A $\phi_{1,3}$-Filtration of the Virasoro Minimal Series $M(p,p')$ with $1 < p'/p < 2$

Dedicated to Professor Heisuke Hironaka on the occasion of his seventy-seventh birthday

By

B. Feigin[*], E. Feigin,[**] M. Jimbo,[***] T. Miwa[†] and Y. Takeyama[‡]

Abstract

The filtration of the Virasoro minimal series representations $M_{r,s}^{(p,p')}$ induced by the $(1,3)$-primary field $\phi_{1,3}(z)$ is studied. For $1 < p'/p < 2$, a conjectural basis of $M_{r,s}^{(p,p')}$ compatible with the filtration is given by using monomial vectors in terms of the Fourier coefficients of $\phi_{1,3}(z)$. In support of this conjecture, we give two results. First, we establish the equality of the character of the conjectural basis vectors with the character of the whole representation space. Second, for the unitary series $(p' = p+1)$, we establish for each m the equality between the character of the degree m monomial basis and the character of the degree m component in the associated graded module $\mathrm{gr}(M_{r,s}^{(p,p+1)})$ with respect to the filtration defined by $\phi_{1,3}(z)$.

§1.   Introduction

In this paper we study a filtration on the Virasoro minimal modules by the $\phi_{1,3}$ primary field. We first state the problem in a general scheme. Let $S = \{\phi_\alpha(z)\}_{\alpha \in A}$ be a set of vertex operators acting on a graded vector space V. In the actual setting, the representation space V is a direct sum $V = \oplus_{\beta \in B} V^{(\beta)}$ such that the index set A is a subset of B, the grading is of the form $V^{(\beta)} = \sum_{i \in \Delta^{(\beta)} + \mathbb{Z}_{\geq 0}} V_i^{(\beta)}$, and the action of the vertex operator

$$\phi_\alpha(z) : V^{(\beta)} \to V^{(\gamma)}$$

is decomposed as $\sum_{n \in \mathbb{Z} + \Delta^{(\gamma)} - \Delta^{(\beta)}} \phi_{\alpha,-n}^{(\gamma,\beta)} z^{n-\Delta^{(\alpha)}}$, where $\phi_{\alpha,-n}^{(\gamma,\beta)} : V_i^{(\beta)} \to V_{i+n}^{(\gamma)}$.

Now for some fixed vector $v_0 \in V^{(\alpha_0)}$, one can define a sequence of subspaces $E_0(V) \subset E_1(V) \subset E_2(V) \subset \cdots \subset V$ by setting

$$(1.1) \quad E_m(V) = \mathrm{span}\{\phi_{\alpha_1,-n_1}^{(\beta_0,\beta_1)} \phi_{\alpha_2,-n_2}^{(\beta_1,\beta_2)} \ldots \phi_{\alpha_k,-n_k}^{(\beta_{k-1},\beta_k)} v_0 \mid$$
$$\alpha_i \in A, \beta_j \in B, \beta_k = \alpha_0, n_i \in \mathbb{Z} + \Delta^{(\beta_{i-1})} - \Delta^{(\beta_i)}, k \leq m\}.$$

In what follows we assume that the Fourier coefficients $\{\phi_{\alpha,-n}^{(\gamma,\beta)}\}$ generate the whole V from v_0. In this case the above construction gives a filtration $E = \{E_m(V)\}_{m=0}^\infty$ on V, which we refer to as the S-filtration. In our examples, V is a representation of the Virasoro or the $\widehat{\mathfrak{sl}}_2$ algebra, and $\phi_\alpha(z)$ are vertex operators from the corresponding conformal field theory.

Let us consider the associated graded space $\mathrm{gr}^E V := \oplus_{m=0}^\infty E_m(V)/E_{m-1}(V)$. Note that the space $\mathrm{gr}^E V$ is bi-graded by the grading $\mathrm{gr}^E V = \oplus_{m=0}^\infty \mathrm{gr}_m^E V$ and that of $V^{(\beta)} = \sum_i V_i^{(\beta)}$. Now there are two natural questions:

(i) Find the bi-graded character

$$\mathrm{ch}_{q,v} \mathrm{gr}^E V^{(\alpha)} := \sum_{m,n} q^n v^m \dim\left(V_n^{(\alpha)} \cap E_m(V)\right)/(V_n^{(\alpha)} \cap E_{m-1}(V)).$$

(ii) Find a monomial basis of V which is compatible with the filtration E. This means that one needs to construct a basis of V of the form

$$\phi_{\alpha_1,-n_1}^{(\beta_0,\beta_1)} \phi_{\alpha_2,-n_2}^{(\beta_1,\beta_2)} \ldots \phi_{\alpha_k,-n_k}^{(\beta_{k-1},\beta_k)} v_0 \quad (\beta_k = \alpha_0)$$

with certain α_i, n_i, β_j such that the images of the basis vectors with $k \leq m$ form a basis of $E_m(V)$.

In the case when V is a Virasoro minimal model and S consists of one field $\phi_{2,1}(z)$, these questions have been studied in [FJMMT1], [FJMMT2] under

certain conditions. In this paper we consider the $\phi_{1,3}(z)$ field. The corresponding filtration on the Virasoro modules is called the $(1,3)$-filtration. We also clarify the connection between the $\phi_{1,3}$ case and the fusion filtration on the representations of $\widehat{\mathfrak{sl}}_2$. Let us describe our results.

Let $p < p'$ be relatively prime positive integers, and let $M_{r,s}^{(p,p')}$ $(1 \leq r < p,$ $1 \leq s < p')$ be the irreducible representations of the Virasoro algebra with the central charge $c = 13 - 6(t + 1/t)$ and highest weight $\Delta_{r,s} = ((rt - s)^2 - (t - 1)^2)/4t$, where $t = p'/p$. We consider the $(1,3)$ primary field $(s - s' = -2, 0, 2)$

$$\phi^{(s',s)}(z) = \sum_{n \in \mathbb{Z} + \Delta_{r,s'} - \Delta_{r,s}} \phi_{-n}^{(s',s)} z^{n - \Delta_{1,3}}, \quad \phi_{-n}^{(s',s)} : M_{r,s}^{(p,p')} \to M_{r,s'}^{(p,p')}.$$

One of our goals is to construct a monomial basis of $M_{r,s}^{(p,p')}$ compatible with the $(1,3)$-filtration under the condition $1 < t < 2$. In this paper we propose a set of vectors as the monomial basis. Let us describe it. Fix a highest weight vector $|r, s\rangle$ of $M_{r,s}^{(p,p')}$. For r and s, we determine $b(r, s)$ by the equality

$$\Delta_{r,b(r,s)} = \min_{\substack{1 \leq a \leq p'-1 \\ a \equiv s \bmod 2}} \Delta_{r,a}.$$

Next we define some rational numbers $w(a, b, c)$ for $1 \leq a, b, c \leq p'$ such that $|a - b|$ and $|b - c|$ are 0 or 2. In the case of $1 < t \leq 5/3$, they are given as follows:

$$w(s \pm 2, s, s \mp 2) = \frac{2}{t},$$

$$w(s, s + 2, s + 2) = w(s + 2, s + 2, s) = 2 - \left\{ \frac{s+1}{t} \right\},$$

$$w(s, s, s + 2) = w(s + 2, s, s) = 1 + \left\{ \frac{s+1}{t} \right\},$$

$$w(s, s + 2, s) = -2 \left\{ \frac{s+1}{t} \right\} + x(s),$$

$$w(s, s, s) = 2,$$

$$w(s, s - 2, s) = 2 \left\{ \frac{s+1}{t} \right\} - \frac{4}{t} + 5 - x(s).$$

Here $\{u\} := u - [u]$, where $[u]$ is the integer part of u, and

$$x(s) = \begin{cases} 2 & (1 \leq s < p'/2) \\ 3 & (p'/2 < s \leq p' - 1). \end{cases}$$

In the case of $5/3 < t < 2$ we have some choices of the definition of the values $w(s, s \pm 2, s)$ and $w(s, s, s)$ under the condition described as (2.23) (see

(2.16)–(2.21) and the text below for the precise definition). It is a part of our conjecture that, for any choice, the vectors given in the following constitute a basis. We call a vector of the form

$$(1.2) \qquad \phi_{-n_1}^{(s_0,s_1)} \ldots \phi_{-n_m}^{(s_{m-1},s_m)} |r, b(r,s)\rangle \quad (s_0 = s, s_m = b(r,s))$$

admissible monomial if it satisfies the condition

$$(1.3) \qquad n_i - n_{i+1} \geq w(s_{i-1}, s_i, s_{i+1}) \qquad (i = 1, \ldots, m-1).$$

Now we state our conjecture:

Conjecture. The set of admissible monomials form a basis of $M_{r,s}^{(p,p')}$.

To support the conjecture, we prove two statements:

A. The character of the proposed basis coincides with the character of $M_{r,s}^{(p,p')}$.

B. In the unitary case $p' = p+1$, the bi-graded character of the proposed basis and that of $\mathrm{gr}^E M_{r,s}^{(p,p')}$ coincide, where $\mathrm{gr}^E M_{r,s}^{(p,p')}$ is the associated graded space with respect to the $(1,3)$-filtration on $M_{r,s}^{(p,p')}$.

The proof of the statement A is based on combinatorics and the Rocha-Caridi character formula. Namely we show that the character of admissible monomials with fixed m can be written in the form

$$(1.4) \qquad \frac{q^{\Delta_{r,s}}}{(q)_m} I_m(q), \qquad (q)_m = \prod_{i=1}^{m}(1 - q^i),$$

where $I_m(q)$ is an alternating sum of the characters for the fusion products [FL]. We also prove that the Rocha-Caridi formula for the character of $M_{r,s}^{(p,p')}$ can be rewritten as $\sum_{m \geq 0} \frac{q^{\Delta_{r,s}}}{(q)_m} I_m(q)$. In order to prove that the admissible monomials form a basis of $M_{r,s}^{(p,p')}$, it is enough to rewrite any monomial $\phi_{-n_1}^{(s_0,s_1)} \cdots \phi_{-n_m}^{(s_{m-1},s_m)} |r, b(r,s)\rangle$ in terms of admissible monomials of length less than or equal to m. This was done in [FJMMT1], [FJMMT2] in the case of the $(2,1)$ field using quadratic relations for its Fourier components. Similar quadratic relations can be written also for the $(1,3)$ field using the results of [DF]. Still it is not clear to us how to rewrite an arbitrary monomial in terms of admissible ones using these relations.

The proof of the statement B is based on the coset construction. For $i = 0, 1$ and integers r, k with $0 \leq r - 1 \leq k$, let

$$(1.5) \qquad L_{i,1} \otimes L_{r-1,k} = \bigoplus_{\substack{1 \leq s \leq k+2 \\ s:\ s+r+i \text{ even}}} M_{r,s}^{(k+2,k+3)} \otimes L_{s-1,k+1}$$

be the decomposition of the tensor product of the irreducible highest weight representations of $\widehat{\mathfrak{sl}}_2$. Here $L_{j,k}$ denotes the level k module with the highest weight j with respect to $\mathfrak{sl}_2 \otimes 1 \hookrightarrow \widehat{\mathfrak{sl}}_2$. We denote by $v_{j,k}$ the highest weight vector. Using a result of [L], we establish the connection between the $(1,3)$ filtration and the fusion filtration on the left hand side of (1.5). Namely, consider the action of the algebra $\widehat{\mathfrak{sl}}_2$ on $L_{i,1}$ and the corresponding S-filtration $G_m(L_{i,1})$, where $S = \{e(z), h(z), f(z)\}$, $x(z) = \sum(x \otimes t^n)z^{-n-1}$, and $\{e, h, f\}$ is the standard basis of $\mathfrak{sl}_2$. We call this filtration the Poincaré-Birkhoff-Witt (PBW) filtration. We show that

$$(1.6) \quad U(\widehat{\mathfrak{sl}}_2) \cdot (G_m(L_{i,1}) \otimes v_{r-1,k}) = \bigoplus_{\substack{1 \leq s \leq k+2 \\ s:\ s+r+i\ \text{even}}} E_m(M_{r,s}^{(k+2,k+3)}) \otimes L_{s-1,k+1},$$

where $E_m(M_{r,s}^{(k+2,k+3)})$ is the $(1,3)$-filtration. Thus the study of this filtration can be reduced to the study of the left hand side of (1.6).

We recall that, for two cyclic $\mathfrak{g}$-modules V_1 and V_2 with cyclic vectors v_1 and v_2, the fusion filtration on the tensor product $V_1(z_1) \otimes V_2(z_2)$ of evaluation representations of $\mathfrak{g} \otimes \mathbb{C}[u]$ is defined by

$$F_m(V_1(z_1) \otimes V_2(z_2)) = \text{span}\{(g^{(1)} \otimes u^{i_1}) \cdots (g^{(s)} \otimes u^{i_s}) \cdot (v_1 \otimes v_2)\},$$

where $g^{(i)} \in \mathfrak{g}$ and $i_1 + \ldots + i_s \leq m$. One can easily show that

$$F_m(L_{i,1} \otimes L_{l,k}) = U(\widehat{\mathfrak{sl}}_2) \cdot (G_m(L_{i,1}) \otimes v_{l,k}).$$

Using (1.6) we express the bi-graded character of the $(1,3)$-filtration via that of the PBW-filtration on $L_{i,1}$. We thus get the bi-graded version (1.4) of the Rocha-Caridi formula as an alternating sum of the bi-graded characters of weight subspaces of $L_{i,1}$.

We finish the introduction with a discussion of possible generalizations. Note that the integer level k in the coset construction (1.5) can be replaced by a fractional one. This generalization leads to the coset realization of the general $M_{r,s}^{(p,p')}$. We expect that the above construction can be applied to the general case.

Now consider the $(1,2)$ field $\phi_{1,2}(z)$. In this case the decomposition (1.5) should be replaced by

$$(L_{0,1} \oplus L_{1,1}) \otimes L_{r-1,k} = \bigoplus_{1 \leq s \leq k+2} M_{r,s}^{(k+2,k+3)} \otimes L_{s-1,k+1},$$

and the algebra $\widehat{\mathfrak{sl}}_2$ by the vertex (intertwining) operator $\mathbb{C}^2(z)$ acting on $L_{0,1} \oplus$

$L_{1,1}$. This vertex operator induces the filtration $G'_m(L_{0,1} \oplus L_{1,1})$. Then

$$U(\widehat{\mathfrak{sl}}_2) \cdot [G'_m(L_{0,1} \oplus L_{1,1}) \otimes L_{r-1,k}] = \bigoplus_{1 \leq s \leq k+2} H'_m(M_{r,s}^{(k+2,k+3)}) \otimes L_{s-1,k+1},$$

where $H'_m(M_{r,s}^{(k+2,k+3)})$ is the $(1,2)$-filtration. As in the $(1,3)$-case, the bi-graded character

$$\sum_{m \geq 0} v^m \mathrm{ch}_q H'_m(M_{r,s}^{(k+2,k+3)})$$

can be expressed as an alternating sum of the bi-graded characters of $\mathbb{C}^2(z)$-filtration on $L_{0,1} \oplus L_{1,1}$.

Our paper is organized as follows: In Section 2, the character of the Virasoro module $M_{r,s}^{(p,p')}$ is written as an alternating sum by using the character of the weight l component of the fusion product π_2^{*m} (Lemma 2.2). The main result in this section is a proof of the statement that for $1 < p'/p < 2$ the alternating sum with fixed m is the character of the admissible monomials of length m (Proposition 2.3). Section 3 prepares some exact sequences of the fusion products and vanishing of the homology groups of the Lie subalgebra $\mathfrak{n}_+ \subset \widehat{\mathfrak{sl}}_2$ generated by $f \otimes 1$ and $e \otimes t^{-1}$ with coefficients in the tensor product of certain finite dimensional modules and irreducible highest weight $\widehat{\mathfrak{sl}}_2$-modules. This is used in Section 5 when the characters of the highest weight vectors in integrable $\widehat{\mathfrak{sl}}_2$-modules are computed. In Section 4, by using Lashkevich's construction of vertex operators in the GKO construction, an isomorphism is given between the fusion product of level 1 and level k irreducible highest weight $\widehat{\mathfrak{sl}}_2$-modules and the associated graded module with respect to the filtration defined by the $(1,3)$ primary field (Proposition 4.4). Section 5 is devoted to the calculation of the characters for the m-th graded components of the Virasoro unitary series with the $(1,3)$-filtration (Theorem 5.15). The result coincides with the combinatorial characters computed in Section 2.

Throughout the text, e, f, h denotes the standard basis of $\mathfrak{sl}_2$, and π_j denotes the $(j+1)$-dimensional irreducible representation.

§2.   Conjectural Monomial Basis by (1,3) Field

§2.1.   Formulation

In this section, we consider Virasoro modules in the minimal series which are not necessarily unitary.

Let Vir be the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}.$$

Throughout this section, we fix relatively prime positive integers p, p' satisfying $3 \le p < p'$. We denote by $M_{r,s}^{(p,p')}$ $(1 \le r \le p-1, 1 \le s \le p'-1)$ the irreducible Vir-module with central charge $c = 13 - 6(t + \frac{1}{t})$ and highest weight

$$\Delta_{r,s} = \frac{(rt - s)^2 - (t-1)^2}{4t} \, ,$$

where

$$t = \frac{p'}{p} \, .$$

We fix $1 \le r \le p-1$, and consider the $(1,3)$ primary field

$$\phi^{(s',s)}(z) = \sum_{n \in \mathbb{Z} + \Delta_{r,s'} - \Delta_{r,s}} \phi_{-n}^{(s',s)} z^{n - \Delta_{1,3}}.$$

The Fourier coefficients $\phi_{-n}^{(s',s)}$ are operators acting as $\left(M_{r,s}^{(p,p')} \right)_d \rightarrow \left(M_{r,s'}^{(p,p')} \right)_{d+n}$, where $\left(M_{r,s}^{(p,p')} \right)_d = \{ |v\rangle \in M_{r,s}^{(p,p')} \mid L_0 |v\rangle = d|v\rangle \}$ stands for the graded component. They are characterized by the intertwining property

$$[L_n, \phi^{(s',s)}(z)] = z^n \left(z \frac{d}{dz} + (n+1)\Delta_{1,3} \right) \phi^{(s',s)}(z).$$

A non-trivial $(1,3)$ primary field exists if and only if $s' = s, s \pm 2$ and $(s', s) \neq (1,1), (p'-1, p'-1)$. Moreover, it is unique up to a constant multiple. We fix the highest weight vector $|r, s\rangle \in \left(M_{r,s}^{(p,p')} \right)_{\Delta_{r,s}}$ and use the normalization $\phi_{\Delta_{r,s} - \Delta_{r,s'}}^{(s',s)} |r, s\rangle = |r, s'\rangle$.

Our problem is to construct a basis of the representation $M_{r,s}^{(p,p')}$ by using the operators $\phi_{-n}^{(s',s)}$. In this paper we restrict to the case

(2.1) $$1 < t < 2,$$

and give a partial answer to this problem.

The form of the basis we propose is similar to the one studied in [FJMMT1], [FJMMT2] using the $(2,1)$ primary field. We define a set of weights

(2.2) $$w(a, b, c) \in \Delta_{r,a} - 2\Delta_{r,b} + \Delta_{r,c} + \mathbb{Z},$$

and consider vectors of the form

(2.3) $$\phi_{-n_1}^{(s_0,s_1)} \cdots \phi_{-n_m}^{(s_{m-1},s_m)} |r, s_m\rangle$$

satisfying

$$(2.4) \qquad n_i - n_{i+1} \geq w(s_{i-1}, s_i, s_{i+1}).$$

The actual form of the weights (2.2) is a little involved, and we postpone their definition to subsection 2.3 (see (2.16)–(2.21) and paragraphs following them). The vectors (2.3) are parametrized by spin-1 and level-p' restricted paths, i.e., sequences of integers

$$\mathbf{s} = (s_0, s_1, \ldots, s_m)$$

satisfying

$$1 \leq s_i \leq p' - 1,$$
$$s_i = s_{i+1} \text{ or } s_{i+1} \pm 2,$$
$$(s_i, s_{i+1}) \neq (1,1), (p'-1, p'-1).$$

We call them simply *paths*. The non-negative integer m is called the length of the path. We denote by $\mathcal{P}^{(p')}_{a,b,m}$ the set of paths $(s_0, s_1, \ldots, s_m)$ of length m satisfying $s_0 = a$ and $s_m = b$. Note that the parity of s_i is common with each path. In particular, we have $a \equiv b \bmod 2$ if $\mathcal{P}^{(p')}_{a,b,m}$ is non-empty.

The set of rational numbers

$$(n_1, \ldots, n_m)$$

in the expression (2.3) is called a rigging associated with the path $(s_0, s_1, \ldots, s_m)$. A rigging satisfies

$$n_i \in \mathbb{Z} + \Delta_{r,s_{i-1}} - \Delta_{r,s_i}.$$

A path with rigging is called a rigged path.

A rigged path of length m is called *admissible* if and only if (2.4) and the following condition hold:

$$(2.5) \qquad n_m \geq \Delta_{r,s_{m-1}} - \Delta_{r,s_m} + \delta_{s_{m-1},s_m}.$$

We denote by $\mathcal{R}^{(p,p')}_{r,a,b,m}$ the set of admissible rigged paths of length m such that $s_0 = a$ and $s_m = b$.

Finally, we fix the boundary of a path $s_m = b(r,s)$ by the following rule: $b = b(r,s)$ is the unique integer satisfying $1 \leq b \leq p' - 1$, $b \equiv a \bmod 2$ and

$$(2.6) \qquad \Delta_{r,b} = \min_{\substack{1 \leq s \leq p'-1 \\ s \equiv a \bmod 2}} \Delta_{r,s}.$$

We will comment on this choice in the next subsection 2.2.

Now we put forward the

Conjecture 2.1. The set of vectors (2.3), where $(n_1, \ldots, n_m)$ runs through the set $\bigcup_{m \geq 0} \mathcal{R}_{r,a,b,m}^{(p,p')}$ and $b = b(r,a)$ as given in (2.6), constitute a basis of $M_{r,a}^{(p,p')}$.

Note that the meaning of the condition (2.5) is clear. If $\Delta_{r,s_m} + n_m < \Delta_{r,s_{m-1}}$, the vector $\phi_{-n_m}^{(s_{m-1},s_m)} |r, s_m\rangle$ is zero because $\left(M_{r,s_{m-1}}^{(p,p')} \right)_{\Delta_{r,s_m}+n_m} = \{0\}$. If $s_{m-1} = s_m$ and $n_m = 0$, the vector $\phi_0^{(s_m,s_m)} |r, s_m\rangle$ is proportional to $|r, s_m\rangle$ because it belongs to $\left(M_{r,s_m}^{(p,p')} \right)_{\Delta_{r,s_m}} = \mathbb{C}|r, s_m\rangle$. We have also $\Delta_{1,1} = 0$ and $\left(M_{1,1}^{(p,p')} \right)_1 = \{0\}$. This is not in contradiction to the condition (2.5) for $r = s_m = s_{m-1} = 1$ because the case $s_m = s_{m-1} = 1$ is prohibited.

In order to support the conjecture, we will show that the set of admissible monomials has the same character as that of $M_{r,a}^{(p,p')}$ (Theorem 2.4). The conjecture will follow if we show further that the above set of vectors span the space $M_{r,a}^{(p,p')}$. So far we have not been able to check the latter point in full generality.

§2.2. A character identity

In this subsection, we rewrite the character of $M_{r,a}^{(p,p')}$ in a form suitable for comparison with the set of paths.

We use the q-supernomial coefficients introduced in [SW]. They are a q-analog of the weight multiplicities of tensor products of various π_k, where π_k denotes the irreducible $\mathfrak{sl}_2$-module of dimension $k+1$. As shown in [FF1], they can be defined as the coefficients of z^l of the graded character of the fusion product (for the definition and properties of fusion product, see Section 3).

$$\mathrm{ch}_{q,z}\left(\pi_1^{*L_1} * \cdots * \pi_N^{*L_N} \right) = \sum_{l \in \mathbb{Z} + \frac{1}{2}\sum_{j=1}^N jL_j} \begin{bmatrix} L_1, \ldots, L_N \\ l \end{bmatrix}_q z^l.$$

Here we will need only the special case $N = 2$, $L_1 = 0$. Set

$$(2.7) \qquad\qquad S_{m,l}(q) := \begin{bmatrix} 0, m \\ l \end{bmatrix}_{q^{-1}}.$$

Formula (2.9) in [SW] gives

$$S_{m,l}(q) = \sum_{\nu \in \mathbb{Z}} q^{(\nu+l-m)(\nu+l)+\nu(\nu-m)} \begin{bmatrix} m \\ \nu \end{bmatrix}_q \begin{bmatrix} \nu \\ m-l-\nu \end{bmatrix}_q.$$

In the right hand side

$$\begin{bmatrix} L \\ a \end{bmatrix}_q = \begin{cases} \frac{(q^{L-a+1})_a}{(q)_a} & (a \in \mathbb{Z}_{\geq 0}, L \in \mathbb{Z}), \\ 0 & (\text{otherwise}), \end{cases}$$

stands for the q-binomial symbol, and $(x)_n = (x)_\infty/(q^n x)_\infty$, $(x)_\infty = \prod_{i=0}^\infty (1 - q^i x)$.

Recall that the character $\chi_{r,s}^{(p,p')}(q)$ of $M_{r,s}^{(p,p')}$ is given by [RC]

$$(2.8) \qquad q^{-\Delta_{r,s}^{(p,p')}} \chi_{r,s}^{(p,p')}(q) = \sum_{\lambda \in \mathbb{Z}} \frac{1}{(q)_\infty} q^{\lambda^2 pp' + \lambda(p'r - ps)}$$
$$- \sum_{\lambda \in \mathbb{Z}} \frac{1}{(q)_\infty} q^{\lambda^2 pp' + \lambda(p'r + ps) + rs}.$$

Lemma 2.2. *For any $b \in \mathbb{Z}$ satisfying $b \equiv a \bmod 2$, the character (2.8) can be written in terms of (2.7) as*

$$(2.9) \qquad q^{-\Delta_{r,a}^{(p,p')}} \chi_{r,a}^{(p,p')}(q) = \sum_{m \geq 0} \frac{1}{(q)_m} I_{r,a,b,m}^{(p,p')}(q),$$

where

$$(2.10) \ I_{r,a,b,m}^{(p,p')}(q) = \sum_{\lambda \in \mathbb{Z}} q^{\lambda^2 pp' + \lambda(p'r - pa) + m^2 - ((a-b)/2 - p'\lambda)^2} S_{m,(a-b)/2 - p'\lambda}(q)$$
$$- \sum_{\lambda \in \mathbb{Z}} q^{\lambda^2 pp' + \lambda(p'r + pa) + ra + m^2 - ((a+b)/2 + p'\lambda)^2} S_{m,(a+b)/2 + p'\lambda}(q).$$

Proof. The q-supernomial coefficients satisfy the recurrence relations ([SW, Lemma 2.3])

$$\begin{bmatrix} L_1, L_2 \\ a \end{bmatrix}_q = q^{L_1 + L_2 - 1} \begin{bmatrix} L_1 - 2, L_2 \\ a \end{bmatrix}_q + \begin{bmatrix} L_1 - 2, L_2 + 1 \\ a \end{bmatrix}_q$$

$$\begin{bmatrix} L_1, 0 \\ a \end{bmatrix}_q = \begin{bmatrix} L_1 \\ a + L_1/2 \end{bmatrix}_q.$$

Iterating this k times, we find

$$\begin{bmatrix} L_1, L_2 \\ a \end{bmatrix}_q = \sum_{m=0}^k q^{(k-m)(L_1 + L_2 - k)} \begin{bmatrix} k \\ m \end{bmatrix}_q \begin{bmatrix} L_1 - 2k, L_2 + m \\ a \end{bmatrix}_q.$$

Choosing $L_1 = 2N, L_2 = 0, k = N$, changing $q \to q^{-1}$ and letting $N \to \infty$ we obtain for all $l \in \mathbb{Z}$

$$\frac{1}{(q)_\infty} = \sum_{m \geq 0} \frac{q^{m^2-l^2}}{(q)_m} S_{m,l}(q).$$

In each summand of the first (resp. second) sum of (2.8), replace $1/(q)_\infty$ by the right hand side of the above identity, choosing $l = (a-b)/2 - p'\lambda$ (resp. $(a+b)/2 + p'\lambda$). The desired identity follows. □

Though (2.9) is an identity valid for any $b \in \mathbb{Z}$, in most cases the polynomial (2.10) comprises negative coefficients. We prove in subsection 2.4 that, if $1 < t < 2$ and (r,b) satisfies (2.6), then the coefficients of $I_{r,a,b,m}^{(p,p')}(q)$ are nonnegative integers. In fact we will show that it can be written as a configuration sum over the set of paths $\mathcal{P}_{a,b,m}^{(p')}$. Define the weight of a path $\mathbf{s} \in \mathcal{P}_{a,b,m}^{(p')}$ by

$$E(\mathbf{s}) = \sum_{i=1}^{m-1} iw(s_{i-1}, s_i, s_{i+1}).$$

Proposition 2.3.   *Under the conditions $1 < t < 2$ and (2.6), we have an equality*

$$(2.11) \qquad I_{r,a,b,m}^{(p,p')}(q) = \sum_{\mathbf{s} \in \mathcal{P}_{a,b,m}^{(p')}} q^{E(\mathbf{s})+m(\Delta_{r,s_{m-1}} - \Delta_{r.b} + \delta_{s_{m-1},b}) + \Delta_{r,b} - \Delta_{r,a}}.$$

We give a proof in Section 2.4. Note that the exponent of q in (2.11) is an integer because of (2.2).

From Proposition 2.3 immediately follows

Theorem 2.4.   *Notation being as above, we have an identity for the character*

$$\chi_{r,a}^{(p,p')}(q) = \sum_{m \geq 0} \frac{1}{(q)_m} \sum_{\mathbf{s} \in \mathcal{P}_{a,b,m}^{(p')}} q^{E(\mathbf{s})+m(\Delta_{r,s_{m-1}} - \Delta_{r.b} + \delta_{s_{m-1},b}) + \Delta_{r,b}}.$$

§2.3.   Definition of $w(a,b,c)$

In this subsection we introduce our weight $w(a,b,c)$.

In (2.11), we fixed $b = b(r,a)$ by the condition (2.6). Conversely, for a given b, r for which (2.6) is valid is either $r_1(b) = \left[\frac{b+1}{t}\right]$ or $r_2(b) = \left[\frac{b-1}{t}\right] + 1$. Set

$$\tau(b) = \left[\frac{b+1}{t}\right] - \left[\frac{b-1}{t}\right].$$

We have $\tau(b) = 1$ or 2. If $\tau(b) = 1$ we have $r_1(b) = r_2(b)$, and if $\tau(b) = 2$ we have $r_1(b) = r_2(b) + 1$. We list a few other properties of $\tau(s)$.

$$(2.12) \qquad\qquad \tau(1) = 1,$$

$$(2.13) \qquad\qquad \tau(2) = \begin{cases} 2 & \text{if } 1 < t < \frac{3}{2}; \\ 1 & \text{if } \frac{3}{2} < t < 2, \end{cases}$$

$$(2.14) \qquad\qquad \tau(p' - 1) = 2,$$

$$(2.15) \qquad\qquad \tau(s) = \tau(p' - s) \text{ if } 1 < s < p' - 1.$$

Set

$$\{x\} = x - [x]$$

where $[x]$ is the integer part of x.

We define the weight $w(a, b, c)$ in the following form:

$$(2.16) \qquad w(s \pm 2, s, s \mp 2) = \frac{2}{t},$$

$$(2.17) \qquad w(s, s+2, s+2) = w(s+2, s+2, s) = 2 - \left\{ \frac{s+1}{t} \right\},$$

$$(2.18) \qquad w(s, s, s+2) = w(s+2, s, s) = 1 + \left\{ \frac{s+1}{t} \right\},$$

$$(2.19) \qquad w(s, s+2, s) = -2 \left\{ \frac{s+1}{t} \right\} + x(s),$$

$$(2.20) \qquad w(s, s, s) = 3 - \tau(s),$$

$$(2.21) \qquad w(s, s-2, s) = 2 \left\{ \frac{s+1}{t} \right\} - \frac{4}{t} + y(s).$$

Here $x(s), y(s)$ are integers defined as follows. If $\tau(s) = 2$, then $x(s) = 2$ and $y(s) = 4$. In the case of $\tau(s) = 1$ we have two possibilities: $(x(s), y(s)) = (2, 3)$ or $(3, 2)$. In the following we give the precise choice of the values $x(s)$ and $y(s)$ when $\tau(s) = 1$, and discuss the motivation for it. To simplify the text we use the notation $C(s) = 1_A, 1_B$ to signify the choice at s as indicated in the following table.

$C(s)$	$\tau(s)$	$x(s)$	$y(s)$
1_A	1	2	3
1_B	1	3	2
2	2	2	4

For example, the wording "$C(s) = 1_B$" means that $\tau(s) = 1$ and we set $(x(s), y(s)) = (3, 2)$. In addition to it we write $C(s) = 2$, which means $\tau(s) = 2$ and hence $(x(s), y(s)) = (2, 4)$.

Now we define $x(s)$ and $y(s)$ for s such that $\tau(s) = 1$. If $1 < t \leq 5/3$ and $\tau(s) = 1$, then we take

$$C(s) = \begin{cases} 1_A & (1 \leq s < \frac{p'}{2}), \\ 1_B & (\frac{p'}{2} < s \leq p'-1). \end{cases}$$

If $5/3 < t < 2$, then we determine $C(s)$'s so that the sequence $C(1), C(2), \ldots,$ $C(p'-1)$ becomes in the form

$$1_A, 1_B, 1_A, 1_B, \ldots, 1_B, 1_A, 2, 1_B, 1_A, 1_B, \ldots, 1_B, 1_A, 2,$$
$$1_B, 1_A, 1_B, \ldots, 1_A, 2, 1_B, 1_A, 1_B, \ldots, 1_B, 1_A, 2,$$

where 1's between successive 2's come always with an even number. Below we will motivate the above assignment and show that it can be made consistently.

Let us seek for the weights $w(a, b, c)$ in the above form (2.16)–(2.21). We will take them independently of the choice $r = r_1(b)$ or $r_2(b)$. We demand further the following.

(i) (2.11) holds for $m = 2$,

(ii) left-right symmetry $w(a, b, c) = w(c, b, a)$,

(iii) symmetry reflecting $M_{r,s}^{(p,p')} \simeq M_{p-r,p'-s}^{(p,p')}$,

(2.22) $$w(a, b, c) = w(p'-a, p'-b, p'-c),$$

(iv)

(2.23) $$\text{if } C(s) = 1_B \text{ then } C(s+2) \neq 1_A.$$

The last condition turns out to be necessary in the course of the proof of (2.11), see subsection 2.4.

The validity of (2.11) for $m = 2$ gives a linear constraint on the weights. There are three cases: $a = b \pm 4, b \pm 2, b$. In the first two cases, the relevant weights are (2.16)–(2.18). They are independent of $\tau(s)$. It is easy to check that the constraint is satisfied in these cases. In the third case, the relevant weights are (2.19)–(2.21). Here the value of $\tau(s)$ matters. If $\tau(s) = 2$, the weights are uniquely given by (2.19)–(2.21), and they satisfy the constraint. If $\tau(s) = 1$, we must specify $C(s) = 1_A$ or 1_B. For $s = 1$ and 2, see (2.12) and (2.13). In these cases, the constraint implies

(2.24) $$C(1) = 1_A,$$

(2.25) $$C(2) = 1_B \quad \text{for} \quad \frac{3}{2} < t < 2.$$

If $2 < s < p' - 2$, the constraint is satisfied for either choice.

The left-right symmetry (ii) is automatically satisfied by the formulas (2.16)–(2.21).

Symmetry (2.22) is also valid for (2.16)–(2.18) and (2.20). It is obvious for (2.16); and follows from

$$\left\{\frac{s}{t}\right\} + \left\{-\frac{s}{t}\right\} = 1 \text{ if } 1 < s < p'$$

for (2.17) $\leftrightarrow$ (2.18), and for (2.20).

We determine the choice of $1_{A/B}$ so that the symmetry (2.22) for (2.19) $\leftrightarrow$ (2.21) is valid. Since $C(1) = 1_A$ and $C(p' - 1) = 2$, we have

$$w(1, 3, 1) = w(p' - 1, p' - 2, p' - 1) = 4 - \frac{4}{t}.$$

For $1 < t < 3/2$, we have $C(2) = C(p' - 2) = 2$, and

$$w(2, 4, 2) = w(p' - 2, p' - 4, p' - 2) = 6 - \frac{6}{t};$$

for $3/2 < t < 2$, we have $C(2) = 1_B$. Setting $C(p' - 2) = 1_A$, we have

$$w(2, 4, 2) = w(p' - 2, p' - 4, p' - 2) = 5 - \frac{6}{t}.$$

For $2 < s < p' - 2$, the symmetry is valid if $\tau(s) = \tau(p' - s) = 2$; if $\tau(s) = \tau(p' - s) = 1$, we need to choose $1_{A/B}$ in such a way that

(2.26) $$C(s) = 1_A, 1_B \quad \Leftrightarrow \quad C(p' - s) = 1_B, 1_A.$$

The last requirement would be inconsistent if $\tau(s) = 1$ for $s = p'/2$. However, we have

Lemma 2.5. *If p' is even, we have $\tau(p'/2) = 2$.*

Proof. If p' is even, then p is odd. We have

$$\tau(p'/2) = [\alpha] - [\beta] \text{ where } \alpha = \frac{p}{2} + \frac{p}{p'}, \ \beta = \frac{p}{2} - \frac{p}{p'}.$$

Since $\alpha + \beta = p$ is odd and $1 < \alpha - \beta = 2p/p' < 2$, we have $\tau(p'/2) = 2$. $\quad\square$

We need also to satisfy the condition (2.23). Let us show the consistency of (2.24), (2.25), (2.26) and (2.23). Suppose that $\tau(s) = 1$. We choose $C(s) = 1_A$ or 1_B as follows.

If $1 < t < 3/2$, we have $C(1) = 1_A$ and $C(2) = 2$. Therefore, the following choice of 1_A or 1_B for s such that $\tau(s) = 1$ satisfies all the constraints.

$$(2.27) \qquad C(s) = \begin{cases} 1_A & \text{if } s < p'/2; \\ 1_B & \text{if } s > p'/2. \end{cases}$$

If $3/2 < t \leq 5/3$, we have $C(1) = 1_A$, $C(2) = 1_B$ and $\tau(4) = 2$. Therefore, the same choice (2.27) will do.

Before going to the case $5/3 < t < 2$, we prepare a few lemmas.

Lemma 2.6. *If $3/2 < t < 2$, we do not have the sequence $(\tau(s), \tau(s + 1)) = (2, 2)$.*

Proof. Since $\frac{1}{2} < \frac{1}{t} < 1$, the increment $\left[\frac{s+1}{t}\right] - \left[\frac{s}{t}\right]$ is either 0 or 1. Therefore, if $\tau(s) = \tau(s + 1) = 2$, we have

$$m \leq \frac{s-1}{t} < m + 1, \quad m + 1 \leq \frac{s}{t} < m + 2,$$
$$m + 2 \leq \frac{s+1}{t} < m + 3, \quad m + 3 \leq \frac{s+2}{t} < m + 4$$

for some integer m. From $\frac{s-1}{t} < m + 1$ and $m + 3 \leq \frac{s+2}{t}$ follows $t < \frac{3}{2}$. $\square$

Lemma 2.7. *Suppose that $\left[\frac{s}{t}\right] = m$, $\left[\frac{s+1}{t}\right] = m$. Then, $\left[\frac{s+2}{t}\right] = m + 1$.*

Proof. The statement follows from $1 < \frac{2}{t} < 2$. $\square$

Lemma 2.8. *Suppose that $\tau(s) = 2$, $\tau(s + 1) = \cdots = \tau(s + k) = 1$ and $\tau(s + k + 1) = 2$. Then, k is even.*

Proof. By Lemma 2.7, we have the sequence

$$\left[\frac{s-1}{t}\right] = m, \left[\frac{s}{t}\right] = m + 1, \left[\frac{s+1}{t}\right] = m + 2, \left[\frac{s+2}{t}\right] = m + 2,$$
$$\left[\frac{s+3}{t}\right] = m + 3, \left[\frac{s+4}{t}\right] = m + 3, \ldots \left[\frac{s+k-1}{t}\right] = m + l, \left[\frac{s+k}{t}\right] = m + l,$$
$$\left[\frac{s+k+1}{t}\right] = m + l + 1, \left[\frac{s+k+2}{t}\right] = m + l + 2.$$

Therefore, $k = 2l$. $\square$

If $5/3 < s < 2$, we have $C(1) = 1_A$ and $C(2) = 1_B$. If we determine the choice for $s < p'/2$, the rest is determined by (2.26). The constraint (2.23) together with the symmetry (2.26) implies that if $s + 2 < p'/2$ and $\tau(s) = \tau(s + 2) = 1$ we have

$$C(s) = 1_A, 1_B \quad \Rightarrow \quad C(s + 2) = 1_A, 1_B.$$

We start from $C(1) = 1_A$ and $C(2) = 1_B$ and continue as $1_A, 1_B, 1_A, 1_B, \ldots$ until 2 appears. By a similar argument as in the proof of Lemma 2.8, we can show that the first appearance of 2 is for even s. Therefore, from Lemmas 2.6 and 2.8 we can define the sequence $C(1), C(2), \ldots$ as

$$1_A, 1_B, 1_A, 1_B, \ldots, 1_B, 1_A, 2, 1_B, 1_A, 1_B, \ldots, 1_B, 1_A, 2, 1_B, 1_A, 1_B, \ldots$$

This sequence does not contain $(C(s), C(s+2)) = (1_B, 1_A)$. The constraint (2.23) is also satisfied.

§2.4.   Proof of Proposition 2.3

In this subsection we fix the weights $w(a, b, c)$ as in the previous section, and prove (2.11). To that end we consider the configuration sum

$$X_{a,b,c,m}(q) := \sum_{\mathbf{s} \in \mathcal{P}_{a,b,c,m}^{(p')}} q^{E(\mathbf{s})},$$

where $\mathcal{P}_{a,b,c,m}^{(p')}$ is the set of paths $(s_0, \ldots, s_{m+1})$ satisfying $s_0 = a, s_m = b$ and $s_{m+1} = c$. From the definition, $X_{a,b,c,m}(q) = 0$ unless $1 \le a, b, c \le p' - 1$, $b = c, c \pm 2$ and $(b, c) \ne (1, 1), (p' - 1, p' - 1)$. Note that $X_{a,b,c,m}(q)$ is uniquely determined from the initial condition $X_{a,b,c,0}(q) = \delta_{a,b}$ and the recurrence relation

$$(2.28) \qquad X_{a,b,c,m+1}(q) = \sum_{d=b,\, b \pm 2} q^{(m+1)w(d,b,c)} X_{a,d,b,m}(q).$$

Let us give an explicit formula for $X_{a,b,c,m}(q)$. As an ingredient we introduce the function $\widetilde{S}_{m,l}(q)$ defined by

$$\widetilde{S}_{m,l}(q) := \sum_{\nu \in \mathbb{Z}} q^{(\nu+l-m)(\nu+l-1)+\nu(\nu-m)} \begin{bmatrix} m \\ \nu \end{bmatrix}_q \begin{bmatrix} \nu \\ m-l-\nu \end{bmatrix}_q.$$

The functions $S_{m,l}(q)$ and $\widetilde{S}_{m,l}(q)$ are related to each other as follows.

Lemma 2.9.   *The following formulae hold:*

(2.29) $S_{m,-l}(q) = S_{m,l}(q), \qquad \widetilde{S}_{m,-l}(q) = q^l \widetilde{S}_{m,l}(q),$

(2.30) $S_{m+1,l}(q) = q^{-m-l-1} S_{m,l+1}(q) + S_{m,l}(q) + q^{-m+l-1} \widetilde{S}_{m,l-1}(q),$

(2.31) $S_{m+1,l}(q) = q^{-m-l-1} S_{m,l+1}(q) + q^{-m} \widetilde{S}_{m,l}(q) + S_{m,l-1}(q),$

(2.32) $S_{m+1,l}(q) = q^{-m} \widetilde{S}_{m,l+1}(q) + S_{m,l}(q) + q^{-m+l-1} S_{m,l-1}(q),$

(2.33) $\widetilde{S}_{m+1,l}(q) = q^{-l} S_{m,l+1}(q) + S_{m,l}(q) + q^{-m+l-1} \widetilde{S}_{m,l-1}(q),$

(2.34) $\widetilde{S}_{m+1,l}(q) = q^{-l} S_{m,l+1}(q) + q^{-m} \widetilde{S}_{m,l}(q) + S_{m,l-1}(q),$

(2.35) $\widetilde{S}_{m+1,l}(q) = q^{-m-l} \widetilde{S}_{m,l+1}(q) + q^{-l} S_{m,l}(q) + S_{m,l-1}(q).$

Proof. We use the notation of q-trinomial

$$\begin{bmatrix} n \\ a\ b\ c \end{bmatrix} := \frac{(q)_n}{(q)_a (q)_b (q)_c} \qquad \text{for} \quad a + b + c = n.$$

Then the product of q-binomials in the definition of $S_{m,l}$ and $\widetilde{S}_{m,l}$ is rewritten as

$$\begin{bmatrix} m \\ \nu \end{bmatrix} \begin{bmatrix} \nu \\ m - \nu - l \end{bmatrix} = \begin{bmatrix} m \\ m - \nu \quad m - l - \nu \quad 2\nu + l - m \end{bmatrix}$$

From this expression it is easy to check (2.29). The other formulae except (2.31) and (2.34) follow directly from the q-trinomial identity:

$$(2.36) \qquad \begin{bmatrix} n \\ a\ b\ c \end{bmatrix} = \begin{bmatrix} n - 1 \\ a - 1\ b\ c \end{bmatrix} + q^a \begin{bmatrix} n - 1 \\ a\ b - 1\ c \end{bmatrix} + q^{a+b} \begin{bmatrix} n - 1 \\ a\ b\ c - 1 \end{bmatrix}.$$

In the following we prove (2.31). The proof of (2.34) is similar.

We start from

$$S_{m+1,l}(q)$$
$$= \sum_{\nu \in \mathbb{Z}} q^{(\nu+l-m-1)(\nu+l)+\nu(\nu-m-1)} \begin{bmatrix} m + 1 \\ m + 1 - l - \nu \quad 2\nu + l - m - 1 \quad m + 1 - \nu \end{bmatrix}.$$

Decompose the right hand side above into three parts by applying (2.36) to the q-trinomial. Then by changing $\nu \to \nu + 1$ we see that the first part is equal to $q^{-m-l-1} S_{m,l+1}(q)$. In the third part we rewrite the q-trinomial as follows:

$$\begin{bmatrix} m \\ m + 1 - l - \nu \quad 2\nu + l - m - 1 \quad m - \nu \end{bmatrix}$$
$$= \left(q^{m+1-l-\nu} + (1 - q^{m+1-l-\nu}) \right) \begin{bmatrix} m \\ m + 1 - l - \nu \quad 2\nu + l - m - 1 \quad m - \nu \end{bmatrix}$$
$$= q^{m+1-l-\nu} \begin{bmatrix} m \\ m + 1 - l - \nu \quad 2\nu + l - m - 1 \quad m - \nu \end{bmatrix}$$
$$+ (1 - q^{2\nu+l-m}) \begin{bmatrix} m \\ m + l - \nu \quad 2\nu + l - m \quad m - \nu \end{bmatrix}.$$

From the first term in the right hand side above, we obtain $q^{-m} \widetilde{S}_{m,l}(q)$. Then

the remaining is

$$\sum_{\nu\in\mathbb{Z}} q^{(\nu+l-m-1)(\nu+l)+\nu(\nu-m-1)+m+1-l-\nu}$$

$$\times \begin{bmatrix} m \\ m+1-l-\nu \;\; 2\nu+l-m-2 \;\; m-\nu+1 \end{bmatrix}$$

$$+\sum_{\nu\in\mathbb{Z}} q^{(\nu+l-m-1)(\nu+l)+\nu(\nu-m-1)+\nu}(1-q^{2\nu+l-m}) \begin{bmatrix} m \\ m+l-\nu \; 2\nu+l-m \; m-\nu \end{bmatrix}.$$

Change $\nu \to \nu+1$ in the first sum, then it is canceled by the part containing $-q^{2\nu+l-m}$ in the second sum. The rest is equal to $S_{m,l-1}(q)$. $\qquad\square$

Now we define the function $f_{a,b,c,m}(q)$ in the following form for $(a,b,c) \in \mathbb{Z}^3$ satisfying $1 \le b, c \le p'-1$, $a \equiv b \,(\mathrm{mod}\,2)$ and $c = b, b \pm 2$:

$$f_{a,b,b+2,m}(q) := q^{l(l+1)/t+m^2-l^2+(m-l)\{(b+1)/t\}}$$

$$\times \begin{cases} \widetilde{S}_{m,l}(q) & \text{if} \quad C(b+2) = 1_A, \\ S_{m,l}(q) & \text{if} \quad C(b+2) = 1_B \text{ or } 2, \end{cases}$$

$$f_{a,b,b,m}(q) := q^{l(l-1)/t+m^2-l^2+l(1-\{(b-1)/t\})}$$

$$\times \begin{cases} q^m S_{m,l}(q) & \text{if} \quad C(b) = 1_A \text{ or } 1_B, \\ q^l \widetilde{S}_{m,l}(q) & \text{if} \quad C(b) = 2, \end{cases}$$

$$f_{a,b,b-2,m}(q) := q^{l(l-1)/t+m^2-l^2+(m+l)(1-\{(b-1)/t\})}$$

$$\times \begin{cases} S_{m,l}(q) & \text{if} \quad C(b-2) = 1_A \text{ or } 2, \\ q^l \widetilde{S}_{m,l}(q) & \text{if} \quad C(b-2) = 1_B. \end{cases}$$

Here we set $l := (b-a)/2$ in the right hand sides. By definition we set $f_{a,b,c,m}(q) = 0$ for other type of triples (a,b,c). Then $X_{a,b,c,m}$ is given in terms of $f_{a,b,c,m}$:

Proposition 2.10. *We have*

$$(2.37) \qquad X_{a,b,c,m}(q) = \sum_{\epsilon=\pm} \epsilon \sum_{n\in\mathbb{Z}} f_{\epsilon(a+2p'n),b,c,m}(q).$$

Proof. It is easy to check that the right hand side of (2.37) satisfies the initial condition, that is, it is equal to $\delta_{a,b}$ in the case of $m = 0$. Now we should check the recurrence relation

$$(2.38) \qquad \sum_{d=b,\,b\pm2} q^{(m+1)w(d,b,c)} \sum_{\epsilon=\pm} \epsilon \sum_{n\in\mathbb{Z}} f_{\epsilon(a+2p'n),d,b,m}(q)$$

$$= \sum_{\epsilon=\pm} \epsilon \sum_{n\in\mathbb{Z}} f_{\epsilon(a+2p'n),b,c,m+1}(q).$$

Divide the cases according to the value of b; (i) $3 \le b \le p' - 3$ (non-boundary), (ii) $b = 2$ or $p' - 2$ (next-to-boundary) and (iii) $b = 1$ or $p' - 1$ (boundary). Then the proof of (2.38) is just case-checking for each combination of the values $c (= b, b \pm 2)$ and the choices of $C(b)$ and $C(c)$. In the following we give a sketch of the calculation in two cases as an example.

First let us consider one of the non-boundary cases; $3 \le b \le p' - 3, c = b - 2$ and $C(b) = 1_A$. Then from the definition of $f_{a,b,c,m}$ we have

$$\sum_{d=b,b\pm2} q^{(m+1)w(d,b,b-2)} f_{a+2p'n,d,b,m}(q)$$
$$= q^{(l-p'n)(l-p'n-1)/t+(m+1)^2-(l-p'n)^2+(m+1+l-p'n)(1-\{(b-1)/t\})}$$
$$\times \left\{ q^{-m-l+p'n-1} S_{m,l-p'n+1}(q) + S_{m,l-p'n}(q) \right.$$
$$\left. + q^{(m+1)(x(b-2)-3)+l-p'n} \widetilde{S}_{m,l-p'n-1}(q) \right\}.$$

Here we set $l = (b-a)/2$. Since $C(b) = 1_A$, $C(b-2)$ is either 1_A or 2. In both cases we have $x(b-2) = 2$, and hence we can apply (2.30) to the right hand side above. Thus we obtain

$$\sum_{d=b,b\pm2} q^{(m+1)w(d,b,b-2)} f_{a+2p'n,d,b,m}(q) = f_{a+2p'n,b,b-2,m+1}(q).$$

This equality still holds after the change of the sign $a, n \to -a, -n$. Therefore we have the equality (2.38).

Next let us consider one of the next-to-boundary cases; $b = 2, c = 4, C(b) = 1_B$. Then we have

$$\sum_{d=2,4} q^{(m+1)w(d,2,4)} f_{a+2p'n,d,2,m}(q)$$
$$= q^{(a/2+p'n)^2/t+m^2-(a/2+p'n)^2+(3m+2)/t+a/2+p'n+m}$$
$$\times \left\{ q^{(m+1)(y(4)-2\tau(4))-m} \widetilde{S}_{m,2-a/2-p'n}(q) + S_{m,1-a/2-p'n}(q) \right\}$$

Now $C(2) = 1_B$, hence we have $C(4) = 1_B$ or 2, and then $y(4) - 2\tau(4) = 0$. Apply (2.32) to the right hand side, and we get

$$\sum_{d=2,4} q^{(m+1)w(d,2,4)} f_{a+2p'n,d,2,m}(q)$$
$$= q^{(a/2+p'n)^2/t+m^2-(a/2+p'n)^2+(3m+2)/t}$$
$$\times \left\{ q^{a/2+p'n+m} S_{m+1,1-a/2+p'n}(q) - S_{m,-a/2-p'n}(q) \right\}.$$

Note that the power of q in the coefficient of $S_{m,-a/2-p'n}(q)$ is invariant under the change of the sign $a, n \to -a, -n$. Therefore after the sum over $n \in \mathbb{Z}$

and $\epsilon = \pm$ in the right hand side of (2.38), the function $S_{m,-a/2-p'n}(q) = S_{m,a/2+p'n}(q)$ disappears. Thus we obtain (2.38). $\qquad\square$

To finish the proof of Proposition 2.3, it suffices to show that

$$(2.39) \qquad I_{r,a,b,m}^{(p,p')}(q) = \sum_{d=b,\,b\pm 2} q^{m(\Delta_{r,d}-\Delta_{r,b}+\delta_{d,b})+\Delta_{r,b}-\Delta_{r,a}} X_{a,d,b,m-1}(q).$$

From (2.37) and the recurrence relations (2.30)–(2.32) we can check (2.39) by direct calculation. $\qquad\square$

Remark. In [FJMMT1], we constructed a monomial basis of $M_{r,s}^{(p,p')}$ with $1 < p'/p < 2$, using (2,1) primary field. To show that the monomials span the space, quadratic exchange relations were employed. In the process of rewriting the monomials, it was necessary to show the non-vanishing of a certain determinant. In the present case of (1,3) primary field, a similar set of quadratic relations can be written explicitly. However it is not clear to us how to derive the spanning property for the proposed set of monomials.

From the next section, we will restrict to the case of unitary series $p' = p+1$.

§3. Preliminaries on the Fusion Product

§3.1. Fusion product

In this section we fix our notation and collect the main properties of the fusion product.

Let $V_1, \ldots, V_n$ be cyclic representations of a Lie algebra $\mathfrak{g}$ with cyclic vectors $v_1, \ldots, v_n$. Fix $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ with $z_i \neq z_j$ for $i \neq j$. Denote by $V_i(z_i)$ the evaluation representation of $\mathfrak{g} \otimes \mathbb{C}[u]$, which is isomorphic to V_i as vector space and the action is defined via the map $\mathfrak{g} \otimes \mathbb{C}[u] \to \mathfrak{g}$, $g \otimes u^j \mapsto z_i^j g$ ($g \in \mathfrak{g}$). Recall (see [FL]) that the fusion product[1] $V_1(z_1) * \cdots * V_n(z_n)$ is the associated graded $\mathfrak{g} \otimes \mathbb{C}[u]$-module with respect to the filtration $\{F_m\}_{m \geq 0}$ on the tensor product $V_1(z_1) \otimes \cdots \otimes V_n(z_n)$:

(3.1)
$$F_m = \mathrm{span}\{(g_1 \otimes u^{k_1} \cdots g_p \otimes u^{k_p})(v_1 \otimes \cdots \otimes v_n) \mid g_1, \ldots, g_p \in \mathfrak{g}, \ k_1 + \cdots + k_p \leq m\}.$$

If $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we have

$$F_m = U(\mathfrak{g}) \cdot \mathrm{span}\{(g_1 \otimes u \cdots g_p \otimes u)(v_1 \otimes \cdots \otimes v_n) \mid g_1, \ldots, g_p \in \mathfrak{g}, \ p \leq m\}.$$

---

[1] In [FL], fusion product is introduced for finite dimensional modules, but the same definition carries over to infinite dimensional modules as well.

We set

(3.2) $(V_1(z_1) * \cdots * V_n(z_n))_m = F_m/F_{m-1},$

so that we have

$$V_1(z_1) * \cdots * V_n(z_n) = \bigoplus_{m=0}^{\infty} (V_1(z_1) * \cdots * V_n(z_n))_m.$$

The most important property of the fusion product is its independence on z in some special cases (see [FL], [FF1], [CL], [FKL], [FoL], [K], [AK]). Among such cases, we will need two cases: $\mathfrak{g} = \mathfrak{sl}_2$ and V_i are irreducible representations, and $\mathfrak{g} = \widehat{\mathfrak{sl}}'_2$ with $n = 2$. Note that, for an arbitrary Lie algebra $\mathfrak{g}$, the fusion product of two representations V_i ($i = 1, 2$) with cyclic vectors v_i, is independent of z_1, z_2 because

(3.3) $F_m = U(\mathfrak{g}) \cdot (G_m \otimes v_2)$ where $G_0 = \mathbb{C}v_1, \ G_{m+1} = G_m + \mathfrak{g} \cdot G_m.$

The filtration G_m on V_1 is called the Poincaré-Birkhoff-Witt (PBW) filtration (see 5.3).

Consider the case where $\mathfrak{g} = \mathfrak{sl}_2$, $V_i = \pi_{a_i}$, and $v_i \in \pi_{a_i}$ is the highest weight vector. We write the corresponding fusion product as

(3.4) $\pi_{a_1} * \cdots * \pi_{a_n}$

as it is independent of the choice of z. The fusion product is also independent of the ordering of the components π_{a_i}. When $a_1 = \cdots = a_n = a$, we use the shorthand notation π_a^{*n}. The fusion product (3.4) is a module over $\mathfrak{sl}_2 \otimes \mathbb{C}[u]$. We set $M^\alpha = \{v \in M \mid h_0 v = \alpha v\}$ and

$$\mathrm{ch}_{q,z} \pi_{a_1} * \cdots * \pi_{a_n} := \sum_{m,\alpha} q^m z^\alpha \dim(\pi_{a_1} * \cdots * \pi_{a_n})_m^\alpha,$$

$$\mathrm{ch}_q (\pi_{a_1} * \cdots * \pi_{a_n})^\alpha := \sum_m q^m \dim(\pi_{a_1} * \cdots * \pi_{a_n})_m^\alpha.$$

For example,

(3.5) $\mathrm{ch}_{q,z} \pi_1^{*m} = \displaystyle\sum_{\substack{-m \le l \le m \\ l \equiv m \bmod 2}} \begin{bmatrix} m \\ \frac{m+l}{2} \end{bmatrix}_q z^l,$

(3.6) $\mathrm{ch}_{q,z} \pi_2^{*m} = \displaystyle\sum_{-m \le l \le m} S_{m,l}(q^{-1}) z^{2l}.$

§3.2. Exact sequences

In this subsection we describe some exact sequences of fusion products of $\mathfrak{sl}_2$-modules proved in [FF2] and [FF3].

Lemma 3.1 ([FF3, Lemma 2.1]). *Let $A = (a_1, \ldots, a_n)$ be a sequence of non-negative integers with $a_1 \leq \cdots \leq a_n$. Then for any $1 \leq i < n$ there exists an $\mathfrak{sl}_2 \otimes \mathbb{C}[u]$-module $S_i(A)$ such that the following sequence is exact:*

$$0 \longrightarrow S_i(A) \longrightarrow \pi_{a_1} * \cdots * \pi_{a_n}$$
$$\longrightarrow \pi_{a_1} * \cdots * \pi_{a_{i-1}} * \pi_{a_i - 1} * \pi_{a_{i+1}+1} * \pi_{a_{i+2}} * \cdots * \pi_{a_n} \longrightarrow 0.$$

Lemma 3.2 ([FF2, Statement 4.1 and Proposition 4.1]). *Under the condition in Lemma 3.1, we have*

$$S_1(A) \simeq \pi_{a_2 - a_1} * \pi_{a_3} * \cdots * \pi_{a_n}.$$

In order to have the above exact sequence to be degree preserving, we must shift the q-degree of the highest weight vector of $S_1(A)$ (with respect to the operator $h \otimes u^0$) to $(n-1)a_1$. This gives the equality

$$(3.7) \qquad \mathrm{ch}_{q,z} \pi_{a_1} * \cdots * \pi_{a_n} = q^{(n-1)a_1} \mathrm{ch}_{q,z} \pi_{a_2 - a_1} * \pi_{a_3} * \cdots * \pi_{a_n}$$
$$+ \mathrm{ch}_{q,z} \pi_{a_1 - 1} * \pi_{a_2 + 1} * \pi_{a_3} * \cdots * \pi_{a_n}.$$

Lemma 3.3 ([FF3, Remark 2.3]). *Under the condition in Lemma 3.1, we have*

$$S_{n-1}(A) \simeq \pi_{a_1} * \cdots * \pi_{a_{n-2}} \otimes \pi_{a_n - a_{n-1}}.$$

Therefore one has an exact sequence of $\mathfrak{sl}_2 \otimes \mathbb{C}[u]$-modules

$$(3.8) \quad 0 \to \pi_{a_1} * \cdots * \pi_{a_{n-2}} \otimes \pi_{a_n - a_{n-1}} \to \pi_{a_1} * \cdots * \pi_{a_n} \to$$
$$\pi_{a_1} * \cdots * \pi_{a_{n-2}} * \pi_{a_{n-1}-1} * \pi_{a_n + 1} \to 0$$

We will also need an exact sequence involving different modules $S_i(A)$:

Lemma 3.4 ([FF3, Proposition 2.1]). *Fix $2 \leq i \leq n - 1$. Let $A = (a_1, \ldots, a_n)$ be a sequence of non-negative integers satisfying $a_i \neq a_{i+1}$ and $a_{i-1} > 0$. Denote by A_i the sequence*

$$(a_1, \ldots, a_{i-1}, a_i - 1, a_{i+1} + 1, a_{i+2}, \ldots, a_n).$$

Then one has an exact sequence of $\mathfrak{sl}_2 \otimes \mathbb{C}[u]$-modules:

$$(3.9)$$
$$0 \to \pi_{a_1} * \cdots * \pi_{a_{i-2}} * \pi_{a_{i-1} - a_i + a_{i+1}} * \pi_{a_{i+2}} * \cdots * \pi_{a_n} \to S_i(A) \to S_i(A_{i-1}) \to 0.$$

By combining Lemma 3.3 and Lemma 3.4, we obtain the following exact sequence when $n \geq 3$, $a_{n-1} \neq a_n$ and $a_{n-2} > 1$:

(3.10)
$$0 \to \pi_{a_1} * \cdots * \pi_{a_{n-3}} * \pi_{a_{n-2}-a_{n-1}+a_n} \to \pi_{a_1} * \cdots * \pi_{a_{n-2}} \otimes \pi_{a_n-a_{n-1}} \to$$
$$\pi_{a_1} * \cdots * \pi_{a_{n-2}-1} \otimes \pi_{a_n-a_{n-1}-1} \to 0.$$

§3.3. $\widehat{\mathfrak{sl}}_2$ and $\mathfrak{n}_+$-homology

We first settle our notation about $\widehat{\mathfrak{sl}}_2$. Let

$$\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where K is a central element and $[d, x_i] = -ix_i$, where we put $x_i = x \otimes t^i$ for $x \in \mathfrak{sl}_2$. Let $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) be the nilpotent subalgebra of creation (resp. annihilation) operators generated by f_0, e_{-1} (resp. e_0, f_1). For a positive integer k, we denote by $L_{l,k}$ ($0 \leq l \leq k$) the set of integrable highest weight representations of $\widehat{\mathfrak{sl}}_2$. We fix a highest weight vector $v_{l,k} \in L_{l,k}$. Then

$$\mathfrak{n}_- v_{l,k} = 0, \ h_0 v_{l,k} = l \, v_{l,k}, \ K v_{l,k} = k \, v_{l,k}, \ d \, v_{l,k} = 0, \ L_{l,k} = U(\mathfrak{n}_+) \cdot v_{l,k}.$$

Representations $L_{l,k}$ are bi-graded by operators d and h_0. We set

$$L_{l,k} = \bigoplus_{\alpha, n \in \mathbb{Z}} (L_{l,k})_n^\alpha, \quad (L_{l,k})_n^\alpha = \{v \mid dv = nv, h_0 v = \alpha v\}.$$

The Virasoro algebra acts on $L_{l,k}$ by the Sugawara operators:

$$L_n = \frac{1}{2(k+2)} \sum_{m \in \mathbb{Z}} : e_{n-m} f_m + f_{n-m} e_m + \frac{1}{2} h_{n-m} h_m :,$$

where $: :$ is the normal ordering sign:

$$: x_i y_j := \begin{cases} x_i y_j, & \text{if } i < j; \\ y_j x_i, & \text{if } i > j; \\ \frac{1}{2}(x_i y_i + y_i x_i), & \text{if } i = j. \end{cases}$$

The central charge is equal to $\frac{3k}{k+2}$. The conformal weight $\Delta(l, k)$ of the highest weight vector $v_{l,k}$ is equal to $\frac{l(l+2)}{4(k+2)}$: $L_0 v_{l,k} = \Delta(l, k) v_{l,k}$.

We now recall the homology result from [FF4]. For an $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-module M, we denote by M' the $\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}]$-module which is isomorphic to M as a vector space and the action is defined via the isomorphism $x_i \mapsto x_{-i}$.

Proposition 3.5 ([FF4, Theorem 2.2]). *Let $a_1 \leq \cdots \leq a_n \leq k$. Then, for $0 \leq l \leq k$, we have*

$$H_p(\mathfrak{n}_+, (\pi_{a_1} * \cdots * \pi_{a_n})' \otimes L_{l,k})^0 = 0 \quad (p > 0),$$

where the superscript 0 denotes the weight zero spaces with respect to the operator h_0.

Proposition 3.6 ([FF4, Corollary 2.3]). *Suppose that $a_1 \leq \cdots \leq a_{n-1} \leq a_n = k+1$. Then, for $0 \leq l \leq k$, we have*

$$H_p(\mathfrak{n}_+, (\pi_{a_1} * \cdots * \pi_{a_n})' \otimes L_{l,k})^0 = 0 \quad (p \geq 0).$$

The above propositions imply that, if $a_1 \leq \cdots \leq a_n \leq k+1$, we have

$$(3.11) \qquad H_p(\mathfrak{n}_+, (\pi_{a_1} * \cdots * \pi_{a_n})' \otimes L_{l,k})^0 = 0 \quad (p > 0).$$

We also note that the characters of the zeroth homology groups of these spaces can be identified with the Kostka polynomials (see [FF4], [FJKLM], [SS]).

The following will be used later.

Lemma 3.7. *For non-negative integers j, k, l, m satisfying $0 \leq j \leq k$, $0 \leq l \leq k+1$, we have*

$$(3.12) \qquad H_p\big(\mathfrak{n}_+, (\pi_1^{*m} \otimes \pi_j)' \otimes L_{l,k+1}\big)^0 = 0 \quad (p > 0),$$

$$(3.13) \qquad H_p\big(\mathfrak{n}_+, (\pi_2^{*m} \otimes \pi_j)' \otimes L_{l,k+1}\big)^0 = 0 \quad (p > 0).$$

Proof. In the case of $j = 0$ the equalities (3.12) and (3.13) follow from (3.11). We assume that $j > 0$ in the following.

First we show (3.12). Consider the exact sequence (3.8) with $(a_1, \ldots, a_n) = (1, \ldots, 1, j+1)$:

$$0 \to \pi_1^{*m} \otimes \pi_j \to \pi_1^{*(m+1)} * \pi_{j+1} \to \pi_1^{*m} * \pi_{j+2} \to 0.$$

Tensoring by $L_{l,k+1}$ we obtain

$$0 \to (\pi_1^{*m} \otimes \pi_j)' \otimes L_{l,k+1} \to (\pi_1^{*(m+1)} * \pi_{j+1})' \otimes L_{l,k+1}$$
$$\to (\pi_1^{*m} * \pi_{j+2})' \otimes L_{l,k+1} \to 0.$$

Taking the homology we get the long exact sequence

$$\cdots \leftarrow H_p(\mathfrak{n}_+, (\pi_1^{*(m+1)} * \pi_{j+1})' \otimes L_{l,k+1})^0 \leftarrow H_p(\mathfrak{n}_+, (\pi_1^{*m} \otimes \pi_j)' \otimes L_{l,k+1})^0$$
$$\leftarrow H_{p+1}(\mathfrak{n}_+, (\pi_1^{*m} * \pi_{j+2})' \otimes L_{l,k+1})^0 \leftarrow \cdots$$

for $p > 0$. Then the assertion (3.12) follows from (3.11).

To show (3.13), we start from the exact sequence (3.8) with $(a_1, \ldots, a_{m+2})$ $= (1, 2, \ldots, 2, j + 1)$:

$$0 \to \pi_1 * \pi_2^{*(m-1)} \otimes \pi_{j-1} \to \pi_1 * \pi_2^{*m} * \pi_{j+1} \to \pi_1^{*2} * \pi_2^{*(m-1)} * \pi_{j+2} \to 0.$$

Arguing as above we obtain

$$(3.14) \qquad H_p\big(\mathfrak{n}_+, (\pi_1 * \pi_2^{*(m-1)} \otimes \pi_{j-1})' \otimes L_{l,k+1}\big)^0 = 0 \quad (p > 0).$$

Now applying a similar argument to the exact sequence (3.10) where $(a_1, \ldots, a_{m+2}) = (2, \ldots, 2, j + 2)$:

$$0 \to \pi_2^{*(m-1)} * \pi_{j+2} \to \pi_2^{*m} \otimes \pi_j \to \pi_1 * \pi_2^{*(m-1)} \otimes \pi_{j-1} \to 0.$$

Consider the associated long exact sequence of homology, then we find (3.13) from (3.11) and (3.14). $\qquad\square$

§4. The $\phi_{1,3}$ Filtration

In what follows we deal with the Virasoro modules $M_{r,s}^{(p,p')}$ in the unitary case, $p = k + 2, p' = k + 3$.

Consider the decomposition of the tensor product of $\widehat{\mathfrak{sl}}_2$-modules (the coset construction [GKO])

$$(4.1) \qquad L_{i,1} \otimes L_{j,k} = \bigoplus_{\substack{0 \leq l \leq k+1 \\ l \equiv i+j \bmod 2}} M_{j+1,l+1}^{(k+2,k+3)} \otimes L_{l,k+1}.$$

On the space $M_{j+1,l+1}^{(k+2,k+3)}$ we have two filtrations: the filtration defined by the (1,3) primary field, and the one induced from the fusion filtration on the left hand side. In this subsection, we show that these two filtrations coincide.

For that purpose we use an operator identity due to [L], which we recall below. Consider the $\widehat{\mathfrak{sl}}_2$ vertex operator associated with π_2,

$$(4.2) \ V^\sigma(z) : L_{l,k+1} \otimes \pi_2 \to L_{l+\sigma,k+1} \otimes z^{\Delta(l+\sigma,k+1)-\Delta(l,k+1)-\Delta(2,k+1)} \mathbb{C}((z)).$$

Here $\sigma = -2, 0, 2$ and $\mathbb{C}((z))$ is the space of Laurent series. We fix a weight basis $v_\tau \in \pi_2$ with $h v_\tau = \tau v_\tau$ $(\tau = -2, 0, 2)$, and write

$$V^\sigma(z) = (V_{-2}^\sigma(z), V_0^\sigma(z), V_2^\sigma(z)),$$

where $V_\tau^\sigma(z)(u) = V^\sigma(z)(u \otimes v_\tau)$. We have the Fourier expansion

$$V_\tau^\sigma(z) = \sum_{n \in \mathbb{Z} + \Delta(l,k+1) - \Delta(l+\sigma,k+1)} z^{-n-\Delta(2,k+1)} V_{\tau,n}^\sigma,$$

$$V_{\tau,n}^\sigma : (L_{l,k+1})_\beta^\alpha \to (L_{l+\sigma,k+1})_{\beta-n}^{\alpha+\tau}.$$

We will also need the (1,3) field for the Virasoro modules. Abbreviating $\phi^{(s+\sigma,s)}(z)$ to $\phi^\sigma(z)$ we write

$$\phi^\sigma(z) = \sum_{n \in \mathbb{Z} + \Delta_{r,s} - \Delta_{r,s+\sigma}} \phi_n^\sigma z^{-n-\Delta_{1,3}},$$

$$\phi_n^\sigma : (M_{r,s}^{(p,p')})_\beta \to (M_{r,s+\sigma}^{(p,p')})_{\beta-n}.$$

In what follows we suppose that some normalization of vertex operators $V_\tau^\sigma(z)$ and $\phi^\sigma(z)$ is fixed.

For $x \in \mathfrak{sl}_2$, set $x(z) = \sum_{n \in \mathbb{Z}} x_n z^{-n-1}$. Introduce further the current

$$j(z) = (j_{-2}(z), j_0(z), j_2(z))$$

acting on the tensor product $L_{i,1} \otimes L_{j,k}$ by

$$j_{-2}(z) = kf(z) \otimes \mathrm{id} - \mathrm{id} \otimes f(z),$$
$$j_0(z) = kh(z) \otimes \mathrm{id} - \mathrm{id} \otimes h(z),$$
$$j_2(z) = ke(z) \otimes \mathrm{id} - \mathrm{id} \otimes e(z).$$

The following proposition is proved in [L].

Proposition 4.1. *There exist non-vanishing constants c_{-2}, c_0, c_2 such that, with respect to the identification* (4.1), *the following equality holds:*

$$(4.3) \qquad j_\tau(z) = c_{-2}\phi^{-2}(z) \otimes V_\tau^{-2}(z) + c_0\phi^0(z) \otimes V_\tau^0(z) + c_2\phi^2(z) \otimes V_\tau^2(z).$$

Noting that $\Delta_{1,3} + \Delta(2,k+1) = 1$, we set

$$j_\tau^\sigma(z) = c_\sigma \phi^\sigma(z) \otimes V_\tau^\sigma(z) = \sum j_{\tau,n}^\sigma z^{-n-1},$$

so that $j_\tau(z) = j_\tau^{-2}(z) + j_\tau^0(z) + j_\tau^2(z)$.

Lemma 4.2. *For each $\gamma \in \mathbb{Z}_{\geq 0}$, we have the equality*

$$U(\widehat{\mathfrak{sl}}_2) \cdot \mathrm{span}\{j_{\tau_1,i_1} \cdots j_{\tau_\gamma,i_\gamma}(|r,s\rangle \otimes v_{r-1,k+1}) \mid \tau_\alpha = -2,0,2, i_\alpha \in \mathbb{Z}\}$$
$$= U(\widehat{\mathfrak{sl}}_2) \cdot \mathrm{span}\{j_{\tau_1,i_1}^{\sigma_1} \cdots j_{\tau_\gamma,i_\gamma}^{\sigma_\gamma}(|r,s\rangle \otimes v_{r-1,k+1}) \mid \tau_\alpha, \sigma_\beta = -2,0,2, i_\alpha \in \mathbb{Z}\}.$$

Proof. We note that for any set of vectors $\{u_l \in L_{l,k+1}\}_{l=0}^{k+1}$ there exist elements $\{x_l \in U(\widehat{\mathfrak{sl}}_2)\}_{l=0}^{k+1}$ such that

$$x_l \cdot u_{l'} = \delta_{l,l'} u_{l'}.$$

Using this fact and the intertwining property of the vertex operator (4.2), Lemma can be proved by induction on γ. $\square$

Proposition 4.3. *Fix r, s and k. For $\sigma_1, \ldots, \sigma_\gamma = -2, 0, 2$, set*

$$W_{\sigma_1,\ldots,\sigma_\gamma} = U(\widehat{\mathfrak{sl}}_2) \cdot \operatorname{span}\{j_{\tau_1,i_1}^{\sigma_1} \cdots j_{\tau_\gamma,i_\gamma}^{\sigma_\gamma}(|r,s\rangle \otimes v_{s-1,k+1})|\tau_\alpha = -2, 0, 2, i_\alpha \in \mathbb{Z}\}$$

and

$$\widetilde{W}_{\sigma_1,\ldots,\sigma_\gamma} = \operatorname{span}\{\phi_{i_1}^{\sigma_1} \cdots \phi_{i_\gamma}^{\sigma_\gamma}|r,s\rangle|i_\alpha \in \mathbb{Z}\} \otimes L_{s+\sigma_1+\cdots+\sigma_\gamma-1,k+1}.$$

Then, the equality $W_{\sigma_1,\ldots,\sigma_\gamma} = \widetilde{W}_{\sigma_1,\ldots,\sigma_\gamma}$ holds.

Proof. Let $a = s + \sigma_1 + \cdots + \sigma_\gamma$. We first note that

$$W_{\sigma_1,\ldots,\sigma_\gamma} \subset \widetilde{W}_{\sigma_1,\ldots,\sigma_\gamma} \subset M_{r,a}^{(k+2,k+3)} \otimes L_{a-1,k+1}.$$

We show the equality

$$\begin{aligned}
W_{\sigma_1,\ldots,\sigma_\gamma} \cap \left((M_{r,a}^{(k+2,k+3)})_n \otimes L_{a-1,k+1}\right) \\
= \widetilde{W}_{\sigma_1,\ldots,\sigma_\gamma} \cap \left((M_{r,a}^{(k+2,k+3)})_n \otimes L_{a-1,k+1}\right).
\end{aligned}$$

Note that the equality follows if we show that for any n_α such that $n_1 + \cdots + n_\gamma = n - \Delta_{r,s}$,

$$\phi_{-n_1}^{\sigma_1} \cdots \phi_{-n_\gamma}^{\sigma_\gamma}|r,s\rangle \otimes v_{a-1,k+1} \in W_{\sigma_1,\ldots,\sigma_\gamma}$$

holds. We prove this statement by induction on n.

Before starting, we note that in view of the equality $\Delta_{1,3} + \Delta(2,k+1) = 1$, we obtain

$$j_{\tau_\alpha,i_\alpha-n_\alpha}^{\sigma_\alpha} = \sum_{i-n=i_\alpha-n_\alpha} \phi_{-n}^{\sigma_\alpha} \otimes V_{\tau_\alpha,i}^{\sigma_\alpha}.$$

For $n = \Delta_{r,a}$ the statement is equivalent to

$$|r,a\rangle \otimes v_{a-1,k+1} \in W_{\sigma_1,\ldots,\sigma_\gamma}.$$

This is a consequence of the fact that for any s_1, s_2 such that $\sigma = s_1 - s_2 = -2, 0, 2$ one can find unique τ, i, i_1, i_2 $(i_1 + i_2 = i)$ such that

$$|r, s_1\rangle \otimes v_{s_1-1,k+1} \in \mathbb{C}j^\sigma_{\tau,i}(|r, s_2\rangle \otimes v_{s_2-1,k+1}) = \mathbb{C}\phi^\sigma_{i_1}|r, s_2\rangle \otimes V^\sigma_{\tau,i_2}v_{s_2-1,k+1}.$$

Now suppose the statement is proved for all $n \le n_0$. Fix τ_α, i_α such that

$$v_{a-1,k+1} \in \mathbb{C}V^{\sigma_1}_{\tau_1,i_1} \cdots V^{\sigma_\gamma}_{\tau_\gamma,i_\gamma} v_{s-1,k+1}.$$

Choose n_α such that $n_1 + \cdots + n_\gamma = n_0 + 1 - \Delta_{r,s}$ and consider

$$w_{n_1,\ldots,n_\gamma} = j^{\sigma_1}_{\tau_1,i_1-n_1} \cdots j^{\sigma_\gamma}_{\tau_\gamma,i_1-n_\gamma}(|r, s\rangle \otimes v_{s-1,k+1}).$$

We have

$$w_{n_1,\ldots,n_\gamma} = c\phi^{\sigma_1}_{-n_1} \cdots \phi^{\sigma_\gamma}_{-n_\gamma}|r, s\rangle \otimes v_{a-1,k+1} + \sum_{n=\Delta_{r,a}}^{n_0} \tilde{w}_n,$$

where c is a non-zero constant and

$$\tilde{w}_n \in \widetilde{W}_{\sigma_1,\ldots,\sigma_\gamma} \cap \left((M^{(k+2,k+3)}_{r,a})_n \otimes L_{a-1,k+1} \right).$$

The statement follows from the induction hypothesis.   □

We now define the $\phi_{1,3}$-filtration on $M^{(k+2,k+3)}_{r,s}$. Note that when $p' = p+1$, the integer $b(r,s)$ determined by the condition (2.6) reads

$$b(r,s) = \begin{cases} r & \text{if } r - s \text{ is even:} \\ r+1 & \text{otherwise.} \end{cases}$$

Now, consider the filtration

$$(4.4) \quad E_m(M^{(k+2,k+3)}_{r,s})$$
$$= \text{span}\{\phi^{\sigma_1}_{i_1} \cdots \phi^{\sigma_\gamma}_{i_\gamma}|r, b(r,s)\rangle | \gamma \le m, \sigma_\alpha = -2, 0, 2\} \cap M^{(k+2,k+3)}_{r,s}.$$

Let

$$\text{gr}^E_m(M^{(k+2,k+3)}_{r,s}) = E_m(M^{(k+2,k+3)}_{r,s})/E_{m-1}(M^{(k+2,k+3)}_{r,s}).$$

Recall the decomposition (see (3.2))

$$L_{i,1} * L_{r-1,k} = \bigoplus_{m=0}^{\infty} (L_{i,1} * L_{r-1,k})_m.$$

Proposition 4.4. *Under the identification* (4.1), *we have an isomorphism of* $\widehat{\mathfrak{sl}}_2$*-modules*

$$(L_{i,1} * L_{r-1,k})_m = \bigoplus_{\substack{1 \le s \le k+2 \\ s \equiv r+i \bmod 2}} \mathrm{gr}_m^E(M_{r,s}^{(k+2,k+3)}) \otimes L_{s-1,k+1}.$$

Proof. We use the definition of the fusion product (3.1) of $\widehat{\mathfrak{sl}}_2$-modules, choosing

$$V_1 = L_{i,1}, \quad V_2 = L_{r-1,k}, \quad v_1 = v_{i,1}, \quad v_2 = v_{r-1,k}, \quad z_1 = k, \quad z_2 = -1.$$

Then the fusion filtration F_m on this tensor product is given by

$$F_m = U(\widehat{\mathfrak{sl}}_2) \cdot \mathrm{span}\{j_{\tau_1,i_1} \cdots j_{\tau_\gamma,i_\gamma} (|r, r+i\rangle \otimes v_{s-1,k+1}) \mid$$
$$\tau_\alpha = -2, 0, 2, \quad i_\alpha \in \mathbb{Z}, \quad \gamma \le m\}.$$

Now our proposition follows from Lemma 4.2 and Proposition 4.3. $\qquad\square$

§5. Decomposition of $\widehat{\mathfrak{sl}}_2$ Fusion Products

Our goal in this section is to determine the character of $\mathrm{gr}_m^E(M_{r,a}^{(k+2,k+3)})$ which appear in the decomposition of the fusion product $L_{i,1} * L_{j,k}$. We show that it is given in terms of $I_{a,b,r,m}^{(k+2,k+3)}(q)$ introduced in (2.10).

§5.1. The functor I_k

Set

$$\mathfrak{g}_\pm = \mathfrak{sl}_2 \otimes \mathbb{C}[t^{\mp 1}] \oplus \mathbb{C}d, \quad \mathfrak{g} = \widehat{\mathfrak{sl}}_2.$$

Let V be a $\mathfrak{g}_-$-module with $\mathbb{Z}_{\ge 0} \times \mathbb{Z}$-grading determined from the degree and the weight with respect to d and h_0, respectively:

$$V = \oplus_{s \ge 0} \oplus_{\alpha \in \mathbb{Z}} V_s^\alpha, \quad V_s^\alpha = \{v \in V \mid d\, v = s\, v, \ h_0 v = \alpha\, v\}.$$

Then the induced $\mathfrak{g}$-module

$$\mathrm{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}} V = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_-)} V$$

is also $\mathbb{Z}_{\ge 0} \times \mathbb{Z}$-graded. We will consider its maximal integrable quotient.

Let k be a positive integer. Since the degree is bounded from below, the operator $e(z)^{k+1} = \sum_n e_n^{(k+1)} z^{-n-1}$ has a well-defined action on $\mathrm{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}} V$. Let $\mathfrak{I}_k$ be the $\mathfrak{g}$-submodule of $\mathrm{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}} V$ generated by

$$(K - k) \cdot \mathrm{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}} V + \sum_n e_n^{(k+1)} \cdot \mathrm{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}} V.$$

We define

$$I_k(V) = \mathrm{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}} V / \mathfrak{I}_k.$$

Then $I_k(V)$ is an integrable $\mathfrak{g}$-module of level k. Moreover, any homomorphism $V \to L$ of $\mathfrak{g}_-$-modules to an integrable $\mathfrak{g}$-module L of level k extends to a homomorphism of $\mathfrak{g}$-modules $I_k(V) \to L$.

Since $I_k(V)$ is integrable of positive level k, it has the decomposition

$$I_k(V) = \bigoplus_{l=0}^{k} I_k^l(V) \otimes L_{l,k},$$

where

(5.1) $$I_k^l(V) = \mathrm{Hom}_{\mathfrak{g}'}(L_{l,k}, I_k(V))$$

is the space of highest weight vectors of weight l. Here we have set $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$. The space $I_k^l(V)$ carries a grading by d. In the next subsection we compute the character of $I_k^l(V)$ for some V.

§5.2.   Tensor product as induced module

In this subsection, we show that the tensor product module $L_{0,1} \otimes L_{j,k}$ can be realized as $I_{k+1}(L_{0,1} \otimes \pi_j)$ (see Proposition 5.4 below). We first prepare two Lemmas.

Let ω denote the involutive automorphism of $\widehat{\mathfrak{sl}}_2$ given by

$$e_i \mapsto f_{-i}, \quad f_i \mapsto e_{-i}, \quad h_i \mapsto -h_{-i}, \quad K \mapsto -K, \quad d \mapsto -d.$$

For a $\mathfrak{g}_-$-module V defined by $\rho_- : \mathfrak{g}_- \to \mathrm{End}(V)$, let V^ω denote the $\mathfrak{g}_+$-module structure on V given by $\rho_+ = \rho_- \circ \omega : \mathfrak{g}_+ \to \mathrm{End}(V)$.

We consider the situation where V is the fusion product of $\mathfrak{sl}_2$-modules or their tensor products. We identify u with t and define the degree operator d appropriately (see Proof of Proposition 5.4). In this case, V^ω and V' (see 3.3) are isomorphic as $\mathfrak{sl}_2 \otimes \mathbb{C}[t^{-1}]$-module.

Recall the definition of $\mathfrak{n}_+$ given in 3.3.

Lemma 5.1. *Notation being as above, we have*

$$\mathrm{ch}_q I_k^l(V) = \mathrm{ch}_{q^{-1}} H_0(\mathfrak{n}_+, V^\omega \otimes L_{l,k})^0.$$

Proof. In order to use the reciprocity law, we rewrite (5.1) as

$$I_k^l(V) \simeq \mathrm{Hom}_{\mathfrak{g}'}(I_k(V), L_{l,k})^*,$$

where $*$ denotes the restricted dual. Setting $\mathfrak{g}'_\pm = \mathfrak{sl}_2 \otimes \mathbb{C}[t^{\mp 1}]$, we have

$$
\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}'}(I_k(V), L_{l,k})^* &\simeq \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{Ind}_{\mathfrak{g}'_-}^{\mathfrak{g}'} V, L_{l,k})^* \\
&\simeq \mathrm{Hom}_{\mathfrak{g}'_-}(V, L_{l,k})^* \\
&\simeq \left((L_{l,k} \otimes V^*)^{\mathfrak{g}'_-} \right)^* \\
&\simeq (V \otimes L_{l,k}^*)/\mathfrak{g}'_-(V \otimes L_{l,k}^*) \\
&\simeq H_0(\mathfrak{g}'_-, V \otimes L_{l,k}^*).
\end{aligned}
$$

Note that as $\mathfrak{g}'_+$-modules we have $(L_{l,k}^*)^\omega \simeq L_{l,k}$. Therefore, we obtain

$$
\begin{aligned}
\mathrm{ch}_q H_0(\mathfrak{g}'_-, V \otimes L_{l,k}^*) &= \mathrm{ch}_{q^{-1}} H_0(\mathfrak{g}'_+, V^\omega \otimes L_{l,k}) \\
&= \mathrm{ch}_{q^{-1}} H_0(\mathfrak{n}_+, V^\omega \otimes L_{l,k})^0.
\end{aligned}
$$

Now our lemma follows. $\qquad\square$

Lemma 5.2. *Let V be a bi-graded $\mathfrak{g}_+$-module. Then*

$$
\begin{aligned}
(5.2) \quad \sum_{p=0}^\infty (-1)^p \mathrm{ch}_q H_p(\mathfrak{n}_+, V \otimes L_{l,k})^0 \\
= \sum_{\lambda \in \mathbb{Z}} q^{(k+2)\lambda^2 - (l+1)\lambda} \left(\mathrm{ch}_q V^{-2\lambda(k+2)+l} - \mathrm{ch}_q V^{-2\lambda(k+2)+l+2} \right).
\end{aligned}
$$

Proof. We first prepare our notation concerning the Weyl group W of $\widehat{\mathfrak{sl}}_2$. It is generated by simple reflections s_0, s_1. The length of $w \in W$ is denoted by $\ell(w)$. Let $\mathfrak{h} = \mathbb{C}h_0 \oplus \mathbb{C}K \oplus \mathbb{C}d$ be the Cartan subalgebra. Define $(i, k, m) \in \mathfrak{h}^*$ by $(i, k, m)h_0 = i, (i, k, m)K = k, (i, k, m)d = m$. We have

$$
\begin{aligned}
s_1(i, k, m) &= (i, k, m) - i\alpha_1, \quad \alpha_1 = (2, 0, 0), \\
s_0(i, k, m) &= (i, k, m) - (k - i)\alpha_0, \quad \alpha_0 = (-2, 0, -1).
\end{aligned}
$$

Let $\rho = (1, 2, 0)$. We write $w * \alpha = w(\alpha + \rho) - \rho$ for the shifted action of the Weyl group on $\mathfrak{h}^*$.

Consider the BGG-resolution of $L_{l,k}$ (see [BGG, Kum])

$$(5.3) \qquad 0 \longleftarrow L_{l,k} \longleftarrow M_0 \longleftarrow M_1 \longleftarrow \cdots,$$
$$M_p = \bigoplus_{\ell(w)=p} M(w*(l,k,0)),$$

where $M(\mu)$ is the Verma module with highest weight μ.

Here we recall that, for a general Lie algebra $\mathfrak{a}$, a $U(\mathfrak{a})$-module V and a free $U(\mathfrak{a})$-module M, the $U(\mathfrak{a})$-module $V \otimes M$ is isomorphic to $V^{\mathrm{triv}} \otimes M$ where V^{triv} is V with trivial $\mathfrak{a}$-action. For example, if $M = U(\mathfrak{a})$, then the isomorphism $V \otimes U(\mathfrak{a}) \simeq V^{\mathrm{triv}} \otimes U(\mathfrak{a})$ is given by $v \otimes x \mapsto \sum_i S^{-1}(x_i^{(1)}) v \otimes x_i^{(2)}$, where $\Delta(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)}$ is the coproduct and S is the antipode on $U(\mathfrak{a})$.

Now, tensoring (5.3) by V, we obtain a $U(\mathfrak{n}_+)$-free resolution for $V \otimes L_{l,k}$. Therefore, the following complex counts $H_*(\mathfrak{n}_+, V \otimes L_{l,k})^0$:

$$(5.4) \quad 0 \longleftarrow (\mathbb{C} \otimes_{U(\mathfrak{n}_+)} (V^{\mathrm{triv}} \otimes M_0))^0 \longleftarrow (\mathbb{C} \otimes_{U(\mathfrak{n}_+)} (V^{\mathrm{triv}} \otimes M_1))^0 \longleftarrow \cdots$$

We can rewrite (5.4) as

$$0 \longleftarrow ((\mathbb{C} \otimes_{U(\mathfrak{n}_+)} M_0) \otimes V^{\mathrm{triv}})^0 \longleftarrow ((\mathbb{C} \otimes_{U(\mathfrak{n}_+)} M_1) \otimes V^{\mathrm{triv}})^0 \longleftarrow \cdots$$

By the Euler-Poincaré principle,

$$\sum_{p=0}^{\infty} (-1)^p \mathrm{ch}_q H_p(\mathfrak{n}_+, V \otimes L_{l,k})^0 = \sum_{p=0}^{\infty} (-1)^p \mathrm{ch}_q \big((\mathbb{C} \otimes_{U(\mathfrak{n}_+)} M_p) \otimes V^{\mathrm{triv}} \big)^0$$
$$= \sum_{w \in W} (-1)^{\ell(w)} q^{(w*(l,k,0))(d)} \mathrm{ch}_q V^{-(w*(l,k,0))(h_0)}.$$

Lemma follows by using the following formulas for the shifted action where $\lambda \in \mathbb{Z}$,

$$(5.5) \qquad (s_1 s_0)^\lambda * (l,k,m) = (-2(k+2)\lambda + l, k, m + (k+2)\lambda^2 - (l+1)\lambda),$$
$$s_0(s_1 s_0)^{\lambda-1} * (l,k,m) = (-l-2+2(k+2)\lambda, k, m + (k+2)\lambda^2 - (l+1)\lambda).$$

$\square$

Corollary 5.3.   If $H_p(\mathfrak{n}_+, V^\omega \otimes L_{l,k}) = 0$ holds for $p > 0$, then

$$\mathrm{ch}_q I_k^l(V) = \sum_{\lambda \in \mathbb{Z}} q^{-(k+2)\lambda^2 + (l+1)\lambda} \left(\mathrm{ch}_q V^{2(k+2)\lambda-l} - \mathrm{ch}_q V^{2(k+2)\lambda-l-2} \right).$$

Proof.   This follows from Lemma 5.1 and Lemma 5.2, with q, V replaced by q^{-1}, V^ω noting that $\mathrm{ch}_{q^{-1}}(V^\omega)^\alpha = \mathrm{ch}_q V^{-\alpha}$. $\square$

Proposition 5.4. *We have an isomorphism of $\widehat{\mathfrak{sl}}_2$-modules*

$$L_{0,1} \otimes L_{j,k} \simeq I_{k+1}(L_{0,1} \otimes \pi_j),$$

where we regard π_j as an $\mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}d$-module by letting $\mathfrak{sl}_2 \otimes t\mathbb{C}[t] \oplus \mathbb{C}d$ act as 0.

Proof. As we noted in 5.1, there exists a natural surjective homomorphism

$$I_{k+1}(L_{0,1} \otimes \pi_j) \to L_{0,1} \otimes L_{j,k}.$$

Therefore, it suffices to check that the multiplicities of irreducible representations in the decomposition of left and right hand sides coincide.

Recall that $L_{0,1}$ is an inductive limit of its Demazure submodules, which are isomorphic to fusion products of 2-dimensional representations [FF2]. Introduce the action of d on π_1^{*2N} by setting

$$d\,v = (N^2 - m)\,v, \quad v \in (\pi_1^{*2N})_m,$$

and denote by $\widetilde{\pi_1^{*2N}}$ the resulting $\mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}d$-module. Then we have an isomorphism of $\mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}d$-modules

$$(5.6) \qquad\qquad L_{0,1} = \lim_{N \to \infty} \widetilde{\pi_1^{*2N}}.$$

Since inductive limit commutes with tensor product and is an exact functor, we have

$$I_{k+1}(L_{0,1} \otimes \pi_j) = \lim_{N \to \infty} I_{k+1}(\widetilde{\pi_1^{*2N}} \otimes \pi_j).$$

Therefore

$$\mathrm{ch}_q I_{k+1}(L_{0,1} \otimes \pi_j) = \lim_{N \to \infty} \mathrm{ch}_q I_{k+1}(\widetilde{\pi_1^{*2N}} \otimes \pi_j).$$

In view of the vanishing of homology (3.12), we can apply Corollary 5.3. We obtain

$$\mathrm{ch}_q I_{k+1}^l(\widetilde{\pi_1^{*2N}} \otimes \pi_j).$$
$$= \sum_{\lambda \in \mathbb{Z}} q^{-(k+3)\lambda^2 + (l+1)\lambda} \left(\mathrm{ch}_q(\widetilde{\pi_1^{*2N}} \otimes \pi_j)^{2(k+3)\lambda - l} - \mathrm{ch}_q(\widetilde{\pi_1^{*2N}} \otimes \pi_j)^{2(k+3)\lambda - l - 2} \right)$$
$$= \sum_{\lambda \in \mathbb{Z}} q^{-(k+3)\lambda^2 + (l+1)\lambda} \left(\mathrm{ch}_q(\widetilde{\pi_1^{*2N}})^{2(k+3)\lambda - l + j} - \mathrm{ch}_q(\widetilde{\pi_1^{*2N}})^{2(k+3)\lambda - l - 2 - j} \right).$$

On the other hand, we have from (3.5)

$$
\operatorname{ch}_q(\widetilde{\pi_1^{*2N}})^\alpha = q^{N^2} \begin{bmatrix} 2N \\ \frac{2N+\alpha}{2} \end{bmatrix}_{q^{-1}} = q^{\alpha^2/4} \begin{bmatrix} 2N \\ \frac{2N+\alpha}{2} \end{bmatrix}_q .
$$

Hence we find

$$
\operatorname{ch}_q I_{k+1}^l (\widetilde{\pi_1^{*2N}} \otimes \pi_j)
$$
$$
= q^{\frac{(l-j)^2}{4}} \left(\sum_{\lambda \in \mathbb{Z}} q^{(k+2)(k+3)\lambda^2 + ((k+3)(j+1)-(k+2)(l+1))\lambda} \begin{bmatrix} 2N \\ \frac{2N-l+j}{2} + (k+3)\lambda \end{bmatrix}_q \right.
$$
$$
\left. - \sum_{\lambda \in \mathbb{Z}} q^{(k+2)(k+3)\lambda^2 - ((k+3)(j+1)+(k+2)(l+1))\lambda + (j+1)(l+1)} \begin{bmatrix} 2N \\ \frac{2N-l-j-2}{2} + (k+3)\lambda \end{bmatrix}_q \right).
$$

The last expression in the above formula coincides with the ABF finitization of the Virasoro minimal model $M_{j+1,l+1}^{(k+2,k+3)}$ up to a power $q^{\Delta_{j+1,l+1} - (l-j)^2/4}$ (see [ABF]). This power comes from conformal dimensions of highest weight vectors of $\widehat{\mathfrak{sl}}_2$-modules:

$$
\Delta_{j+1,l+1} - \frac{(l-j)^2}{4} = \Delta(j,k) - \Delta(l,k+1),
$$

the right hand side being the difference between eigenvalue of L_0 and degree. Letting $N \to \infty$ we conclude that

$$
\operatorname{ch}_q I_{k+1}^l (L_{0,1} \otimes \pi_j) = q^{\Delta(l,k+1) - \Delta(j,k)} \times \operatorname{ch}_q M_{j+1,l+1}^{(k+2,k+3)}.
$$

This completes the proof of Proposition. □

§5.3. PBW-filtration on $L_{0,1}$

Consider the fusion filtration F_m on the tensor product $L_{0,1} \otimes L_{j,k}$ given by (3.3). In order to study F_m/F_{m-1}, we consider in this subsection the structure of the PBW-filtration G_m on $L_{0,1}$. Denote the associated graded space by $Q_m = G_m/G_{m-1}$. Since G_m is invariant under the action of d and h_0, the space Q_m is bi-graded by the degree and the weight. Note that for any $n \geq 0$

$$
(Q_m)_n^{2m} = (G_m)_n^{2m}.
$$

It is known [LP, FS] that this space has a monomial basis

$$
(5.7) \qquad\qquad\qquad e_{-n_1} \cdots e_{-n_m} v_{0,1}
$$

where $n_j \geq n_{j+1} + 2$, $n_m > 0$ and $\sum_{j=1}^m n_j = n$. In particular, we have

$$\mathrm{ch}_q(Q_m)^{2m} = \frac{q^{m^2}}{(q)_m}.$$

There is a canonical action of $\mathfrak{g}_-$ on Q_m. Define a $\mathfrak{g}_-$-invariant filtration on Q_m by setting

$$J_n(Q_m) = \sum_{i \leq n} U(\mathfrak{g}_-)(Q_m)_{m^2+i}^{2m}.$$

Proposition 5.5.   *We have*

$$Q_m = \cup_{n=0}^\infty J_n(Q_m).$$

The associated graded space $\mathrm{gr}_n^J(Q_m)$ is isomorphic to a direct sum of the fusion product π_2^{*m}. Each monomial vector (5.7) generates an $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-module isomorphic to π_2^{*m}.

Obviously, $\cup_{n=0}^\infty J_n(Q_m) \hookrightarrow Q_m$. For the proof of this proposition we need the following lemma.

Lemma 5.6.   *We have*

$$\mathrm{ch}_q Q_m = \frac{q^{m^2}}{(q)_m} \mathrm{ch}_{q^{-1}} \pi_2^{*m}.$$

The proof of this lemma is given below.

We also need the dual functional realization for the space Q_m. Set $K_m = \mathbb{C}[z_1, \ldots, z_m]$. Consider the restricted dual space

$$Q_m^* = \oplus_n (Q_m)_n^*.$$

For $\varphi \in Q_m^*$ one can define an element $F_\varphi \in K_m \otimes (\mathfrak{sl}_2^{\otimes m})^*$ by

$$(5.8) \qquad F_\varphi(x^{(1)} \otimes \cdots \otimes x^{(m)}) = \varphi(x_+^{(1)}(z_1) \cdots x_+^{(m)}(z_m) v_{0,1}).$$

Here for $x \in \mathfrak{sl}_2$ we set $x_+(z) = \sum_{n \geq 1} x_{-n} z^n$. The map $\varphi \mapsto F_\varphi$ is injective. We will give a characterization of the image of this mapping. First we recall some results of [FF1] on the dual functional realization of the fusion product. Note the $\mathfrak{sl}_2$ decomposition

$$\pi_2 \otimes \pi_2 = \pi_0 \oplus \pi_2 \oplus \pi_4.$$

Define

$$(5.9) \qquad V_m = \{F(z_1, \ldots, z_m) \in K_m \otimes \pi_2^{\otimes m} |$$

$$F|_{z_i=z_j} \in \sigma^{(i,j)} \left(\pi_0 \otimes \pi_2^{\otimes(m-2)} \right),$$

$$(5.10) \qquad (\partial_{z_i} F)|_{z_i=z_j} \in \sigma^{(i,j)} \left((\pi_0 \oplus \pi_2) \otimes \pi_2^{\otimes(m-2)} \right) \}.$$

Here

$$\sigma^{(i,j)} (v_i \otimes v_j \otimes v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes \widehat{v_j} \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n.$$

The space V_m is a $\mathfrak{g}_-$-module by the action

$$x_n \cdot F = \sum_{j=1}^{m} z_j^n \varpi_j(x) F.$$

Here $\varpi_j(x)$ is the action of $x \in \mathfrak{sl}_2$ on the j-th component of the tensor product $\pi_2^{\otimes m}$. The homogeneous degree of elements in K_m is counted by $-d \in \mathfrak{g}_-$. We denote by $(K_m)_n$ the subspace with homogeneous degree n.

Multiplication by elements of K_m commutes with this action. Let K_m^0 be the maximal ideal of K_m generated by $z_1, \ldots, z_m$. Combining the results in [FF1] (see Theorem 3.1, Theorem 4.1 and Proposition 4.1), we obtain the following proposition:

Proposition 5.7. *We have an isomorphism of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$-modules*

$$V_m / K_m^0 V_m \simeq (\pi_2^{*m})^*.$$

We need a variant of this result in the symmetric case. The symmetric group $\mathfrak{S}_m$ acts on V_m by

$$(5.11) \qquad (\sigma F)(z_1, \ldots, z_m) = (1 \otimes \sigma) \cdot F(z_{\sigma(1)}, \ldots, z_{\sigma(m)}),$$

where the action of σ on $\pi_2^{\otimes m}$ is $\sigma(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$. Set

$$W_m = V_m \cap (K_m \otimes \pi_2^{\otimes m})^{\mathfrak{S}_m}.$$

Let $S_m = K_m^{\mathfrak{S}_m}$ and $(S_m)_n = (K_m)_n^{\mathfrak{S}_m}$. Note that

$$(5.12) \qquad (W_m)^{-2m} = \prod_{i<j} (z_i - z_j)^2 S_m \otimes v^{\otimes m},$$

where v is the lowest weight vector of π_2.

Set

$$S_m^0 = K_m^0 \cap S_m.$$

This is the ideal of S_m consisting of symmetric polynomials vanishing at the origin.

Lemma 5.8. *The natural map*

$$W_m/S_m^0 W_m \to V_m/K_m^0 V_m$$

is an isomorphism.

Proof. First we prove

$$W_m \cap (K_m^0 V_m) \subset S_m^0 W_m.$$

It is enough to show that for any $n > 0$

(5.13) $\mathrm{Sym}(gF) \in S_m^0 W_m$ for any $g \in (K_m^0)_n$ and $F \in V_m$.

Here $\mathrm{Sym}\, F = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \sigma F$, and $(K_m^0)_n$ is the homogeneous component of degree n. We prove (5.13) by a decreasing induction on n. If $\deg g$ is large enough, we can write $g = \sum_{i=1}^N a_i g_i$ with some $a_i \in S_m^0$ and $g_i \in K_m$. The assertion is evident in this case. Now take $g \in (K_m^0)_n$ and suppose that (5.13) is true for degree higher than n. Set $\bar{g} = \mathrm{Sym}\, g$. Then there exist $g_{ij} \in (K_m)_n$ such that

$$g - \bar{g} = \sum_{i<j} g_{ij}, \quad s_{ij} g_{ij} = -g_{ij},$$

where $s_{ij} \in \mathfrak{S}_m$ is the transposition of i and j. We have

$$\mathrm{Sym}(gF) = \bar{g} \cdot \mathrm{Sym}\, F + \frac{1}{2} \sum_{i<j} \mathrm{Sym}\{g_{ij}(1 - s_{ij})F\}.$$

The first term in the right hand side belongs to $S_m^0 W$. Using (5.9), (5.10) and that

$$\pi_0 \oplus \pi_4 \simeq S^2(\pi_2), \quad \pi_2 \simeq \wedge^2 \pi_2$$

one can write

$$(1 - s_{ij})F = (z_i - z_j)F_{i,j} \quad \text{for some } F_{ij} \in V_m.$$

By the induction hypothesis we obtain that

$$\mathrm{Sym}\{g_{ij}(z_i - z_j)F_{ij}\} \in S_0 W.$$

Second, we show that

$$V_m \subset W_m + K_m^0 V_m.$$

In fact, we have seen that for any $F \in V_m$

$$(1 - s_{ij})F \in K_m^0 V_m.$$

Therefore, we have

$$F = \mathrm{Sym}\, F + (F - \mathrm{Sym}\, F) \in W_m + K_m^0 V_m.$$

$\square$

Define a decreasing filtration on W_m by

$$J^n(W_m) = \sum_{i \geq n} (S_m)_i \cdot W_m.$$

Lemma 5.9.   *The associated graded space* $\mathrm{gr}_J^n(W_m) = J^n(W_m)/ J^{n+1}(W_m)$ *is isomorphic to a direct sum of* $(\pi_2^{*m})^*$:

$$\mathrm{gr}_J(W_m) = S_m \otimes (\pi_2^{*m})^*.$$

For the proof of this lemma we prepare a character identity.

Lemma 5.10.   *We have the identity*

$$(5.14) \qquad \mathrm{ch}_{q,z} L_{0,1} = \sum_{m \geq 0} \frac{q^{m^2}}{(q)_m} \mathrm{ch}_{q^{-1},z} \pi_2^{*m}.$$

Proof.   Recall the isomorphism (5.6). We show that

$$(5.15) \qquad q^{N^2} \mathrm{ch}_{q^{-1},z} \pi_1^{*2N} = \sum_{m=0}^{N} q^{m^2} \begin{bmatrix} N \\ m \end{bmatrix}_q \mathrm{ch}_{q^{-1},z} \pi_2^{*m}.$$

Using the relation (see (3.7))

$$(5.16)$$
$$\mathrm{ch}_{q,z} \pi_1^{*2k_1} * \pi_2^{*k_2} = \mathrm{ch}_{q,z} \pi_1^{*2(k_1-1)} * \pi_2^{*(k_2+1)} + q^{2k_1+k_2-1} \mathrm{ch}_{q,z} \pi_1^{*2(k_1-1)} * \pi_2^{*k_2},$$

we rewrite

$$
\begin{aligned}
&q^{N^2}\mathrm{ch}_{q^{-1},z}\pi_1^{*2N}\\
={}&q^{N^2}(\mathrm{ch}_{q^{-1},z}\pi_1^{*2(N-1)}*\pi_2+q^{-2N+1}\mathrm{ch}_{q^{-1},z}\pi_1^{*2(N-1)})\\
={}&q^{N^2}(\mathrm{ch}_{q^{-1},z}\pi_1^{*2(N-2)}*\pi_2^{*2}+(q^{-2N+1}+q^{-2N+2})\mathrm{ch}_{q^{-1},z}\pi_1^{*2(N-2)}*\pi_2\\
&+q^{-4N+4}\mathrm{ch}_{q^{-1},z}\pi_1^{*2(N-2)}).
\end{aligned}
$$

Repeating this procedure N times we obtain (5.15). Obviously,

$$
\lim_{N\to\infty}\begin{bmatrix}N\\m\end{bmatrix}_q=\frac{1}{(q)_m}.
$$

Lemma is proved. $\qquad\square$

We fix an isomorphism of $\mathfrak{sl}_2$ modules: $\mathfrak{sl}_2^*\simeq\pi_2$. Then, we have

Lemma 5.11. *The mapping $\varphi\mapsto F_\varphi$ is an injection $Q_m^*\to z_1\cdots z_m W_m$.*

Proof. The graded action of x_n ($x\in\mathfrak{sl}_2$) on $\mathrm{gr}^G(L_{0,1})=\oplus_{m=0}^\infty Q_m$ is mutually commutative and zero if $n\geq 0$. The former property implies that $F_\varphi(z)$ is symmetric with respect to the action (5.11), and the latter implies that the action of $x_+(z)$ is equal to that of $x(z)=\sum_{n\in\mathbb{Z}}x_{-n}z^n$ on $\mathrm{gr}^G(L_{0,1})$. The integrability of the representation $L_{0,1}$ implies that $e(z)^2=0$. Therefore, by applying $\mathrm{ad}\,f_0$, we see that the following operators act as zero on $\mathrm{gr}^G(L_{0,1})$:

$$
e_+(z)^2,\ e_+(z)h_+(z),\ 2e_+(z)f_+(z)-h_+(z)^2,\ h_+(z)f_+(z),\ f_+(z)^2.
$$

All these properties imply (5.9) and (5.10). $\qquad\square$

Proof of Lemmas 5.6 *and* 5.9. Proposition 5.7 and Lemma 5.8 imply that for any symmetric polynomial g we have an isomorphism

$$
gW_m/gS_m^0 W_m\simeq(\pi_2^{*m})^*.
$$

Suppose that a vector space B is a subspace of a sum of vector spaces $A_1+\cdots+A_k$. Then, there is a natural surjection.

$$
(5.17)\qquad A_1/(A_1\cap B)\oplus\cdots\oplus A_k/(A_k\cap B)\to(A_1+\cdots+A_k)/B.
$$

Take a basis $\{g_1,\ldots,g_k\}$ of $(S_m)_n$. Set $A_i=g_iW_m$ ($1\leq i\leq k$) and $B=J^{n+1}(W_m)$. Then we have $A_1+\cdots+A_k=J^n(W_m)$ and $A_i\cap B=g_iS_m^0 W_m$.

Using (5.17) in this setting, we obtain

$$\mathrm{ch}_q L_{0,1} = \sum_{m=0}^{\infty} \mathrm{ch}_q Q_m$$
$$= \sum_{m=0}^{\infty} \mathrm{ch}_{q^{-1}} Q_m^* \leq \sum_{m=0}^{\infty} q^m \mathrm{ch}_{q^{-1}} W_m \leq \sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m} \mathrm{ch}_{q^{-1}} \pi_2^{*m}.$$

Here we used (5.12) to obtain the last inequality. Because of Lemma 5.10, the left end and the right end are equal. Lemma 5.6 follows from this. Moreover, for each n, we have the isomorphism

$$\oplus_{i=1}^k g_i W_m / g_i S_m^0 W_m \simeq \oplus_{i=1}^k g_i \otimes (\pi_2^{*m})^* \simeq \mathrm{gr}_J^n(W_m)$$

in the above notation. Lemma 5.9 follows from this. $\square$

Proof of Proposition 5.5.   Consider the dual $\mathfrak{g}_-$-action on Q_m^*. The mapping in Lemma 5.11 is $\mathfrak{g}_-$-linear. Set

$$(5.18) \qquad\qquad Q_m' = U(\mathfrak{g}_-)(Q_m)^{2m}.$$

Define a coupling between Q_m' and $z_1 \cdots z_m W_m$ as follows. Take $x \cdot w \in Q_m'$ and $g \in z_1 \cdots z_m W_m$. Here $x \in U(\mathfrak{g}_-)$ and $w \in (Q_m)^{2m}$. There exists a non-degenerate coupling [FS] between $(Q_m)^{2m}$ and $z_1 \cdots z_m (W_m)^{-2m}$ induced from the non-degenerate coupling between the subspace of the tensor algebra over the vector space $\oplus_{i \in \mathbb{Z}_{<0}} \mathbb{C} e_i$ which is spanned by $e_{i_1} \cdots e_{i_m}$ $(i_1, \ldots, i_m \in \mathbb{Z}_{<0})$, and the space of polynomials in $z_1, \cdots, z_m$ divisible by $z_1 \cdots z_m$:

$$\langle e_{i_1} \cdots e_{i_m}, z_1^{n_1} \cdots z_m^{n_m} \rangle = \prod_{j=1}^m \delta_{i_j + n_j, 0}.$$

Using this coupling we define

$$\langle x \cdot w, g \rangle = \langle w, S(x) \cdot g \rangle.$$

Here $S(x)$ is the antipode of $U(\mathfrak{g}_-)$. Using the scaling operator we see that the above coupling further induces a coupling

$$(5.19) \qquad\qquad \mathrm{gr}_n^J(Q_m') \times \mathrm{gr}_J^n(W_m) \to \mathbb{C}.$$

The $\mathfrak{g}_-$-module π_2^{*m} is cyclic and generated by the highest weight vector:

$$\pi_2^{*m} \subset U(\mathfrak{g}_-) \cdot (\pi_2^{*m})^{2m}.$$

The dual module $(\pi_2^{*m})^*$ is cocyclic:

$$U(\mathfrak{g}_-) \cdot w \supset \big((\pi_2^{*m})^*\big)^{2m} \text{ for any } 0 \ne w \in (\pi_2^{*m})^*.$$

Now we know that $\mathrm{gr}_J^n(W_m)$ is a direct sum of cocyclic modules from Lemma 5.9, and Q_m' is a cyclic module by definition (5.18). Therefore, the coupling (5.19) defines an inclusion

$$(5.20) \qquad\qquad \mathrm{gr}_J^n(W_m) \subset \mathrm{gr}_n^J(Q_m')^*.$$

Comparing this result with Lemmas 5.9 and 5.10, we obtain the equality

$$Q_m' = Q_m$$

and that the inclusion (5.20) is bijective. $\qquad\qquad\qquad\qquad\qquad\square$

§5.4. Fusion filtration

In this section we conclude our discussion on the fusion filtration on the tensor product of $\widehat{\mathfrak{sl}}_2$-modules.

Lemma 5.12.   *The following formula holds*:

$$(5.21) \qquad \mathrm{ch}_q I_{k+1}^l\big(\mathrm{gr}^J(Q_m) \otimes \pi_j\big)$$
$$= \frac{q^{m^2}}{(q)_m} \sum_{\lambda \in \mathbb{Z}} q^{-(k+3)\lambda^2 + (l+1)\lambda}$$
$$\times \Big(\mathrm{ch}_{q^{-1}}(\pi_2^{*m})^{2(k+3)\lambda - l + j} - \mathrm{ch}_{q^{-1}}(\pi_2^{*m})^{2(k+3)\lambda - l - j - 2} \Big).$$

For $j \equiv l \bmod 2$, we have

$$(5.22) \quad q^{\Delta(j,k) - \Delta(l,k+1)} \mathrm{ch}_q M_{j+1,l+1}^{(k+2,k+3)} = \sum_{m \ge 0} \mathrm{ch}_q I_{k+1}^l\big(\mathrm{gr}^J(Q_m) \otimes \pi_j\big).$$

Proof.   In view of Proposition 5.5 and the vanishing of homology (3.13), we can apply Corollary 5.3 to $V = \mathrm{gr}^J(Q_m) \otimes \pi_j$. A simple calculation leads to formula (5.21). To show (5.22), we apply the identity of characters Lemma 2.2 taking $p = k+2$, $p' = k+3$, $r = b = j+1$, $a = l+1$ and noting the relation

$$(5.23) \qquad\qquad S_{m,\alpha}(q) = \mathrm{ch}_{q^{-1}}(\pi_2^{*m})^\alpha.$$

Comparing the result with (5.21) we obtain (5.22). $\qquad\qquad\qquad\square$

Let F_m be the fusion filtration (3.1) on the tensor product $L_{0,1}(1) \otimes L_{j,k}(0)$.

Theorem 5.13.   *We have*

$$F_m/F_{m-1} \simeq I_{k+1}(Q_m \otimes \pi_j).$$

For the character we have

(5.24) $\mathrm{ch}_q I_{k+1}^l(Q_m \otimes \pi_j) = \mathrm{ch}_q I_{k+1}^l(\mathrm{gr}^J(Q_m) \otimes \pi_j),$

which is given explicitly by (5.21).

Proof.   Since there is a surjection $I_{k+1}(Q_m \otimes \pi_j) \to F_m/F_{m-1}$, we are to show that both sides have the same character. We note that

(5.25)
$$\mathrm{ch}_q I_{k+1}^l(L_{0,1} \otimes \pi_j) \leq \sum_{m \geq 0} \mathrm{ch}_q I_{k+1}^l(Q_m \otimes \pi_j) \leq \sum_{m \geq 0} \mathrm{ch}_q I_{k+1}^l(\mathrm{gr}^J(Q_m) \otimes \pi_j).$$

On the other hand, $I_{k+1}(L_{0,1} \otimes \pi_j) = L_{0,1} \otimes L_{j,k}$ implies that

$$q^{\Delta(j,k)-\Delta(l,k+1)}\mathrm{ch}_q I_{k+1}^l(L_{0,1} \otimes \pi_j) = \mathrm{ch}_q M_{j+1,l+1}^{k+2,k+3}$$

(the factor $q^{\Delta(j,k)-\Delta(l,k+1)}$ comes from the difference between eigenvalue of L_0 and degree). Because of Lemma 5.12, we obtain equalities in (5.25).   □

Finally, let us discuss the fusion product

$$L_{1,1} * L_{j,k} = \bigoplus_{\substack{0 \leq l \leq k+1 \\ l \not\equiv j \bmod 2}} \mathrm{gr}^E(M_{j+1,l+1}^{(k+2,k+3)}) \otimes L_{l,k+1}.$$

Let $\tilde{F}_m$ be the fusion filtration (3.1) on the tensor product $L_{1,1}(1) \otimes L_{j,k}(0)$ of $\widehat{\mathfrak{sl}}_2 \otimes \mathbb{C}[u]$-modules, and set

$$\tilde{F}_m/\tilde{F}_{m-1} = \bigoplus_{\substack{0 \leq l \leq k+1 \\ l \not\equiv j \bmod 2}} \tilde{M}_{j+1,l+1,m}^{(k+2,k+3)} \otimes L_{l,k+1}.$$

Proposition 5.14. $\mathrm{ch}_q \tilde{M}_{j+1,l+1,m}^{(k+2,k+3)} = \mathrm{ch}_q \mathrm{gr}_m^E(M_{k-j+1,k-l+2}^{(k+2,k+3)}).$

Proof.   Consider an automorphism $\imath : \widehat{\mathfrak{sl}}_2 \to \widehat{\mathfrak{sl}}_2$ defined by $E_i \mapsto E_{1-i}$, $F_i \mapsto F_{1-i}$, where E_0, E_1, F_0, F_1 are the Chevalley generators of $\widehat{\mathfrak{sl}}_2$. Then we have an isomorphism of $\widehat{\mathfrak{sl}}_2$-modules $L_{j,k} \to \tilde{L}_{k-j,k}$, where the action of $\widehat{\mathfrak{sl}}_2$ on the right hand side is a composition of $\imath$ and of the natural action. This proves our proposition.   □

Let us summarize the conclusion.

Theorem 5.15.   *For the unitary series $M_{r,s}^{(k+2,k+3)}$, the character of the (1,3) filtration (4.4) is given by the formula introduced in Lemma 2.2:*

$$(5.26) \qquad \mathrm{ch}_q \mathrm{gr}_m^E(M_{r,s}^{(k+2,k+3)}) = \frac{q^{\Delta_{r,s}}}{(q)_m} I_{r,s,r+i,m}^{(k+2,k+3)}(q).$$

Here $i = 0,1$ is given by $i \equiv r - s \bmod 2$.

Proof.   First we give the proof in the case of $i = 0$. From Proposition 4.4 we have

$$\bigoplus_{\substack{1 \le s \le k+2 \\ s \equiv r \bmod 2}} \mathrm{gr}_m^E(M_{r,s}^{(k+2,k+3)}) \otimes L_{s-1,k+1} = (L_{0,1} * L_{r-1,k})_m = F_m/F_{m-1}.$$

Here F_m is the fusion filtration on the tensor product $L_{0,1}(0) \otimes L_{r-1,k}(1)$. From Theorem 5.13 we have

$$\mathrm{gr}_m^E(M_{r,s}^{(k+2,k+3)}) = \mathrm{Hom}_{\mathfrak{g}'}(L_{s-1,k}, F_m/F_{m-1}) \simeq I_{k+1}^{s-1}(Q_m \otimes \pi_{r-1}).$$

Take the character of the both ends above and use (5.24). Then we obtain

$$\mathrm{ch}_q \mathrm{gr}_m^E(M_{r,s}^{(k+2,k+3)}) = \mathrm{ch}_q I_{k+1}^{s-1}(\mathrm{gr}^J(Q_m) \otimes \pi_{r-1}).$$

The character in the right hand side is given by (5.21). Then, from the relation (5.23), we finally get the equality (5.26). To show the theorem in the case of $i = 1$, use the automorphism of $\widehat{\mathfrak{sl}}_2$ given in the proof of Proposition 5.14. $\square$

Acknowledgments

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