

# Classification of Deformation Quantization Algebroids on Complex Symplectic Manifolds

By

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## Abstract

A (holomorphic) deformation quantization algebroid over a complex symplectic manifold  $X$  is a stack locally equivalent to the ring of WKB operators, that is, microdifferential operators with an extra central parameter  $\tau$ . In this paper, we will show that the (holomorphic) deformation quantization algebroids endowed with an anti-involution are classified by  $H^2(X; k_X^*)$ , where  $k^*$  is a subgroup of the group of invertible series in  $\mathbb{C}[[\tau^{-1}]]$ . In the formal case, the analogous classification is given by  $H^2(X; \mathbb{C}_X)[[\hbar]]^{\text{odd}}$ , where one sets  $\hbar = \tau^{-1}$ .

## Introduction

Let  $M$  be a complex manifold,  $T^*M$  its cotangent bundle endowed with the canonical symplectic structure and  $\mathcal{O}_{T^*M}$  the sheaf of holomorphic functions. Let  $\mathcal{W}_{T^*M}$  be the sheaf of algebras on  $T^*M$  of WKB operators, that is, microdifferential operators with an extra central parameter  $\tau$ . Recall that the order of the operators defines a filtration on  $\mathcal{W}_M$  such that its associated graded algebra is isomorphic to  $\mathcal{O}_{T^*M}[\tau^{-1}, \tau]$ . The product is given by the Leibniz rule not involving  $\tau$ -derivatives and it is compatible with the filtration and with the canonical Poisson structure on  $\mathcal{O}_{T^*M}$ . It follows that  $\mathcal{W}_{T^*M}$  or,

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more generally, any  $\mathcal{W}$ -algebra, *i.e.* a filtered sheaf of algebras which is locally isomorphic to  $\mathcal{W}_{T^*M}$  and whose associated graded sheaf is  $\mathcal{O}_{T^*M}[\tau^{-1}, \tau]$ , provides a (holomorphic) deformation quantization of  $T^*M$ .

On a complex symplectic manifold  $X$ , the  $\mathcal{W}$ -algebras exist on any symplectic local chart  $f: X \supset U \rightarrow T^*M$ , but there may not exist a  $\mathcal{W}$ -algebra globally defined on  $X$ . However, replacing sheaves of algebras by algebroid stacks, one may show that the sheaves  $f^{-1}\mathcal{W}_M$  glue together as an algebroid stack  $\mathfrak{W}_X$  (see [16, 18, 26, 12]). Again,  $\mathfrak{W}_X$  is endowed with a filtration and its associated graded stack is the trivial algebroid stack  $\mathcal{O}_X[\tau^{-1}, \tau]$ . Then we may mimic the definition of a  $\mathcal{W}$ -algebra, and define a  $\mathcal{W}$ -algebroid on  $X$  as a filtered algebroid stack which is locally equivalent to  $\mathfrak{W}_X$  and whose associated graded algebroid stack is  $\mathcal{O}_X[\tau^{-1}, \tau]$ . These objects play the role of (holomorphic) deformation quantizations of  $X$ .

The purpose of this paper (appeared as an e-print in [24]) is to show that the  $\mathcal{W}$ -algebroids on  $X$  which are endowed with an anti-involution are classified by  $H^2(X; k_X^*)$ , where  $k^*$  is a subgroup of the group of invertible series in  $\mathbb{C}[[\tau^{-1}]]$ . In the formal case, the classification is given by the familiar group  $H^2(X; \mathbb{C}_X)[[\hbar]]^{\text{odd}}$  of odd formal power series in  $\hbar = \tau^{-1}$  with coefficients in  $H^2(X; \mathbb{C}_X)$ . Note that this is compatible with the classification of the deformation quantization algebroids (not assumed endowed with an anti-involution) given in [8].

The classification of deformation quantization algebras on real symplectic manifolds has been considered by many authors (see for example [10, 21, 2, 4]). We refer to [22] (see also [8]) for the analogous classification in the complex setting, where these algebras are constructed under suitable hypothesis.

The paper is organized as follows: in Section 1 we recall the definition of WKB operator, that of  $\mathcal{W}$ -algebra and their classification. In Section 2 we give the main definitions and properties of filtered and graded stacks, focusing on algebroid stacks. In Section 3 we define the  $\mathcal{W}$ -algebroids on a complex symplectic manifold  $X$ . In Section 4 we recall the cohomology theory with values in a stack. In Section 5 we classify the  $\mathcal{W}$ -algebroids (endowed with an anti-involution) on  $X$ .

**Notations and conventions.** All the filtrations are over  $\mathbb{Z}$ , increasing and exhaustive. If  $A$  (resp.  $\mathcal{A}$ ) is a filtered algebra (resp. sheaf of filtered algebras), we will denote by  $\text{Gr}(A)$  (resp.  $\mathcal{G}r(\mathcal{A})$ ) its associated graded algebra (resp. sheaf of graded algebras), and by  $\text{Gr}_0(A)$  (resp.  $\mathcal{G}r_0(\mathcal{A})$ ) the algebra (resp. sheaf of algebras) of homogeneous elements of degree 0. We will use similar notations for morphisms.

We will use the upper index  $^{\text{op}}$  to denote opposite structure, when referring either to (sheaves of) algebras, to categories or to stacks. If  $F$  is a functor between categories, resp. stacks,  $F^{\text{op}}$  will denote the induced functor between the corresponding opposite categories, resp. stacks. (Note that a natural transformation of functors  $F \Rightarrow G$  induces a natural transformation  $G^{\text{op}} \Rightarrow F^{\text{op}}$ .)

If  $\mathcal{A}$  is a sheaf of algebras, we will denote by  $\mathcal{A}^\times$  the sheaf of groups of its invertible elements and, for each section  $a \in \mathcal{A}^\times$ , by  $\text{ad}(a): \mathcal{A} \rightarrow \mathcal{A}$  the algebra isomorphism  $b \mapsto aba^{-1}$ .

§1.  $\mathcal{W}$ -Algebras

The relation between Sato’s microdifferential operators and WKB operators<sup>1</sup> is discussed in [23, 1, 26]. We follow here the presentation in [26], and we refer to [27, 15, 17] for the theory of microdifferential operators.

Let  $M$  be an  $n$ -dimensional complex manifold,  $\pi: T^*M \rightarrow M$  its cotangent bundle and  $\rho: J^1M \rightarrow T^*M$  the natural projection from the 1-jets bundle. Let  $(t, \tau)$  be the system of homogeneous symplectic coordinates on  $T^*\mathbb{C}$ , and recall that  $J^1M$  is identified with the open subset of the projective cotangent bundle  $P^*(M \times \mathbb{C})$  defined by  $\tau \neq 0$ . Denote by  $\mathcal{E}_{M \times \mathbb{C}}$  (resp.  $\mathcal{E}_{M \times \mathbb{C}}(m)$ ,  $m \in \mathbb{Z}$ ) the sheaf of finite order (resp. of order  $\leq m$ ) microdifferential operators on  $P^*(M \times \mathbb{C})$  and consider the subring  $\mathcal{E}_{M \times \mathbb{C}, \partial_t}$  of operators commuting with  $\partial_t$ . The ring of finite order WKB operators is defined by

$$\mathcal{W}_{T^*M} = \rho_*(\mathcal{E}_{M \times \mathbb{C}, \partial_t}|_{J^1M}),$$

and its subsheaf of operators of order  $\leq m$  by

$$\mathcal{W}_{T^*M}(m) = \rho_*(\mathcal{E}_{M \times \mathbb{C}, \partial_t}(m)|_{J^1M}).$$

In a local coordinate system  $(x)$  on  $M$ , with associated symplectic local coordinates  $(x, u)$  on  $T^*M$ , a WKB operator  $P$  of order  $\leq m$  defined on an open subset  $U$  of  $T^*M$  has a total symbol

$$\sigma_{\text{tot}}(P) = \sum_{j=-\infty}^m p_j(x, u)\tau^j,$$

where the  $p_j$ ’s are holomorphic functions on  $U$  subject to the estimates

$$(1.1) \quad \begin{cases} \text{for any compact subset } K \text{ of } U \text{ there exists a constant} \\ C_K > 0 \text{ such that for all } j < 0, \sup_K |p_j| \leq C_K^{-j}(-j)!. \end{cases}$$

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<sup>1</sup>WKB stands for Wentzel-Kramer-Brillouin.

The product structure on  $\mathcal{W}_{T^*M}$  is given by the Leibniz formula not involving  $\tau$ -derivatives: if  $Q$  is another WKB operator defined on  $U$  of total symbol  $\sigma_{\text{tot}}(Q)$ , then

$$\sigma_{\text{tot}}(PQ) = \sum_{\alpha \in \mathbb{N}^n} \frac{\tau^{-|\alpha|}}{\alpha!} \partial_u^\alpha \sigma_{\text{tot}}(P) \partial_x^\alpha \sigma_{\text{tot}}(Q).$$

Note that  $\mathcal{W}_{T^*M}$  contains  $\pi^{-1}\mathcal{D}_M$  as a sub- $\mathbb{C}$ -algebra, where  $\mathcal{D}_M$  denotes the ring of differential operators on  $M$ . A WKB operator in  $\pi^{-1}\mathcal{D}_M$  has a total symbol  $\sum_{j=0}^m p_j(x, u)\tau^j$  with  $p_j(x, u)$  a  $j$ -homogeneous polynomial in the fiber variables.

The center of  $\mathcal{W}_{T^*M}$  is the constant sheaf  $k_{T^*M}$  with stalk the subfield  $k = \mathcal{W}_{\text{pt}} \subset \mathbb{C}[[\tau^{-1}, \tau]]$  of WKB operators over a point, *i.e.* series  $\sum_{j=-\infty}^m a_j \tau^j$  which satisfy the estimates:

$$\left\{ \begin{array}{l} \text{there exists a constant } C > 0 \text{ such that} \\ \text{for all } j < 0, |a_j| \leq C^{-j}(-j)!. \end{array} \right.$$

The algebra  $\mathcal{W}_{T^*M}$  is filtered by the  $\mathbb{C}$ -modules  $\mathcal{W}_{T^*M}(m)$ , as well as the field  $k$  by  $k(m) = \mathcal{W}_{\text{pt}}(m)$ . Note that  $k(0)$  is a commutative ring,  $\mathcal{W}_{T^*M}(0)$  is a  $k(0)$ -algebra, and there is an isomorphism of filtered  $k$ -algebras

$$\mathcal{W}_{T^*M} \simeq \mathcal{W}_{T^*M}(0) \otimes_{k(0)} k_{T^*M}.$$

We denote by

$$\sigma_m(\cdot) : \mathcal{W}_{T^*M}(m) \rightarrow \mathcal{O}_{T^*M}, \quad P \mapsto p_m(x, u)$$

the symbol map of order  $m$ , which does not depend on the local coordinate system on  $T^*M$ . If  $\sigma_m(P)$  is not identically zero, then one says that  $P$  has order  $m$  and  $\sigma_m(P)$  is called the principal symbol of  $P$ . In particular, an element  $P$  in  $\mathcal{W}_{T^*M}$  is invertible if and only if its principal symbol is nowhere vanishing. Note that  $\text{Gr}(k) \simeq \mathbb{C}[[\tau^{-1}, \tau]]$  and that the principal symbol maps induce an isomorphism of graded  $\mathbb{C}[[\tau^{-1}, \tau]]$ -algebras

$$\text{Gr}(\mathcal{W}_{T^*M}) \simeq \mathcal{O}_{T^*M}[[\tau^{-1}, \tau]].$$

*Remark 1.1.*

- (i) The algebra  $\mathcal{W}_{T^*M}$  is a (holomorphic) deformation quantization of  $T^*M$  in the following sense. Denote by  $\mathcal{O}_{T^*M}^\tau$  the subsheaf of  $\mathcal{O}_{T^*M}[[\tau^{-1}, \tau]]$  of series  $f(x, u; \tau) = \sum_{j=-\infty}^m f_j(x, u)\tau^j$  satisfying the estimates (1.1). This is a filtered  $k$ -module whose associated graded  $\mathbb{C}[[\tau^{-1}, \tau]]$ -module is  $\mathcal{O}_{T^*M}[[\tau^{-1}, \tau]]$ .

The algebra  $\mathcal{W}_{T^*M}$  has thus the same graded sheaf as  $\mathcal{O}_{T^*M}^\tau$  and it is locally isomorphic to  $\mathcal{O}_{T^*M}^\tau$  as filtered  $k$ -modules (via the total symbol  $\sigma_{\text{tot}}$ ), in such a way that the Leibniz rule induces the Wick star-product on  $\mathcal{O}_{T^*M}^\tau$ . Note that most of the authors use to set  $\hbar = \tau^{-1}$  and consider the  $\mathbb{C}[[\hbar]]$ -algebra  $\widehat{\mathcal{W}}_{T^*M}(0)$  of formal WKB operators of degree  $\leq 0$ , obtained by dropping the estimates (1.1).

(ii) Note that the Borel transform defines an isomorphism of  $\mathbb{C}$ -modules

$$B: \mathcal{O}_{T^*M}^\tau(0) \xrightarrow{\sim} \mathcal{O}_{T^*M}\{t\},$$

where  $\mathcal{O}_{T^*M}\{t\}$  denotes the sheaf of germs in  $t = 0$  of functions in  $\mathcal{O}_{J^1M}$ . Recall that, if  $f(x, u; \tau) = \sum_{j=-\infty}^0 f_j(x, u)\tau^j$ , then  $B(f)(x, u; t)$  is given formally by  $\frac{1}{2\pi i} \int_\gamma f(x, u; \tau) \frac{e^{t\tau}}{\tau} d\tau$  for  $\gamma$  a counter clockwise oriented circle around 0 in  $\mathbb{C}$ . In particular,  $k(0) \simeq \mathbb{C}\{t\}$  as  $\mathbb{C}$ -modules.

In order to define the canonical anti-involution<sup>2</sup> of a WKB operator, which is of use in microlocal analysis (see [27, 15, 17]), we need to consider a slight modification of  $\mathcal{W}_{T^*M}$ .

Let  $\Omega_M$  be the canonical line bundle on  $M$ , that is, the sheaf of differential forms of top degree. Each locally defined volume form  $\theta \in \Omega_M$  gives rise to a local isomorphism  $*_\theta: \mathcal{W}_{T^*M}^{\text{op}} \xrightarrow{\sim} \mathcal{W}_{T^*M}$ , which sends a WKB operator  $P$  to its formal adjoint  $P^{*\theta}$  with respect to  $\theta$ . In a local coordinate system  $(x)$  satisfying  $\theta = dx$ , with associated symplectic local coordinates  $(x, u)$ , one has

$$\sigma_{\text{tot}}(P^{*\theta}) = \sum_{\alpha \in \mathbb{N}^n} \frac{(-\tau)^{-|\alpha|}}{\alpha!} \partial_u^\alpha \partial_x^\alpha \hat{\sigma}_{\text{tot}}(P),$$

where  $\hat{\cdot}: \mathcal{O}_{T^*M}^\tau \rightarrow \mathcal{O}_{T^*M}^\tau$  is defined by  $\hat{f}(x, u; \tau) = f(x, u; -\tau)$ .

Twisting  $\mathcal{W}_{T^*M}$  by  $\Omega_M$ , one then gets a globally defined anti- $k$ -linear<sup>3</sup> isomorphism of algebras

$$\mathcal{W}_{T^*M}^{\text{op}} \xrightarrow{\sim} \pi^{-1}\Omega_M \otimes \mathcal{W}_{T^*M} \otimes \pi^{-1}\Omega_M^{\otimes -1} \quad P \mapsto \theta \otimes P^{*\theta} \otimes \theta^{\otimes -1},$$

which does not depend on the choice of the volume form. (Here the tensor product is over  $\pi^{-1}\mathcal{O}_M$  and  $\theta^{\otimes -1}$  denotes the unique section of  $\Omega_M^{\otimes -1}$ , the dual of  $\Omega_M$ , satisfying  $\theta^{\otimes -1} \otimes \theta = 1$ .) This leads to replace the algebra  $\mathcal{W}_{T^*M}$  by its twisted version by half-forms

$$\mathcal{W}_{T^*M}^{\sqrt{v}} = \pi^{-1}\Omega_M^{\otimes 1/2} \otimes \mathcal{W}_{T^*M} \otimes \pi^{-1}\Omega_M^{\otimes -1/2}.$$

<sup>2</sup>An anti-involution is an isomorphism of rings  $\iota: \mathcal{A}^{\text{op}} \xrightarrow{\sim} \mathcal{A}$  such that  $\iota^2 = \text{id}$ .

<sup>3</sup>A map between  $k$ -modules is anti- $k$ -linear if it is  $\mathbb{C}$ -linear and sends  $\tau$  to  $-\tau$ .

Recall that the sections of  $\mathcal{W}_{T^*M}^{\sqrt{v}}$  are locally defined by  $\theta^{\otimes 1/2} \otimes P \otimes \theta^{\otimes -1/2}$  for a volume form  $\theta$  and a WKB operator  $P$ , with the equivalence relation  $\theta_1^{\otimes 1/2} \otimes P_1 \otimes \theta_1^{\otimes -1/2} = \theta_2^{\otimes 1/2} \otimes P_2 \otimes \theta_2^{\otimes -1/2}$  if and only if  $P_2 = (\theta_1/\theta_2)^{1/2} P_1 (\theta_1/\theta_2)^{-1/2}$ .

The sheaf  $\mathcal{W}_{T^*M}^{\sqrt{v}}$  is a filtered  $k$ -algebra on  $T^*M$  locally isomorphic to  $\mathcal{W}_{T^*M}$  and satisfying the following properties:

- (i) there is an isomorphism of graded  $\mathbb{C}[\tau^{-1}, \tau]$ -algebras

$$\sigma: \mathcal{G}r(\mathcal{W}_{T^*M}^{\sqrt{v}}) \xrightarrow{\sim} \mathcal{O}_{T^*M}[\tau^{-1}, \tau]$$

preserving the Poisson structures (induced by the commutator on  $\mathcal{W}_{T^*M}^{\sqrt{v}}$  and by the Poisson bracket on  $T^*M$ , respectively);

- (ii) it is endowed with a filtered anti- $k$ -linear anti-involution

$$*: (\mathcal{W}_{T^*M}^{\sqrt{v}})^{\text{op}} \xrightarrow{\sim} \mathcal{W}_{T^*M}^{\sqrt{v}},$$

such that  $\mathcal{G}r_0(*)$  is the identity (if  $P \in \mathcal{W}_{T^*M}^{\sqrt{v}}(m)$ , one has  $\sigma_m(P^*) = (-1)^m \sigma_m(P)$ ).

It follows that  $\mathcal{W}_{T^*M}^{\sqrt{v}}$  is again a deformation quantization of  $T^*M$ .

This suggests the following definition:

**Definition 1.2.** A  $\mathcal{W}$ -algebra with anti-involution on  $T^*M$ , a  $(\mathcal{W}, *)$ -algebra for short, is a sheaf of filtered  $k$ -algebras  $\mathcal{A}$  together with

- (i) an isomorphism of graded  $\mathbb{C}[\tau^{-1}, \tau]$ -algebras  $\nu: \mathcal{G}r(\mathcal{A}) \xrightarrow{\sim} \mathcal{O}_{T^*M}[\tau^{-1}, \tau]$ ;
- (ii) an anti-involution  $\iota$ ;

such that the triplet  $(\mathcal{A}, \nu, \iota)$  is locally isomorphic to  $(\mathcal{W}_{T^*M}^{\sqrt{v}}, \sigma, *)$ .

An isomorphism  $\varphi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$  of  $(\mathcal{W}, *)$ -algebras is a filtered  $k$ -algebra isomorphism commuting with the anti-involutions and such that  $\nu_2 \circ \mathcal{G}r(\varphi) = \nu_1$ .

Isomorphisms of  $(\mathcal{W}, *)$ -algebras translate to WKB operators the notion of equivalence between star-products. Indeed, the isomorphism  $\nu$  in (i) preserves the Poisson structures, so that any  $(\mathcal{W}, *)$ -algebra provides a (holomorphic) deformation quantization of  $T^*M$ . Note also that the anti-involution  $\iota$  in (ii) is filtered anti- $k$ -linear and  $\mathcal{G}r_0(\iota)$  is the identity. We refer to [3, 4] for similar definitions in the context of microdifferential and Toeplitz operators.

*Remark 1.3.* One may show that any  $k$ -algebra automorphism  $\varphi$  of  $\mathcal{W}_{T^*M}$  is filtered with  $\mathcal{G}r(\varphi) = \text{id}$  (see [27], or also [13], for the microdifferential case). It follows that any sheaf of  $k$ -algebras  $\mathcal{A}$  locally isomorphic to  $\mathcal{W}_{T^*M}$  is filtered and endowed with an isomorphism of graded  $\mathbb{C}[\tau^{-1}, \tau]$ -algebras  $\mathcal{G}r(\mathcal{A}) \simeq \mathcal{O}_{T^*M}[\tau^{-1}, \tau]$ , so that property (i) in Definition 1.2 could in fact be dropped. However, this is no more true when replacing sheaves of algebras by algebroid stacks (see Remark 3.4), so we prefer to leave it in the definition.

**Example 1.4.** Let  $f: T^*M \rightarrow T^*M$  be a symplectic transformation. Then the sheaf-theoretical inverse image  $f^{-1}\mathcal{W}_{T^*M}^{\sqrt{v}}$  is a  $k$ -algebra on  $T^*M$  and inherits an anti-involution and a filtration from  $\mathcal{W}_{T^*M}^{\sqrt{v}}$ , in such a way that  $f$  induces a graded isomorphism  $\sigma_f: \mathcal{G}r(f^{-1}\mathcal{W}_{T^*M}^{\sqrt{v}}) \xrightarrow{\sim} \mathcal{O}_{T^*M}[\tau^{-1}, \tau]$ . By [26], locally there exists a Quantized Symplectic Transformation over  $f$ , that is, a filtered  $k$ -algebra isomorphism  $f^{-1}\mathcal{W}_{T^*M}^{\sqrt{v}} \xrightarrow{\sim} \mathcal{W}_{T^*M}^{\sqrt{v}}$  preserving the anti-involutions and compatible with  $\sigma_f$ . It follows that  $f^{-1}\mathcal{W}_{T^*M}^{\sqrt{v}}$  is a  $(\mathcal{W}, *)$ -algebra.

Denote by  $\text{Aut}_{(\mathcal{W}, *)}(\mathcal{W}_{T^*M}^{\sqrt{v}})$  the group of  $(\mathcal{W}, *)$ -algebra automorphisms of  $\mathcal{W}_{T^*M}^{\sqrt{v}}$  and set

$$\begin{aligned} \mathcal{W}_{T^*M}^{\sqrt{v},*} &= \{P \in \mathcal{W}_{T^*M}^{\sqrt{v}}(0); \sigma_0(P) = 1 \text{ and } PP^* = 1\}, \\ k^* &= \{s(\tau) \in k(0); s(\tau) = 1 + \sum_{j < 0} a_j \tau^j \text{ and } s(\tau)s(-\tau) = 1\}. \end{aligned}$$

Note that  $\mathcal{W}_{T^*M}^{\sqrt{v},*}$  is a subgroup of the group of invertible WKB operators of order 0, and that  $k^* = \mathcal{W}_{\text{pt}}^{\sqrt{v},*}$ .

**Lemma 1.5** (cf. [26]). *There is an exact sequence of sheaves of groups*

$$(1.2) \quad 1 \rightarrow k_{T^*M}^* \rightarrow \mathcal{W}_{T^*M}^{\sqrt{v},*} \xrightarrow{\text{ad}} \text{Aut}_{(\mathcal{W}, *)}(\mathcal{W}_{T^*M}^{\sqrt{v}}) \rightarrow 1.$$

The set of isomorphism classes of  $(\mathcal{W}, *)$ -algebras on  $T^*M$  is in bijection with  $H^1(T^*M; \text{Aut}_{(\mathcal{W}, *)}(\mathcal{W}_{T^*M}^{\sqrt{v}}))$ . Hence we get:

**Corollary 1.6.** *The  $(\mathcal{W}, *)$ -algebras on  $T^*M$  are classified by the pointed set  $H^1(T^*M; \mathcal{W}_{T^*M}^{\sqrt{v},*}/k_{T^*M}^*)$ .*

## §2. Filtered and Graded Stacks

In order to define the  $\mathcal{W}$ -algebroids on a complex symplectic manifold, we need to translate the notions of filtration and graduation from sheaves to

stacks. In this section we define what a filtered category is and show how to associate a graded category to a filtered one. Then we stackify these definitions. Finally, we recall the notion of algebroid stack and give a cocycle description for the graded stack associated to a filtered algebroid stack. We assume that the reader is familiar with the basic notions of the theory of stacks, which are, roughly speaking, sheaves of categories. (The classical reference is [14], and a short presentation is given *e.g.* in [16, 11].)

Let  $R$  be a filtered commutative ring.

**Definition 2.1.** A filtered  $R$ -category is an  $R$ -category<sup>4</sup>  $\mathcal{C}$  satisfying the following properties:

- for any objects  $P, Q \in \mathcal{C}$ , the  $R$ -module  $\text{Hom}_{\mathcal{C}}(P, Q)$  is a filtered  $R$ -module ( $F_n \text{Hom}_{\mathcal{C}}(P, Q)$  will denote its subgroup of morphisms of degree  $\leq n$ );
- for any  $P, Q, R \in \mathcal{C}$  and any morphisms  $f$  in  $F_m \text{Hom}_{\mathcal{C}}(Q, R)$  and  $g$  in  $F_n \text{Hom}_{\mathcal{C}}(P, Q)$ , the composed morphism  $f \circ g$  is in  $F_{m+n} \text{Hom}_{\mathcal{C}}(P, R)$ ;
- for each  $P \in \mathcal{C}$ , the identity morphism  $\text{id}_P$  is in  $F_0 \text{Hom}_{\mathcal{C}}(P, P)$ .

An  $R$ -functor  $\Phi: \mathcal{C} \rightarrow \mathcal{C}'$  between filtered  $R$ -categories is filtered if the  $R$ -module morphism  $\text{Hom}_{\mathcal{C}}(P, Q) \rightarrow \text{Hom}_{\mathcal{C}'}(\Phi(P), \Phi(Q))$  is filtered for any objects  $P, Q \in \mathcal{C}$ .

A natural transformation  $\alpha = (\alpha_P)_{P \in \mathcal{C}}: \Phi \Rightarrow \Phi'$  between filtered  $R$ -functors is filtered of degree  $\leq n$  if  $\alpha_P$  is in  $F_n \text{Hom}_{\mathcal{C}'}(\Phi(P), \Phi'(P))$  for each  $P \in \mathcal{C}$ .

In a similar way one defines the graded categories, the graded functors and the graded natural transformations. (Note that a graded category amounts to a DG-category with zero differential.) If  $\mathcal{C}$  is graded, we denote by  $G_n \text{Hom}_{\mathcal{C}}(P, Q)$  the subgroup of  $\text{Hom}_{\mathcal{C}}(P, Q)$  of morphisms homogeneous of degree  $n$ .

If  $\Phi, \Phi': \mathcal{C} \rightarrow \mathcal{C}'$  are filtered  $R$ -functors, we define  $F_n \text{Hom}(\Phi, \Phi')$  as the set of those natural transformations of functors which are filtered of degree  $\leq n$  and we denote by  $\text{Fct}_{FR}(\mathcal{C}, \mathcal{C}')$  the filtered  $R$ -category thereby obtained. Similarly, one defines the graded category of graded functors between two graded categories.

To any filtered  $R$ -category  $\mathcal{C}$  there is an associated graded  $\text{Gr}(R)$ -category  $\text{Gr}(\mathcal{C})$ , whose objects are the same of those of  $\mathcal{C}$  and for any objects  $P, Q$  the set of morphisms is defined by  $\text{Hom}_{\text{Gr}(\mathcal{C})}(P, Q) = \text{Gr}(\text{Hom}_{\mathcal{C}}(P, Q))$ . Similarly,

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<sup>4</sup>An  $R$ -category is a category whose sets of morphisms are endowed with an  $R$ -module structure, so that the composition is bilinear. An  $R$ -functor is a functor between  $R$ -categories which is linear at the level of morphisms.



to any filtered functor  $\Phi$ , one associates a graded functor  $\text{Gr}(\Phi)$ . In this way, we get a functor

$$\text{Gr}: \{\text{Filtered } R\text{-categories}\} \longrightarrow \{\text{Graded Gr}(R)\text{-categories}\}.$$

Note that, if  $\Phi, \Phi'$  are filtered functors, then for each  $n$  there is a natural injective morphism

$$\text{Gr}_n(\text{Hom}(\Phi, \Phi')) = \frac{F_n \text{Hom}(\Phi, \Phi')}{F_{n-1} \text{Hom}(\Phi, \Phi')} \rightarrow G_n \text{Hom}(\text{Gr}(\Phi), \text{Gr}(\Phi')).$$

To any filtered natural transformation  $\alpha: \Phi \Rightarrow \Phi'$  we may thus associate a graded natural transformation  $\text{Gr}(\Phi) \Rightarrow \text{Gr}(\Phi')$ , denoted by  $\text{Gr}(\alpha)$ . One checks that  $\text{Gr}(\cdot)$  defines a 2-functor from the 2-category of filtered  $R$ -categories, filtered  $R$ -functors and filtered natural transformations to that of graded  $\text{Gr}(R)$ -categories, graded  $\text{Gr}(R)$ -functors and graded natural transformations.

If  $\mathbb{C}$  is a filtered  $R$ -category,  $\text{Gr}_0(\mathbb{C})$  will denote the sub- $\text{Gr}_0(R)$ -category of  $\text{Gr}(\mathbb{C})$  with the same objects but only those morphisms which are homogeneous of degree 0, that is,  $\text{Hom}_{\text{Gr}_0(\mathbb{C})}(P, Q) = \text{Gr}_0(\text{Hom}_{\mathbb{C}}(P, Q))$ . As above, we get a functor  $\text{Gr}_0(\cdot)$  from filtered  $R$ -categories to  $\text{Gr}_0(R)$ -categories, which extends to a 2-functor if we restrict to filtered natural transformations of degree  $\leq 0$ .

Recall that there is a fully faithful functor  $(\cdot)^+$  from filtered (resp. graded) algebras to filtered (resp. graded) categories, which sends an algebra  $A$  to the category  $A^+$  with a single object  $\bullet$  and  $\text{End}(\bullet) = A$  as set of morphisms. If  $f, g: A \rightarrow B$  are filtered (resp. graded) algebra morphisms, then each filtered (resp. graded) natural transformation of degree  $\leq n$  (resp. of degree  $n$ )  $f^+ \Rightarrow g^+$  corresponds to an element  $b$  in  $F_n B$  (resp. in  $G_n B$ ) such that  $bf(a) = g(a)b$  for any  $a \in A$ . Clearly, for any filtered  $R$ -algebra  $A$  one has  $\text{Gr}(A^+) = \text{Gr}(A)^+$ . Moreover, if  $f, g: A \rightarrow B$  are filtered  $R$ -algebra morphisms and  $\alpha: f^+ \Rightarrow g^+$  a filtered natural transformation defined by  $b \in F_n B$ , then the graded natural transformation  $\text{Gr}(\alpha): \text{Gr}(f)^+ \Rightarrow \text{Gr}(g)^+$  is given by the symbol of  $b$ , that is, the image of  $b$  via the natural map  $F_n B \rightarrow \text{Gr}_n B = F_n B / F_{n-1} B$ .

Let  $A$  be a filtered  $R$ -algebra. For any filtered (left)  $A$ -modules  $M$  and  $N$ , let  $F_m \text{Hom}_{FA}(M, N)$  be the set of those  $A$ -linear morphisms  $M \rightarrow N$  which send  $F_n M$  to  $F_{n+m} N$ . We denote by  $\text{Mod}_F(A)$  the filtered  $R$ -category thereby obtained. One easily checks that it is equivalent to the category  $\text{Fct}_{FR}(A^+, \text{Mod}_F(R))$ , and that the Yoneda embedding

$$A^+ \rightarrow \text{Fct}_{FR}((A^+)^{\text{op}}, \text{Mod}_F(R)) \approx \text{Mod}_F(A^{\text{op}})$$

identifies  $A^+$  with the full subcategory of filtered right  $A$ -modules which are

isomorphic to  $A$ . Everything remains true replacing filtered algebras and categories by graded ones. Note that  $\text{Gr}(\cdot)$  induces a faithful graded  $\text{Gr}(R)$ -functor

$$\text{Gr}: \text{Gr}(\text{Mod}_F(A)) \rightarrow \text{Mod}_G(\text{Gr}(A)),$$

which sends a filtered  $A$ -module  $L$  to its associated graded module  $\text{Gr}(L)$ .

Let  $X$  be a topological space. As for categories, there are natural notions of filtered (resp. graded) stack, of filtered (resp. graded) functor between filtered (resp. graded) stacks and of filtered (resp. graded) natural transformation between filtered (resp. graded) functors.

As above, we denote by  $(\cdot)^+$  the faithful and locally full functor from sheaves of filtered (resp. graded) algebras to filtered (resp. graded) stacks, which sends an algebra  $\mathcal{A}$  to the stack  $\mathcal{A}^+$  defined as follows: it is the stack associated with the pre-stack  $X \supset U \mapsto \mathcal{A}(U)^+$ . If  $f, g: \mathcal{A} \rightarrow \mathcal{B}$  are filtered (resp. graded) algebra morphisms, filtered (resp. graded) natural transformations  $f^+ \Rightarrow g^+$  are locally described as above.

Let  $\mathcal{R}$  be a sheaf of filtered commutative rings.

Given a filtered  $\mathcal{R}$ -algebra  $\mathcal{A}$ , the  $\mathcal{R}$ -stack<sup>5</sup>  $\text{Mod}_F(\mathcal{A})$  of filtered (left)  $\mathcal{A}$ -modules is filtered and equivalent to the stack  $\mathfrak{Fct}_{F\mathcal{R}}(\mathcal{A}^+, \text{Mod}_F(\mathcal{R}))$  of filtered  $\mathcal{R}$ -functors. Moreover the Yoneda embedding gives a fully faithful functor

$$\mathcal{A}^+ \rightarrow \mathfrak{Fct}_{F\mathcal{R}}((\mathcal{A}^+)^{\text{op}}, \text{Mod}_F(\mathcal{R})) \approx \text{Mod}_F(\mathcal{A}^{\text{op}})$$

into the stack of filtered right  $\mathcal{A}$ -modules. This identifies  $\mathcal{A}^+$  with the full substack of filtered right  $\mathcal{A}$ -modules which are locally isomorphic to  $\mathcal{A}$ . As above, everything remains true replacing filtered algebras and stacks by graded ones.

Let  $\mathfrak{S}$  be a filtered  $\mathcal{R}$ -stack. We denote by  $\mathfrak{Gr}(\mathfrak{S})$  the graded  $\mathcal{G}r(\mathcal{R})$ -stack associated to the pre-stack  $X \supset U \mapsto \text{Gr}(\mathfrak{S}(U))$ . As above, this defines a functor (which is, in fact, a 2-functor)

$$\mathfrak{Gr}: \{\text{Filtered } \mathcal{R}\text{-stacks}\} \longrightarrow \{\text{Graded } \mathcal{G}r(\mathcal{R})\text{-stacks}\}.$$

As for categories, we will also make use of the 2-functor  $\mathfrak{Gr}_0(\cdot)$ .

**Proposition 2.2.** *Let  $\mathcal{A}$  be a filtered  $\mathcal{R}$ -algebra. Then there is an equivalence of graded  $\mathcal{G}r(\mathcal{R})$ -stacks  $\mathfrak{Gr}(\mathcal{A}^+) \approx \mathcal{G}r(\mathcal{A})^+$ .*

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<sup>5</sup>The  $\mathcal{R}$ -stacks and  $\mathcal{R}$ -functors are defined in a similar way as for categories. When  $\mathcal{R} = \mathbb{Z}$ , they are usually called linear stacks.

*Proof.* By the Yoneda embedding, the functor of  $\mathcal{G}r(\mathcal{R})$ -stacks

$$\mathcal{G}r: \mathfrak{Gr}(\mathfrak{Mod}_F(\mathcal{A}^{\text{op}})) \rightarrow \mathfrak{Mod}_G(\mathcal{G}r(\mathcal{A})^{\text{op}})$$

restricts to a functor  $\mathfrak{Gr}(\mathcal{A}^+) \rightarrow \mathcal{G}r(\mathcal{A})^+$ . Since at each  $x \in X$  this reduces to the equality  $\mathfrak{Gr}(\mathcal{A}_x^+) = \mathfrak{Gr}(\mathcal{A}_x)^+$ , it follows that this last functor is a local, hence global, equivalence.  $\square$

We will use the notion of algebroid stack introduced by Kontsevich [18], as developed in [12].

**Definition 2.3.** An  $\mathcal{R}$ -algebroid stack is an  $\mathcal{R}$ -stack  $\mathfrak{A}$  which is locally non-empty and locally connected by isomorphisms.

Equivalently, an  $\mathcal{R}$ -algebroid stack is an  $\mathcal{R}$ -stack such that there exist an open covering  $X = \bigcup_{i \in I} U_i$ ,  $\mathcal{R}|_{U_i}$ -algebras  $\mathcal{A}_i$  on  $U_i$  and  $\mathcal{R}|_{U_i}$ -equivalences  $\mathfrak{A}|_{U_i} \approx \mathcal{A}_i^+$ . Note that, an  $\mathcal{R}$ -algebroid stack  $\mathfrak{A}$  has a global object if and only if it is globally equivalent to  $\mathcal{A}^+$  for an  $\mathcal{R}$ -algebra  $\mathcal{A}$  (if  $\mathcal{L} \in \mathfrak{A}(X)$  is a global object, take  $\mathcal{A} = \mathcal{E}nd_{\mathfrak{A}}(\mathcal{L})$ , the sheaf of endomorphisms of  $\mathcal{L}$ ).

**Corollary 2.4.** Let  $\mathfrak{A}$  be a filtered  $\mathcal{R}$ -stack. If  $\mathfrak{A}$  is an  $\mathcal{R}$ -algebroid stack, then its associated graded stack  $\mathfrak{Gr}(\mathfrak{A})$  is a  $\mathcal{G}r(\mathcal{R})$ -algebroid stack.

Proceeding as in [12], one gets that a filtered  $\mathcal{R}$ -algebroid stack  $\mathfrak{A}$  on  $X$  is given, up to equivalence, by the datum of

$$(\{U_i\}, \{\mathcal{A}_i\}, \{f_{ij}\}, \{a_{ijk}\}),$$

where  $X = \bigcup_{i \in I} U_i$  is an open covering,  $\mathcal{A}_i = \bigcup_{m \in \mathbb{Z}} F_m \mathcal{A}_i$  are filtered  $\mathcal{R}$ -algebras on  $U_i$ ,  $f_{ij}: \mathcal{A}_j \xrightarrow{\sim} \mathcal{A}_i$  are filtered  $\mathcal{R}$ -algebra isomorphisms on  $U_{ij}$  and  $a_{ijk} \in F_{m_{ijk}} \mathcal{A}_i^\times(U_{ijk})$  are invertible sections such that

$$\begin{cases} f_{ij} f_{jk} = \text{ad}(a_{ijk}) f_{ik}, & \text{as morphisms } \mathcal{A}_k \rightarrow \mathcal{A}_i \text{ on } U_{ijk}, \\ a_{ijk} a_{ikl} = f_{ij}(a_{jkl}) a_{ijl} & \text{in } \mathcal{A}_i^\times(U_{ijkl}). \end{cases}$$

(Here and in the sequel we set  $U_{ij} = U_i \cap U_j$  and so on.)

**Proposition 2.5.** Let  $\mathfrak{A}$  be as above. Then the associated graded  $\mathcal{G}r(\mathcal{R})$ -algebroid stack  $\mathfrak{Gr}(\mathfrak{A})$  is given, up to equivalence, by the datum of

$$(\{U_i\}, \{\mathcal{G}r(\mathcal{A}_i)\}, \{\mathcal{G}r(f_{ij})\}, \{\sigma_{m_{ijk}}^i(a_{ijk})\}),$$

where  $\sigma_{m_{ijk}}^i$  denotes the symbol map  $F_{m_{ijk}} \mathcal{A}_i \rightarrow \mathcal{G}r_{m_{ijk}}(\mathcal{A}_i)$ .

Recall that, given an  $\mathcal{R}$ -algebroid stack  $\mathfrak{A}$ , the stack of  $\mathfrak{A}$ -modules is by definition the  $\mathcal{R}$ -stack of  $\mathcal{R}$ -functors  $\mathfrak{Fct}_{\mathcal{R}}(\mathfrak{A}, \mathfrak{Mod}(\mathcal{R}))$ . It is an example of stack of twisted sheaves (see for example [11]). Similarly, one defines the stack of filtered (resp. graded) modules over a filtered (resp. graded) algebroid stack.

### §3. $\mathcal{W}$ -Algebroids

Let  $(X, \omega)$  be a complex symplectic manifold. Recall that a local model for  $X$  is an open subset  $V$  of the cotangent bundle  $T^*M$  of a complex manifold  $M$ , equipped with the canonical symplectic form. Hence it is possible to define the  $(\mathcal{W}, *)$ -algebras on  $X$  in the same way as in Definition 1.2.

Although there may not exist a globally defined  $(\mathcal{W}, *)$ -algebra on  $X$ , Polesello-Schapira [26] defined a canonical stack  $\mathfrak{Mod}(\mathcal{W}; X)$  of WKB-modules on  $X$ . Following [12], this result may be restated as:

**Theorem 3.1** (cf. [26]). *On any complex symplectic manifold  $X$  there exists a canonical  $k$ -stack  $\mathfrak{W}_X$  which is locally equivalent to  $(f^{-1}\mathcal{W}_{T^*M}^{\sqrt{v}})^+$  for any symplectic local chart  $f: X \supset U \rightarrow T^*M$ .*

Note that  $\mathfrak{W}_X$  is a  $k$ -algebroid stack and the stack of  $\mathfrak{W}_X$ -modules equivalent to  $\mathfrak{Mod}(\mathcal{W}; X)$ . In particular, if  $X = T^*M$  for a complex manifold  $M$ , then  $\mathfrak{W}_{T^*M}$  is by definition equal to  $(\mathcal{W}_{T^*M}^{\sqrt{v}})^+$  and it is canonically equivalent to  $(f^{-1}\mathcal{W}_{T^*M}^{\sqrt{v}})^+$  for any symplectic transformation  $f: T^*M \rightarrow T^*M$ .

The next proposition follows directly from the construction of  $\mathfrak{W}_X$  given in [26] and by Proposition 2.5. It allows us to say that the algebroid stack  $\mathfrak{W}_X$  provides a (holomorphic) deformation quantization of  $X$ .

**Proposition 3.2.** *The  $k$ -algebroid stack  $\mathfrak{W}_X$  is filtered and satisfies the following properties:*

(i)<sup>+</sup> *there is an equivalence of graded  $\mathbb{C}[\tau^{-1}, \tau]$ -stacks*

$$\sigma: \mathfrak{Gr}(\mathfrak{W}_X) \xrightarrow{\cong} (\mathcal{O}_X[\tau^{-1}, \tau])^+;$$

(ii)<sup>+</sup> *it is endowed with a filtered anti- $k$ -linear (strict) anti-involution<sup>6</sup>*

$$*: \mathfrak{W}_X^{\text{op}} \xrightarrow{\cong} \mathfrak{W}_X$$

*such that  $\sigma \circ \mathfrak{Gr}_0(*) = \sigma^{\text{op}}$ , where we identify  $(\mathcal{O}_Y^+)^{\text{op}}$  with  $\mathcal{O}_Y^+$ .*

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<sup>6</sup>An anti-involution on a linear stack  $\mathfrak{S}$  is a linear equivalence  $\iota: \mathfrak{S}^{\text{op}} \xrightarrow{\cong} \mathfrak{S}$  endowed with an invertible transformation  $\epsilon: \iota^2 \Rightarrow \text{id}_{\mathfrak{S}}$  such that the transformations  $\epsilon \circ \text{id}_{\iota}: \iota^3 \Rightarrow \iota$  and  $\text{id}_{\iota} \circ \epsilon: \iota \Rightarrow \iota^3$  are inverse one to each other. It is strict if  $\epsilon = \text{id}$ .

We may thus mimic the definition of  $(\mathcal{W}, *)$ -algebra and get:

**Definition 3.3.** A  $\mathcal{W}$ -algebroid with an anti-involution on  $X$ , a  $(\mathcal{W}, *)$ -algebroid for short, is a filtered  $k$ -stack  $\mathfrak{A}$  endowed with

(i)<sup>+</sup> an equivalence of graded  $\mathbb{C}[\tau^{-1}, \tau]$ -stacks  $\nu: \mathfrak{Gr}(\mathfrak{A}) \xrightarrow{\cong} (\mathcal{O}_X[\tau^{-1}, \tau])^+$ ;

(ii)<sup>+</sup> a (strict) anti-involution  $\iota$ ;

such that the triplet  $(\mathfrak{A}, \nu, \iota)$  is locally equivalent to  $(\mathfrak{W}_X, \sigma, *)$ .

A (strict) equivalence of  $(\mathcal{W}, *)$ -algebroids  $(\mathfrak{A}_1, \nu_1, \iota_1) \rightarrow (\mathfrak{A}_2, \nu_2, \iota_2)$  is a filtered  $k$ -equivalence  $\Phi: \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$  satisfying  $\iota_2 \circ \Phi^{\text{op}} = \Phi \circ \iota_1$  and  $\nu_2 \circ \mathfrak{Gr}(\Phi) = \nu_1$ .

An invertible transformation between equivalences of  $(\mathcal{W}, *)$ -algebroids is a filtered invertible transformation of functors  $\alpha: \Phi_1 \Rightarrow \Phi_2$  satisfying  $\text{id}_{\nu_2} \circ \mathfrak{Gr}(\alpha) = \text{id}_{\nu_1}$  (as natural transformations  $\nu_2 \circ \mathfrak{Gr}(\Phi_1) = \nu_1 \Rightarrow \nu_1$ ) and such that the natural transformations  $\alpha \circ \text{id}_{\iota_1}: \Phi_1 \circ \iota_1 \Rightarrow \iota_2 \circ \Phi_2^{\text{op}}$  and  $\text{id}_{\iota_2} \circ \alpha: \iota_2 \circ \Phi_2^{\text{op}} \Rightarrow \Phi_1 \circ \iota_1$  are inverse one to each other.

As the  $(\mathcal{W}, *)$ -algebras give the (holomorphic) deformation quantizations of  $T^*M$ , we may say that the  $(\mathcal{W}, *)$ -algebroids provide the (holomorphic) deformation quantizations of  $X$ . We call  $\mathfrak{W}_X$  the canonical  $(\mathcal{W}, *)$ -algebroid on  $X$ .

*Remark 3.4.*

- (i) In fact, it is possible to show that any  $k$ -stack  $\mathfrak{A}$  which is locally  $k$ -equivalent to  $\mathfrak{W}_X$  is filtered. However, contrarily to the case of  $\mathcal{W}$ -algebras (see Remark 1.3), its associated graded stack  $\mathfrak{Gr}(\mathfrak{A})$  is in general not globally equivalent to  $(\mathcal{O}_X[\tau^{-1}, \tau])^+$ . These objects are (holomorphic) deformation quantizations of gerbes in the sense of [8], and are studied in [13].
- (ii) Note that any anti-involution of  $\mathfrak{W}_X$  is isomorphic to a strict one. Hence it is not restrictive to suppose that the anti-involution  $\iota$  in (ii)<sup>+</sup> is strict. Similarly, it is not restrictive to take equalities instead of invertible natural transformations in the definition of equivalence of  $(\mathcal{W}, *)$ -algebroids (see the proof of [25, Theorem 3.3] for the microdifferential case).

If  $\mathcal{A}$  is a  $(\mathcal{W}, *)$ -algebra on  $X$ , then  $\mathcal{A}^+$  is a  $(\mathcal{W}, *)$ -algebroid with a global object. Conversely:

**Proposition 3.5.** *Let  $(\mathfrak{A}, \nu, \iota)$  be a  $(\mathcal{W}, *)$ -algebroid on  $X$  with a global object. Then there exists a  $(\mathcal{W}, *)$ -algebra  $\mathcal{A}$  on  $X$  and an equivalence of  $(\mathcal{W}, *)$ -algebroids  $\mathcal{A}^+ \approx \mathfrak{A}$ .*

Note that, if such  $\mathcal{A}$  exists, in general it is not unique. Moreover, any other  $(\mathcal{W}, *)$ -algebra on  $X$  is locally isomorphic to  $\mathcal{A}$ .

*Proof.* Let  $\mathcal{L} \in \mathfrak{A}(X)$  be a global object and set  $\mathcal{B} = \mathcal{E}nd_{\mathfrak{A}}(\mathcal{L})$ . Then  $\mathcal{B}$  is a  $\mathcal{W}$ -algebra on  $X$ , that is, it is a filtered  $k$ -algebra locally isomorphic to  $f^{-1}\mathcal{W}_{T^*M}^{\overline{\nu}}$  for any symplectic local chart  $f: X \supset U \rightarrow T^*M$ , and whose graded algebra is isomorphic to  $\mathcal{O}_X[\tau^{-1}, \tau]$ . Since there is a  $k$ -equivalence  $\mathcal{B}^+ \approx \mathfrak{A}$ , it follows that  $\iota$  defines an anti-involution on  $\mathcal{B}^+$  and, as the functor  $(\cdot)^+$  is locally full (see Section 2), one may suppose that this is locally isomorphic to  $\iota^+$ , for a locally defined anti-involution  $\iota$  on  $\mathcal{B}$ . One thus gets an open covering  $X = \bigcup_{i \in I} U_i$ , anti-involutions  $\iota_i$  on  $\mathcal{B}|_{U_i}$  and invertible operators  $P_{ij} \in \mathcal{B}^\times(U_{ij})$  satisfying  $\iota_i = \text{ad}(P_{ij}) \circ \iota_j$ ,  $\iota_i(P_{ij}) = P_{ij}$  on  $U_{ij}$  and  $P_{ij}P_{jk} = P_{ik}$  on  $U_{ijk}$ . Moreover, since  $\nu \circ \mathfrak{G}r_0(\iota) = \nu^{\text{op}}$ , the  $P_{ij}$ 's must be of order 0 and principal symbol 1. Choose  $Q_{ij} \in \mathcal{B}^\times(U_{ij})$  of order 0 and principal symbol 1 such that  $P_{ij} = Q_{ij}^2$  and  $\iota_i(Q_{ij}) = Q_{ij}$  (see [26, Lemma 5.3] for the construction of such operators in the microdifferential case). One checks easily that the  $Q_{ij}$ 's satisfy  $\iota_i \circ \text{ad}(Q_{ij}) = \text{ad}(Q_{ij}) \circ \iota_j$  on  $U_{ij}$  and  $Q_{ij}Q_{jk} = Q_{ik}$  on  $U_{ijk}$ . It follows that the  $\mathcal{W}$ -algebra  $\mathcal{A}$  defined by the 1-cocycle  $\{\text{ad}(Q_{ij})\}$  is endowed with an anti-involution, *i.e.* it is an  $(\mathcal{W}, *)$ -algebra, and that there is an equivalence of  $(\mathcal{W}, *)$ -algebroids  $\mathcal{A}^+ \approx \mathfrak{A}$ . □

#### §4. Cohomology with Values in a Stack

As for classifying  $\mathcal{W}$ -algebras one uses cohomology with values in a sheaf of groups, so to classify  $\mathcal{W}$ -algebroids we need a cohomology theory with values in a stack with group-like properties. In this section we briefly recall the definition of cohomology with values in a stack and show how to describe it explicitly by means of crossed modules. References are made to [6, 7]<sup>7</sup>. We assume that the reader is familiar with the notions of monoidal category, monoidal functor and monoidal transformation, and also with their stack counterpart. (The classical reference is [20] and a more recent one is [19]. See for example [SGA4, exposé XVIII] for the stack case.)

Let  $X$  be a topological space.

<sup>7</sup>We prefer to follow here the terminology of Baez-Lauda [*Higher-dimensional algebra. V. 2-groups*, Theory Appl. Categ. 12 (2004), 423–491], which seems to us more friendly than the classical one of *gr*-category as in *loc.cit.*

**Definition 4.1.**

- (i) A 2-group is a monoidal category  $(\mathbb{G}, \otimes, I)$  whose morphisms are all invertible and such that for each object  $P \in \mathbb{G}$  there exist an object  $Q \in \mathbb{G}$  and a morphism  $P \otimes Q \simeq I$  (and hence a morphism  $Q \otimes P \simeq I$ ). A functor (resp. a natural transformation of functors) of 2-groups is a monoidal functor (resp. a monoidal transformation of functors) between the underlying monoidal categories.
- (ii) A pre-stack (resp. stack) of 2-groups on  $X$  is a monoidal pre-stack (resp. stack)  $(\mathfrak{G}, \otimes, I)$  such that for each open subset  $U \subset X$ , the monoidal category  $(\mathfrak{G}(U), \otimes, I)$  is a 2-group. Functors (resp. natural transformations of functors) of stacks of 2-groups are monoidal functors (resp. monoidal natural transformations).

In the sequel, if there is no risk of confusion, a stack of 2-groups  $(\mathfrak{G}, \otimes, I)$  on  $X$  will be called a 2-group on  $X$  and denoted by  $\mathfrak{G}$ .

Let  $\mathcal{G}$  be a (pre-)sheaf of groups on  $X$ . We denote by  $\mathcal{G}[0]$  the discrete (pre-)stack defined by trivially enriching  $\mathcal{G}$  with identity arrows, and by  $\mathcal{G}[1]$  the stack associated to the pre-stack whose category on an open subset  $U \subset X$  has a single object  $\bullet$  and  $\text{End}(\bullet) = \mathcal{G}(U)$  as set of morphisms. Note that the Yoneda embedding identifies  $\mathcal{G}[1]$  with the stack of right  $\mathcal{G}$ -torsors. Clearly  $\mathcal{G}[0]$  is a (pre-)stack of 2-groups, while  $\mathcal{G}[1]$  defines a stack of 2-groups if and only if  $\mathcal{G}$  is commutative<sup>8</sup>.

Let  $\mathfrak{G}$  be a pre-stack of 2-groups on  $X$ . We define the 0-th cohomology group of  $X$  with values in  $\mathfrak{G}$  by

$$H^0(X; \mathfrak{G}) = \varinjlim_{\mathcal{U}} H^0(\mathcal{U}; \mathfrak{G}),$$

where  $\mathcal{U}$  ranges over open coverings of  $X$ . For an open covering  $\mathcal{U} = \{U_i\}_{i \in I}$ , the elements of  $H^0(\mathcal{U}; \mathfrak{G})$  are represented by pairs  $(\{\mathcal{P}_i\}, \{\alpha_{ij}\})$  (the 0-cocycles), where  $\mathcal{P}_i$  is an object in  $\mathfrak{G}(U_i)$  and  $\alpha_{ij}: \mathcal{P}_j \xrightarrow{\sim} \mathcal{P}_i$  is an isomorphism on double intersection  $U_{ij}$ , such that  $\alpha_{ij} \circ \alpha_{jk} = \alpha_{ik}$  on triple intersection  $U_{ijk}$ , with the relation  $(\{\mathcal{P}_i\}, \{\alpha_{ij}\})$  is equivalent to  $(\{\mathcal{P}'_i\}, \{\alpha'_{ij}\})$  if and only if there exists an isomorphism  $\delta_i: \mathcal{P}'_i \xrightarrow{\sim} \mathcal{P}_i$  compatible with  $\alpha_{ij}$  and  $\alpha'_{ij}$  on  $U_{ij}$ .

Note that, if  $\mathfrak{G}$  is a stack, then  $H^0(X; \mathfrak{G})$  is isomorphic to the group of isomorphism classes of objects in  $\mathfrak{G}(X)$ . If  $\mathcal{G}$  is a (pre-)sheaf of groups on  $X$ , then  $H^0(X; \mathcal{G}[0])$  is nothing but the usual Čech cohomology group  $H^0(X; \mathcal{G})$ .

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<sup>8</sup>Recall that for any (pre-)stack of 2-groups  $\mathfrak{G}$ , the (pre-)sheaf of groups  $\text{Aut}_{\mathfrak{G}}(I)$  is commutative.

Similarly, the 1-st cohomology (pointed) set of  $X$  with values in  $\mathfrak{G}$  is defined by

$$H^1(X; \mathfrak{G}) = \varinjlim_{\mathcal{U}} H^1(\mathcal{U}; \mathfrak{G}),$$

where  $\mathcal{U}$  ranges over open coverings of  $X$ . For an open covering  $\mathcal{U} = \{U_i\}_{i \in I}$ , the elements of  $H^1(\mathcal{U}; \mathfrak{G})$  are given by pairs  $(\{\mathcal{P}_{ij}\}, \{\alpha_{ijk}\})$  (the 1-cocycles), where  $\mathcal{P}_{ij}$  is an object in  $\mathfrak{G}(U_{ij})$  and  $\alpha_{ijk}: \mathcal{P}_{ij} \otimes \mathcal{P}_{jk} \xrightarrow{\sim} \mathcal{P}_{ik}$  is an isomorphism on  $U_{ijk}$  such that the diagram on quadruple intersection  $U_{ijkl}$

$$\begin{array}{ccc} \mathcal{P}_{ij} \otimes \mathcal{P}_{jk} \otimes \mathcal{P}_{kl} & \xrightarrow{\alpha_{ijk} \otimes \text{id}_{\mathcal{P}_{kl}}} & \mathcal{P}_{ik} \otimes \mathcal{P}_{kl} \\ \downarrow \text{id}_{\mathcal{P}_{ij}} \otimes \alpha_{jkl} & & \downarrow \alpha_{ikl} \\ \mathcal{P}_{ij} \otimes \mathcal{P}_{jl} & \xrightarrow{\alpha_{ijl}} & \mathcal{P}_{il} \end{array}$$

commutes. The 1-cocycles  $(\{\mathcal{P}_{ij}\}, \{\alpha_{ijk}\})$  and  $(\{\mathcal{P}'_{ij}\}, \{\alpha'_{ijk}\})$  are equivalent if and only if there exists a pair  $(\{\mathcal{Q}_i\}, \{\delta_{ij}\})$ , with  $\mathcal{Q}_i$  an object of  $\mathfrak{G}(U_i)$  and  $\delta_{ij}: \mathcal{P}'_{ij} \otimes \mathcal{Q}_j \xrightarrow{\sim} \mathcal{Q}_i \otimes \mathcal{P}_{ij}$  an isomorphism on  $U_{ij}$  such that the diagram on  $U_{ijk}$

$$\begin{array}{ccccc} \mathcal{P}'_{ij} \otimes \mathcal{P}'_{jk} \otimes \mathcal{Q}_k & \xrightarrow{\text{id}_{\mathcal{P}'_{ij}} \otimes \delta_{jk}} & \mathcal{P}'_{ij} \otimes \mathcal{Q}_j \otimes \mathcal{P}_{jk} & \xrightarrow{\delta_{ij} \otimes \text{id}_{\mathcal{P}_{jk}}} & \mathcal{Q}_i \otimes \mathcal{P}_{ij} \otimes \mathcal{P}_{jk} \\ \downarrow \alpha'_{ijk} \otimes \text{id}_{\mathcal{Q}_k} & & & & \downarrow \text{id}_{\mathcal{Q}_i} \otimes \alpha_{ijk} \\ \mathcal{P}'_{ik} \otimes \mathcal{Q}_k & \xrightarrow{\delta_{ik}} & & & \mathcal{Q}_i \otimes \mathcal{P}_{ik} \end{array}$$

commutes.

Note that, if  $\mathfrak{G}$  is a stack, then  $H^1(X; \mathfrak{G})$  classifies the right  $\mathfrak{G}$ -torsors on  $X$  (see for example [6]). As before, if  $\mathcal{G}$  is a (pre-)sheaf of groups on  $X$ , then  $H^1(X; \mathcal{G}[0])$  gives the usual Čech cohomology  $H^1(X; \mathcal{G})$ .

**Definition 4.2.** A crossed module on  $X$  is a complex<sup>9</sup> of sheaves of groups  $\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0$  endowed with a left action of  $\mathcal{G}^0$  on  $\mathcal{G}^{-1}$  such that, for any local sections  $g \in \mathcal{G}^0$  and  $h, h' \in \mathcal{G}^{-1}$ , one has

$$d(g h) = \text{ad}(g)(d(h)) \quad \text{and} \quad d(h') h = \text{ad}(h')(h).$$

A morphism of crossed modules is a morphism of complexes compatible with the actions in the natural way.

To each crossed module  $\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0$  there is an associated 2-group on  $X$ , which we denote by  $[\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0]$ , defined as follows: it is the stack associated

<sup>9</sup>Here we use the convention as in [7, SGA4, exposé XVIII] for which  $\mathcal{G}^i$  is in  $i$ -th degree.



to the pre-stack whose objects on an open subset  $U \subset X$  are the sections  $g \in \mathcal{G}^0(U)$  and whose morphisms  $g \rightarrow g'$  are given by sections  $h \in \mathcal{G}^{-1}(U)$  satisfying  $g' = d(h)g$ . The 2-group structure for objects is  $g \otimes g' = gg'$  and for morphisms is given by the rule

$$(g_1 \xrightarrow{h_1} g'_1) \otimes (g_2 \xrightarrow{h_2} g'_2) = g_1 g_2 \xrightarrow{h_1 g_1 h_2} g'_1 g'_2.$$

In a similar way, a morphism of crossed modules induces a functor of the corresponding 2-groups.

*Remark 4.3.* In fact, it is true that any 2-group on  $X$  comes from a crossed module. However, this result is not of practical use. We refer to [SGA4, exposé XVIII] for the proof of this fact in the commutative case and to [9] for the non commutative case on  $X = \text{pt}$ .

By definition, an object  $\mathcal{P}$  of  $[\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0]$  on an open subset  $U \subset X$  is described by an open covering  $U = \bigcup_{i \in I} U_i$ , sections  $\{g_i\} \in \mathcal{G}^0(U_i)$  and sections  $\{h_{ij}\} \in \mathcal{G}^{-1}(U_{ij})$ , subject to the relations  $g_i = d(h_{ij})g_j$  on  $U_{ij}$  and  $h_{ij}h_{jk} = h_{ik}$  on  $U_{ijk}$ . In particular, if  $\mathcal{G}$  is a sheaf of groups, then  $[1 \rightarrow \mathcal{G}]$  is naturally identified with  $\mathcal{G}[0]$ . If moreover  $\mathcal{G}$  is commutative, the complex  $\mathcal{G} \rightarrow 1$  is a crossed module and the associated 2-group is identified with  $\mathcal{G}[1]$ . We denote by  $\mathbf{1}$  the trivial 2-group  $1[0] = 1[1]$  with (one object and) one morphism.

Let  $\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0$  be a crossed module on  $X$  and consider the exact sequence of groups

$$(4.1) \quad 1 \rightarrow \ker d \rightarrow \mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0 \rightarrow \text{coker } d \rightarrow 1.$$

**Proposition 4.4.** *The sequence (4.1) induces an extension of 2-groups on  $X$*

$$\mathbf{1} \rightarrow \ker d[1] \xrightarrow{i} [\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0] \xrightarrow{\pi} \text{coker } d[0] \rightarrow \mathbf{1},$$

that is,  $\pi$  is surjective on objects and  $i$  factors through an equivalence with the kernel of  $\pi$  (the full sub-2-group of  $[\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0]$  whose objects satisfy  $\pi(\mathcal{P}) = 1 \in \text{coker } d$ ).

In particular, if  $d$  is surjective (resp. injective), then there is a 2-group equivalence  $\ker d[1] \approx [\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0]$  (resp.  $[\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0] \approx \text{coker } d[0]$ ).

*Proof.* Since  $\ker d$  is commutative, the sequence (4.1) induces a (non-exact) sequence of crossed modules and one then takes the associated sequence of 2-groups

$$[\ker d \rightarrow 1] \xrightarrow{i} [\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0] \xrightarrow{\pi} [1 \rightarrow \text{coker } d].$$

In particular, the functor  $i$  sends the object  $\bullet$  of  $\ker d[1]$  to the unit object  $1 \in \mathcal{G}^0$ . It is fully faithful and its essential image is identified with the sub-2-group of  $[\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0]$  of objects locally isomorphic to 1. Note that  $\text{coker } d$  is identified with the sheaf associated to the pre-sheaf whose sections on an open subset  $U \subset X$  are isomorphism classes of objects in the 2-group  $[\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0](U)$ , and  $\pi$  with the functor which sends an object  $\mathcal{P} \in [\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0]$  to its isomorphism class. Hence  $\pi$  is surjective on objects and  $\pi(\mathcal{P}) = 1 \in \text{coker } d$  if and only if  $\mathcal{P}$  is locally isomorphic to  $1 \in \mathcal{G}^0$ . □

Let  $\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0$  be a crossed module on  $X$ . Then the cohomology of  $X$  with values in the 2-group  $[\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0]$  admits a very explicit description, which we recall below. This is usually referred as the (hyper-)cohomology of  $X$  with values in  $\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0$  and it will be denoted by  $H^i(X; \mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0)$ ,  $i = 0, 1$ .

Up to a refinement of the open covering  $\mathcal{U} = \{U_i\}_{i \in I}$  of  $X$ , the 0-cocycles on  $\mathcal{U}$  with values in  $[\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0]$  may be described by pairs  $(\{g_i\}, \{h_{ij}\})$ , where  $g_i \in \mathcal{G}^0(U_i)$  and  $h_{ij} \in \mathcal{G}^{-1}(U_{ij})$  are sections satisfying the relations

$$\begin{cases} g_i = d(h_{ij})g_j & \text{in } \mathcal{G}^0(U_{ij}) \\ h_{ij}h_{jk} = h_{ik} & \text{in } \mathcal{G}^{-1}(U_{ijk}), \end{cases}$$

and  $(\{g_i\}, \{h_{ij}\})$  is equivalent to  $(\{g'_i\}, \{h'_{ij}\})$  if and only if there exist sections  $\{k_i\} \in \mathcal{G}^{-1}(U_i)$  such that the following relations hold

$$\begin{cases} g'_i = d(k_i)g_i & \text{in } \mathcal{G}^0(U_i) \\ h'_{ij}k_j = k_i h_{ij} & \text{in } \mathcal{G}^{-1}(U_{ij}). \end{cases}$$

The same description for 1-cocycles needs some care, since one has to consider open coverings for any double intersection  $U_{ij}$ . In other words, one has to replace coverings by hyper-coverings. Indices become thus very cumbersome, and we will not write them explicitly<sup>10</sup>. Hence the 1-cocycles on  $\mathcal{U}$  with values in  $[\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0]$  may be described by pairs  $(\{g_{ij}\}, \{h_{ijk}\})$ , where  $g_{ij} \in \mathcal{G}^0(U_{ij})$  and  $h_{ijk} \in \mathcal{G}^{-1}(U_{ijk})$  are sections satisfying the relations

$$\begin{cases} g_{ij}g_{jk} = d(h_{ijk})g_{ik} & \text{in } \mathcal{G}^0(U_{ijk}) \\ h_{ijk}h_{ikl} = {}^{g_{ij}}h_{jkl}h_{ijl} & \text{in } \mathcal{G}^{-1}(U_{ijkl}). \end{cases}$$

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<sup>10</sup>Recall that, for a paracompact space, usual coverings are cofinal among hyper-coverings.

Moreover,  $(\{g_{ij}\}, \{h_{ijk}\})$  is equivalent to  $(\{g'_{ij}\}, \{h'_{ijk}\})$  if and only if there exists a pair  $(\{l_i\}, \{k_{ij}\})$ , with  $k_{ij} \in \mathcal{G}^{-1}(U_{ij})$  and  $l_i \in \mathcal{G}^0(U_i)$  such that

$$\begin{cases} g'_{ij}l_j = d(k_{ij})l_i g_{ij} & \text{in } \mathcal{G}^0(U_{ij}) \\ h'_{ijk}k_{ik} = g'_{ij}k_{jk}k_{ij}^{l_i}h_{ijk} & \text{in } \mathcal{G}^{-1}(U_{ijk}). \end{cases}$$

Note that, if  $\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0$  is a complex of sheaves of commutative groups (with trivial action) then  $H^i(X; \mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0)$  is the usual Čech hyper-cohomology of  $X$  with values in  $\mathcal{G}^{-1} \xrightarrow{d} \mathcal{G}^0$ . In particular, if  $\mathcal{G}$  is a sheaf of commutative groups, then there is an isomorphism

$$(4.2) \quad H^i(X; \mathcal{G}[1]) \simeq H^{i+1}(X; \mathcal{G}), \quad i = 0, 1.$$

**§5. Classification of  $\mathcal{W}$ -Algebroids**

Let  $X$  be a complex symplectic manifold and  $\mathfrak{W}_X$  the canonical  $(\mathcal{W}, *)$ -algebroid on  $X$ . We denote by  $\mathfrak{Aut}_{(\mathcal{W}, *)}(\mathfrak{W}_X)$  the stack of 2-groups of auto-equivalences of  $\mathfrak{W}_X$  as  $(\mathcal{W}, *)$ -algebroid. Its objects are filtered  $k$ -equivalences  $\Phi: \mathfrak{W}_X \rightarrow \mathfrak{W}_X$  satisfying  $\sigma \circ \mathfrak{G}\mathfrak{r}(\Phi) = \sigma$  and  $* \circ \Phi^{\text{op}} = \Phi \circ *$ , and its morphisms are the filtered invertible transformations  $\alpha: \Phi_1 \Rightarrow \Phi_2$  satisfying  $\text{id}_\sigma \circ \mathfrak{G}\mathfrak{r}(\alpha) = \text{id}_\sigma$  and such that the natural transformations  $\alpha \circ \text{id}_* : \Phi_1 \circ * \Rightarrow * \circ \Phi_2^{\text{op}}$  and  $\text{id}_* \circ \alpha : * \circ \Phi_2^{\text{op}} \Rightarrow \Phi_1 \circ *$  are inverse one to each other.

Let  $\mathfrak{A}$  be a  $(\mathcal{W}, *)$ -algebroid. By definition, there exists an open covering  $X = \bigcup_{i \in I} U_i$  such that  $\mathfrak{A}|_{U_i}$  is equivalent to  $\mathfrak{W}_X|_{U_i}$  as  $(\mathcal{W}, *)$ -algebroids. Let  $\Phi_i: \mathfrak{A}|_{U_i} \rightarrow \mathfrak{W}_X|_{U_i}$  and  $\Psi_i: \mathfrak{W}_X|_{U_i} \rightarrow \mathfrak{A}|_{U_i}$  be quasi-inverse to each other. On  $U_{ij}$  there are  $(\mathcal{W}, *)$ -algebroid equivalences  $\Phi_{ij} = \Phi_i \circ \Psi_j: \mathfrak{W}_X|_{U_{ij}} \rightarrow \mathfrak{W}_X|_{U_{ij}}$ , and on  $U_{ijk}$  there are invertible transformations of  $(\mathcal{W}, *)$ -algebroid equivalences  $\alpha_{ijk}: \Phi_{ij} \circ \Phi_{jk} \Rightarrow \Phi_{ik}$ . On  $U_{ijkl}$  the following diagram commutes

$$(5.1) \quad \begin{array}{ccc} \Phi_{ij} \circ \Phi_{jk} \circ \Phi_{kl} & \xrightarrow{\alpha_{ijk} \circ \text{id}_{\Phi_{kl}}} & \Phi_{ik} \circ \Phi_{kl} \\ \Downarrow \text{id}_{\Phi_{ij}} \circ \alpha_{jkl} & & \Downarrow \alpha_{ikl} \\ \Phi_{ij} \circ \Phi_{jl} & \xrightarrow{\alpha_{ijl}} & \Phi_{il}. \end{array}$$

It follows that the  $(\mathcal{W}, *)$ -algebroids are described by 1-cocycles  $(\Phi_{ij}, \alpha_{ijk})$  with values in  $\mathfrak{Aut}_{(\mathcal{W}, *)}(\mathfrak{W}_X)$ . In a similar way one describes the equivalences between  $(\mathcal{W}, *)$ -algebroids. Hence we have:

**Proposition 5.1.** *The set of the equivalence classes of  $(\mathcal{W}, *)$ -algebroids on  $X$ , pointed by the class of  $\mathfrak{W}_X$ , is isomorphic to  $H^1(X; \mathfrak{Aut}_{(\mathcal{W}, *)}(\mathfrak{W}_X))$ .*

Let us briefly recall how to describe more explicitly the 1-cocycle  $(\Phi_{ij}, \alpha_{ijk})$  attached to a  $(\mathcal{W}, *)$ -algebroid  $\mathfrak{A}$ . We follow here [26, 12]. By definition,  $\mathfrak{W}_X$  is locally equivalent to  $(f^{-1}\mathcal{W}_{T^*M}^{\sqrt{v}})^+$  for any symplectic local chart  $f: X \supset U \rightarrow T^*M$ . Hence, up to a refinement of the open covering  $X = \bigcup_{i \in I} U_i$ , one may suppose that there exists a symplectic embedding  $f_i: U_i \rightarrow T^*M$  such that  $\mathfrak{W}_X|_{U_i}$  is equivalent to  $(\mathcal{W}_{U_i}^{\sqrt{v}})^+ = (f_i^{-1}\mathcal{W}_{T^*M}^{\sqrt{v}})^+$ . Since the functor  $(\cdot)^+$  is locally full (see Section 2), up to shrinking again the open covering, one may suppose that  $\Phi_{ij}$  is isomorphic to  $\varphi_{ij}^+$ , for a filtered  $k$ -algebra isomorphism  $\varphi_{ij}: \mathcal{W}_{U_i}^{\sqrt{v}} \rightarrow \mathcal{W}_{U_j}^{\sqrt{v}}$  on  $U_{ij}$ . From  $\sigma \circ \mathfrak{G}\mathfrak{r}(\Phi_{ij}) = \sigma$  and  $* \circ \Phi_{ij}^{\text{op}} = \Phi_{ij} \circ *$  it follows that  $\varphi_{ij}$  is a  $(\mathcal{W}, *)$ -algebra isomorphism. On  $U_{ijk}$  the invertible transformation  $\alpha_{ijk}: \varphi_{ij}^+ \circ \varphi_{jk}^+ \Rightarrow \varphi_{ik}^+$  is (locally) given by an invertible operator  $P_{ijk} \in \mathcal{W}_{U_{ijk}}^{\sqrt{v}}$  satisfying

$$\varphi_{ij} \circ \varphi_{jk} = \text{ad}(P_{ijk}) \circ \varphi_{ik}.$$

Since  $(\text{id}_* \circ \alpha_{ijk}) \circ (\alpha_{ijk} \circ \text{id}_*) = \text{id}$  and  $\text{id}_\sigma \circ \mathfrak{G}\mathfrak{r}(\alpha_{ijk}) = \text{id}_\sigma$ , it follows that  $P_{ijk}$  must satisfies  $P_{ijk}^* P_{ijk} = 1$  (hence it is of order 0) and  $\sigma_0(P_{ijk}) = 1$ , so that it is a section of  $\mathcal{W}_{U_i}^{\sqrt{v},*}$ . Finally, on  $U_{ijkl}$  the diagram (5.1) corresponds to the equality

$$P_{ijk} P_{ikl} = \varphi_{ij}(P_{jkl}) P_{ijl}.$$

The datum of

$$(\{f_i\}, \{\varphi_{ij}\}, \{P_{ijk}\})$$

as above is enough to reconstruct the  $(\mathcal{W}, *)$ -algebroid  $\mathfrak{A}$  (up to equivalence).

In the particular case of  $X = T^*M$ , one has  $f_i = \text{id}$ . We thus proved:

**Lemma 5.2.** *The assignment  $\varphi \mapsto \varphi^+$  defines an equivalence of 2-groups on  $T^*M$*

$$\left[ \mathcal{W}_{T^*M}^{\sqrt{v},*} \xrightarrow{\text{ad}} \text{Aut}_{(\mathcal{W},*)}(\mathcal{W}_{T^*M}^{\sqrt{v}}) \right] \xrightarrow{\approx} \mathfrak{Aut}_{(\mathcal{W},*)}((\mathcal{W}_{T^*M}^{\sqrt{v}})^+)$$

We are now ready to prove our main result.

**Theorem 5.3.** *The set of the equivalence classes of  $(\mathcal{W}, *)$ -algebroids on  $X$  is isomorphic to  $H^2(X; k_X^*)$ .*

Note that  $H^2(X; k_X^*)$  has a group structure, contrarily to the set of equivalence classes of  $(\mathcal{W}, *)$ -algebroids. One may thus state the Theorem 5.3 by saying that there is a free and transitive action of  $H^2(X; k_X^*)$ .

*Proof.* Consider the natural functor of 2-groups

$$i: k_X^*[1] \longrightarrow \mathfrak{Aut}_{(\mathcal{W},*)}(\mathfrak{W}_X)$$

induced by the functor of pre-stacks which sends the unique object  $\bullet$  to the identity functor  $\text{id}_{\mathfrak{W}_X}$ . At any point  $p \in X$ , we may find a symplectic local chart  $f: X \supset U \rightarrow T^*M$  around  $p$ , such that  $\mathfrak{W}_X|_U$  is equivalent to  $(\mathcal{W}_U^{\sqrt{v}})^+ = (f^{-1}\mathcal{W}_{T^*M}^{\sqrt{v}})^+$  as  $(\mathcal{W}, *)$ -algebroids. We thus have a chain of equivalences of 2-groups

$$\begin{aligned} \mathfrak{Aut}_{(\mathcal{W},*)}(\mathfrak{W}_X)|_U &\approx \mathfrak{Aut}_{(\mathcal{W},*)}(\mathfrak{W}_X|_U) \\ &\approx \mathfrak{Aut}_{(\mathcal{W},*)}((\mathcal{W}_U^{\sqrt{v}})^+) \\ &\approx \left[ \mathcal{W}_U^{\sqrt{v},*} \xrightarrow{\text{ad}} \mathcal{A}ut_{(\mathcal{W},*)}(\mathcal{W}_U^{\sqrt{v}}) \right], \end{aligned}$$

(the last one follows from Lemma 5.2) and hence the functor  $i$  restricts on  $U$  to

$$i|_U: k_U^*[1] \longrightarrow \left[ \mathcal{W}_U^{\sqrt{v},*} \xrightarrow{\text{ad}} \mathcal{A}ut_{(\mathcal{W},*)}(\mathcal{W}_U^{\sqrt{v}}) \right].$$

By Lemma 1.5 and Proposition 4.4, this is an equivalence of 2-groups. It follows that the functor  $i$  is locally, and hence globally, an equivalence. We thus get a chain of isomorphisms of pointed sets

$$H^1(X; \mathfrak{Aut}_{(\mathcal{W},*)}(\mathfrak{W}_X)) \simeq H^1(X; k_X^*[1]) \simeq H^2(X; k_X^*),$$

where the latter follows by (4.2). □

Consider the exponential sequence<sup>11</sup>

$$0 \rightarrow \mathbb{Z} \rightarrow k(0) \rightarrow k(0)^\times \rightarrow 1.$$

It induces an isomorphism of groups  $k(0)^{\text{odd}} \simeq k^*$ , where  $k(0)^{\text{odd}} = \{s(\tau) \in k(0); s(\tau) + s(-\tau) = 0\}$ . In the formal case, *i.e.* dropping the estimates (1.1), this is nothing but the group  $\mathbb{C}[[\hbar]]^{\text{odd}}$  of odd formal power series in  $\hbar = \tau^{-1}$ . Since Theorems 3.1 and 5.3 are still valid in the formal case, it follows that the formal  $(\mathcal{W}, *)$ -algebroids on  $X$  are classified by  $H^2(X; \mathbb{C}[[\hbar]]_X^{\text{odd}}) \simeq H^2(X; \mathbb{C}_X) [[\hbar]]^{\text{odd}}$ .

Note that the inclusion  $k(0)^{\text{odd}} \hookrightarrow \mathbb{C}[[\hbar]]^{\text{odd}}$  defines a split exact sequence of  $\mathbb{C}$ -modules. Therefore  $H^2(X; k(0)_X^{\text{odd}})$  is a direct summand of  $H^2(X; \mathbb{C}_X) [[\hbar]]^{\text{odd}}$ , and it vanishes whenever  $H^2(X; \mathbb{C}_X) = 0$ . Since  $H^2(X; k_X^*) \simeq H^2(X; k(0)_X^{\text{odd}})$ , we have:

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<sup>11</sup>For any WKB operator  $P$  of order  $\leq 0$ , the operator  $\exp P$  is well defined, since it satisfies the estimates (1.1). This follows, for example, by using the quasi-norm introduced by Boutet de Monvel-Krée in [5].

**Corollary 5.4.** *Suppose that  $H^2(X; \mathbb{C}_X) = 0$ . Then  $\mathfrak{W}_X$  is the unique (up to equivalence)  $(\mathcal{W}, *)$ -algebroid on  $X$ .*

*Remark 5.5.*

- (i) By taking  $H^0$  instead of  $H^1$  in the proof of Theorem 5.3, we get an isomorphism between  $H^1(X; k_X^*)$  and the group of isomorphism classes of  $(\mathcal{W}, *)$ -algebroid autoequivalences of  $\mathfrak{W}_X$ .
- (ii) If  $X = T^*M$  for a complex manifold  $M$ , the coboundary map

$$\delta: H^1(T^*M; \mathcal{W}_{T^*M}^{\sqrt{v},*}/k_{T^*M}^*) \rightarrow H^2(T^*M; k_{T^*M}^*)$$

associated to the exact sequence (1.2), may be interpreted as the map which sends the class  $[\mathcal{A}]$  of a  $(\mathcal{W}, *)$ -algebra to the class  $[\mathcal{A}^+]$  of the corresponding  $(\mathcal{W}, *)$ -algebroid. Note that the  $(\mathcal{W}, *)$ -algebras as in Example 1.4 are sent by  $\delta$  to the unit element.

- (iii) One may show that Theorem 5.3 holds in the framework of real symplectic manifolds, replacing  $(\mathcal{W}, *)$ -algebras with (sheaves of) deformation quantization algebras endowed with an anti-involution, and  $k$  by  $\mathbb{C}[[\hbar]]$ . This follows from three well-known facts (see for example [10, 4]): the existence of a deformation quantization algebra endowed with an anti-involution (and, *a fortiori*, of a deformation quantization algebroid endowed with an anti-involution); that any two such algebras are locally isomorphic; the existence of an exact sequence similar to (1.2). However, in that case, any deformation quantization algebroid has always a global object and all global objects give rise to isomorphic deformation quantization algebras (see [10]). It follows that in the real case there are no distinctions between the deformation quantizations given by sheaves of algebras and those provided by algebroid stacks. In particular, one recovers that the deformation quantization algebras with an anti-involution on a real symplectic manifold  $X$  are classified by  $H^2(X; \mathbb{C}_X)[[\hbar]]^{\text{odd}}$ .
- (iv) Using the same techniques, in [25] it is shown that on a complex contact manifold the stack defined by Kashiwara in [16] is the unique (up to equivalence) quantization endowed with an anti-involution.

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