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Lattice Cohomology of Normal Surface Singularities

Dedicated to Professor Heisuke Hironaka on his 77th birthday

By

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Abstract

For any negative definite plumbed 3-manifold M we construct from its plumbed graph a graded $\mathbb{Z}[U]$-module. This, for rational homology spheres, conjecturally equals the Heegaard-Floer homology of Ozsváth and Szabó, but it has even more structure. If M is a complex singularity link then the normalized Euler-characteristic can be compared with the analytic invariants. The Seiberg-Witten Invariant Conjecture of [16], [13] is discussed in the light of this new object.

§1. Introduction

The article is a symbiosis of singularity theory and low-dimensional topology. Accordingly, it is preferable to separate its goals in two categories.

From the point of view of 3-*dimensional topology*, the article contains the following main result. For every negative definite plumbed 3-manifold it constructs a graded $\mathbb{Z}[U]$-module from the combinatorics of the plumbing graph. This for rational homology spheres conjecturally equals the Heegaard-Floer homology of Ozsváth and Szabó. In fact, it has more structure (e.g. instead of a

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$\mathbb{Z}_2$, odd/even grading it has a $\mathbb{Z}$ grading like a usual homology, see (5.2.6)(c)). The existence of these extra structures for arbitrary 3-manifolds might be an interesting subject for further investigation.

The motivations and aims from the point of view of *singularity theory* are the following.

In [16] L. Nicolaescu and the author formulated a conjecture which relates the geometric genus of a complex analytic normal surface singularity $(X, 0)$ — whose link M is a rational homology sphere — with the Seiberg-Witten invariant of M associated with the canonical *spin*^{c}-structure. The conjecture generalized a conjecture of Neumann and Wahl [20] which formulated the relationship for complete intersection singularities with integral homology sphere links. The conjecture [16] was verified in different cases, see [3],[12],[16],[17],[18],[19].

Since the Seiberg-Witten theory provides a rational number for *any spin*^{c}-structure, it was a natural challenge to search for a complete set of conjecturally valid identities, which involve all *spin*^{c}-structures. The preprint [13] proposed such identities, connecting the sheaf-cohomology of holomorphic line bundles associated with the analytic type of the singularity with the Seiberg-Witten invariants of the link. The identities were supported by a proof valid for rational singularities.

But, a few months later, [10] appeared with a list of counterexamples. This posed a lot of questions: what kind of guiding principles were wrongly interpreted in the original conjectures? How can one 'correct' them?

The present manuscript aims to answer some of them.

First, let us recall in short the original conjecture (for canonical *spin*^{c}-structure). One fixes a topological type (identified by a rational homology sphere link) and considers the Seiberg-Witten invariant of this link (normalized with a certain invariant K^2+s, see below). About this the conjecture predicted two things: First, that it is an upper bound for the geometric genus of all the possible analytic structures supported by the fixed topological type. Second, that this bound is optimal, and it is realized by all $\mathbb{Q}$-Gorenstein analytic structures.

Well, both expectations were wrong, but the nature of the two errors are completely different. Regarding the second part, the 'Seiberg-Witten invariant identity', the error can be localized easily. Indeed, the conjecture was over-optimistic: the identity is not valid for *every* $\mathbb{Q}$-Gorenstein singularity. Nevertheless, it is proved for large classes of singularities, and we expect that the list will be continued. Hence, the form of the identity shouldn't be modified, just we expect its validity for a *subclass* of $\mathbb{Q}$-Gorenstein singularities. At this moment, it is hopeless to identify exactly this subclass, nevertheless, in [3] it has a description exclusively in terms of the analytic structure — independently of the Seiberg-Witten theory; and [23] suggests that it can be identified by some vanishing properties.

In fact, we were more concern about the inequality part: Laufer type computation sequences identified a possible topological upper bound for the geometric genus, which in all cases explicitly analysed (at the time of [16],[13]) coincided with the Seiberg-Witten invariant, and the computation sequence technique resonated perfectly with the theory of Ozsváth and Szabó from [25]. Then, which part of this line of argument fails in general? The present article gives the following answer: There exists a cohomology theory $\{\mathbb{H}^q\}_{q\geq 0}$, such that its normalized Euler-characteristic (conjecturally) equals the Seiberg-Witten invariant. On the other hand, its 0^{th} normalized 'Betti-number' (or invariants related with it) serves as topological upper bound for the geometric genus (and fits with computation sequence constructions). In simpler cases (e.g. for rational, elliptic or star-shaped resolution graphs) one has a vanishing $\mathbb{H}^q = 0$ for all $q \geq 1$, hence the Seiberg-Witten invariant was able to serve as an upper bound. But, in general, this is not the case: the geometric genus of those analytic structures for which the 'Seiberg-Witten invariant identity' holds, are not extremal.

The article starts with the construction of this cohomology theory: the *lattice cohomology.* Here, we do not restrict ourselves to the rational homology sphere case. The construction provides from the plumbing graph of the link M (or, from the associated intersection lattice) a graded $\mathbb{Z}[U]$-module $\mathbb{H}^q(M,\sigma)$ for each $q\geq 0$, and for all torsion $spin^c$-structures σ of M.

We emphasize and exemplify more the case $\mathbb{H}^0$. $\mathbb{H}^0$, as a combinatorial $\mathbb{Z}[U]$-module associated with the link, is not new in the literature: it was considered by Ozsváth and Szabó in [25] in Heegaard-Floer homology computations of some special plumbed 3-manifolds (under the notation $\mathbb{H}^+$). Later, in [12], the author computed $\mathbb{H}^0$ for a larger class of 3-manifolds ('almost rational' graph-manifolds). In the present article, in Section 4, we prove similar characterization and structure results for $\mathbb{H}^*$ valid for rational, elliptic and almost rational graphs. Moreover, we analyze examples with $\mathbb{H}^1 \neq 0$ too. Section 5 connects $\mathbb{H}^*$ with the Heegaard-Floer homology.

Section 6 deals with the theory of line bundles associated with surface singularities. (It contains some parts from the unpublished [13] and from the lecture notes [15]. Some similar h^1-computations for the case of rational singularities were also found independently by T. Okuma [22].) In this section we determine a topological upper bound for the dimension of the sheaf-cohomologies of these line bundles in terms of their Chern classes. The description sits in $\mathbb{H}^0$.

The last section 7 presents the 'Seiberg-Witten invariant conjecture' (the unmodified conjectured identities), with examples and more comments.

§2. Preliminaries

§2.1. Negative definite plumbing graphs

2.1.1. Let $(X,0)$ be a complex analytic normal surface singularity with link M. Fix a sufficiently small Stein representative X of the germ $(X,0)$ and let $\pi:\tilde{X}\to X$ be a *good* resolution of the singular point $0\in X$. Let $E:=\pi^{-1}(0)$ be the exceptional divisor with irreducible components $\{E_j\}_{j\in\mathcal{J}}$ and write $\Gamma(\pi)$ for the dual resolution graph associated with π. Recall that $\Gamma(\pi)$ is connected and the intersection matrix $I:=\{(E_j,E_i)\}_{j,i}$ is negative definite. We write e_j for E_j^2, g_j for the genus of E_j $(j\in\mathcal{J})$, and $g:=\sum_j g_j$. Moreover, let c be the number of independent cycles in (the topological realization of) Γ. E.g., $c=0$ if and only if $\Gamma(\pi)$ is a tree. The rank of $H_1(M,\mathbb{Z})$ is $c+2g$. Hence, M is a rational homology sphere (i.e. $H_1(M,\mathbb{Q})=0$) if and only if $g=c=0$.

2.1.2. Since π identifies $\partial\tilde{X}$ with M, the graph $\Gamma(\pi)$ can be viewed as a plumbing graph and M as the associated S^1-plumbed manifold. In the sequel Γ will denote either a good resolution graph as above, or a negative definite plumbing graph of M. Similarly, $\tilde{X}$ denotes either the space of a good resolution, or the oriented 4-manifold obtained by plumbing disc-bundles corresponding to Γ.

§2.2. The combinatorics of the plumbing

2.2.1. Definition. The lattices L and L'. The image of the boundary operator $\partial: H_2(\tilde{X},M,\mathbb{Z})\to H_1(M,\mathbb{Z})$ is the torsion subgroup H of $H_1(M,\mathbb{Z})$. The exact sequence of $\mathbb{Z}$-modules

$$0\to L\xrightarrow{i} L'\to H\to 0 \tag{1}$$

will stand for the homological exact sequence

$$0\to H_2(\tilde{X},\mathbb{Z})\to H_2(\tilde{X},M,\mathbb{Z})\xrightarrow{\partial} Tors(H_1(M,\mathbb{Z}))\to 0,$$

(or for its Poincaré dual). Here L is freely generated by the homology classes $\{E_j\}_{j\in\mathcal{J}}$ and is equipped with the intersection form $(\cdot,\cdot)$. For each j, consider a small transversal disc D_j in $\tilde{X}$ with $\partial D_j \subset \partial \tilde{X}$. Then L' is freely generated by the (relative homology) classes $\{D_j\}_{j\in\mathcal{J}}$. Notice that the morphism $i : L \to L'$ can be identified with $L \to \mathrm{Hom}(L, \mathbb{Z})$ given by $l \mapsto (l, \cdot)$. The intersection form has a natural extension to $L_{\mathbb{Q}} = L \otimes \mathbb{Q}$, and we will regard $\mathrm{Hom}(L, \mathbb{Z})$ as a sub-lattice of $L_{\mathbb{Q}}$: $\alpha \in \mathrm{Hom}(L, \mathbb{Z})$ corresponds with the unique $l_\alpha \in L_{\mathbb{Q}}$ which satisfies $\alpha(l) = (l_\alpha, l)$ for any $l \in L$. Hence, the exact sequence (1) can be recovered completely from the lattice L.

2.2.2. Characteristic elements. *Spin*^{c}-structures. The set of characteristic elements are defined by

$$Char = Char(L) := \{k \in L' : (k, x) + (x, x) \in 2\mathbb{Z} \text{ for any } x \in L\}.$$

The unique rational cycle $K \in L'$ which satisfies the system (of adjunction relations) $(K, E_j) = -(E_j, E_j) - 2 + 2g_j$ for all j is called the *canonical cycle*. Then $Char = K + 2L'$. There is a natural action of L on $Char$ by $l * k := k + 2l$ whose orbits are of type $k + 2L$. Obviously, H acts freely and transitively on the set of orbits by $[l'] * (k + 2L) := k + 2l' + 2L$.

If $\tilde{X}$ is a 4-manifold as above, then $H^2(\tilde{X}, \mathbb{Z})$ has no 2-torsion. Therefore, the first Chern class (of the associated determinant line bundle) realizes an identification between the *spin*^{c}-structures $Spin^c(\tilde{X})$ on $\tilde{X}$ and $Char \subset L' = H^2(\tilde{X}, \mathbb{Z})$ (see e.g. [4, 2.4.16]). On the other hand, the *spin*^{c}-structures on M form an $H_1(M, \mathbb{Z})$ torsor. In the image of the restriction $Spin^c(\tilde{X}) \to Spin^c(M)$ are exactly those *spin*^{c}-structures of M whose Chern classes are restrictions $L' \to H^2(M, \mathbb{Z}) = H_1(M, \mathbb{Z})$, i.e. are torsion elements sitting in H. We call them *torsion* structures, and we denote them by $Spin^c_t(M)$. One has an identification of $Spin^c_t(M)$ with the set of L-orbits of $Char$, and this identification is compatible with the action of H on both sets. In the sequel, we think about $Spin^c_t(M)$ by this identification: any torsion *spin*^{c}-structure of M will be represented by an orbit $[k] := k + 2L \subset Char$. The *canonical spin*^{c}-structure (is torsion and) corresponds to $[K]$.

We write $\hat{H}$ for the Pontrjagin dual $\mathrm{Hom}(H, S^1)$ of H. One has a natural isomorphism

$$\theta : H \to \hat{H}, \quad \text{induced by} \quad [l'] \mapsto e^{2\pi i (l', \cdot)}.$$

2.2.3. Positive cones. One can consider two types of 'positivity conditions' for rational cycles. The first one is considered in L. A cycle $x = \sum_j r_j E_j \in L_{\mathbb{Q}}$ is called *effective*, denoted by $x \geq 0$, if $r_j \geq 0$ for all j. Their collection is denoted by $L_{\mathbb{Q},e}$, while $L'_e := L_{\mathbb{Q},e} \cap L'$ and $L_e := L_{\mathbb{Q},e} \cap L$.

The second is the *numerical effectiveness* of the rational cycles, i.e. positivity considered in L'. We define $L_{\mathbb{Q},ne} := \{x \in L_{\mathbb{Q}} : (x, E_j) \geq 0 \text{ for all } j\}$. In fact, $L_{\mathbb{Q},ne}$ is the positive cone in $L_{\mathbb{Q}}$ generated by $\{D_j\}_j$, i.e. it is exactly $\{\sum_j r_j D_j, r_j \geq 0 \text{ for all } j\}$. Since I is negative definite, all the entries of D_j are *strictly* negative. In particular, $-L_{\mathbb{Q},ne} \subset L_{\mathbb{Q},e}$. Similarly as above, write $L_{ne} := L \cap L_{\mathbb{Q},ne}$.

2.2.4. Liftings. We will consider some 'liftings' (set theoretical sections) of the element of H into L'. They correspond to the positive cones in $L_{\mathbb{Q}}$ considered in (2.2.3).

More precisely, for any $l' + L = h \in H$, let $l'_e(h) \in L'$ be the unique minimal effective rational cycle in $L_{\mathbb{Q},e}$ whose class is h. Clearly, the set $\{l'_e(h)\}_{h \in H}$ is exactly $Q := \{\sum_j r_j E_j \in L' ; 0 \leq r_j < 1\}$.

Similarly, for any $h = l' + L$, the intersection $(l' + L) \cap L_{\mathbb{Q},ne}$ has a unique maximal element $l'_{ne}(h)$, and the intersection $(l' + L) \cap (-L_{\mathbb{Q},ne})$ has a unique minimal element $\bar{l}'_{ne}(h)$ (cf. [12, 5.4]). By their definitions $\bar{l}'_{ne}(h) = -l'_{ne}(-h)$.

For some h, $\bar{l}'_{ne}(h)$ might be situated in Q, but, in general, this is not the case. In general, the characterization of all the elements $\bar{l}'_{ne}(h)$ is not simple (see e.g. [12]).

2.2.5. The χ-functions (Riemann-Roch formula). For any characteristic element $k \in Char$ one defines

$$\chi_k : L' \to \mathbb{Q} \quad \text{by} \quad \chi_k(l') := -(l', l' + k)/2.$$

Clearly, $\chi_k(L) \subset \mathbb{Z}$. For the interpretation of χ_k as (twisted) Riemann-Roch formula, consider the following. Let $\tilde{X}$ be a resolution as in (2.1.1), and fix a holomorphic line bundle $\mathcal{L} \in Pic(\tilde{X})$, and write $c_1(\mathcal{L}) = l' \in L'$ for its Chern class. Set $k := K - 2l' \in Char$. For any $l \in L$ with $l > 0$ one defines the sheaf $\mathcal{O}_l := \mathcal{O}_{\tilde{X}}/\mathcal{O}_{\tilde{X}}(-l)$ supported by E (see e.g. 6.1.1). Consider the sheaf $\mathcal{L} \otimes \mathcal{O}_l$ and let $\chi(\mathcal{L} \otimes \mathcal{O}_l) = h^0(\mathcal{L} \otimes \mathcal{O}_l) - h^1(\mathcal{L} \otimes \mathcal{O}_l)$ be its (holomorphic) Euler-characteristic. The Riemann-Roch theorem states that this can be computed combinatorially, namely

$$\chi(\mathcal{L} \otimes \mathcal{O}_l) = \chi_k(l).$$

§3. The Lattice Cohomology Associated with L

§3.1. Lattice cohomology associated with $\mathbb{Z}^s$ and a system of weights

3.1.1. We consider a free $\mathbb{Z}$-module, with a fixed basis $\{E_j\}_j$, denoted by $\mathbb{Z}^s$. It is also convenient to fix a total ordering of the index set $\mathcal{J}$, which in the sequel will be denoted by $\{1,\dots,s\}$.

Our goal is to define a graded $\mathbb{Z}[U]$-module associated with the pair $(\mathbb{Z}^s,\{E_j\}_j)$ and a system of weights, which will be introduces in (3.1.4). First we set some notations regarding $\mathbb{Z}[U]$-modules.

3.1.2. $\mathbb{Z}[U]$-modules. Consider the graded $\mathbb{Z}[U]$-module $\mathcal{T}:=\mathbb{Z}[U,U^{-1}]$, and (following [25]) denote by $\mathcal{T}_0^+$ its quotient by the submodule $U\cdot\mathbb{Z}[U]$. This has a grading in such a way that $\deg(U^{-d})=2d$ $(d\geq 0)$. Similarly, for any $n\geq 1$, the quotient of $\mathbb{Z}\langle U^{-(n-1)},U^{-(n-2)},\dots,1,U,\dots\rangle$ by $U\cdot\mathbb{Z}[U]$ (with the same grading) defines the graded module $\mathcal{T}_0(n)$. Hence, $\mathcal{T}_0(n)$, as a $\mathbb{Z}$-module, is freely generated by $1,U^{-1},\dots,U^{-(n-1)}$, and has finite $\mathbb{Z}$-rank n.

More generally, for any graded $\mathbb{Z}[U]$-module P with d-homogeneous elements P_d, and for any $r\in\mathbb{Q}$, we denote by $P[r]$ the same module graded (by $\mathbb{Q}$) in such a way that $P[r]_{d+r}=P_d$. Then set $\mathcal{T}_r^+:=\mathcal{T}_0^+[r]$ and $\mathcal{T}_r(n):=\mathcal{T}_0(n)[r]$. (Hence, for $m\in\mathbb{Z}$, $\mathcal{T}_{2m}^+=\mathbb{Z}\langle U^{-m},U^{-m-1},\dots\rangle$.)

3.1.3. The cochain complex. $\mathbb{Z}^s\otimes\mathbb{R}$ has a natural cellular decomposition into cubes. The set of zero-dimensional cubes is provided by the lattice points $\mathbb{Z}^s$. Any $l\in\mathbb{Z}^s$ and subset $I\subset\mathcal{J}$ of cardinality q defines a q-dimensional cube, which has its vertices in the lattice points $(l+\sum_{j\in I'}E_j)_{I'}$, where I' runs over all subsets of I. On each such cube we fix an orientation. This can be determined, e.g., by the order $(E_{j_1},\dots,E_{j_q})$, where $j_1<\cdots<j_q$, of the involved base elements $\{E_j\}_{j\in I}$. The set of oriented q-dimensional cubes defined in this way is denoted by $\mathcal{Q}_q$ $(0\leq q\leq s)$.

Let $\mathcal{C}_q$ be the free $\mathbb{Z}$-module generated by oriented cubes $\square_q\in\mathcal{Q}_q$. Clearly, for each $\square_q\in\mathcal{Q}_q$, the oriented boundary $\partial\square_q$ has the form $\sum_k\varepsilon_k\,\square_{q-1}^k$ for some $\varepsilon_k\in\{-1,+1\}$. Here, in this sum, we write only those $(q-1)$-cubes which appear with non-zero coefficient. These are called *faces* of $\square_q$.

It is clear that $\partial\circ\partial=0$. But, obviously, the homology of the chain complex $(\mathcal{C}_*,\partial)$ (or, of the cochain complex $(\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}_*,\mathbb{Z}),\delta)$) is not very interesting: it is just the (co)homology of $\mathbb{R}^s$. A more interesting (co)homology can be constructed as follows. For this, we consider a set of compatible *weight functions* $w_q:\mathcal{Q}_q\to\mathbb{Z}$ $(0\leq q\leq s)$.

3.1.4. Definition. A set of functions $w_q : \mathcal{Q}_q \to \mathbb{Z}$ $(0 \le q \le s)$ is called a *set of compatible weight functions* if the following hold:

(a) For any integer $k \in \mathbb{Z}$, the set $w_0^{-1}(\,(-\infty, k]\,)$ is finite;

(b) for any $\square_q \in \mathcal{Q}_q$ and for any of its faces $\square_{q-1} \in \mathcal{Q}_{q-1}$ one has $w_q(\square_q) \ge w_{q-1}(\square_{q-1})$.

(In the sequel sometimes we will omit the index q of w_q.)

Assume that we already fixed a set of compatible weight functions $\{w_q\}_q$. Then we set $\mathcal{F}^q := \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}_q, \mathcal{T}_0^+)$. Notice that $\mathcal{F}^q$ is, in fact, a $\mathbb{Z}[U]$-module by $(p * \phi)(\square_q) := p(\phi(\square_q))$ $(p \in \mathbb{Z}[U])$. Moreover, $\mathcal{F}^q$ has a $\mathbb{Z}$-grading: $\phi \in \mathcal{F}^q$ is homogeneous of degree $d \in \mathbb{Z}$ if for each $\square_q \in \mathcal{Q}_q$ with $\phi(\square_q) \neq 0$, $\phi(\square_q)$ is a homogeneous element of $\mathcal{T}_0^+$ of degree $d - 2 \cdot w(\square_q)$. (In fact, the grading is $2\mathbb{Z}$-valued; hence, the reader interested only in the present construction may divide all the degrees by two. Nevertheless, we prefer to keep the present form in our presentation because of its resonance with the Heegaard-Floer homology of the link.)

Next, we define $\delta_w : \mathcal{F}^q \to \mathcal{F}^{q+1}$. For this, fix $\phi \in \mathcal{F}^q$ and we show how $\delta_w\phi$ acts on a cube $\square_{q+1} \in \mathcal{Q}_{q+1}$. First write $\partial\square_{q+1} = \sum_k \varepsilon_k \square_q^k$, then set

$$(\delta_w\phi)(\square_{q+1}) := \sum_k \varepsilon_k\, U^{w(\square_{q+1}) - w(\square_q^k)}\, \phi(\square_q^k).$$

3.1.5. Lemma. *$\delta_w \circ \delta_w = 0$, i.e. $(\mathcal{F}^*, \delta_w)$ is a cochain complex.*

Proof. With the obvious notations, $(\delta_w^2\phi)(\square_{q+2}^j)$ equals

$$\sum_k \varepsilon_k^j U^{w(\square_{q+2}^j) - w(\square_{q+1}^k)} \sum_l \varepsilon_l^k U^{w(\square_{q+1}^k) - w(\square_q^l)} \phi(\square_q^l)$$
$$= \sum_l U^{w(\square_{q+2}^j) - w(\square_q^l)} \Big(\sum_k \varepsilon_k^j \varepsilon_l^k \Big)\, \phi(\square_q^l).$$

But, for any l, $\sum_k \varepsilon_k^j \varepsilon_l^k = 0$ since $\partial^2 = 0$. □

3.1.6. In fact, $(\mathcal{F}^*, \delta_w)$ has a natural **augmentation** too. Indeed, set $m_w := \min_{l \in \mathbb{Z}^s} w_0(l)$ and choose $l_w \in \mathbb{Z}^s$ such that $w_0(l_w) = m_w$. Then one defines the $\mathbb{Z}[U]$-linear map

$$\epsilon_w : \mathcal{T}_{2m_w}^+ \longrightarrow \mathcal{F}^0$$

such that $\epsilon_w(U^{-m_w - s})(l)$ is the class of $U^{-m_w + w_0(l) - s}$ in $\mathcal{T}_0^+$ for any integer $s \ge 0$.

3.1.7. Lemma. *ϵ_w is injective, and $\delta_w \circ \epsilon_w = 0$.*

Proof. Since $\epsilon_w(U^{-m_w-s})(l_w) = U^{-s}$, the injectivity is clear. Take $\square \in \mathcal{Q}_1$ with $\partial\square = a - b$. Then

$$(\delta_w\epsilon_w)(t)(\square) = U^{w(\square)-w(a)}\epsilon_w(t)(a) - U^{w(\square)-w(b)}\epsilon_w(t)(b)$$
$$= U^{w(\square)}t - U^{w(\square)}t = 0. \qquad \square$$

We invite the reader to verify that ϵ_w and δ_w are morphisms of $\mathbb{Z}[U]$-modules, and are homogeneous of degree zero.

3.1.8. Definitions. The homology of the cochain complex $(\mathcal{F}^*, \delta_w)$ is called the *lattice cohomology* of the pair $(\mathbb{R}^s, w)$, and it is denoted by $\mathbb{H}^*(\mathbb{R}^s, w)$. The homology of the augmented cochain complex

$$0 \longrightarrow \mathcal{T}^+_{2m_w} \xrightarrow{\epsilon_w} \mathcal{F}^0 \xrightarrow{\delta_w} \mathcal{F}^1 \xrightarrow{\delta_w} \dots$$

is called the *reduced lattice cohomology* of the pair $(\mathbb{R}^s, w)$, and it is denoted by $\mathbb{H}^*_{red}(\mathbb{R}^s, w)$. If the pair $(\mathbb{R}^s, w)$ is clear from the context, we omit it from the notation. Clearly, for any $q \geq 0$, both $\mathbb{H}^q$ and $\mathbb{H}^q_{red}$ admit an induced graded $\mathbb{Z}[U]$-module structure and $\mathbb{H}^q = \mathbb{H}^q_{red}$ for $q > 0$. Moreover, the $\mathbb{Z}$-grading of $\mathcal{F}^q$ induces a $\mathbb{Z}$-grading on $\mathbb{H}^q$ and $\mathbb{H}^q_{red}$; the homogeneous part of degree d is denoted by $\mathbb{H}^q_d$, or $\mathbb{H}^q_{red,d}$.

It is easy to see that $\mathbb{H}^*(\mathbb{R}^s, w)$ depends essentially on the choice of w.

3.1.9. Lemma. *One has a graded $\mathbb{Z}[U]$-module isomorphism $\mathbb{H}^0 = \mathcal{T}^+_{2m_w} \oplus \mathbb{H}^0_{red}$.*

Proof. Consider the isomorphism $U^{-m_w} : \mathcal{T}^+_0 \to \mathcal{T}^+_{2m_w}$. Then define $r_w : \mathbb{H}^0 \to \mathcal{T}^+_{2m_w}$ by $r_w(\phi) := U^{-m_k}\phi(l_w)$. Since $r_w \circ \epsilon_w = 1$, the exact sequence $0 \to \mathcal{T}^+_{2m_w} \xrightarrow{\epsilon_w} \mathbb{H}^0 \to \mathbb{H}^0_{red} \to 0$ splits. $\square$

3.1.10. Next, we present another realization of the modules $\mathbb{H}^*$.

3.1.11. Definitions. For each $n \in \mathbb{Z}$, define $S_n = S_n(w) \subset \mathbb{R}^s$ as the union of all the cubes $\square_q$ (of any dimension) with $w(\square_q) \leq n$. Clearly, $S_n = \emptyset$, whenever $n < m_w$. For any $q \geq 0$, set

$$\mathbb{S}^q(\mathbb{R}^s, w) := \oplus_{n \geq m_w} H^q(S_n, \mathbb{Z}).$$

Then $\mathbb{S}^q$ is $\mathbb{Z}$ (in fact, $2\mathbb{Z}$)-graded, the $d = 2n$-homogeneous elements $\mathbb{S}^q_d$ consist of $H^q(S_n, \mathbb{Z})$. Also, $\mathbb{S}^q$ is a $\mathbb{Z}[U]$-module; the U-action is given by the restriction map $r_{n+1} : H^q(S_{n+1}, \mathbb{Z}) \to H^q(S_n, \mathbb{Z})$. Namely, $U * (\alpha_n)_n = (r_{n+1}\alpha_{n+1})_n$. Moreover, for $q = 0$, the fixed base-point $l_w \in S_n$ provides an augmentation (splitting) $H^0(S_n,\mathbb{Z}) = \mathbb{Z} \oplus \tilde{H}^0(S_n,\mathbb{Z})$, hence an augmentation of the graded $\mathbb{Z}[U]$-modules

$$\mathbb{S}^0 = \mathcal{T}^+_{2m_w} \oplus \mathbb{S}^0_{red} = (\oplus_{n\geq m_w}\mathbb{Z}) \oplus (\oplus_{n\geq m_w}\tilde{H}^0(S_n,\mathbb{Z})).$$

3.1.12. Theorem.

(*a*) *There exists a graded* $\mathbb{Z}[U]$*-module isomorphism, compatible with the augmentations*:

$$\mathbb{H}^*(\mathbb{R}^s, w) = \mathbb{S}^*(\mathbb{R}^s, w).$$

(*b*) *For any degree* d*, there exists an integer* $N(d) \geq 0$*, such that*

$$\mathbb{H}^*_{red,d} \cap im(U^{N(d)}) = 0.$$

(*c*) *For any* n *one has* $U^{n-m_w+1}\mathbb{H}^*_{2n} = 0$*. If there exists* N *such that* S_n *is contractible for any* $n \geq N$*, then* $U^{N-m_w}\mathbb{H}^*_{red} = 0$*.*

Proof. (a) Let $\mathcal{F}^q_d$ be the set of $d = 2n$-homogeneous elements $\phi \in \mathcal{F}^q$. Since δ_w is homogeneous of degree zero, $(\mathcal{F}^*_d, \delta_w)$ is a complex. Let $(\mathcal{C}^*(S_n), \delta)$ be the usual cochain complex of S_n. Then the two complexes can be naturally identified. Indeed, take $\phi \in \mathcal{F}^q_d$. Then, for any $\square_q$, $\phi(\square_q)$ has the form $a_\phi(\square_q)U^{w(\square_q)-n}$. Hence $a_\phi(\square_q) \in \mathbb{Z}$ is well-defined for any q-cube $\square_q$ of S_n, and the correspondence $\phi \mapsto a_\phi$ realizes the bijection $\mathcal{F}^*_d \to \mathcal{C}^*(S_n)$.

Since $\tilde{H}_q(\mathbb{R}^s,\mathbb{Z}) = 0$, for any n there exists N such that $\tilde{H}_q(S_n) \to \tilde{H}_q(S_{n+N})$ is trivial. (b) is the dual statement of this. (c) follows from (a). □

3.1.13. *Remark.* Although $\mathbb{H}^*_{red}(\mathbb{R}^s, w)$ has finite $\mathbb{Z}$-rank in any fixed homogeneous degree, in general, it is not finitely generated over $\mathbb{Z}[U]$. E.g., set $s = 1$, and define w_0 by

$$w_0(-n) = w_0(n) = [n/2] + 4\{n/2\} \quad \text{for any } n \in \mathbb{Z}_{\geq 0},$$

where $[\,]$ and $\{\,\}$ are the integral, respectively the fractional parts; and let w_1 on the segment $[n, n+1]$ take the value $\max\{w_0(n), w_0(n+1)\}$. Then $\mathbb{H}^0_{red} = \oplus_{k\geq 1}\mathcal{T}_k(1)^2$.

3.1.14. Restrictions. Assume that $T \subset \mathbb{R}^s$ is a subspace of $\mathbb{R}^s$ consisting of union of some cubes (from $\mathcal{Q}_*$). Let $\mathcal{C}_q(T)$ be the free $\mathbb{Z}$-module generated by q-cubes of T, $\mathcal{F}^q(T) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}_q(T), \mathcal{T}^+_0)$. Then $(\mathcal{F}^*(T), \delta_w)$ is a complex, whose homology will be denoted by $\mathbb{H}^*(T, w)$. It has a natural graded $\mathbb{Z}[U]$-module structure. The restriction map induces a natural graded $\mathbb{Z}[U]$-module homogeneous homomorphism (of degree zero)

$$r^* : \mathbb{H}^*(\mathbb{R}^s, w) \to \mathbb{H}^*(T, w).$$

§3.2. Lattice cohomology associated with Γ and $k \in Char$

3.2.1. We consider a graph Γ as in Section 2 and we fix a characteristic element $k \in Char$. Notice that Γ automatically provides a free $\mathbb{Z}$-module $L = \mathbb{Z}^s$ with a fixed bases $\{E_j\}_j$. Using Γ and k, we define a set of compatible weight functions $\{w_q\}_q$.

The definition reflects our effort to connect the topology of a singularity-link (e.g. the lattice cohomology) with analytic invariants. For more detailed motivation, see (4.2.4) and (6.2).

For any $g \geq 0$ and $n \geq 0$, let $M^g(n)$ be the maximum of all possible dimensions of sheaf-cohomologies $H^1(C, \mathcal{L})$, where C runs over all Riemann surfaces of genus g and $\mathcal{L}$ is a holomorphic line bundle on C with holomorphic Euler-characteristic $\chi(\mathcal{L}) = n$. (This number exists, in fact $M^g(n) \leq g$.)

Now, we define $\{w_q\}_q$ as follows. For $q = 0$ we set $w_0 := \chi_k$ (cf. 2.2.5). Since the intersection form is negative definite, (3.1.4)(a) is satisfied.

Next, we define w_1. Consider a segment $S \in \mathcal{Q}_1$ with vertices l and $l + E_j$ for some $l \in L$ and $j \in \mathcal{J}$. We set

$$w_1(S) := \max\{\, \chi_k(l)\,,\, \chi_k(l + E_j)\,\} + M^{g_j}(\,|\chi_k(l) - \chi_k(l + E_j)|\,).$$

Finally, for any $\square_q \in \mathcal{Q}_q$ $(q \geq 2)$ set

$$w_q(\square_q) := \max\{\, w_1(S) \,:\, S \text{ is a segment of } \square_q\,\}.$$

3.2.2. Examples. (a) Assume that $g_j = 0$ for all j. Since $M^0(n) = 0$ for any $n \geq 0$, for any q

$$w(\square_q) = \max\{\chi_k(v) \,:\, v \text{ is a vertex of } \square_q\}.$$

(b) Assume that $g_j \leq 1$ for any j. Since $M^1(n) = 0$ for $n \geq 1$ and $M^1(0) = 1$, the definition of w_1 might be modified into

$$w_1(S) = \max\{\, \chi_k(l)\,,\, \chi_k(l + E_j)\,,\, \min\{\, \chi_k(l)\,,\, \chi_k(l + E_j)\,\} + g_j\,\}.$$

(c) By a vanishing theorem, in general, $M^g(n) = 0$ whenever $n \geq g$.

3.2.3. Definition. The $\mathbb{Z}[U]$-modules $\mathbb{H}^*(\mathbb{R}^s, w)$ and $\mathbb{H}^*_{red}(\mathbb{R}^s, w)$ obtained by these weight functions are called the *lattice cohomologies* associated with the pair (Γ, k) and are denoted by $\mathbb{H}^*(\Gamma, k)$, respectively $\mathbb{H}^*_{red}(\Gamma, k)$. We also write $m_k := m_w = \min_{l\in L} \chi_k(l)$.

3.2.4. Theorem. *$\mathbb{H}^*_{red}(\Gamma, k)$ is finitely generated over $\mathbb{Z}$.*

Proof. We start the proof with the following statement:

Fact. *There exist $X \in L_e$ and an increasing infinite sequence of cycles $\{x_i\}_{i\geq 0}$ with $x_0 = X$, such that*

(a) $x_{i+1} = x_i + E_{j(i)}$ for some $j(i)$, $i \geq 0$,

(b) if $x_i = \sum_j m_{i,j}E_j$, then for all j, $m_{i,j}$ tends to infinity as i tends to infinity,

(c) $\chi_k(x_{i+1}) - \chi_k(x_i) \geq g_{j(i)}$.

Similarly, there exists $Y \in L_e$ and an increasing infinite sequence of cycles $\{y_i\}_{i\geq 0}$, with $y_0 = Y$ and similar properties as in (a)–(b), and (c) $\chi_k(-y_{i+1}) - \chi_k(-y_i) \geq g_{j(i)}$.

Indeed, take a cycle $Z \in L$ such that $(Z, E_j) < 0$ for any j. Let $\{z_i\}_{i=0}^t$ be an increasing sequence with $z_0 = 0$, $z_t = Z$, $z_{i+1} = z_i + E_{j(i)}$ $(0 \leq i < t)$. Then for m sufficiently large, $X = mZ$, and the sequence $\{m'Z + z_i\}$ (where $m' \geq m$ and $0 \leq i < t$) works. A similar statement is valid for $Y = mZ$ (and similar type of sequence) with $m \gg 0$.

Fix $X, Y \in L_e$, such that $-Y \leq l_w \leq X$. Let $T(-Y, X) = \{r \in \mathbb{R}^s : -Y \leq r \leq X\}$. $T(-Y, X)$ has a natural cube-decomposition compatible with the decomposition of $\mathbb{R}^s$, hence by (3.1.14), one has a map $r^*_{-Y,X} : \mathbb{H}^*_{red}(\mathbb{R}^s, w) \to \mathbb{H}^*_{red}(T(-Y, X), w)$.

Set X and Y as in Fact; clearly we may assume that $-Y \leq l_w \leq X$. We claim that $r^*_{-Y,X}$ is an isomorphism. Indeed, consider the restriction map $r^*_{l,i} : \mathbb{H}^*_{red}(T(-y_l, x_{i+1}), w) \to \mathbb{H}^*_{red}(T(-y_l, x_i), w)$. If $l \in T(-y_l, x_{i+1}) \setminus T(-y_l, x_i)$ then $l = z + E_{j(i)}$, $z \leq x_i$ and the coefficients of $E_{j(i)}$ in z and x_i are the same. Hence, $(x_i, E_{j(i)}) \geq (z, E_{j(i)})$. This implies that

$$\chi_k(z + E_{j(i)}) - \chi_k(z) \geq \chi_k(x_{i+1}) - \chi_k(x_i) \geq g_{j(i)},$$

which also shows (via 3.2.2(c)) that $w_1[z, z + E_{j(i)}] = w_0(z + E_{j(i)}) \geq w_0(z)$. Hence, the retract $T(-y_l, x_{i+1}) \to T(-y_l, x_i)$, which sends cycles of type $z + E_{j(i)}$ (as above) to z (and preserves all cycles of different type) induces an isomorphism $r^*_{l,i}$. Similar argument works if we move from y_l to y_{l+1}. Now, property (b) guarantees that $r^*_{-Y,X}$ is an isomorphism. On the other hand, $\mathbb{H}^*_{red}(T(-Y, X), w)$ is finitely generated over $\mathbb{Z}$. $\square$

3.2.5. Corollary. *For any pair* (Γ, k)*, the space* S_n *is contractible for* n *sufficiently large.*

Proof. Fix X, Y as in Fact of the proof of (3.2.4). Let n be so large that $T(-Y, X) \subset S_n$. Then, the same argument as in the proof of (3.2.4) shows that $S_n \cap T(-y_l, x_i) \hookrightarrow S_n \cap T(-y_l, x_{i+1})$ admits a deformation retract. Hence, by induction, $T(-Y, X) \subset S_n$ have the same homotopy type. □

If $g = 0$, one may also prove the contractibility of S_n for $n \gg 0$ by verifying that S_n is a deformation retract in the real ellipsoid $\{x \in \mathbb{R}^s : \chi_k(x) \leq n\}$, which is obviously contractible.

3.2.6. Definitions. We will consider the following (euler-characteristic type) numerical invariants:

$$eu(\mathbb{H}^0(\Gamma, k)) := -m_k + \operatorname{rank}_{\mathbb{Z}}(\mathbb{H}^0_{red}(\Gamma, k)),$$

$$eu(\mathbb{H}^*(\Gamma, k)) := -m_k + \textstyle\sum_q (-1)^q \operatorname{rank}_{\mathbb{Z}}(\mathbb{H}^q_{red}(\Gamma, k)).$$

3.2.7. *Remark.* There is a symmetry present in the picture. Indeed, the involution $x \mapsto -x$ $(x \in L')$ induces identities $\chi_{-k}(-l) = \chi_k(l)$, hence isomorphisms

$$\mathbb{H}^*(\Gamma, k) = \mathbb{H}^*(\Gamma, -k) \quad \text{and} \quad \mathbb{H}^*_{red}(\Gamma, k) = \mathbb{H}^*_{red}(\Gamma, -k).$$

Notice that the involution $[k] \mapsto [-k]$ corresponds to the natural involution of $Spin^c_t(M) \subset Spin^c(M)$.

Regarding the canonical structure, $[K] = [-K]$ if and only if $K \in L$. In singularity theory, such graphs are called 'numerical Gorenstein' (when the tangent bundle on $X \setminus 0$ is topologically trivial). On the other hand, this happens if and only if the canonical *spin*^{c}-structure is *spin*.

§3.3. Dependence of $\mathbb{H}^*(\Gamma, k)$ on $k \in Char$

3.3.1. Fix Γ as above. Above we defined for any $k \in Char$ a graded $\mathbb{Z}[U]$-module $\mathbb{H}^*(\Gamma, k)$. Some of these graded roots are not very different. Indeed, assume that $[k] = [k']$ (cf. 2.2.2), hence $k' = k + 2l$ for some $l \in L$. Then $\chi_{k'}(x - l) = \chi_k(x) - \chi_k(l)$ for any $x \in L$. Therefore, the transformation $x \mapsto x' := x - l$ realizes the following identification:

3.3.2. Lemma. *If* $k' = k + 2l$ *for some* $l \in L$*, then:* $\mathbb{H}^*(\Gamma, k') = \mathbb{H}^*(\Gamma, k)[-2\chi_k(l)]$.

In fact, there is an easy way to choose one module from the multitude $\{\mathbb{H}^*(\Gamma,k)\}_{k\in[k]}$. Indeed, set $m_k=\min_{l\in L}\chi_k(l)$ as above. Since $(k+2l)^2=k^2-8\chi_k(l)$, we get

$$8m_k := k^2 - \max_{k'\in[k]}(k')^2 \leq 0.$$

Set $M_{[k]} := \{k\in[k] : m_k = 0\}$. Hence, if k_0 and $k_0+2l\in M_{[k]}$, then $-\chi_{k_0}(l)=0$. In particular, for any fixed orbit $[k]$, any choice of $k_0\in M_{[k]}$ provides the same module $\mathbb{H}^*(\Gamma,k_0)$. In the sequel we will denote this module by $\mathbb{H}^*(\Gamma,[k])$. Notice that with this notation, for any $k\in[k]$

$$\mathbb{H}^*(\Gamma,k)=\mathbb{H}^*(\Gamma,[k])[2m_k].$$

Recall that the set of orbits $[k]$ is the index set of the torsion *spin*^{c}-structures of M, cf. (2.2.2).

3.3.3. Distinguished representative. There is another more sophisticated way to choose a representative from a class $[k]$. Let $[k]=K+2(l'+L)$. Then in the class $l'+L$ (corresponding to an element of H) one can chose $\bar{l}'_{ne}\in L'$, cf. (2.2.4). The distinguished representative of $[k]$ is, by definition, $k_r := K+2\bar{l}'_{ne}$. For example, the distinguished characteristic element in $[K]$ is K itself. In [12], the elements k_r had a key role. The following result basically was proved there:

3.3.4. Proposition. *Fix a representative $k_r=K+2\bar{l}'_{ne}$ as above. Then in Fact (cf. proof of (3.2.4)) one may take $Y=0$. This means that there exists an increasing sequence $\{y_i\}_{i\geq 0}$ with $y_0=0$, $y_{i+1}=y_i+E_{j(i)}$ for some $j(i)\in\mathcal{J}$ for all $i\geq 0$, all the coefficients of y_i tend to infinity, and finally, for any $i\geq 0$ one has*

$$\chi_{k_r}(-y_{i+1})-\chi_{k_r}(-x_i)\geq g_{j(i)}.$$

Proof. Notice that $\chi_{k_r}(-y_{i+1})-\chi_{k_r}(-y_i) = -e_{j(i)}-1+g_{j(i)}+(\bar{l}'_{ne}-y_i,E_{j(i)})$.

Therefore, if in a graph with $g=0$ we can find a sequence with the wanted properties, then the same sequence will work if we decorate the vertices of the graph with some g_j. Hence, we may assume that $g=0$. In this case the statement follows from [12, (6.1)(b)], and its proof. In short, the argument is the following. Take $Y>0$ (arbitrary large) provided by Fact. Then one can connect $-Y$ to 0 with an increasing sequence along which χ_{k_r} is decreasing. Indeed, for any $y<0$ there exists j so that E_j is in the support of y, and $\chi_{k_r}(y+E_j)\leq\chi_{k_r}(y)$. (If not, then $(E_j,y+\bar{l}'_{ne})\leq 0$ for all E_j supported by y. But the same inequality automatically works for all other components. Hence $y+\bar{l}'_{ne}\in -L_{\mathbb{Q},ne}$ with $y<0$, a contradiction.) □

3.3.5. Corollary.
(a) $\mathbb{H}^*(\Gamma, k_r) = \mathbb{H}^*((\mathbb{R}_{\geq 0})^s, k_r)$, *i.e., in the construction of* $\mathbb{H}^*(\Gamma, k_r)$ *one may only work with effective cycles from* L_e *instead of* L *(in other words, only with cubes sitting in* $\mathbb{R}^s_{\geq 0}$*).*

(b) With respect to the canonical characteristic element K, $S_n(K)$ *is connected for all* $n \geq 1$.

Proof. (a) follows from a combination of (3.3.4) with the proof of (3.2.4). (b) was proved in [12, (6.1)(d)] under the assumption $c = g = 0$. The very same proof (based on part (a)) can be adopted. □

§3.4. (In)dependence of Γ

3.4.1. Clearly, many different negative definite plumbing graphs can provide the same 3-manifold M. But all these plumbing graphs can be connected by each other by a finite sequence of blowups/downs of (-1)-vertices with genus zero and whose number of incident edges is ≤ 2.

3.4.2. Proposition. *The set* $\mathbb{H}^*(\Gamma, [k])$, *where* $[k]$ *runs over* $Spin^c_t(M)$, *depends only on* M *and is independent of the choice of the (negative definite) plumbing graph* Γ *which provides* M.

Proof. First we assume that Γ' is obtained from Γ by 'blowing up a smooth point of one of the exceptional curves'. More precisely, Γ' denotes a graph with one more vertex and one more edge than Γ: we glue to a vertex j_0 by the new edge the new vertex with decoration -1 and genus 0, while the decoration of E_{j_0} is modified from e_{j_0} into $e_{j_0} - 1$, and we keep all the other decorations. We will use the notations $L(\Gamma)$, $L(\Gamma')$, $L'(\Gamma)$, $L'(\Gamma')$. Similarly, write I, I' for the corresponding intersection forms. Set E_{new} for the new base element in $L(\Gamma')$. The following facts can be verified:

• Consider the maps $\pi_* : L(\Gamma') \to L(\Gamma)$ defined by $\pi_*(\sum x_j E_j + x_{new} E_{new}) = \sum x_j E_j$, and $\pi^* : L(\Gamma) \to L(\Gamma')$ defined by $\pi^*(\sum x_j E_j) = \sum x_j E_j + x_{j_0} E_{new}$. Then $I'(\pi^* x, x') = I(x, \pi_* x')$. This shows that $I'(\pi^* x, \pi^* y) = I(x, y)$ and $I'(\pi^* x, E_{new}) = 0$ for any $x, y \in L(\Gamma)$.

• Set the (nonlinear) map: $c : L'(\Gamma) \to L'(\Gamma')$, $c(l') := \pi^*_{\mathbb{Q}}(l') + E_{new}$. Then $c(Char(\Gamma)) \subset Char(\Gamma')$ and c induces an isomorphism between the orbit spaces $Char(\Gamma)/2L(\Gamma)$ and $Char(\Gamma')/2L(\Gamma')$.

• Consider $k \in Char(\Gamma)$ and write $k' := c(k) \in Char(\Gamma')$. Then for any $x \in L(\Gamma)$ one has: $\chi_k(x) = \chi_{k'}(\pi^* x)$. Moreover, for any $z \in L(\Gamma')$, write z in the form $\pi^*\pi_* z + aE_{new}$ for some $a \in \mathbb{Z}$. Then $\chi_{k'}(z) = \chi_{k'}(\pi^*\pi_* z) +$

$\chi_{k'}(aE_{new}) = \chi_k(\pi_*(z)) + a(a+1)/2$. Hence, the projection in the direction E_{new} provides a homotopy equivalence $S_n(\Gamma', k') \to S_n(\Gamma, k)$.

In fact, this can be done in two steps. Let Π^* be the union of cubes of Γ' with all vertices in $\pi^*(L(\Gamma)) \cup (\pi^*(L(\Gamma)) - E_{new})$. Then $S_n(\Gamma', k')$ has a deformation retract (via projection in E_{new} direction) into $\Pi^* \cap S_n(\Gamma', k')$. On the other hand, the projection of the later one onto $S_n(\Gamma, k)$ is a homotopy equivalence (by checking the liftings of the cubes).

There is a similar verification in the case when one blows up "an intersection point" corresponding to two indices i_0 and j_0 with $(E_{i_0}, E_{j_0}) = 1$. (The only difference is that $\pi^*(\sum x_j E_j) = \sum x_j E_j + (x_{j_0} + x_{i_0})E_{new}$.) The details are left to the reader. □

3.4.3. *Remarks.*

(a) **Lattice homology.** Obviously, there exists a parallel homological theory as well (already used in [25] for $q = 0$). Indeed, take $\mathcal{F}_q := \mathcal{C}_q \otimes_{\mathbb{Z}} \mathcal{T}_0^+$, and define $\partial_w : \mathcal{F}_q \to \mathcal{F}_{q-1}$ by

$$\partial_w(\square_q \otimes t) = \sum_k \varepsilon_k \cdot \square_{q-1}^k \otimes U^{w(\square_q) - w(\square_{q-1}^k)} t.$$

Then $\mathbb{H}_*(\mathcal{F}_*, \partial_w)$ is the corresponding *lattice homology* of the pair $(\mathbb{Z}^s, w)$. Similarly as above, it equals $\oplus_n H_*(S_n, \mathbb{Z})$. If w is given as in (3.2.1), then we get the lattice homology $\mathbb{H}_*(\Gamma, k)$ of (Γ, k).

(b) **Graded root.** For each Γ whose plumbed manifold is rational homology sphere, and $k \in Char(\Gamma)$, the author in [12] constructed a *graded root*, from which one recovers by a natural procedure $\mathbb{H}^0(\Gamma, k)$. Using the weight functions w_0 and w_1 of (3.2.3), one can define in a similar way a graded root for any Γ (whose vertices of degree n correspond to the connected components of S_n) with similar properties to those from [12].

(c) It might happen, that some non-empty real ellipsoids $\{x \in \mathbb{R}^s : (\chi_k \otimes \mathbb{R})(x) \leq n\}$ contain no lattice points at all. In fact, $\min(\chi_k \otimes \mathbb{R}) - m_k$ can be arbitrarily large. Take for example the rational -2 curve and blow up in n different points. Then $\min \chi_K \otimes \mathbb{R} = -n/8$, but $m_K = 0$.

§3.5. Path cohomology

3.5.1. Construction. Fix $\mathbb{Z}^s$ and compatible weight functions w_0 and w_1 as in (3.1.3–3.1.4).

We consider a sequence $\gamma := \{x_i\}_{i=0}^t$ so that $x_0 = 0$, $x_i \neq x_j$ for $i \neq j$, and $x_{i+1} = x_i \pm E_{j(i)}$ for $0 \leq i < t$. We write T for the union of 0-cubes marked by the points $\{x_i\}_i$ and segments of type $[x_i, x_{i+1}]$. Then, by (3.1.14), we get a graded $\mathbb{Z}[U]$-module $\mathbb{H}^*(T, w)$, which is called the *path-cohomology* associated with the 'path' γ and weights $\{w_q\}_q$. It is denoted by $\mathbb{H}^*(\gamma, w)$. It has an augmentation with $\mathcal{T}^+_{2m_\gamma}$, where $m_\gamma := \min_i w_0(x_i)$, and one gets the *reduced path cohomology* $\mathbb{H}^0_{red}(\gamma, w)$ with

$$\mathbb{H}^0(\gamma, w) = \mathcal{T}^+_{2m_\gamma} \oplus \mathbb{H}^0_{red}(\gamma, w).$$

Similarly as in (3.2.6), we consider its 'Euler-invariant'

$$eu(\mathbb{H}^0(\gamma, w)) := -m_\gamma + \mathrm{rank}_{\mathbb{Z}}\, \mathbb{H}^0_{red}(\gamma, w).$$

3.5.2. Lemma. *$\mathbb{H}^q(\gamma, w) = 0$ for $q \geq 1$, and*

$$eu(\mathbb{H}^0(\gamma, w)) = -w_0(0) + \sum_{i=0}^{t-1} w_1([x_i, x_{i+1}]) - w_0(x_{i+1}).$$

Proof. Use induction comparing the paths $\{x_i\}_{i=0}^{n-1}$ and $\{x_i\}_{i=0}^{n}$ $(0 < n \leq t)$. □

3.5.3. Restriction map. Examples. In general, the restriction map $r^0 : \mathbb{H}^0(\mathbb{R}^s, w) \to \mathbb{H}^0(\gamma, w)$ is not onto. Indeed, let us fix a graph Γ and weights as in (3.2.1), and we will study different paths connecting $x_0 = 0$ with $x_t = Z_{min}$, the Artin's cycle (the unique minimal element of $-L_{ne} \setminus 0$). In order to simplify the picture, we assume that $c + g = 0$, $k = K$, and $\chi(Z_{min}) \leq 0$ (i.e. Γ is not rational, cf. 4.1.1). We write $\chi := \chi_K$. There is an 'optimal' way to find Z_{min}, given by Laufer's algorithm [7]: start with $x_0 = 0$, take for x_1, say, E_1 arbitrarily; if x_i (already constructed) is in $-L_{ne}$, then stop, set $t = i$, and $x_t = Z_{min}$; if $(x_i, E_{j(i)}) > 0$ for some $j(i)$ then take $x_{i+1} = x_i + E_{j(i)}$ and continue the algorithm with x_{i+1}.

If one considers any path γ_L connecting 0 and Z_{min} provided by Laufer's algorithm, then $\chi(x_1) = \chi(E_1) = 1$, and after that χ will decrease to $\chi(Z_{min})$, hence $\mathbb{H}^0(\gamma_L, K) = \mathcal{T}^+_{2\chi(Z_{min})} \oplus \mathcal{T}_0(1)$.

Assume that the multiplicity of E_1 in Z_{min} is ≥ 2. Then one may take the 'non-optimal' increasing path γ connecting 0 by Z_{min}, by taking $x_0 = 0$, $x_1 = E_1$, $x_2 = 2E_1$, and after that we proceed according to Laufer's algorithm. Then the maximum χ-value reached is $\chi(2E_1) = 2 - e_1 \geq 3$ and $\mathbb{H}^0(\gamma, K) = \mathcal{T}^+_{2\chi(Z_{min})} \oplus \mathcal{T}_0(2 - e_1)$.

One may verify that in the first case of γ_L the restriction r^0 is onto, while in the second case it is not. E.g., if Γ is minimally elliptic (see 4.2), then

$\mathbb{H}^0(\Gamma, K) = \mathbb{H}^0(\gamma_L, K) = \mathcal{T}_0^+ \oplus \mathcal{T}_0(1)$ and r^0 is an isomorphism, while in the second case r^0 is not onto (by a rank argument). Moreover, in this second case, $eu(\mathbb{H}^0(\gamma, K)) > eu(\mathbb{H}^0(\Gamma, K))$.

3.5.4. Lemma. *Fix two end-points, say* 0 *and* $l \in L$. *We consider all the paths* $\mathcal{P}(l)$ *as in* (3.5.1) *with* $x_0 = 0$ *and* $x_t = l$.

(*a*) *There exists* $\gamma \in \mathcal{P}(l)$ *such that* $r^0 : \mathbb{H}^0(\Gamma, w) \to \mathbb{H}^0(\gamma, w)$ *is onto.*

(*b*) *If for some* $\gamma \in \mathcal{P}(l)$ *the restriction* r^0 *is onto then* $eu(\mathbb{H}^0(\gamma, w)) \leq eu(\mathbb{H}^0(\Gamma, w))$.

In particular,

$$\min_{\gamma \in \mathcal{P}(l)} eu(\mathbb{H}^0(\gamma, w)) \leq eu(\mathbb{H}^0(\Gamma, w)).$$

Proof. (a) Take any γ from $\mathcal{P}(l)$. If $n \gg 0$ then $S_n \cap \gamma$ contains all the vertices and segments of Γ, hence it is contractible, $H^0(S_n \cap \gamma, \mathbb{Z}) = \mathbb{Z}$ and the restriction $r_n^0 : H^0(S_n, \mathbb{Z}) \to H^0(S_n \cap \gamma, \mathbb{Z})$ is onto. If r^0 is not onto, then let n be the largest integer for which r_n^0 is not onto. This means that there exists x_i and x_j $(i < j)$ so that the path $[x_i, x_j]$ of γ is not in S_n, but there is a path γ_{ij} connecting x_i with x_j in S_n. Then replace $[x_i, x_j]$ by γ_{ij}. Notice that the higher degree homologies (for $n' > n$) remain unmodified. Repeating this procedure after a finite step we get the wished path.

The proof of (b) is left to the reader. □

§4. Examples

The $\mathbb{Z}[U]$-module $\mathbb{H}^0(\Gamma, k)$ is not new, it is the combinatorial module considered in [25], [12] (where it was denoted by $\mathbb{H}^+$). In fact, regarding $\mathbb{H}^0(\Gamma, k)$, [12] is one of our main sources of examples.

§4.1. The case of rational graphs

4.1.1. Definition. By its very definition, a singularity is rational of its geometric genus p_g is vanishing. By [1], [2], this can be characterized combinatorially: a singularity is rational if and only if its graph satisfies $\min_{l \in L_e \setminus 0} \chi_K(l) > 0$ (or, equivalently, $\chi_K(Z_{min}) = 1$, or $\chi_K(Z_{min}) > 0$). Therefore, a (connected, negative definite) graph with this property is called rational. For them $c = g = 0$ automatically.

In [12] the author have given a different characterization:

4.1.2. Proposition. *Assume that* Γ *is a negative definite connected plumbing graph with* $c = g = 0$*. Then* Γ *is rational if and only if* $\mathbb{H}^0_{red}(\Gamma, K) = 0$*. Moreover, in this case,* $\mathbb{H}^0_{red}(\Gamma, k) = 0$*, and also* $m_k = 0$*, for any* $k \in Char$*.*

Even if one drops the assumption $c = g = 0$, one can prove:

4.1.3. Proposition. *If* $\mathbb{H}^0(\Gamma, K) = \mathcal{T}_0^+$ *then* Γ *is rational.*

Proof. Using (3.3.5) we get $\chi_K|L_e \geq 0$. Hence $\chi_K(E_j) = 1 - g_j \geq 0$. Assume that $\chi_K(Z_{min}) = 0$. Then Z_{min} cannot be connected by 0 in $S_0 \cap L_e$ since $w_1([0, E_j]) = 1$ by (3.2.2)(b). This contradicts the assumption, hence $\chi_K(Z_{min}) > 0$, i.e. Γ is rational. □

We add to this the following vanishing result:

4.1.4. Proposition. *If* Γ *is rational then* $\mathbb{H}^*_{red}(\Gamma, k) = 0$ *for any* $k \in Char$*.*

Proof. By (3.3.2) we may replace k with any characteristic element in its class. Take the distinguished representative k_r, cf. (3.3.3). The result follows from the proof of (3.2.4) once we show that one may take $X = Y = 0$ in Fact. By (3.3.4) we may take $Y = 0$. Hence, we have to show that for Γ rational there exists an increasing sequence $\{x_i\}_{i \geq 0}$ with $x_0 = 0$, $x_{i+1} = x_i + E_{j(i)}$, all the coefficients of x_i tend to infinity, and $\chi_{k_r}(x_{i+1}) \geq \chi_{k_r}(x_i)$. For this take a sequence $\{z_i\}_{i=0}^t$ which connects 0 and Z_{min} provided by Laufer's algorithm, cf. (3.5.3). Then $z_0 = 0$, $z_1 = E_1$, and $(E_{j(i)}, z_i) = 1$ for $1 \leq i < t$ [7]. Hence $\chi_{k_r}(z_1) = 1 - (\bar{l}'_{ne}, E_1) \geq 1$ and $\chi_{k_r}(z_{i+1}) - \chi_{k_r}(z_i) = -(\bar{l}'_{ne}, E_{j(i)}) \geq 0$. Hence, the sequence $\{mZ_{min} + z_i\}$ with $m \geq 0$ and $0 \leq i < t$ works. □

Therefore, the above proof combined with the proof of (3.2.5) gives

4.1.5. Corollary. *If* Γ *is rational, then* S_n *is contractible for any* $k \in Char$ *whenever is non-empty.*

§4.2. The case of elliptic graphs

4.2.1. Definition ([8], [31])**.** A connected negative definite graph is elliptic if $\min_{l \in L_e \setminus 0} \chi_K(l) = 0$.

In this case Γ might have a cycle or a vertex with genus one, but in any case $c+g \leq 1$. The next characterization result was proved in [12] for $c = g = 0$, here we verify the general situation.

4.2.2. Proposition. Γ *is elliptic if and only if* $\mathbb{H}^0(\Gamma, K) = \mathcal{T}_0^+ \oplus \mathcal{T}_0(1)^\ell$ *for some* $\ell \geq 1$.

Proof. Notice that, by (3.3.5), $\mathbb{H}^0(\Gamma, K) = \mathbb{H}^0(\mathbb{R}^s_{\geq 0}, K)$. Assume that Γ is elliptic. Then $\chi_K|L_e \geq 0$. Moreover, by (3.3.5), $S_n(K)$ is connected for $n \geq 1$, hence $\mathbb{H}^0(\Gamma, K) = \mathcal{T}_0^+ \oplus \mathcal{T}_0(1)^\ell$ for some $\ell \geq 0$. Since Γ is not rational, $\ell \neq 0$. Conversely, if $\mathbb{H}^0$ has that form, then $\chi_K|L_e \geq 0$ and there exists a cycle $x \in L_e \setminus 0$ with $\chi_K(x) = 0$, hence Γ is elliptic. □

In fact, in the 'classical' theory of elliptic singularities, there is a combinatorial integer which guides the main topological and analytical properties, namely, *the length of the elliptic sequence* ℓ^{es}, introduced by Laufer and S. S.-T. Yau (see e.g. [32], [33]). E.g., Yau proved that $\ell^{es} + 1$ is a topological upper bound for the geometric genus, and [11] shows that it is realized by any Gorenstein singularity when $c = g = 0$. The 'simplest' elliptic singularities, the minimally elliptic ones, are characterized by $\ell^{es} = 0$, or by the identity $Z_{min} = -K$ [8].

The point is that the above integer ℓ provided by (4.2.2), in fact, equals $\ell^{es} + 1$. In particular, for minimally elliptic singularities one has $\ell = 1$.

We exemplify the above proposition (4.2.2) for three minimally elliptic singularity. The simplest case, when $c = g = 0$, is the hypersurface singularity $\{x^2 + y^3 + z^7 = 0\}$, whose minimal good resolution graph has four vertices and three edges with $E_1^2 = -1$, $E_2^2 = -2$, $E_3^2 = -3$, $E_4^2 = -7$, and E_1 is connected with the others. Then $\ker(U)$ has rank 2, the generating lattice points are the zero cycle and $Z_{min} = 6E_1 + 3E_2 + 2E_3 + E_4$. (In general, the generators of $\ker(U)$ correspond to lattice points important in singularity theory as well, cf. [12], [14].)

4.2.3. Example. Assume that Γ consists of three vertices, each pair of vertices is connected by an edge, the self-intersections are -2, -2, -3, and $g = 0$. In this case $-K = \sum E_j = Z_{min}$, hence Γ is minimally elliptic [8]. With the notation $l = \sum_j x_j E_j$, $2\chi_K(l) = (x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2+x_3^2-x_3$, which is always non-negative on L. It is zero at $(0,0,0)$ and $(1,1,1)$. They cannot be connected by a segment, hence S_0 consists of two points. On the other hand, there are a lot of lattice points l in S_1: with the third coordinate $x_3 = -1$ one has the pair $(x_1, x_2) = (-1, -1)$, with $x_3 = 0$ the pairs $P = \{(-1,-1),\ (-1,0),\ (0,-1),\ (0,0),\ (1,0),\ (0,1),\ (1,1)\}$; with $x_3 = 1$ the pairs $P + (1,1)$, and for $x_3 = 2$ the pair $(2,2)$. (They are situated symmetrically with respect to $-K/2$.) Hence, S_1 consists of 16 lattice points. One can verify that they can be connected by segments, S_1 contains eight 2-cubes and one 3-cube, and S_1 is contractible. And this is the case for all S_n with $n \geq 1$.

4.2.4. Example. Assume that Γ has only one vertex with self-intersection -1 and $g = 1$. In this case again $K = -E$ and the graph is minimally elliptic (it is the 'simple-elliptic' singularity $\tilde{E}_8$: $\{x^2 + y^3 + z^6 = 0\}$). Notice that there are only two lattice points l with $\chi_K(l) = 0$, namely the zero cycle and E, and they are connected by the segment $[0, E]$ from $\mathcal{Q}_1$. Hence, $w_1[0, E] = 0$ would imply $\mathbb{H}^0_{red}(\Gamma, K) = 0$. Therefore, in order to have the 'right' result, we are forced to put 1 for the weight of this segment, a fact compatible with (3.2.1)–(3.2.2).

Then, with this weight functions, one has: $\mathbb{H}^q = 0$ for $q > 0$ and $\mathbb{H}^0(\Gamma, K) = \mathcal{T}_0^+ \oplus \mathcal{T}_0(1)$.

Notice that in the case of a minimally elliptic singularity, K is integral [8], hence $[K] = [-K]$ (cf. 3.2.7). We may add the following vanishing result for the other *spin*^{c}-structures:

4.2.5. Proposition. *If* Γ *is minimally elliptic, and the associated minimal* (*resolution*) *graph is good, then* $\mathbb{H}^*_{red}(\Gamma, [k]) = 0$ *for any* $[k] \neq [K]$.

Proof. By (3.4.2) we may assume that Γ is minimal. Then the proof of (4.1.4) can be adopted. Indeed, consider the representative k_r. Since $[k_r] \neq [K]$, $k_r \neq 0$, hence there exists at least one j with $(\bar{l}'_{ne}, E_j) < 0$. By [8, p. 1261–1262], there exists a computation sequence $\{z_i\}_{i=0}^t$ for Z_{min} so that the last $E_{j(t-1)}$ is E_j, $(z_i, E_{j(i)}) = 1$ for $1 \leq i \leq t-2$, and $(z_{t-1}, E_{j(t-1)}) = 2$ (this fact uses the minimality of Γ). Then the proof of (4.1.4) works in this case too. □

§4.3. The case of almost rational graphs

4.3.1. Definition ([12])**.** Assume that the graph Γ is connected and negative definite with $c + g = 0$. We say that Γ is *almost-rational* if there exists a vertex $j_0 \in \mathcal{J}$ of Γ such that replacing its Euler number e_{j_0} by some $e'_{j_0} \leq e_{j_0}$ we get a rational graph. (In general, the choice of j_0 is not unique.)

4.3.2. Examples. Almost rational graphs include: rational graphs, elliptic graphs (with $c + g = 0$), star-shaped graphs (with central vertex of genus zero). But there are more 'exotic' ones as well; e.g. the plumbing graph of the rational surgery 3-manifolds $S^3_r(K)$, where $r \in \mathbb{Q}_{<0}$ and K is an algebraic knot in S^3 (see e.g. [14], [15]). On the other hand, not every graph is almost rational. For example, if Γ has two (or more) vertices j with $-e_j + 2$ less than or equal to the valency of the vertex j, then Γ is not almost rational (e.g. the graph from (4.4.1)).

For almost rational graphs, $\mathbb{H}^0$ might be rather complicated module (see e.g. [12] for the explicit description in the case of star-shaped graphs). On the other hand, we have:

4.3.3. Theorem. *For any almost rational graph,* $\mathbb{H}^q(\Gamma, k) = 0$ *for any* $q > 0$ *and* $k \in Char$.

Proof. The complete proof is rather technical and long, and we will omit it. It is based on the results of [12, §9]. In fact, one can construct an infinite increasing path γ: $\{x_i\}_{i\geq 0}$ with $x_0 = 0$, so that the restriction $r^* : \mathbb{H}^*(\Gamma, k) \to \mathbb{H}^*(\gamma, k)$ is an isomorphism (the fact that r^0 is an isomorphism is the main result of [12, §9]). The isomorphism is induced by a deformation retract whose existence is proved by a combination of results from [12] with the proof of (4.1.4). $\square$

§4.4. Examples with non-vanishing $\mathbb{H}^1$

4.4.1. Example. Consider the following graph:

On the right hand side we give names to the base elements. Set $\chi := \chi_K$. We prefer to write any $l \in L$ in the form $l = l_x + zE + l_y$, where $l_x = \sum x_i E_i$, $l_y = \sum y_i E_i'$, $x_i,\ y_i,\ z \in \mathbb{Z}$ $(i = 1, 2, 3)$; or in the form $(x_1, x_2, x_3; z; y_1, y_2, y_3)$. Then $-K = (7, 14, 5; 3; 7, 14, 5)$ and $Z_{min} = (3, 6, 2; 1; 3, 6, 2)$, with $\chi(Z_{min}) = \chi(2Z_{min}) = -1$. In fact, $m_K = -1$ too. Then, it turns out that

$$\mathbb{H}^0(\Gamma, K) = \mathcal{T}^+_{-2} \oplus \mathcal{T}_{-2}(1) \oplus \mathcal{T}_0(1) \oplus \mathcal{T}_0(1),$$

where the generators of $\ker(U)$ with homogeneous degree -2 are (the dual classes of) Z_{min} and $2Z_{min}$, while with degree 0 are the (dual classes of) zero cycle and $-K$. Moreover, there exists a non-trivial class in $\mathbb{H}^1$ of homogeneous degree 0. In fact,

$$\mathbb{H}^1(\Gamma, K) = \mathcal{T}_0(1), \quad \text{and} \quad \mathbb{H}^q(\Gamma, K) = 0 \quad \text{for } q \geq 2.$$

In order to see (at least part of) these, we will analyse S_{-1} and S_0. Since $\chi(l) = \chi(-K - l)$, we can use all the time the χ-symmetry of the lattice points with respect to $-K/2$. If $z = 0$ then $\chi(l) = \chi(l_x) + \chi(l_y)$, and since l_x and l_y are supported by rational subgraphs, $\chi(l_x) \geq 0$, $\chi(l_y) \geq 0$. Hence, the lattice points in S_{-1} have $z = 1$ or $z = 2$, and they correspond by the above symmetry.

Let us assume that $z = 1$. Then $\chi(l) = \chi(l_x) + \chi(l_y) - x_2 - y_2 + 1$. Therefore, with the notation

$$f(x) := \chi(l_x) - x_2 = (\ 2x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 - x_2 - x_3\ )/2,$$

we have $\chi(l) = f(x) + f(y) + 1$. By real calculus the minimum of f over $\mathbb{R}^3$ is > -2, hence its minimum over $\mathbb{Z}^3$ is ≥ -1. Therefore, $\chi(l) = -1$ if and only if $f(x) = f(y) = -1$. By a computation, the integral solutions of $f(x) = -1$ are the triplets

$$A := \{(1,2,1),(1,3,1),(2,3,1),(2,4,1),(2,4,2),(2,5,2),(3,5,2),(3,6,2)\}.$$

Therefore, points $(x,1,y)$ with $x \in A$ and $y \in A$ (denoted simply by $(A,1,A)$) are in S_{-1}. Let $B = (7,14,5) - A$. Then, by symmetry, we get that the set of lattice points of S_{-1} is $(A,1,A) \cup (B,2,B)$. They determine two contractible connected components of S_{-1} in which Z_{min} and $2Z_{min}$ are 'representatives'.

Next, we plan to solve the equation $\chi(l) = 0$ with $z = 1$. Then, $f(x) + f(y) = -1$. $f(x) = 0$ has 24 integral solutions, namely union of the triplets $A' :=$

$$\{(0,0,0),(0,1,0),(1,1,0),(1,2,0),(0,1,1),(1,1,1),$$
$$(0,2,1),(2,2,1),(1,4,1),(3,4,1),(2,5,1),(3,5,1)\},$$

and the triplets of type $A'' = (4,8,3) - A'$. Set $\tilde{A} := A \cup A' \cup A''$, and $\tilde{B} = (7,14,5) - \tilde{A}$. Then the points of type

$$X := (A,1,\tilde{A}) \cup (\tilde{A},1,A) \cup (B,2,\tilde{B}) \cup (\tilde{B},2,B)$$

are in S_0. Since $\tilde{A} \cap B$ is not empty, all the points from X can be connected by segments. In fact, S_0 has three connected components, one of them contains the zero cycle, the other contains $-K$, and the third one, CS_0, contains all the points from X.

Finally, notice that the two intersection points $P = \tilde{A} \cap B = (4,8,3)$ and $Q = A \cap \tilde{B} = (3,6,2)$ create a loop in CS_0. Indeed, half of it is the connecting path of $(P;1;Q)$ and $(Q;1;P)$ through points in X with $z = 1$, the other half connects $(P;2;Q)$ with $(Q;2;P)$ through points in X with $z = 2$. This loop can be contracted only in S_1 (which is contractible).

4.4.2. Example. In the above example the subgraph of $\{E_1, E_2, E_3\}$ is a 'cusp', Γ was obtained by gluing two cusps to the 'central' curve E. One may create non-trivial higher dimensional modules by gluing k cusps to a central curve E which has self-intersection $-6k - 1$.

§5. Heegaard-Floer Homology and Singularity Links

§5.1. Heegaard-Floer homology

In this section we will assume that M is an oriented rational homology 3-sphere.

5.1.1. Review. Heegaard-Floer homology $HF^+(M)$ was introduced by Ozsváth and Szabó in [24] (and intensively studied in a series of articles). $HF^+(M)$ is a $\mathbb{Z}[U]$-module with a $\mathbb{Q}$-grading compatible with the $\mathbb{Z}[U]$-action, where $\deg(U) = -2$. Additionally, $HF^+(M)$ also has an (absolute) $\mathbb{Z}_2$-grading; $HF^+_{even}(M)$, respectively $HF^+_{odd}(M)$, denote the part of $HF^+(M)$ with the corresponding parity. Moreover, $HF^+(M)$ has a natural direct sum decomposition of $\mathbb{Z}[U]$-modules (compatible with all the gradings) corresponding to the *spin*^{c}-structures of M:

$$HF^+(M) = \oplus_{\sigma\in Spin^c(M)}\ HF^+(M,\sigma).$$

For any *spin*^{c}-structure σ, one has a graded $\mathbb{Z}[U]$-module isomorphism

$$HF^+(M,\sigma) = \mathcal{T}^+_{d(M,\sigma)} \oplus HF^+_{red}(M,\sigma),$$

where $HF^+_{red}(M,\sigma)$ has a finite $\mathbb{Z}$-rank and an induced (absolute) $\mathbb{Z}_2$-grading. One also considers

$$\chi(HF^+(M,\sigma)) := \operatorname{rank}_{\mathbb{Z}} HF^+_{red,even}(M,\sigma) - \operatorname{rank}_{\mathbb{Z}} HF^+_{red,odd}(M,\sigma).$$

Then one recovers the Seiberg-Witten topological invariant of (M,σ) (see [28]) via

$$\mathbf{sw}(M,\sigma) := \chi(HF^+(M,\sigma)) - d(M,\sigma)/2.$$

With respect to the change of orientation the above invariants behave as follows: The *spin*^{c}-structures $Spin^c(M)$ and $Spin^c(-M)$ are canonically identified (where $-M$ denotes M with the opposite orientation). Moreover, $d(M,\sigma) = -d(-M,\sigma)$ and $\chi(HF^+(M,\sigma)) = -\chi(HF^+(-M,\sigma))$. Notice also that one can recover $HF^+(M,\sigma)$ from $HF^+(-M,\sigma)$ via [24, (7.3)] and [26, (1.1)].

5.1.2. Example. If M is an integral homology sphere then for the unique (=canonical) *spin*^{c}-structure σ_{can}, $\mathbf{sw}(M,\sigma_{can})$ equals the Casson invariant $\lambda(M)$ (normalized as in [9, (4.7)]).

§5.2. Lattice homology and Heegaard-Floer homology

5.2.1. Assume that Γ is a connected negative definite plumbing graph whose associated plumbed 3-manifold is a *rational homology sphere*. Our goal is to recover the Heegaard-Floer homology of M in a purely combinatorial way from Γ. We write $\#\mathcal{J} = s$.

5.2.2. Theorem ([25], [12]). *Assume that Γ is an almost rational graph. Then, for any $[k] \in Spin^c(M)$*

$$HF^+_{odd}(-M,[k]) = 0,$$

and

$$HF^+_{even}(-M,[k]) = \mathbb{H}^0(\Gamma,[k])\Big[-\max_{k'\in[k]}\frac{(k')^2+s}{4}\Big].$$

In particular (cf. 3.3.2), for any $k \in [k]$ one has

$$(*) \qquad d(M,[k]) = \max_{k'\in[k]}\frac{(k')^2+s}{4} = \frac{k^2+s}{4} - 2\min\chi_k.$$

5.2.3. Corollary. *If Γ is an almost rational graph, then for any $k \in Char$:*

$$-\mathbf{sw}(M,[k]) - \frac{k^2+s}{8} = -\min\chi_k + \mathrm{rank}_{\mathbb{Z}}\,\mathbb{H}^0_{red}(\Gamma,k) = eu(\mathbb{H}^0(\Gamma,k)).$$

5.2.4. Conjecture. Let M be a plumbed rational homology sphere associated with a connected negative definite graph Γ. Then for any $k \in Char$ the identity $(*)$ of (5.2.2) is valid, and

$(**)$

$$-\mathbf{sw}(M,[k]) - \frac{k^2+s}{8} = -\min\chi_k + \sum_q(-1)^q\,\mathrm{rank}_{\mathbb{Z}}\,\mathbb{H}^q_{red}(\Gamma,k) = eu(\mathbb{H}^*(\Gamma,k)).$$

In fact, we predict that with $d = d(M,[k])$:

$$HF^+_{red,even}(-M,[k]) = \bigoplus_{q\ even}\mathbb{H}^q_{red}(\Gamma,[k])[-d], \quad \text{and}$$

$$HF^+_{red,odd}(-M,[k]) = \bigoplus_{q\ odd}\mathbb{H}^q_{red}(\Gamma,[k])[-d].$$

5.2.5. Example. Take Γ from (4.4.1), and $k = K$. Then $(*)$ is true by [25, Corollary 1.5]. Moreover, by (4.4.1), $eu(\mathbb{H}^*) = -(-1) + 3 - 1 = 3$,

$(K^2+s)/8=-1$. On the other hand, the Casson invariant of M is -2 (using, e.g., the formula of Raţiu, see [16, (5.3)]). Hence, $(**)$ is valid as well.

5.2.6. *Remarks.* (a) The above identities are not valid (in this form) when $c+g>0$.

(b) (4.1.2–4.1.4), or (5.2.2) shows that if Γ is rational then M is an L-space (in the sense of Ozsváth and Szabó, i.e. $HF^+_{red}(M)=0$). From the perspective of Conjecture (5.2.4), we expect that this is an 'if and only if' correspondence: Γ is rational if and only if M is an L-space. Notice that by (4.1.2), if $\mathbb{H}_{red}=0$ then Γ is rational.

(c) Although we expect an identification of the $\mathbb{H}^*$ modules with the Heegaard-Floer modules HF^+, the lattice cohomology (apparently) contains more structure (at least, the author is not able to recover them in HF^+). For their existence the explanation is, maybe, that the involved 3-manifolds are rather special. We list here three such extra properties.

(i) The (absolute) grading of $\{\mathbb{H}^q\}_{q\geq 0}$ (with respect to q) is indexed by $\mathbb{Z}$ in contrast with the $\mathbb{Z}_2$ (even/odd) grading of HF^+.

(ii) Consider from (3.1.12) the identity $\mathbb{H}^*=\oplus_n H^*(S_n,\mathbb{Z})$. How can the ring structure of each $H^*(S_n,\mathbb{Z})$ exploited?

(iii) For a fixed graph Γ, consider any distinguished representative k_r. Since $\chi_{k_r}(l)\geq \chi_K(l)$ for any $l\in L$, we get $S_n(k_r)\subset S_n(K)$, hence a natural ring homomorphism $H^*(S_n(K))\to H^*(S_n(k_r))$, or $R:\mathbb{H}^*(\Gamma,K)\to\mathbb{H}^*(\Gamma,k_r)$. Notice that in the case of rational or elliptic graphs it happens that properties of the module associated with the canonical *spin*^{c}-structure 'dominates' all the others, cf. (4.1.2) or (4.2.5). Is it possible to say something similar in general? Is R onto?

§6. Line Bundles Associated with Surface Singularities

Starting from this section, we start to analyse the analytic aspects of the singularity $(X,0)$ as well. The analytic type is preserved in the complex manifold structure of the resolution $\tilde{X}$. Holomorphic line bundles on $\tilde{X}$ codify a lot of information about it.

§6.1. Cohomological computations

6.1.1. Let $\pi:(\tilde{X},E)\to(X,0)$ be a fixed good resolution of $(X,0)$. Let $Pic(\tilde{X})$ be the group of isomorphism classes of holomorphic line bundles on $\tilde{X}$ and $c_1: Pic(\tilde{X})\to L'$, $c_1(\mathcal{L})=\sum_j \deg(\mathcal{L}|E_j)\,D_j$ the set of Chern classes of $\mathcal{L}$. We prefer to use the same notation for $l = \sum n_j E_j \in L$ and divisors $\sum n_j E_j$ of $\tilde{X}$ supported by E. Hence, we can consider the line bundle $\mathcal{O}_{\tilde{X}}(l) := \mathcal{O}_{\tilde{X}}(\sum n_j E_j)$. If $l > 0$, we write $\chi(l)$ for $\chi_K(l) = \chi(\mathcal{O}_l)$ (cf. (2.2.5)). We write $|l|$ for the support of l.

In this subsection we analyse $h^1(\mathcal{L}) := \dim H^1(\tilde{X}, \mathcal{L})$ for any $\mathcal{L} \in Pic(\tilde{X})$. First, recall the following general (Grauert-Riemenschneider type) vanishing theorem (cf. [27, page 119, Ex. 15]):

6.1.2. *If $c_1(\mathcal{L}) \in K + L_{\mathbb{Q},ne}$, then $h^1(l, \mathcal{L}|_l) = 0$ for any $l \in L$, $l > 0$, hence $h^1(\tilde{X}, \mathcal{L}) = 0$.*

The next statement is an improvement of it, valid for rational singularities:

6.1.3. *Assume that $(X, 0)$ is a rational singularity. If $c_1(\mathcal{L}) \in L_{\mathbb{Q},ne}$, then $h^1(l, \mathcal{L}|_l) = 0$ for any $l > 0$, $l \in L$, hence $h^1(\tilde{X}, \mathcal{L}) = 0$ too.*

Proof. From the point of view of the next discussion, it is instructive to see the proof. For any $l > 0$ there exists $E_j \subset |l|$ such that $(E_j, l + K) < 0$. Indeed, $(E_j, l + K) \geq 0$ for any j would imply $\chi(l) = -(l, l + K)/2 \leq 0$, which would contradict the rationality of $(X, 0)$ [1]. Then, using

$$0 \to \mathcal{L} \otimes \mathcal{O}_{E_j}(-l + E_j) \to \mathcal{L}|_l \to \mathcal{L}|_{l-E_j} \to 0$$

one gets $h^1(\mathcal{L}|_l) = h^1(\mathcal{L}|_{l-E_j})$, hence by induction $h^1(\mathcal{L}|_l) = 0$. □

This will be generalized in two different ways. First we show that the computation of any $h^1(\mathcal{L})$ can be reduced to the computation of some $h^1(\mathcal{L}')$ with $c_1(\mathcal{L}') \in L_{\mathbb{Q},ne}$.

6.1.4. Proposition. *Let $\tilde{X} \to X$ be a good resolution of a normal singularity $(X, 0)$ as above.*

(a) For any $l' \in L'$ there exists a unique minimal element $l_{l'} \in L_e$ with $e(l') := l' - l_{l'} \in L_{\mathbb{Q},ne}$.

(b) $l_{l'}$ can be found by the following (generalized Laufer's) algorithm. One constructs a sequence $x_0, x_1, \ldots, x_t \in L_e$ with $x_0 = 0$ and $x_{i+1} = x_i + E_{j(i)}$, where each index $j(i)$ is determined by the following principle. Assume that x_i is already constructed. Then, if $l' - x_i \in L_{\mathbb{Q},ne}$, then one stops, and $t = i$. Otherwise, there exists at least one j with $(l' - x_i, E_j) < 0$. Take for $j(i)$ one of these j's. Then this algorithm stops after a finitely many steps, and $x_t = l_{l'}$.

(c) For any $\mathcal{L} \in Pic(\tilde{X})$ with $c_1(\mathcal{L}) = l'$ one has:

$$h^1(\mathcal{L}) = h^1(\mathcal{L} \otimes \mathcal{O}_{\tilde{X}}(-l_{l'})) - (l', l_{l'}) - \chi(l_{l'}).$$

In particular (*since* $c_1(\mathcal{L}\otimes\mathcal{O}_{\tilde X}(-l_{l'}))\in L_{\mathbb{Q},ne}$), *the computation of any* $h^1(\mathcal{L})$ *can be reduced* (*modulo the combinatorics of* L) *to the computation of some* $h^1(\mathcal{L}')$ *with* $c_1(\mathcal{L}')\in L_{\mathbb{Q},ne}$.

Proof. (a) Since $(\,,\,)$ is negative definite, there exists $l\in L_e$ with $l'-l\in L_{\mathbb{Q},ne}$ (take e.g. a large multiple of some Z with $(Z,E_j)<0$ for any j). Next, we prove that if $l'-l_i\in L_{\mathbb{Q},ne}$ for $l_i\in L_e$, $i=1,2$, and $l:=\min\{l_1,l_2\}$, then $l'-l\in L_{\mathbb{Q},ne}$ as well. For this, write $x_i:=l_i-l\in L_e$. Then $|x_1|\cap|x_2|=\emptyset$, hence for any fixed j, $E_j\not\subset|x_i|$ for at least one of the i's. Therefore, $(l'-l,E_j)=(l'-l_i,E_j)+(x_i,E_j)\geq 0$.

(b) First we prove that $x_i\leq l_{l'}$ for any i. For $i=0$ this is clear. Assume that it is true for some i but not for $i+1$, i.e. $E_{j(i)}\not\subset|l_{l'}-x_i|$. But this would imply $(l'-x_i,E_{j(i)})=(l'-l_{l'},E_{j(i)})+(l_{l'}-x_i,E_{j(i)})\geq 0$, a contradiction. The fact that $x_i\leq l_{l'}$ for any i implies that the algorithm must stop, and $x_t\leq l_{l'}$. But then by the minimality of $l_{l'}$ (part a) $x_t=l_{l'}$. (Cf. [7].)

(c) For any $0\leq i<t$, consider the exact sequence

$$0\to\mathcal{L}\otimes\mathcal{O}_{\tilde X}(-x_{i+1})\to\mathcal{L}\otimes\mathcal{O}_{\tilde X}(-x_i)\to\mathcal{L}\otimes\mathcal{O}_{E_{j(i)}}(-x_i)\to 0.$$

Since $\deg(\mathcal{L}\otimes\mathcal{O}_{E_{j(i)}}(-x_i))=(l'-x_i,E_{j(i)})<0$, one gets $h^0(\mathcal{L}\otimes\mathcal{O}_{E_{j(i)}}(-x_i))=0$. Therefore

$$h^1(\mathcal{L}\otimes\mathcal{O}_{\tilde X}(-x_i))-h^1(\mathcal{L}\otimes\mathcal{O}_{\tilde X}(-x_{i+1}))=-\chi(\mathcal{L}\otimes\mathcal{O}_{E_{j(i)}}(-x_i))$$

which equals $-(l',x_{i+1}-x_i)+\chi(x_i)-\chi(x_{i+1})$. Hence the result follows by induction. □

6.1.5. Examples. Rational singularities. If $(X,0)$ is rational then $c_1:Pic(\tilde X)\to L'$ is an isomorphism. Moreover, using (6.1.3) and (6.1.4)(c), one has $h^1(\mathcal{L})=-(l',l_{l'})-\chi(l_{l'})$. In particular, $h^1(\mathcal{L})$ *depends only on* Γ *and it is independent of the analytic structure of* $(X,0)$.

§6.2. Path cohomology and upper bounds for $h^1(\mathcal{L})$

6.2.1. For the next result, we start with the following set of data and notations: $\mathcal{L}\in Pic(\tilde X)$, $l':=c_1(\mathcal{L})$, $k:=K-2l'$ (cf. 2.2.5). We consider a 'path' γ: $\{x_i\}_{i=0}^t$, where $x_0=0$, $x_t\in l'-K-L_{\mathbb{Q},ne}$, and $x_{i+1}=x_i\pm E_{j(i)}$ for some $j(i)\in\mathcal{J}$ $(0\leq i<t)$.

Using the exact sequence $0\to\mathcal{L}\otimes\mathcal{O}(-x_t)\to\mathcal{L}\to\mathcal{L}|_{x_t}\to 0$, and the Grauert-Riemenschneider vanishing (6.1.2), we get $h^1(\mathcal{L})=h^1(\mathcal{L}|_{x_t})$ (this motivates the corresponding restriction for x_t). In the next proposition the 'symbol' $h^1(\mathcal{L}|_{x_0})$ will stand for zero.

6.2.2. Proposition. *With the above notations, for any* $0 \leq i < t$ *with* $x_{i+1} > x_i$ *one has*:

$$h^1(\mathcal{L}|_{x_{i+1}}) - h^1(\mathcal{L}|_{x_i}) \leq \begin{cases} -\Delta_i + M^{g_{j(i)}}(-\Delta_i) & \text{if } \Delta_i < 0, \\ M^{g_{j(i)}}(\Delta_i) & \text{if } \Delta_i \geq 0, \end{cases}$$

where $\Delta_i := \chi_k(x_{i+1}) - \chi_k(x_i)$. *If* $x_{i+1} < x_i$ *then* $h^1(\mathcal{L}|_{x_{i+1}}) - h^1(\mathcal{L}|_{x_i}) \leq 0$.

In particular, adding all these inequalities, we get a topological upper bound for $h^1(\mathcal{L})$.

6.2.3. Example. Assume that $g = 0$ and γ is increasing. Since $M^0(n) = 0$ for all $n \geq 0$, we get

$$h^1(\mathcal{L}) \leq \sum_{i=0}^{t-1} \max\{\, 0\,,\, \chi_k(x_i) - \chi_k(x_{i+1})\,\}.$$

Proof of (6.2.2). Assume that $x_{i+1} > x_i$ (the other case is trivial). Write $\mathcal{M}_i$ for the line bundle $\mathcal{L} \otimes \mathcal{O}_{E_{j(i)}}(-x_i)$ on $E_{j(i)}$. From the cohomological exact sequence

$$\cdots \to H^1(E_{j(i)}, \mathcal{M}_i) \to H^1(\mathcal{L}|_{x_{i+1}}) \to H^1(\mathcal{L}|_{x_i}) \to 0$$

we have to estimate $h^1(\mathcal{M}_i)$. Notice that $\chi(\mathcal{M}_i) = \Delta_i$ by (2.2.5). Hence, if $\Delta_i \geq 0$, then $h^1(\mathcal{M}_i) \leq M^{g_{j(i)}}(\Delta_i)$, by the very definition of $M^g(n)$. Assume that $\Delta_i < 0$. Then, by Serre duality

$$h^1(\mathcal{M}_i) = -\Delta_i + h^0(\mathcal{M}_i) = -\Delta_i + h^1(\mathcal{M}_i^{-1}(K + E_{j(i)})) \leq -\Delta_i + M^{g_{j(i)}}(-\Delta_i).$$

□

6.2.4. *Remark.* Assume that we add another term $x_{t+1} = x_t + E_{j(t)}$ to the sequence $\{x_i\}_{i=1}^t$ with similar restriction $x_{t+1} \in l' - K - L_{\mathbb{Q},ne}$. Then $deg_{E_{j(t)}} \mathcal{M}_t > 2g_{j(t)} - 2$, $\Delta_t \geq g_{j(i)}$ and $M^{g_{j(t)}}(\Delta_t) = h^1(\mathcal{M}_t) = 0$. Therefore, even if one continues the sequence arbitrarily long inside of $l' - K - L_{\mathbb{Q},ne}$, nothing will be changed (e.g. the upper bound accumulates no more contribution). Sometimes we will just say and write that $x_t = \infty$, which means that x_t is in the 'right' region $l' - K - L_{\mathbb{Q},ne}$.

Next we reinterpret (6.2.2) in terms of path cohomology. Let $\mathcal{P}$ be the set of paths with $x_0 = 0$ and $x_t = \infty$, in the sense of (6.2.4). Moreover, consider the weight functions $\{w_q\}_q$ associated with (Γ, k) as in (3.2.1), and write $\mathbb{H}^0(\gamma;\Gamma,k)$ for $\mathbb{H}^0(\gamma,w)$. Then from (6.2.2) and (3.5.2) we get

6.2.5. Corollary. *For any $\gamma\in\mathcal{P}$ one has $h^1(\mathcal{L})\leq eu\,(\mathbb{H}^0(\gamma;\Gamma,k))$. Hence*

$$h^1(\mathcal{L})\leq \min_{\gamma\in\mathcal{P}}\ eu\,(\mathbb{H}^0(\gamma;\Gamma,k)).$$

6.2.6. *Remark.* Recall that by (3.5.4) one has: $\min_{\gamma\in\mathcal{P}}\ eu\,(\mathbb{H}^0(\gamma;\Gamma,k))\leq eu\,(\mathbb{H}^0(\Gamma,k))$.

6.2.7. Example. If Γ is almost rational (cf. 4.3.1), a consequence of the results of [12] is that

$$\min_{\gamma\in\mathcal{P}}\ eu\,(\mathbb{H}^0(\gamma;\Gamma,k))\ =\ eu\,(\mathbb{H}^0(\Gamma,k)),$$

and, in fact, the minimum $\min_{\gamma\in\mathcal{P}}$ is realized by an increasing path. The point is that $\ker U\in\mathbb{H}^0(\Gamma,k)$ admits 'representative' lattice points which are totally ordered (with respect to $<$) sitting on an increasing path. In fact, $\mathbb{H}^0(\Gamma,k)$ is determined in [12] from the values of χ_k along this path.

6.2.8. Example. The situation from (6.2.7), in general, is not true. I.e., one may have

$$\min_{\gamma\in\mathcal{P}}\ eu\,(\mathbb{H}^0(\gamma;\Gamma,k))\ <\ eu\,(\mathbb{H}^0(\Gamma,k)),$$

i.e., the path cohomology may provides a strict better upper bound for $h^1(\mathcal{L})$ than the lattice cohomology (cf. 6.2.6). To see this, construct Γ with $c=g=0$ as follows. Let E and E' be two vertices, both with self-intersection -14, and connected by an edge. Attach to both of them two-two cusps as in (4.4.1–4.4.2). Take $k=K$. Then $\chi(Z_{min})=\chi(3Z_{min})=-3$ and $m_K=\chi(2Z_{min})=-4$. By a computation

$$\mathbb{H}^0(\Gamma,K)=\mathcal{T}^+_{-8}\oplus\mathcal{T}_{-6}(1)^6\oplus\mathcal{T}_0(1)^2,$$

where the generators in degree zero are 0 and $-K$, in degree -4 is $2Z_{min}$, while in degree -3 the cycles $Z_{min},L,R,L',R',3Z_{min}$. Here, the cycles L and R are symmetric with respect to the natural symmetry compatible with $E\leftrightarrow E'$, for both $Z_{min}<L,R<2Z_{min}$, but L and R are not comparable by $<$. Hence, when one travels from Z_{min} to $2Z_{min}$ by a Laufer type path, then one has to make a choice (left-right) to pass through L or R, but one doesn't have to touch both of them. The situation is similar with L' and R' which sit between $2Z_{min}$

and $3Z_{min}$. Hence, it turns out that the module for a minimal increasing path (with end-point at K, or at ∞) is

$$\mathbb{H}^0(\gamma_{min}, K) = \mathcal{T}^+_{-8} \oplus \mathcal{T}_{-6}(1)^4 \oplus \mathcal{T}_0(1)^2,$$

which has *eu* two less than $\mathbb{H}^0(\Gamma, K)$.

6.2.9. Example. We may ask how sharp is the topological upper bound (6.2.5). Although it is not very easy to provide abundant examples for $h^1(\mathcal{L})$, for the geometric genus $p_g := h^1(\mathcal{O}_{\tilde{X}})$ more examples are available. In this case, in many graphs the inequality (6.2.5) is optimal, i.e. the topological upper bound is realized by the p_g of some analytic structure. Nevertheless, this is not the case all the time. For the graph Γ discussed in (4.4.1), both the lattice and path cohomologies provide the same upper bound $p_g \leq 4$ (cf. 6.2.6). On the other hand, by a (not simple) line of arguments, one finds out that there is no analytic structure supported on this topological type with $p_g = 4$ ($p_g = 3$ can be realized by a splice type complete intersection). The reader may decide if this example is 'generic' or 'pathological'.

(Note that $p_g \geq h^1(Z_{min}) = 1 - \chi(Z_{min})$, hence $p_g \geq 2$ for any analytic structure, while $p_g = 3$ for any Gorenstein structure.)

§7. The Seiberg-Witten Invariant Conjecture

§7.1. Line bundles on $\tilde{X}$ revisited

7.1.1. The bundles $\mathcal{O}_{\tilde{X}}(l')$. Start with the data of (6.1.1) and assume that *M is rational homology sphere*. The 'exponential exact sequence' $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$ on $\tilde{X}$ induces the exact sequence

$$0 \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to Pic(\tilde{X}) \xrightarrow{c_1} L' \to 0.$$

For any $l \in L$ one has $c_1(\mathcal{O}_{\tilde{X}}(l)) = l$. Hence $l \mapsto \mathcal{O}_{\tilde{X}}(l)$ is a group section of c_1 above the subgroup L of L'. Since L'/L is torsion, and $H^1(\mathcal{O}_{\tilde{X}}) = \mathbb{C}^{p_g}$ is torsion-free, this can be extended in unique way to a group section $s : L' \to Pic(\tilde{X})$ of c_1. We write $\mathcal{O}_{\tilde{X}}(l')$ for $s(l')$.

7.1.2. Relation with coverings. The next theorem (7.1.3) illuminates a different aspect of the line bundles $\mathcal{O}_{\tilde{X}}(l')$. Notice that $\tilde{X} \setminus E \approx X \setminus \{0\}$ has the homotopy type of M, hence the abelianization map $\pi_1(\tilde{X} \setminus E) = \pi_1(M) \to H$ defines a regular Galois covering of $\tilde{X} \setminus E$. This has a unique extension $p : Z \to \tilde{X}$ with Z normal and p finite [5]. The (reduced) branch locus of p is included in E, and the Galois action of H extends to Z as well. Since E is a normal crossing divisor, the only singularities that Z might have are cyclic quotient singularities.

7.1.3. Theorem. *Consider the finite covering $p: Z \to \tilde{X}$, and set $Q \subset L'$ as in* (2.2.4)*. Then the H-eigenspace decomposition of $p_*\mathcal{O}_Z$ has the form*:

$$p_*\mathcal{O}_Z = \oplus_{\chi \in \hat{H}} \mathcal{L}_\chi,$$

where $\mathcal{L}_{\theta(h)} = \mathcal{O}_{\tilde{X}}(-l'_e(h))$ for any $h \in H$. In particular, $p_*\mathcal{O}_Z = \oplus_{l' \in Q}\mathcal{O}_{\tilde{X}}(-l')$.

The proof is based on a similar statement of Kollár valid for cyclic coverings, see e.g. [6, §9]. For details, see [13], [15] or [22].

§7.2. The conjectured identities

7.2.1. The next expected property is a generalization of the conjecture of [16], where only the case of canonical *spin*^{c}-structure was considered. The generalization to any *spin*^{c}-structure appeared in [13], where it was formulated for any $\mathbb{Q}$-Gorenstein singularity (with rational homology sphere link). The article [10] shows that we cannot expect the validity of the identities in this generality. Nevertheless, we expect that it is true for a large class of normal surface singularities (subclass of $\mathbb{Q}$-Gorenstein singularities with rational homology sphere links). In the next paragraphs we will present two (equivalent) versions.

In this section we assume that the link M of $(X,0)$ is a rational homology sphere. We fix a good resolution $\pi: \tilde{X} \to X$ with $s := \#\mathcal{J}$. Also, we set

$$\mathbb{L}' := \{l' \in L' : e(l') = l'_{ne}(h) \text{ for some } h \in H\} = \bigcup_{h \in H} l'_{ne}(h) + L_e.$$

(For notations, see (2.2.4) and (6.1.4).) One can verify that $L'_e \subset \bigcup_{l' \in Q} -l' + L_e \subset \mathbb{L}'$.

7.2.2. Property A. *Consider an arbitrary $l' \in \mathbb{L}'$ and define a characteristic element by $k := K - 2l' \in Char$. Then, we say that $(X,0)$ satisfies Property A if*

$$h^1(\mathcal{O}_{\tilde{X}}(l')) = -\mathbf{sw}(M,[k]) - \frac{k^2 + s}{8}. \quad (1)$$

7.2.3. *Remark.* In order to prove the property, it is enough to verify it for line bundles $\mathcal{L}$ with $c_1(\mathcal{L}) = l'$ of type $l' = l'_{ne}(h)$ (for some $h \in H$). Indeed, write l' in the form $l' = l'_1 + l$ where $l'_1 = e(l') = l'_{ne}(l' + L)$ and $l \in L_e$. Let $RHS(l')$, resp. $RHS(l'_1)$, be the right hand side of (1) for l', resp. l'_1. Since $[K - 2l'] = [K - 2l'_1]$, the Seiberg-Witten invariants are the same, hence

$$RHS(l') - RHS(l'_1) = \frac{-(K - 2l')^2 + (K - 2l'_1)^2}{8} = -(l, l') - \chi(l).$$

This combined with (6.1.4)(c) shows that (7.2.2)(1) for $\mathcal{L}$ and $\mathcal{L} \otimes \mathcal{O}_{\tilde{X}}(-l)$ are equivalent.

In fact, consider *any* set of representatives $\{l'\}_{l' \in R}$ ($R \subset \mathbb{L}'$) of the classes H, i.e. $\{l' + L\}_{l' \in R} = H$. Then the above argument applied for elements from R shows that the validity of the property (7.2.2) follows from the verification of (1) for line bundles $\mathcal{L}$ with $c_1(\mathcal{L}) \in R$. The possibility $R = -Q$ is emphasized by (7.1.3) and will be exploited in the second version of the property.

7.2.4. Universal abelian cover. Let $(X_{ab}, 0)$ be the universal abelian cover of $(X, 0)$ with its natural H-action. Namely, $(X_{ab}, 0)$ is the unique normal singularity with a finite projection $(X_{ab}, 0) \to (X, 0)$, regular over $X \setminus 0$ corresponding to the abelianization map $\pi_1(X \setminus 0) = \pi_1(M) \to H$. Then the space Z considered in (7.1.2–7.1.3) is a partial resolution of $(X_{ab}, 0)$ with only cyclic quotient singularities. The geometric genus $p_g(X_{ab}, 0)$ of $(X_{ab}, 0)$ can be computed as the dimension of $H^1(Z, \mathcal{O}_Z)$, but this space has a natural eigenspace decomposition $\oplus_{\chi \in \hat{H}} H^1(Z, \mathcal{O}_Z)_\chi$ too. Hence one may consider the invariants

$$p_g(X_{ab}, 0)_\chi := \dim_{\mathbb{C}} H^1(Z, \mathcal{O}_Z)_\chi \quad \text{(for any } \chi \in \hat{H}).$$

Notice that (7.1.3) reads as

$$p_g(X_{ab}, 0)_{\theta(h)} = h^1(\mathcal{O}_{\tilde{X}}(-l'_e(h))) \quad \text{(for any } h \in H).$$

Since the set $\{-l'_e(h)\}_{h \in H}$ is a set of representatives for H, by (7.2.3) the previous Property A (7.2.2) is *equivalent* with the following.

7.2.5. Property B. *For any $h \in H$ consider $k := K + 2l'_e(h) \in Char$. Then for any $h \in H$*

$$p_g(X_a, 0)_{\theta(h)} = -\mathbf{sw}(M, [k]) - \frac{k^2 + s}{8}. \tag{2}$$

§7.3. Examples

7.3.1. Example. Property A (hence B too) **is true for any rational singularity.** Indeed, by (7.2.3), we can assume that $l' = l'_{ne}(h)$ for some h. Then, by (6.1.3), $h^1(\mathcal{O}_{\tilde{X}}(l')) = 0$. On the other hand, by [12], $-\mathbf{sw}(M, [k]) = (k_r^2+s)/8$, where $k_r = K+2\bar{l}'_{ne}(-l'+L)$. Since $\bar{l}'_{ne}(-l'+L) = -l'_{ne}(l'+L) = -l'$ one gets $k_r = k$. Hence the right hand side of (7.2.2)(1) is also vanishing.

This proof also shows that for $(X, 0)$ rational, and for any $h \in H$, one has

$$p_g(X_{ab}, 0)_{\theta(h)} = \frac{(K + 2\bar{l}'_{ne}(h))^2 - (K + 2l'_e(h))^2}{8} = -\chi(\bar{l}'_{ne}(h)) + \chi(l'_e(h)).$$

In particular, $(X_{ab}, 0)$ is rational if and only if $\chi(\bar{l}'_{ne}(h)) = \chi(l'_e(h))$ for all $h \in H$. One can find rational graphs whose universal abelian covers are not rational, a fact which stresses the differences between the 'liftings' $l'_e(h)$ and $\bar{l}'_{ne}(h)$.

7.3.2. Example. Splice quotients. The validity of Property A for rational singularities (cf. 7.2.5), the surgery formulas of [3] regarding the Seiberg-Witten invariants, and the result of Okuma from [23] lead in [3] to the verification of Property A for all splice quotients. (The case of trivial line bundle was verified earlier in [19].) Splice quotient singularities were introduced by Neumann and Wahl (see e.g. [21]), they include all the rational, minimal elliptic singularities, and all singularities which admit a good $\mathbb{C}^*$-action.

Assume now that $(X, 0)$ is a splice quotient, and additionally, its topological type is also almost rational. Set $l' \in \mathbb{L}'$ and $k = K - 2l'$ as in Property A. Then Property A, (5.2.3) and (6.2.7) read as

$$h^1(\mathcal{O}_{\tilde{X}}(l')) = eu(\mathbb{H}^0(\Gamma, k)) = \min_{\gamma \in \mathcal{P}} eu(\mathbb{H}^0(\gamma; \Gamma, k)),$$

which (by 6.2.5) is a topological upper bound for $h^1(\mathcal{L})$, where $\mathcal{L}$ is an *any bundle* with $c_1(\mathcal{L}) = l'$.

In particular, if Γ is almost rational, and the topological type admits a splice quotient analytic structure, then the geometric genus of the splice quotient analytic structure (which satisfies Property A) is an upper bound for the geometric genera of all the possible analytic structures.

7.3.3. Example. One can find even hypersurface singularities when Property A is not true for p_g (i.e. for $l' = 0$). Such examples are provided in [10] by super-isolated singularities. In the examples of [10, (4.1)], p_g is strict higher then the expected value $-\mathbf{sw}(M,[K])-(K^2+s)/8$. Now, using our previous discussions, this phenomenon can be explained as follows.

In general, in the light of Conjecture (5.2.4), Property A/B is equivalent to

$$(1)\qquad p_g = eu(\mathbb{H}^*(\Gamma,k)) = -\min\chi_K + \textstyle\sum_q (-1)^q \operatorname{rank}_{\mathbb{Z}} \mathbb{H}^q_{red}(\Gamma,K).$$

On the other hand, the inequalities from subsection (6.2) read as

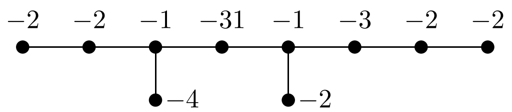
(2)

$$p_g \le \min_{\gamma\in\mathcal{P}} eu\,(\mathbb{H}^0(\gamma;\Gamma,K)) \;\le\; eu(\mathbb{H}^0(\Gamma,K)) = -\min\chi_K + \operatorname{rank}_{\mathbb{Z}} \mathbb{H}^0_{red}(\Gamma,K).$$

Assume that three things are happening simultaneously: (a) in (2) the second inequality is equality, (b) for some analytic structure the first inequality in (2) is sharp (hence $p_g = eu(\mathbb{H}^0(\Gamma,K)))$, and (c) $\mathbb{H}^q_{red}\neq 0$ for $q\ge 2$, creating the situation $eu(\mathbb{H}^0) > eu(\mathbb{H}^*)$. Then Property A fails, and in fact $p_g > eu(\mathbb{H}^*(\Gamma,k))$ for that analytic structure.

This is the case for all the examples of [10, (4.1)].

Let us analyse a little bit more the case C_4 of [10]. The corresponding graph Γ is



In this case $\mathbb{H}^0(\Gamma,K)=\mathcal{T}^+_{-10}\oplus\mathcal{T}_{-10}(3)\oplus\mathcal{T}_0(1)^2$, hence $eu(\mathbb{H}^0)=10$, but $eu(\mathbb{H}^*)=8$. (Strictly speaking, the author verified that $-\mathbf{sw}(M,[K])-(K^2+s)/8=8$, cf. (5.2.4).) Hence the topological bound given by (2) is $p_g\le 10$. This topological type admits two, very natural, but rather different analytic structures. The first is the super-isolated hypersurface singularity mentioned above: it has $p_g=10$ [10]. On the other hand, there is a splice quotient singularity which satisfies Property A, hence with $p_g=8$ [19]. This is the $\mathbb{Z}_5$-factor of the complete intersection $\{z_1^3+z_2^4+z_3^5z_4 = z_3^7+z_4^2+z_1^4z_2=0\}\subset(\mathbb{C}^4,0)$ by the diagonal action $(\alpha^2,\alpha^4,\alpha,\alpha)$ $(\alpha^5=1)$.

Therefore, in general, the geometric genus of those analytic structures which satisfy Property A is not 'extremal' (in contrast with the almost rational case (7.3.2)). In [3], Property A is reformulated completely in terms of the analytic structure (independently of any Seiberg-Witten type theory). [23] suggests that in the heart of the its validity there is a cohomological vanishing result.

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