) is proved.

Now we treat the case (10.1.2) from here till the end of (10.9).

(10.7) Proposition. *In notation and assumptions of* (10.1.2) (X, C) *has no* (IIA) *type points.*

Proof. Assume that P is of type (IIA) (and P' is of type (IA) by Lemma (10.5)). Let $d := \gcd(4, m')$. By [Mor88, Th. 4.9, (i)] $w_P(0) = 3/4$. Then by Lemma (2.8) and (3.1.1) we have

$$w_{P'}(0) = 1 - w_P(0) + K_X \cdot C = 1/4 - d/(4m') < 1/2.$$

Hence, $a'_1 = 1$ (see [Mor88, Prop. 5.1]). Moreover,

$$d + 3m' = 4a'_2 \tag{10.7.1}$$

(because $w_P(0) = 1 - a'_2/m'$ by [Mor88, Th. 4.9, (i)]). Applying L-deformations at P' and L' deformation at P (see [Mor88, 4.12.2]) we may assume that P' is ordinary and $(X, P) \simeq \{y_1y_2 + y_3^2 + y_4^3 = 0\}/\boldsymbol{\mu}_4(1, 1, 3, 2)$, where $C^\sharp$ is the y_1-axis. This new (X, C) is again a $\mathbb{Q}$-conic bundle germ by [Mor88, 9.4]. Applying [Mor88, 9.4.3-9.4.5] we get an ℓ-splitting $\operatorname{gr}^1_C \mathcal{O} = \mathcal{L} \tilde{\oplus} \mathcal{M}$, where $\mathcal{L} \simeq \mathcal{O}$ and $\mathcal{M} \simeq \mathcal{O}(-1)$. Moreover, $ql_C(\mathcal{L}) = P^\sharp + P'^\sharp$ and $ql_C(\mathcal{M}) = -1 + 2P^\sharp + (m' - a'_2)P'^\sharp$ and we may assume that y_3 is an ℓ-free ℓ-basis of $\mathcal{L}$ at P.

Now one can see that computations of [Mor88, 9.4.6] apply and give us $\chi(F^1(\omega, J)/F^4(\omega, J)) < 0$. If $2 \not| m'$, then we can apply the second half of the proof of (10.6.14) to get a contradiction:

$$2 = -K_X \cdot f^{-1}(o) < -10K_X \cdot C = 10/(4m'), \quad m' = 1.$$

Thus $2|m'$. If $m' = 2m''$ and $2 \not| m''$, then considering diagram (2.4.1) one has $H^1(\omega_{X'}/\operatorname{Sat}_{\omega_{X'}}(J^2\omega_{X'})) \neq 0$ and similarly gets:

$$2 = -K_{X'} \cdot f'^{-1}(o') < -10K_{X'} \cdot C' = -20K_X \cdot C = 20/2m' = 5/m''.$$

Thus $m'' = 1$, and one sees $d = 2$ and $a'_2 = 2$ by (10.7.1), which contradicts $\gcd(m', a'_2) = 1$, a condition on (IA) points. Hence $4|m'$ and we set $m' = 4m''$.

Take a Weil divisor ξ such that $ql_C(\xi) = P^\sharp - m''P'^\sharp$. Then ξ is a 4-torsion in $\operatorname{Cl}^{\mathrm{sc}} X$. By [Mor88, 9.4.5] we have

$$ql_C(\mathcal{M}) = -1 + 2P^\sharp + (4m'' - a'_2)P'^\sharp$$

Recall that $a'_2 = 3m'' + 1$. Taking this into account we obtain

$$ql_C(\mathcal{M} \tilde{\otimes} \xi) = -1 + 3P^\sharp - P'^\sharp = -2 + 3P^\sharp + (4m'' - 1)P'^\sharp.$$

The last expression is normalized. By [Mor88, 8.9.1 (iii)] $\deg_C \mathcal{M} \tilde{\otimes} \xi = -2$. Note that $\operatorname{gr}^1_C \mathcal{O} \tilde{\otimes} \xi = (\mathcal{L} \tilde{\otimes} \xi) \tilde{\oplus} (\mathcal{M} \tilde{\otimes} \xi)$. Hence, $H^1(\operatorname{gr}^1_C \mathcal{O} \tilde{\otimes} \xi) \neq 0$. This is a contradiction. □

(10.7.2) Corollary ([Mor88, 9.4.7]). *In notation and assumptions of* (10.1.2) *points P and P' are type* (IA) *points such that $a_1 = a'_1 = 1$, and moreover $\ell(P) < m$ and $\ell(P') < m'$.*

Proof. By (10.5) and (10.7) P is of type (IA) and P' is of type (IA) or ($\mathrm{IA}^\vee$). Replacing (X, C) with L-deformation we may assume that both P and P' are ordinary. Then by (10.6.9), (10.6.1), and [Mor88, 9.3, 9.4] P' is of type (IA) and $a_1 = a'_1 = 1$ (L-deformation does not change a_i's because P and P' are of type (IA) or ($\mathrm{IA}^\vee$)). If $\ell(P) \ge m$, then an L'-deformation (X_λ, C_λ) (see [Mor88, 4.12.2]) has at least one Gorenstein singular point besides P_λ and P'_λ. This contradicts (9.1) and [Mor88, 6.2]. Thus $\ell(P) < m$. By symmetry we also have $\ell(P') < m'$. □

(10.7.3) Corollary ([Mor88, 9.4.8]). *In notation and assumptions of* (10.1.2) *we have $i_P(1) = i_{P'}(1) = 1$ and an isomorphism $\mathrm{gr}^1_C \mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$.*

Proof. Since $\ell(P) < m$ and $\ell(P') < m'$, by [Mor88, 2.16 (ii)] we have $i_P(1) = i_{P'}(1) = 1$. Hence $\deg \mathrm{gr}^1_C \mathcal{O} = -1$, see (3.1.2). Taking account of $H^1(\mathrm{gr}^1_C \mathcal{O}) = 0$ we obtain the last statement. □

(10.8) Proposition ([Mor88, 9.8]). *We have*

$$\ell(P) + ql\deg(\mathcal{L}, P) = \ell(P') + ql\deg(\mathcal{L}, P') = 1.$$

Proof. Similar to the proof of [Mor88, Theorem 9.8]. □

(10.9) Proposition ([Mor88, 9.9.1]). *In the case* (10.1.2)*, there is an ℓ-isomorphism $\mathcal{M} \simeq \mathrm{gr}^0_C \omega$.*

Proof. Since $\mathrm{gr}^0_C \omega \simeq \mathcal{O}(-1)$, it is sufficient to show that $ql\deg(\mathcal{M}, P) = ql\deg(\mathrm{gr}^0_C \omega, P) = R(a_2)$ and $ql\deg(\mathcal{M}, P') = ql\deg(\mathrm{gr}^0_C \omega, P') = R'(a'_2)$. By symmetry it is sufficient to prove for example the first equality. According to (10.8) there are two cases.

(10.9.1) Case: $\ell(P) = 0$, $ql\deg(\mathcal{L}, P) = 1$. Then (X, P) is a cyclic quotient singularity of type $\frac{1}{m}(1, a_2, -1)$. If u_1 is an ℓ-free ℓ-basis of $\mathcal{L}$ at P, then $R(\mathrm{wt}\, u_1) = 1$, so $\mathrm{wt}\, u_1 \equiv -1 \mod m$. An ℓ-free ℓ-basis of $\mathrm{gr}^1_C \mathcal{O}$ at P is x_2, x_3. Hence we can put $u_1 = x_3$ and x_2 is an ℓ-free ℓ-basis of $\mathcal{M}$. Therefore, $ql\deg(\mathcal{M}, P) = R(\mathrm{wt}\, x_2) = R(a_2)$.

(10.9.2) Case: $\ell(P) = 1$, $ql\deg(\mathcal{L}, P) = 0$. Then we can choose a some coordinate system so that $(X^\sharp, P^\sharp)$ is given by $\phi = 0$ with $\phi \equiv x_1 x_3 \mod (x_2, x_3, x_4)^2$

and $C^\sharp$ is the x_1-axis (see [Mor88, 2.16]). If u_1 is an ℓ-free ℓ-basis of $\mathcal{L}$ at P, then $R(\operatorname{wt} u_1) = 0$, so $\operatorname{wt} u_1 \equiv 0 \bmod m$. Again an ℓ-free ℓ-basis of $\operatorname{gr}^1_C \mathcal{O}$ at P is x_2, x_4. Hence we can put $u_1 = x_4$ and x_2 is an ℓ-free ℓ-basis of $\mathcal{M}$. Therefore, $ql \deg(\mathcal{M}, P) = R(\operatorname{wt} x_2) = R(a_2)$.

□

Taking Propositions (10.4) and (10.9) into account one can see that all the arguments and computations from [Mor88, 9.9.2-9.9.10] apply in our case. This proves (iii) of the following theorem (cf. [KM92, 2.2.4]).

(10.10) Theorem (cf. [Mor88, Th. 9.10], [KM92, 2.2.4])**.** *Let $(X, C \simeq \mathbb{P}^1)$ be a $\mathbb{Q}$-conic bundle germ having two non-Gorenstein points P, P' of indices m, $m' \geq 2$ and no other singularities. Then P, P' are of type* (IA) *by* (10.5) *and* (10.7)*, and the following assertions hold.*

(i) *If $m' = 2$ and m is odd, then the general member of $|-2K_X|$ does not contain C and has only log terminal singularities.*

(ii) *If (X, C) is toroidal, then a general member $F \in |-K_X|$ does not contain C. It has two connected components. Each of them is a Du Val singularity of A-type.*

(iii) *If (X, C) is not toroidal, we further assume either m, $m' \geq 3$ or $m' = 2$ and m is even. Then a general member $F \in |-K_X|$ is a normal surface containing C, smooth outside of $\{P, P'\}$, with Du Val points of A-type at P, P'. Furthermore C on F is contractible to a Du Val point of A-type.*

Note that (i) and (ii) of (10.10) are easy (cf. [Mor88, Th. 7.3]). In (ii) one can also take as F the sum of two horizontal toric divisors.

§11. Two Non-Gorenstein Points Case: the Classification

The following is the main result of this section.

(11.1) Theorem. *Let $(X, C \simeq \mathbb{P}^1)$ be a $\mathbb{Q}$-conic bundle germ having two points of indices m, $m' \geq 2$ and no other singularities. Assume either m, $m' \geq 3$ or $m' = 2$ and m is even. Then (X, C) is either toroidal or as in* (1.2.2).

The above theorem is an easy consequence of Theorem (10.10) and Proposition (11.2) below.

(11.2) Proposition (cf. [Pro97a, §4]). *Let $f\colon (X, C) \to (Z, o)$ be a non-Gorenstein $\mathbb{Q}$-conic bundle germ with $C \simeq \mathbb{P}^1$. Assume that the general element*

$F \in |-K_X|$ containing C has only Du Val singularities. Let $F \xrightarrow{f_1} \bar{F} \to Z$ be the Stein factorization and let $\bar{P} = f_1(C)$. Assume that $(\bar{F}, \bar{P})$ is a singularity of type A. Then one of the following holds:

(i) *f is as in* (1.2.2), *or*

(ii) *X is of index* 2 *and (Z, o) is smooth.*

Proof of Proposition (11.2). By the inversion of adjunction [Kol92, 17.6] the log divisor $K_X + F$ is plt. Consider diagram (2.4.1) and put $F' := g^*F$. We may assume that $Z' \simeq \mathbb{C}^2$ and $Z \simeq \mathbb{C}^2/\boldsymbol{\mu}_d(1, q)$, where $\gcd(d, q) = 1$. By [Kol92, 20.3] $K_{X'} + F' = g^*(K_X + F) \sim 0$ is plt. In particular, F' is normal and irreducible. Further, diagram (2.4.1) induces the following diagram

(11.2.1)
$$\begin{array}{ccc} (F', C') & \xrightarrow{g_{F'}} & (F, C) \\ {\scriptstyle f_1'}\downarrow & & {\scriptstyle f_1}\downarrow \\ (\bar{F}', \bar{P}') & \xrightarrow{\bar{g}} & (\bar{F}, \bar{P}) \\ {\scriptstyle f_2'}\downarrow & & {\scriptstyle f_2}\downarrow \\ (Z', o') & \xrightarrow{h} & (Z, o) \end{array}$$

where the vertical arrows are Stein factorizations of restrictions $f'|_{F'}$ and $f|_F$. It is clear that f_2' and f_2 are double covers. By adjunction $K_{F'} \sim 0$ and f_1' is a crepant morphism contracting C'. Since $\bar{g}$ is étale in codimension one and (F, P) is a singularity of type A, $(\bar{F}', \bar{P}')$ is also of type A. Note that $(\bar{F}', \bar{P}')$ cannot be smooth (because f_1' is non-trivial).

Consider the case $d \geq 2$. Then by Lemma (7.5.1) $(\bar{F}', \bar{P}')$ is of type A_1. In this case, F' is smooth and so is X' (see, e.g., [Pro97a, Lemma 1.4]). Therefore, f is the quotient of a smooth conic bundle by $\boldsymbol{\mu}_d$. By Proposition (2.9) we get the case (1.2.2).

Thus we may assume that $d = 1$ (and $X' = X$). Let $R \subset Z$ be the ramification divisor of f_2. Since $(\bar{F}, \bar{P})$ is of type A, in some coordinate system on $Z = \mathbb{C}^2$, R is given by the equation $x^k + y^2 = 0$. Let $\Gamma \subset Z$ is given by $x = 0$ and let $S := f^*\Gamma$. By the inversion of adjunction the log divisor $K_Z + \Gamma + \frac{1}{2}R$ is log canonical (lc). So are $K_{\bar{F}} + f_2^*\Gamma = f_2^*(K_Z + \Gamma + \frac{1}{2}R)$ and $K_F + f^*\Gamma = K_F + S|_F$. Again by the inversion of adjunction $K_X + F + S$ is lc near F. Shrinking X we may assume that $K_X + F + S$ is lc everywhere. Replacing Γ with a general hyperplane section through o, we may assume that S is smooth outside of C. Then $K_X + S$ is plt. In particular, S is normal and has only log terminal singularities of type T [KSB88]. Let $D := F|_S$. Then $K_S + D \sim 0$ is lc and $D \supset C$. By the classification of two-dimensional log canonical singularities [Kaw88], [Kol92, Ch. 3] $K_S + C$ is plt.

The restriction $f_S : S \to \Gamma$ is a rational curve fibration such that $-K_S$ is f_S-ample. If C is a Cartier divisor on S, then S is smooth and so is X. Take the minimal positive n such that nC is Cartier. Then $nC \sim 0$. This induces an étale in codimension one $\boldsymbol{\mu}_n$-cover $\pi: S^\natural \to S$ such that $C^\natural := \pi^*C \sim 0$. The divisor $K_{S^\natural} + C^\natural = \pi^*(K_S + C)$ is plt (see, e.g., [Kol92, 20.3]). Hence, $C^\natural$ is smooth and so is $S^\natural$. Thus S is a quotient of $S^\natural \simeq \mathbb{C} \times \mathbb{P}^1$ by $\boldsymbol{\mu}_n$. It is easy to see that S has singular points of types $\frac{1}{n}(1, q)$ and $\frac{1}{n}(-1, q)$, where $\gcd(n, q) = 1$. These points are of type T if and only if

$$(q + 1)^2 \equiv (q - 1)^2 \equiv 0 \bmod n$$

(see [KSB88]). This implies $n = 2$ or 4. If $n = 2$, then S is Gorenstein and so is X, a contradiction. Hence $n = 4$, so the singularities of S are of types $\frac{1}{4}(1, 1)$ and A_3. By [KSB88] (X, C) has exactly one non-Gorenstein point which is of index 2. □

§12. Index Two $\mathbb{Q}$-conic Bundles

Index two $\mathbb{Q}$-conic bundles were classified in [Pro97a, §3]. Under the condition that the base (Z, o) is smooth, these are quotients of some elliptic fibrations by an involution. Here we propose an alternative description and sketch a different proof. (Note that a $\mathbb{Q}$-conic bundle of index two over a singular base is either of type (1.2.4) or toroidal [Pro97a, §3]).

(12.1) Theorem. *Let $f: (X, C) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ of index two. Assume that (Z, o) is smooth. Fix an isomorphism $(Z, o) \simeq (\mathbb{C}^2, 0)$. Then there is an embedding*

(12.1.1)
$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}(1,1,1,2) \times \mathbb{C}^2 \\ & {}_{f}\searrow & \downarrow p \\ & & \mathbb{C}^2 \end{array}$$

such that X is given by two equations

(12.1.2)
$$\begin{aligned} q_1(y_1, y_2, y_3) - \psi_1(y_1, \dots, y_4; u, v) &= 0, \\ q_2(y_1, y_2, y_3) - \psi_2(y_1, \dots, y_4; u, v) &= 0, \end{aligned}$$

where ψ_i *and* q_i *are weighted quadratic in* $y_1, \dots, y_4$ *with respect to* $\mathrm{wt}(y_1, \dots, y_4) = (1,1,1,2)$ *and* $\psi_i(y_1, \dots, y_4; 0, 0) = 0$. *The only non-Gorenstein point of* X *is* $(0,0,0,1;0,0)$. *Up to projective transformations, the following are the possibilities for* q_1 *and* q_2:

(12.1.3) $q_1 = y_1^2 - y_2^2$ *and* $q_2 = y_1 y_2 - y_3^2$; *then* $f^{-1}(o)$ *is reduced and has exactly four irreducible components*;

(12.1.4) $q_1 = y_1 y_2$ *and* $q_2 = (y_1 + y_2) y_3$; *then* $f^{-1}(o)$ *has three irreducible components, one of them has multiplicity* 2;

(12.1.5) $q_1 = y_1 y_2 - y_3^2$ *and* $q_2 = y_1 y_3$; *then* $f^{-1}(o)$ *has two irreducible components, one of them has multiplicity* 3;

(12.1.6) $q_1 = y_1^2 - y_2^2$ *and* $q_2 = y_3^2$; *then* $f^{-1}(o)$ *has two irreducible components, both of multiplicity* 2;

(12.1.7) $q_1 = y_1 y_2 - y_3^2$ *and* $q_2 = y_1^2$; *then* $f^{-1}(o)$ *is irreducible of multiplicity* 4;

(12.1.8) $q_1 = y_1^2$ *and* $q_2 = y_2^2$; *then* $f^{-1}(o)$ *is also irreducible of multiplicity* 4.

Conversely, if $X \subset \mathbb{P}(1,1,1,2) \times \mathbb{C}^2$ *is given by equations of the form* (12.1.2) *and singularities of* X *are terminal, then the projection* $f \colon (X, f^{-1}(0)_{\mathrm{red}}) \to (\mathbb{C}^2, 0)$ *is a* $\mathbb{Q}$*-conic bundle of index* 2.

Sketch of the proof. First we prove the last statement. Assume that X has only terminal singularities. Then X does not contain the surface $\{y_1 = y_2 = y_3 = 0\} = \operatorname{Sing} \mathbb{P} \times \mathbb{C}^2$ (otherwise both ψ_1 and ψ_2 do not depend on y_4). By the adjunction formula, $K_X = -L|_X$, where L is a Weil divisor on $\mathbb{P} \times \mathbb{C}^2$ such that the restriction $L|_{\mathbb{P}}$ is $\mathcal{O}_{\mathbb{P}}(1)$. Therefore, $X \to \mathbb{C}^2$ is a $\mathbb{Q}$-conic bundle. It is easy to see that the only non-Gorenstein point of X is $(0,0,0,1;0,0)$ and it is of index 2.

Now let $f \colon (X, C) \to (Z, o) \simeq (\mathbb{C}^2, 0)$ be a $\mathbb{Q}$-conic bundle germ of index two. Let $P \in X$ be a point of index 2. We claim that P is the only non-Gorenstein point. Indeed, if C is irreducible, the assertion follows by Corollary (2.7.2), (iii). If $C = \cup C_i$ is reducible, the same holds by Lemma (4.4.2) and [KM92, Th. 4.2, Prop. 4.6]. Thus P is the only non-Gorenstein point on X. By (5.2.1) each (X, C_i) is of type (IA) at P. (The case (IB) is excluded by Proposition (8.3) and [Mor88, Th. 6.3]). Hence the general member $F \in |-K_X|$ satisfies $F \cap C = \{P\}$ and has only Du Val singularity at P (see [Mor88, Th. 7.3]).

Let $\pi\colon (X^\sharp, P^\sharp) \to (X, P)$ be the index-one cover and let $F^\sharp = \pi^{-1}(F)_{\mathrm{red}}$ be the pull-back of F. Let $\Gamma := f^{-1}(o)$ be the scheme fiber and let $\Gamma^\sharp = \pi^{-1}(\Gamma)$.

(12.1.9) Lemma.

$$\mathcal{O}_{F^\sharp\cap\Gamma^\sharp} \simeq \mathbb{C}[x, y]/(xy,\, x^2 + y^2).$$

Furthermore $\boldsymbol{\mu}_2$*-action is given by* $\mathrm{wt}(x, y) \equiv (1, 1) \bmod 2$.

Proof. Since $(F^\sharp, P^\sharp)$ is a Du Val singularity, we may assume that $(F^\sharp, P^\sharp) \subset (\mathbb{C}^3_{x,y,z}, 0)$. The scheme $F^\sharp \cap \Gamma^\sharp$ is defined in $\mathbb{C}^3_{x,y,z}$ by three equations $\alpha = \beta = \gamma = 0$, where two of them are coordinates on $Z = \mathbb{C}^2$, and the rest is the defining equation of $F^\sharp \subset \mathbb{C}^3$. Since the morphism $F^\sharp \to Z$ is flat and of degree 4, we have

$$\mathcal{O}_{F^\sharp\cap\Gamma^\sharp} \simeq \mathbb{C}\{x, y, z\}/(\alpha, \beta, \gamma)$$

is of length 4. Furthermore $\boldsymbol{\mu}_2$ acts on the ring so that $\mathrm{wt}(x, y, z, \alpha, \beta, \gamma) \equiv (1, 1, 0, 0, 0, 0) \bmod 2$ because the quotient $(F^\sharp, P^\sharp)/\boldsymbol{\mu}_2$ is Du Val, and, in particular, Gorenstein. If $\alpha, \beta, \gamma \in (x, y, z)^2$, then $\operatorname{len} \mathbb{C}\{x, y, z\}/(\alpha, \beta, \gamma) \geq 8$, which is a contradiction. Hence, in view of the weights, we may assume that $\alpha = (\mathrm{unit}) \cdot z + \alpha_1 \cdot \beta + \alpha_2 \cdot \gamma$ modulo permutation of α, β, γ. Thus we have

$$\mathcal{O}_{F^\sharp\cap\Gamma^\sharp} \simeq \mathbb{C}\{x, y\}/(\beta,\, \gamma).$$

Since $\mathrm{wt}(x, y, \beta, \gamma) \equiv (1, 1, 0, 0) \bmod 2$, we see $\beta, \gamma \in (x, y)^2$. Hence we may assume that $\beta \equiv xy \bmod (x, y)^3$ modulo coordinate change of x, y and change of β, γ. Modulo analytic change of coordinates x, y, we may assume $(\beta, \gamma) = (xy,\, x^a + y^b)$ for some $a, b \geq 2$. Since the ring $\mathcal{O}_{F^\sharp\cap\Gamma^\sharp}$ is of length 4, we have $4 = a + b$ and hence $a = b = 2$. □

Using this lemma one can apply arguments of [Mor75, pp. 631–633] to get the desired embedding $X \subset \mathbb{P}(1, 1, 1, 2) \times Z$ considering the graded anti-canonical $\mathcal{O}_Z$-algebra

$$\mathcal{R} := \bigoplus_{i\geq 0} \mathcal{R}_i, \quad \text{where} \quad \mathcal{R}_i := H^0(\mathcal{O}_X(-iK_X)).$$

We sketch the main idea.

Let w be a local generator of $\mathcal{O}_{X^\sharp}(-K_X)$ at $P^\sharp$, let u, v be coordinates on $Z = \mathbb{C}^2$, and let $z = 0$ be the local equation of $F^\sharp$ in $(X^\sharp, P^\sharp)$. Using the vanishing of $H^1(\mathcal{O}_X(-K_X))$ for $i > 0$ and the exact sequence

$$0 \to \mathcal{O}_X(-(i-1)K_X) \to \mathcal{O}_X(-iK_X) \to \mathcal{O}_F(-iK_X) \to 0$$

one can see

$$\mathcal{R}_i/(zw)\mathcal{R}_{i-1} \simeq H^0(\mathcal{O}_F(-iK_X)), \quad i > 0.$$

Therefore,

$$\mathcal{R}_i/(zw)\mathcal{R}_{i-1} + (u,\, v)\mathcal{R}_i = \left(\mathcal{O}_{F^\sharp \cap \Gamma^\sharp}(-iK_X)\right)^{\boldsymbol{\mu}_2}.$$

By Lemma (12.1.9) we have an embedding

$$\mathcal{R}/(zw,\, u,\, v)\mathcal{R} \hookrightarrow \left(\mathbb{C}[x,\, y,\, w]/(xy,\, x^2+y^2)\right)^{\boldsymbol{\mu}_2}.$$

Using $R_0/(u,v)R_0 = \mathbb{C}$, one can easily see that

$$\mathcal{R}/(zw,\, u,\, v)\mathcal{R} = \mathbb{C}[y_1, y_2, y_4]/(y_1y_2,\, y_1^2 + y_2^2),$$

where $y_1 = xw$, $y_2 = yw$, $y_4 = w^2$. Put $y_3 := zw$. Then similarly to [Mor75, pp. 631–633] we obtain

$$\mathcal{R} \simeq \mathcal{O}_Z[y_1, y_2, y_3, y_4]/\mathcal{I},$$

where $\mathcal{I}$ is generated by the following regular sequence

$$\begin{aligned} &y_1y_2 + y_3\ell_1(y_1, \dots, y_3) &&+\psi_1(y_1, \dots, y_4; u, v),\\ &y_1^2 + y_2^2 + y_3\ell_2(y_1, \dots, y_3) &&+\psi_2(y_1, \dots, y_4; u, v) \end{aligned}$$

with $\psi_i(y_1, \dots, y_4; 0, 0) = 0$. □

As is seen in Theorem (1.2), a $\mathbb{Q}$-conic bundle is often constructed as a quotient of one of index two by a cyclic group. Theorem (12.1) is useful in such a context. Finally we provide facts which are used in the study of $\mathbb{Q}$-conic bundles with imprimitive points (cf. Proposition (7.6)).

(12.1.10) Proposition. *Assume that in the notation of Theorem* (12.1) *a cyclic group $\boldsymbol{\mu}_d$ acts on X and Z so that f is $\boldsymbol{\mu}_d$-equivariant. Then the diagram* (12.1.1) *can be chosen to be $\boldsymbol{\mu}_d$-equivariant.*

Proof. The sheaf $\mathcal{O}_X(-K_X)$ has a natural $\boldsymbol{\mu}_d$-linearization. Hence the embedding $X = \operatorname{Proj} \mathcal{R} \hookrightarrow \mathbb{P}(1,1,1,2)$ is $\boldsymbol{\mu}_d$-equivariant. □

The following is obvious.

(12.1.11) Lemma. *In notation of Theorem* (12.1) *assume that f has an equivariant $\boldsymbol{\mu}_d$-action and that $f^{-1}(o)$ has two irreducible components, both of multiplicity* 2 (*i.e., we are in case* (12.1.6)), *which are permuted by some element of $\boldsymbol{\mu}_d$. Then the coordinates $y_1, \dots, y_4, u, v$ can be chosen so that they and the equations* (12.1.2) *are semi-invariant.*

Proof. Indeed, by (12.1.10) the action of $\boldsymbol{\mu}_d$ preserves the pencil $\lambda_1(q_1 - \psi_1) + \lambda_2(q_2 - \psi_2)$. It remains to note that in (12.1.6) q_1 and q_2 are the only degenerate quadratic forms in this pencil and they cannot be interchanged. □

(12.1.12) Lemma. *In notation and assumptions of Theorem* (12.1) *and Lemma* (12.1.11) *assume additionally that* $d = 2$*. Furthermore assume that* $\boldsymbol{\mu}_2$ *acts on* X *and* Z *so that the action is free in codimension one, has a unique fixed point* $P = (0, 0, 0, 1; 0, 0)$*, and the quotient* $(X, P)/\boldsymbol{\mu}_2$ *is a terminal singularity. Then modulo change of coordinates, we are in case* (12.1.6) *with the action written as follows*:

$$y_1 \mapsto y_1, \quad y_2 \mapsto -y_2, \quad y_3 \mapsto y_3, \quad y_4 \mapsto -y_4, \quad u \mapsto -u, \quad v \mapsto -v.$$

Proof. By Lemma (12.1.11) we can choose the coordinates $y_1, \dots, y_4, u, v$ so that they and the equations (12.1.2) are semi-invariant. Since the action of $\boldsymbol{\mu}_2$ on $Z \simeq \mathbb{C}^2$ is free outside of o, this action is given by $u \mapsto -u$, $v \mapsto -v$. Modulo multiplication of ± 1 on the $\boldsymbol{\mu}_2$-linearization of $\mathcal{O}(-K_X)$, we may assume also that $y_3 \mapsto y_3$. Then $y_i \mapsto \pm y_i$ for all i.

Recall that we are in the case (12.1.6) with $X \subset \mathbb{P}(1, 1, 1, 2)_{y_1, y_2, y_3, y_4} \times \mathbb{C}^2_{u,v}$. Since $(1, 1, 0, 0; 0, 0) \in X$, the point is not $\boldsymbol{\mu}_2$-fixed by the assumption. Hence $y_1 y_2 \mapsto -y_1 y_2$. Modulo permutation of y_1, y_2, we have $y_1 \mapsto y_1$ and $y_2 \mapsto -y_2$. It remains to show only that $y_4 \mapsto -y_4$. Assume to the contrary that $y_4 \mapsto y_4$.

Let $U \subset \mathbb{P}(1, 1, 1, 2)$ be the chart $y_4 \neq 0$. Then $U \simeq \mathbb{C}^3_{z_1, z_2, z_3}/\boldsymbol{\mu}_2(1, 1, 1)$. Let $X^\sharp$ be the pull-back of $X \cap (U \times \mathbb{C}^2_{u,v})$ on $\mathbb{C}^3_{z_1, z_2, z_3} \times \mathbb{C}^2_{u,v}$ and let $P^\sharp \in X^\sharp$ be the preimage of P.

Since the induced map $X^\sharp \to X$ is étale in codimension one, $(X^\sharp, P^\sharp) \to (X, P)$ is the index-one cover. Hence $(X^\sharp, P^\sharp) \to (X, P)/\boldsymbol{\mu}_2$ is also the index-one cover of the terminal point $(X, P)/\boldsymbol{\mu}_2$ of index 4 (the last is true because the action of $\boldsymbol{\mu}_2$ is free in codimension one). Hence the morphism is a $\boldsymbol{\mu}_4$-covering by the structure of terminal singularities. However $(X, P)/\boldsymbol{\mu}_2$ is the quotient of $(X^\sharp, P^\sharp)$ by commuting $\boldsymbol{\mu}_2$-actions:

$$(z_1, z_2, z_3, u, v) \mapsto (-z_1, -z_2, -z_3, u, v), (z_1, -z_2, z_3, -u, -v)$$

This is a contradiction, and we have $y_4 \mapsto -y_4$ as claimed. □

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