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Symmetric Crystals for $\mathfrak{gl}_\infty$

Dedicated to Professor Heisuke Hironaka on the occasion of his seventy-seventh birthday

By

Naoya Enomoto* and Masaki Kashiwara**

Abstract

In the preceding paper, we formulated a conjecture on the relations between certain classes of irreducible representations of affine Hecke algebras of type B and symmetric crystals for $\mathfrak{gl}_\infty$. In the present paper, we prove the existence of the symmetric crystal and the global basis for $\mathfrak{gl}_\infty$.

§1. Introduction

Lascoux-Leclerc-Thibon ([LLT]) conjectured the relations between the representations of Hecke algebras of *type A* and the crystal bases of the affine Lie algebras of type A. Then, S. Ariki ([A]) observed that it should be understood in the setting of affine Hecke algebras and proved the LLT conjecture in a more general framework. Recently, we presented the notion of symmetric crystals and conjectured that certain classes of irreducible representations of the affine Hecke algebras of *type B* are described by symmetric crystals for $\mathfrak{gl}_\infty$ ([EK]).

The purpose of the present paper is to prove the existence of symmetric crystals in the case of $\mathfrak{gl}_\infty$.

Let us recall the Lascoux-Leclerc-Thibon-Ariki theory. Let $\mathrm{H}_n^{\mathrm{A}}$ be the affine Hecke algebra of type A of degree n. Let $\mathrm{K}_n^{\mathrm{A}}$ be the Grothendieck group

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*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606–8502, Japan.
e-mail: henon@kurims.kyoto-u.ac.jp
**e-mail: masaki@kurims.kyoto-u.ac.jp

of the abelian category of finite-dimensional $\mathrm{H}_n^{\mathrm{A}}$-modules, and $\mathrm{K}^{\mathrm{A}} = \oplus_{n \geqslant 0} \mathrm{K}_n^{\mathrm{A}}$. Then it has a structure of Hopf algebra by the restriction and the induction. The set $I = \mathbb{C}^*$ may be regarded as a Dynkin diagram with I as the set of vertices and with edges between $a \in I$ and ap_1^2. Here p_1 is the parameter of the affine Hecke algebra usually denoted by q. Let $\mathfrak{g}_I$ be the associated Lie algebra, and $\mathfrak{g}_I^-$ the unipotent Lie subalgebra. Let U_I be the group associated to $\mathfrak{g}_I^-$. Hence $\mathfrak{g}_I$ is isomorphic to a direct sum of copies of $A_{\ell-1}^{(1)}$ if p_1^2 is a primitive ℓ-th root of unity and to a direct sum of copies of $\mathfrak{gl}_\infty$ if p_1 has an infinite order. Then $\mathbb{C} \otimes \mathrm{K}^{\mathrm{A}}$ is isomorphic to the algebra $\mathscr{O}(U_I)$ of regular functions on U_I. Let $U_q(\mathfrak{g}_I)$ be the associated quantized enveloping algebra. Then $U_q^-(\mathfrak{g}_I)$ has an upper global basis $\{\mathrm{G}^{\mathrm{up}}(b)\}_{b \in \mathrm{B}(\infty)}$. By specializing $\bigoplus \mathbb{C}[q, q^{-1}]\mathrm{G}^{\mathrm{up}}(b)$ at $q = 1$, we obtain $\mathscr{O}(U_I)$. Then the LLTA-theory says that the elements associated to irreducible H^{A}-modules corresponds to the image of the upper global basis.

In [EK], we gave analogous conjectures for affine Hecke algebras of type B. In the type B case, we have to replace $U_q^-(\mathfrak{g}_I)$ and its upper global basis with symmetric crystals (see § 2.3). It is roughly stated as follows. Let $\mathrm{H}_n^{\mathrm{B}}$ be the affine Hecke algebra of type B of degree n. Let $\mathrm{K}_n^{\mathrm{B}}$ be the Grothendieck group of the abelian category of finite-dimensional modules over $\mathrm{H}_n^{\mathrm{B}}$, and $\mathrm{K}^{\mathrm{B}} = \oplus_{n \geqslant 0} \mathrm{K}_n^{\mathrm{B}}$. Then K^{B} has a structure of a Hopf bimodule over K^{A}. The group U_I has the anti-involution θ induced by the involution $a \mapsto a^{-1}$ of $I = \mathbb{C}^*$. Let U_I^θ be the θ-fixed point set of U_I. Then $\mathscr{O}(U_I^\theta)$ is a quotient ring of $\mathscr{O}(U_I)$. The action of $\mathscr{O}(U_I) \simeq \mathbb{C} \otimes \mathrm{K}^{\mathrm{A}}$ on $\mathbb{C} \otimes \mathrm{K}^{\mathrm{B}}$, in fact, descends to the action of $\mathscr{O}(U_I^\theta)$.

We introduce $V_\theta(\lambda)$ (see § 2.3), a kind of the q-analogue of $\mathscr{O}(U_I^\theta)$. The conjecture in [EK] is then:

(i) $V_\theta(\lambda)$ has a crystal basis and a global basis.

(ii) K^{B} is isomorphic to a specialization of $V_\theta(\lambda)$ at $q = 1$ as an $\mathscr{O}(U_I)$-module, and the irreducible representations correspond to the upper global basis of $V_\theta(\lambda)$ at $q = 1$.

Remark. In [KM], Miemietz and the second author gave an analogous conjecture for the affine Hecke algebras of type D.

In the present paper, we prove that $V_\theta(\lambda)$ has a crystal basis and a global basis for $\mathfrak{g} = \mathfrak{gl}_\infty$ and $\lambda = 0$.

More precisely, let $I = \mathbb{Z}_{\mathrm{odd}}$ be the set of odd integers. Let α_i $(i \in I)$ be the simple roots with

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let θ be the involution of I given by $\theta(i) = -i$. Let $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$ be the algebra over $\mathbf{K}{:=}\mathbb{Q}(q)$ generated by E_i, F_i, and invertible elements T_i ($i \in I$) satisfying the following defining relations:

(i) the T_i's commute with each other,

(ii) $T_{\theta(i)} = T_i$ for any i,

(iii) $T_i E_j T_i^{-1} = q^{(\alpha_i+\alpha_{\theta(i)}, \alpha_j)} E_j$ and $T_i F_j T_i^{-1} = q^{(\alpha_i+\alpha_{\theta(i)}, -\alpha_j)} F_j$ for $i, j \in I$,

(iv) $E_i F_j = q^{-(\alpha_i, \alpha_j)} F_j E_i + (\delta_{i,j} + \delta_{\theta(i),j} T_i)$ for $i, j \in I$,

(v) the E_i's and the F_i's satisfy the Serre relations (see Definition 2.1 (4)).

Then there exists a unique irreducible $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$-module $V_\theta(0)$ with a generator ϕ satisfying $E_i\phi = 0$ and $T_i\phi = \phi$ (Proposition 2.11). We define the endomorphisms $\widetilde{E}_i$ and $\widetilde{F}_i$ of $V_\theta(0)$ by

$$\widetilde{E}_i a = \sum_{n \geqslant 1} F_i^{(n-1)} a_n, \quad \widetilde{F}_i a = \sum_{n \geqslant 0} f_i^{(n+1)} a_n,$$

when writing

$$a = \sum_{n \geqslant 0} F_i^{(n)} a_n \quad \text{with } E_i a_n = 0.$$

Here $F_i^{(n)} = F_i^n/[n]!$ is the divided power. Let $\mathbf{A}_0$ be the ring of functions $a \in \mathbf{K}$ which do not have a pole at $q = 0$. Let $L_\theta(0)$ be the $\mathbf{A}_0$-submodule of $V_\theta(0)$ generated by the elements $\widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell}\phi$ ($\ell \geqslant 0$, $i_1, \ldots, i_\ell \in I$). Let $\mathrm{B}_\theta(0)$ be the subset of $L_\theta(0)/qL_\theta(0)$ consisting of the $\widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell}\phi$'s. In this paper, we prove the following theorem.

Theorem (Theorem 4.15)**.**

(i) $\widetilde{F}_i L_\theta(0) \subset L_\theta(0)$ *and* $\widetilde{E}_i L_\theta(0) \subset L_\theta(0)$,

(ii) $\mathrm{B}_\theta(0)$ *is a basis of* $L_\theta(0)/qL_\theta(0)$,

(iii) $\widetilde{F}_i \mathrm{B}_\theta(0) \subset \mathrm{B}_\theta(0)$, *and* $\widetilde{E}_i \mathrm{B}_\theta(0) \subset \mathrm{B}_\theta(0) \sqcup \{0\}$,

(iv) $\widetilde{F}_i\widetilde{E}_i(b) = b$ *for any* $b \in \mathrm{B}_\theta(0)$ *such that* $\widetilde{E}_i b \neq 0$, *and* $\widetilde{E}_i\widetilde{F}_i(b) = b$ *for any* $b \in \mathrm{B}_\theta(0)$.

By this theorem, $\mathrm{B}_\theta(0)$ has a similar structure to the crystal structure. Namely, we have operators $\widetilde{F}_i\colon \mathrm{B}_\theta(0) \to \mathrm{B}_\theta(0)$ and $\widetilde{E}_i\colon \mathrm{B}_\theta(0) \to \mathrm{B}_\theta(0) \sqcup \{0\}$, which satisfy (iv). Moreover $\varepsilon_i(b) := \max\left\{n \in \mathbb{Z}_{\geqslant 0} \mid \widetilde{E}_i^n b \in \mathrm{B}_\theta(0)\right\}$ is finite. We call it the *symmetric crystal* associated with (I, θ). Contrary to the usual crystal case, $\widetilde{E}_{\theta(i)}b$ may coincide with $\widetilde{E}_i b$ in the symmetric crystal case.

Let $-$ be the bar operator of $V_\theta(0)$. Namely, $-$ is a unique endomorphism of $V_\theta(0)$ such that $\overline{\phi} = \phi$, $\overline{av} = \bar{a}\bar{v}$ and $\overline{F_i v} = F_i\bar{v}$ for $a \in \mathbf{K}$ and $v \in V_\theta(0)$. Here $\bar{a}(q) = a(q^{-1})$. Let $V_\theta(0)_{\mathbf{A}}$ be the smallest submodule of $V_\theta(0)$ over $\mathbf{A} := \mathbb{Q}[q, q^{-1}]$ such that it contains ϕ and is stable by the $F_i^{(n)}$'s.

Then we prove the existence of global basis:

Theorem (Theorem 5.5)**.**

(i) *For any* $b \in \mathrm{B}_\theta(0)$, *there exists a unique* $\mathrm{G}_\theta^{\mathrm{low}}(b) \in V_\theta(0)_{\mathbf{A}} \cap L_\theta(0)$ *such that* $\overline{\mathrm{G}_\theta^{\mathrm{low}}(b)} = \mathrm{G}_\theta^{\mathrm{low}}(b)$ *and* $b = \mathrm{G}_\theta^{\mathrm{low}}(b) \mod qL_\theta(0)$,

(ii) $\{\mathrm{G}_\theta^{\mathrm{low}}(b)\}_{b\in\mathrm{B}_\theta(0)}$ *is a basis of the* $\mathbf{A}_0$*-module* $L_\theta(0)$, *the* $\mathbf{A}$*-module* $V_\theta(0)_{\mathbf{A}}$ *and the* $\mathbf{K}$*-vector space* $V_\theta(0)$.

We call $\mathrm{G}_\theta^{\mathrm{low}}(b)$ the *lower global basis*. The $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$-module $V_\theta(0)$ has a unique symmetric bilinear form $(\bullet, \bullet)$ such that $(\phi, \phi) = 1$ and E_i and F_i are transpose to each other. The dual basis to $\{\mathrm{G}_\theta^{\mathrm{low}}(b)\}_{b\in\mathrm{B}_\theta(0)}$ with respect to $(\bullet, \bullet)$ is called an *upper global basis*.

Let us explain the strategy of our proof of these theorems. We first construct a PBW type basis $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m}}$ of $V_\theta(0)$ parametrized by the θ-restricted multisegments $\mathfrak{m}$. Then, we explicitly calculate the actions of E_i and F_i in terms of the PBW basis $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m}}$. Then, we prove that the PBW basis gives a crystal basis by the estimation of the coefficients of these actions. For this we use a criterion for crystal bases (Theorem 4.1).

§2. General Definitions and Conjectures

§2.1. Quantized universal enveloping algebras and its reduced q-analogues

We shall recall the quantized universal enveloping algebra $U_q(\mathfrak{g})$. Let I be an index set (for simple roots), and Q the free $\mathbb{Z}$-module with a basis $\{\alpha_i\}_{i\in I}$.

Let $(\bullet, \bullet)\colon Q \times Q \to \mathbb{Z}$ be a symmetric bilinear form such that $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}_{>0}$ for any i and $(\alpha_i^\vee, \alpha_j) \in \mathbb{Z}_{\leqslant 0}$ for $i \neq j$ where $\alpha_i^\vee := 2\alpha_i/(\alpha_i, \alpha_i)$. Let q be an indeterminate and set $\mathbf{K} := \mathbb{Q}(q)$. We define its subrings $\mathbf{A}_0$, $\mathbf{A}_\infty$ and $\mathbf{A}$ as follows.

$$\begin{aligned} \mathbf{A}_0 &= \{f \in \mathbf{K} \mid f \text{ is regular at } q = 0\}, \\ \mathbf{A}_\infty &= \{f \in \mathbf{K} \mid f \text{ is regular at } q = \infty\}, \\ \mathbf{A} &= \mathbb{Q}[q, q^{-1}]. \end{aligned}$$

Definition 2.1. The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is the $\mathbf{K}$-algebra generated by elements e_i, f_i and invertible elements t_i $(i \in I)$ with the following defining relations.

(1) The t_i's commute with each other.

(2) $t_j e_i t_j^{-1} = q^{(\alpha_j, \alpha_i)} e_i$ and $t_j f_i t_j^{-1} = q^{-(\alpha_j, \alpha_i)} f_i$ for any $i, j \in I$.

(3) $[e_i, f_j] = \delta_{ij} \dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ for i, $j \in I$. Here $q_i := q^{(\alpha_i, \alpha_i)/2}$.

(4) (*Serre relation*) For $i \neq j$,

$$\sum_{k=0}^{b} (-1)^k e_i^{(k)} e_j e_i^{(b-k)} = 0, \ \sum_{k=0}^{b} (-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0.$$

Here $b = 1 - (\alpha_i^\vee, \alpha_j)$ and

$$\begin{aligned} &e_i^{(k)} = e_i^k/[k]_i!\,, \ f_i^{(k)} = f_i^k/[k]_i!\,, \\ &[k]_i = (q_i^k - q_i^{-k})/(q_i - q_i^{-1})\,, \ [k]_i! = [1]_i \cdots [k]_i\,. \end{aligned}$$

Let us denote by $U_q^-(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$) the $\mathbf{K}$-subalgebra of $U_q(\mathfrak{g})$ generated by the f_i's (resp. the e_i's).

Let e_i' and e_i^* be the operators on $U_q^-(\mathfrak{g})$ (see [K1, 3.4]) defined by

$$[e_i, a] = \frac{(e_i^* a) t_i - t_i^{-1} e_i' a}{q_i - q_i^{-1}} \quad (a \in U_q^-(\mathfrak{g})).$$

These operators satisfy the following formulas similar to derivations:

$$\begin{aligned} e_i'(ab) &= e_i'(a) b + (\mathrm{Ad}(t_i) a) e_i' b, \\ e_i^*(ab) &= a e_i^* b + (e_i^* a)(\mathrm{Ad}(t_i) b). \end{aligned} \tag{2.1}$$

Note that in [K1], the operator e''_i was defined. It satisfies $e''_i = - \circ e'_i \circ -$, while e^*_i satisfies $e^*_i = * \circ e'_i \circ *$. They are related by $e^*_i = \mathrm{Ad}(t_i) \circ e''_i$.

The algebra $U_q^-(\mathfrak{g})$ has a unique symmetric bilinear form $(\bullet, \bullet)$ such that $(1,1) = 1$ and

$$(e'_i a, b) = (a, f_i b) \quad \text{for any } a, b \in U_q^-(\mathfrak{g}).$$

It is non-degenerate and satisfies $(e^*_i a, b) = (a, b f_i)$. The left multiplication of f_j, e'_i and e^*_i have the commutation relations

$$e'_i f_j = q^{-(\alpha_i, \alpha_j)} f_j e'_i + \delta_{ij}, \;\; e^*_i f_j = f_j e^*_i + \delta_{ij}\, \mathrm{Ad}(t_i),$$

and both the e'_i's and the e^*_i's satisfy the Serre relations.

Definition 2.2. The reduced q-analogue $\mathcal{B}(\mathfrak{g})$ of $\mathfrak{g}$ is the $\mathbf{K}$-algebra generated by e'_i and f_i.

§2.2. Review on crystal bases and global bases

Since e'_i and f_i satisfy the q-boson relation, any element $a \in U_q^-(\mathfrak{g})$ can be uniquely written as

$$a = \sum_{n \geqslant 0} f_i^{(n)} a_n \quad \text{with } e'_i a_n = 0.$$

Here $f_i^{(n)} = \dfrac{f_i^n}{[n]_i!}$.

Definition 2.3. We define the modified root operators $\widetilde{e}_i$ and $\widetilde{f}_i$ on $U_q^-(\mathfrak{g})$ by

$$\widetilde{e}_i a = \sum_{n \geqslant 1} f_i^{(n-1)} a_n, \quad \widetilde{f}_i a = \sum_{n \geqslant 0} f_i^{(n+1)} a_n.$$

Theorem 2.4 ([K1]). *We define*

$$L(\infty) = \sum_{\ell \geqslant 0,\, i_1, \ldots, i_\ell \in I} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \subset U_q^-(\mathfrak{g}),$$
$$\mathrm{B}(\infty) = \Big\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \bmod qL(\infty) \mid \ell \geqslant 0, i_1, \cdots, i_\ell \in I \Big\} \subset L(\infty)/qL(\infty).$$

Then we have

(i) $\widetilde{e}_i L(\infty) \subset L(\infty)$ *and* $\widetilde{f}_i L(\infty) \subset L(\infty)$,

(ii) $\mathrm{B}(\infty)$ *is a basis of* $L(\infty)/qL(\infty)$,

(iii) $\widetilde{f}_i \mathrm{B}(\infty) \subset \mathrm{B}(\infty)$ *and* $\widetilde{e}_i \mathrm{B}(\infty) \subset \mathrm{B}(\infty) \cup \{0\}$.

We call $(L(\infty), \mathrm{B}(\infty))$ *the* crystal basis *of* $U_q^-(\mathfrak{g})$.

Let $-$ be the automorphism of $\mathbf{K}$ sending q to q^{-1}. Then $\overline{\mathbf{A}_0}$ coincides with $\mathbf{A}_\infty$.

Let V be a vector space over $\mathbf{K}$, L_0 an $\mathbf{A}_0$-submodule of V, L_∞ an $\mathbf{A}_\infty$-submodule, and $V_{\mathbf{A}}$ an $\mathbf{A}$-submodule. Set $E := L_0 \cap L_\infty \cap V_{\mathbf{A}}$.

Definition 2.5 ([K1], [K2, 2.1])**.** We say that $(L_0, L_\infty, V_{\mathbf{A}})$ is *balanced* if each of L_0, L_∞ and $V_{\mathbf{A}}$ generates V as a $\mathbf{K}$-vector space, and if one of the following equivalent conditions is satisfied.

(i) $E \to L_0/qL_0$ is an isomorphism,

(ii) $E \to L_\infty/q^{-1}L_\infty$ is an isomorphism,

(iii) $(L_0 \cap V_{\mathbf{A}}) \oplus (q^{-1}L_\infty \cap V_{\mathbf{A}}) \to V_{\mathbf{A}}$ is an isomorphism,

(iv) $\mathbf{A}_0 \otimes_{\mathbb{Q}} E \to L_0$, $\mathbf{A}_\infty \otimes_{\mathbb{Q}} E \to L_\infty$, $\mathbf{A} \otimes_{\mathbb{Q}} E \to V_{\mathbf{A}}$ and $\mathbf{K} \otimes_{\mathbb{Q}} E \to V$ are isomorphisms.

Let $-$ be the ring automorphism of $U_q(\mathfrak{g})$ sending q, t_i, e_i, f_i to q^{-1}, t_i^{-1}, e_i, f_i.

Let $U_q(\mathfrak{g})_{\mathbf{A}}$ be the $\mathbf{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$, $f_i^{(n)}$ and t_i. Similarly we define $U_q^-(\mathfrak{g})_{\mathbf{A}}$.

Theorem 2.6. $(L(\infty), L(\infty)^-, U_q^-(\mathfrak{g})_{\mathbf{A}})$ *is balanced.*

Let

$$\mathrm{G}^{\mathrm{low}} \colon L(\infty)/qL(\infty) \xrightarrow{\sim} E := L(\infty) \cap L(\infty)^- \cap U_q^-(\mathfrak{g})_{\mathbf{A}}$$

be the inverse of $E \xrightarrow{\sim} L(\infty)/qL(\infty)$. Then $\left\{\mathrm{G}^{\mathrm{low}}(b) \mid b \in \mathrm{B}(\infty)\right\}$ forms a basis of $U_q^-(\mathfrak{g})$. We call it a (lower) *global basis.* It is first introduced by G. Lusztig ([L]) under the name of "canonical basis" for the A, D, E cases.

Definition 2.7. Let

$$\{\mathrm{G}^{\mathrm{up}}(b) \mid b \in \mathrm{B}(\infty)\}$$

be the dual basis of $\left\{\mathrm{G}^{\mathrm{low}}(b) \mid b \in \mathrm{B}(\infty)\right\}$ with respect to the inner product $(\,\bullet\,,\,\bullet\,)$. We call it the upper global basis of $U_q^-(\mathfrak{g})$.

§2.3. Symmetric crystals

Let θ be an automorphism of I such that $\theta^2 = \mathrm{id}$ and $(\alpha_{\theta(i)}, \alpha_{\theta(j)}) = (\alpha_i, \alpha_j)$. Hence it extends to an automorphism of the root lattice Q by $\theta(\alpha_i) = \alpha_{\theta(i)}$, and induces an automorphism of $U_q(\mathfrak{g})$.

Definition 2.8. Let $\mathcal{B}_\theta(\mathfrak{g})$ be the $\mathbf{K}$-algebra generated by E_i, F_i, and invertible elements T_i $(i \in I)$ satisfying the following defining relations:

(i) the T_i's commute with each other,

(ii) $T_{\theta(i)} = T_i$ for any i,

(iii) $T_i E_j T_i^{-1} = q^{(\alpha_i + \alpha_{\theta(i)}, \alpha_j)} E_j$ and $T_i F_j T_i^{-1} = q^{(\alpha_i + \alpha_{\theta(i)}, -\alpha_j)} F_j$ for $i, j \in I$,

(iv) $E_i F_j = q^{-(\alpha_i, \alpha_j)} F_j E_i + (\delta_{i,j} + \delta_{\theta(i),j} T_i)$ for $i, j \in I$,

(v) the E_i's and the F_i's satisfy the Serre relations (Definition 2.1 (4)).

We set $E_i^{(n)} = E_i^n/[n]_i!$ and $F_i^{(n)} = F_i^n/[n]_i!$.

Lemma 2.9. *Identifying $U_q^-(\mathfrak{g})$ with the subalgebra of $\mathcal{B}_\theta(\mathfrak{g})$ by the morphism $f_i \mapsto F_i$, we have*

$$T_i a = \big(\mathrm{Ad}(t_i t_{\theta(i)}) a\big) T_i, \tag{2.2}$$

$$E_i a = \big(\mathrm{Ad}(t_i) a\big) E_i + e'_i a + \big(\mathrm{Ad}(t_i)(e^*_{\theta(i)} a)\big) T_i \tag{2.3}$$

for $a \in U_q^-(\mathfrak{g})$.

Proof. The first relation is obvious. In order to prove the second, it is enough to show that if a satisfies (2.3), then $f_j a$ satisfies (2.3). We have

$$\begin{aligned} E_i(f_j a) &= (q^{-(\alpha_i, \alpha_j)} f_j E_i + \delta_{i,j} + \delta_{\theta(i),j} T_i) a \\ &= q^{-(\alpha_i, \alpha_j)} f_j \big(\big(\mathrm{Ad}(t_i) a\big) E_i + e'_i a + \big(\mathrm{Ad}(t_i)(e^*_{\theta(i)} a)\big) T_i\big) \\ &\qquad\qquad + \delta_{i,j} a + \delta_{\theta(i),j} \big(\mathrm{Ad}(t_i t_{\theta(i)}) a\big) T_i \\ &= \big(\big(\mathrm{Ad}(t_i)(f_j a)\big) E_i + e'_i(f_j a) + \big(\mathrm{Ad}(t_i)(e^*_{\theta(i)}(f_j a)\big) T_i. \end{aligned}$$

□

The following lemma can be proved in a standard manner and we omit the proof.

Lemma 2.10. *Let $\mathbf{K}[T_i^{\pm}; i \in I]$ be the commutative $\mathbf{K}$-algebra generated by invertible elements T_i $(i \in I)$ with the defining relations $T_{\theta(i)} = T_i$. Then the map $U_q^-(\mathfrak{g}) \otimes \mathbf{K}[T_i^{\pm}; i \in I] \otimes U_q^+(\mathfrak{g}) \to \mathcal{B}_\theta(\mathfrak{g})$ induced by the multiplication is bijective.*

Let $\lambda \in P_+ := \{\lambda \in \mathrm{Hom}(Q, \mathbb{Q}) \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}_{\geqslant 0}$ for any $i \in I\}$ be a dominant integral weight such that $\theta(\lambda) = \lambda$.

Proposition 2.11.

(i) *There exists a $\mathcal{B}_\theta(\mathfrak{g})$-module $V_\theta(\lambda)$ generated by a non-zero vector ϕ_λ such that*

  (a) $E_i\phi_\lambda = 0$ *for any* $i \in I$,

  (b) $T_i\phi_\lambda = q^{(\alpha_i,\lambda)}\phi_\lambda$ *for any* $i \in I$,

  (c) $\{u \in V_\theta(\lambda) \mid E_i u = 0$ *for any* $i \in I\} = \mathbf{K}\phi_\lambda$.

  Moreover such a $V_\theta(\lambda)$ is irreducible and unique up to an isomorphism.

(ii) *there exists a unique symmetric bilinear form $(\,\bullet\,,\bullet\,)$ on $V_\theta(\lambda)$ such that $(\phi_\lambda, \phi_\lambda) = 1$ and $(E_i u, v) = (u, F_i v)$ for any $i \in I$ and $u, v \in V_\theta(\lambda)$, and it is non-degenerate.*

Remark 2.12. Set $P_\theta = \{\mu \in P \mid \theta(\mu) = \mu\}$. Then $V_\theta(\lambda)$ has a weight decomposition

$$V_\theta(\lambda) = \bigoplus_{\mu \in P_\theta} V_\theta(\lambda)_\mu,$$

where $V_\theta(\lambda)_\mu = \{u \in V_\theta(\lambda) \mid T_i u = q^{(\alpha_i,\mu)} u\}$. We say that an element u of $V_\theta(\lambda)$ has a θ-weight μ and write $\mathrm{wt}_\theta(u) = \mu$ if $u \in V_\theta(\lambda)_\mu$. We have $\mathrm{wt}_\theta(E_i u) = \mathrm{wt}_\theta(u) + (\alpha_i + \alpha_{\theta(i)})$ and $\mathrm{wt}_\theta(F_i u) = \mathrm{wt}_\theta(u) - (\alpha_i + \alpha_{\theta(i)})$.

In order to prove Proposition 2.11, we shall construct two $\mathcal{B}_\theta(\mathfrak{g})$-modules, analogous to Verma modules and dual Verma modules.

Lemma 2.13. *Let $U_q^-(\mathfrak{g})\phi'_\lambda$ be a free $U_q^-(\mathfrak{g})$-module with a generator ϕ'_λ. Then the following action gives a structure of a $\mathcal{B}_\theta(\mathfrak{g})$-module on $U_q^-(\mathfrak{g})\phi'_\lambda$:*

$$\begin{cases} T_i(a\phi'_\lambda) = q^{(\alpha_i,\lambda)}(\mathrm{Ad}(t_i t_{\theta(i)})a)\phi'_\lambda, \\ E_i(a\phi'_\lambda) = \big(e'_i a + q^{(\alpha_i,\lambda)}\,\mathrm{Ad}(t_i)(e^*_{\theta(i)}a)\big)\phi'_\lambda, \\ F_i(a\phi'_\lambda) = (f_i a)\phi'_\lambda \end{cases} \tag{2.4}$$

for any $i \in I$ and $a \in U_q^-(\mathfrak{g})$.

Moreover $\mathcal{B}_\theta(\mathfrak{g})/\sum_{i\in I}(\mathcal{B}_\theta(\mathfrak{g})E_i + \mathcal{B}_\theta(\mathfrak{g})(T_i - q^{(\alpha_i,\lambda)})) \to U_q^-(\mathfrak{g})\phi'_\lambda$ is an isomorphism.

Proof. We can easily check the defining relations in Definition 2.8 except the Serre relations for the E_i's.

For $i \neq j \in I$, set $S = \sum_{n=0}^{b}(-1)^n E_i^{(n)} E_j E_i^{(b-n)}$ where $b = 1 - \langle h_i, \alpha_j \rangle$. It is enough to show that the action of S on $U_q^-(\mathfrak{g})\phi'_\lambda$ is equal to 0. We can easily check that $SF_k = q^{-(b\alpha_i+\alpha_j,\alpha_k)}F_k S$. Since $S\phi'_\lambda = 0$, we have $SU_q^-(\mathfrak{g})\phi'_\lambda = 0$.

Hence $U_q^-(\mathfrak{g})\phi'_\lambda$ has a $\mathcal{B}_\theta(\mathfrak{g})$-module structure.

The last statement is obvious. □

Lemma 2.14. *Let $U_q^-(\mathfrak{g})\phi''_\lambda$ be a free $U_q^-(\mathfrak{g})$-module with a generator ϕ''_λ. Then the following action gives a structure of a $\mathcal{B}_\theta(\mathfrak{g})$-module on $U_q^-(\mathfrak{g})\phi''_\lambda$:*

$$\begin{cases} T_i(a\phi''_\lambda) = q^{(\alpha_i,\lambda)}(\mathrm{Ad}(t_i t_{\theta(i)})a)\phi''_\lambda, \\ E_i(a\phi''_\lambda) = (e'_i a)\phi''_\lambda, \\ F_i(a\phi''_\lambda) = \big(f_i a + q^{(\alpha_i,\lambda)}(\mathrm{Ad}(t_i)a)f_{\theta(i)}\big)\phi''_\lambda \end{cases} \tag{2.5}$$

for any $i \in I$ and $a \in U_q^-(\mathfrak{g})$. Moreover, there exists a non-degenerate bilinear form $\langle\,\bullet\,,\bullet\,\rangle\colon U_q^-(\mathfrak{g})\phi'_\lambda \times U_q^-(\mathfrak{g})\phi''_\lambda \to \mathbf{K}$ such that $\langle F_i u, v\rangle = \langle u, E_i v\rangle$, $\langle E_i u, v\rangle = \langle u, F_i v\rangle$, $\langle T_i u, v\rangle = \langle u, T_i v\rangle$ for $u \in U_q^-(\mathfrak{g})\phi'_\lambda$ and $v \in U_q^-(\mathfrak{g})\phi''_\lambda$, and $\langle \phi'_\lambda, \phi''_\lambda\rangle = 1$.

Proof. There exists a unique symmetric bilinear form $(\,\bullet\,,\bullet\,)$ on $U_q^-(\mathfrak{g})$ such that $(1,1) = 1$ and f_i and e'_i are transpose to each other. Let us define $\langle\,\bullet\,,\bullet\,\rangle\colon U_q^-(\mathfrak{g})\phi'_\lambda \times U_q^-(\mathfrak{g})\phi''_\lambda \to \mathbf{K}$ by $\langle a\phi'_\lambda, b\phi''_\lambda\rangle = (a,b)$ for $a \in U_q^-(\mathfrak{g})$ and $b \in U_q^-(\mathfrak{g})$. Then we can easily check $\langle F_i u, v\rangle = \langle u, E_i v\rangle$, $\langle T_i u, v\rangle = \langle u, T_i v\rangle$. Since e_i^* is transpose to the right multiplication of f_i, we have $\langle E_i u, v\rangle = \langle u, F_i v\rangle$. Hence the action of E_i, F_i, T_i on $U_q^-(\mathfrak{g})\phi''_\lambda$ satisfy the defining relations in Definition 2.8. □

Proof of Proposition 2.11. Since $E_i\phi''_\lambda = 0$ and ϕ''_λ has a θ-weight λ, there exists a unique $\mathcal{B}_\theta(\mathfrak{g})$-linear morphism $\psi\colon U_q^-(\mathfrak{g})\phi'_\lambda \to U_q^-(\mathfrak{g})\phi''_\lambda$ sending ϕ'_λ to ϕ''_λ. Let $V_\theta(\lambda)$ be its image $\psi(U_q^-(\mathfrak{g})\phi'_\lambda)$.

(i) (c) follows from $\big\{u \in U_q^-(\mathfrak{g}) \mid e'_i u = 0 \text{ for any } i\big\} = \mathbf{K}$ and $U_q^-(\mathfrak{g})\phi''_\lambda \supset V_\theta(\lambda)$. The other properties (a), (b) are obvious. Let us show that $V_\theta(\lambda)$ is irreducible. Let S be a non-zero $\mathcal{B}_\theta(\mathfrak{g})$-submodule. Then S contains a non-zero vector v such that $E_i v = 0$ for any i. Then (c) implies that v is a constant multiple of ϕ_λ. Hence $S = V_\theta(\lambda)$.

Let us prove (ii). For $u, u' \in U_q^-(\mathfrak{g})\phi'_\lambda$, set $(\!(u, u')\!) = \langle u, \psi(u')\rangle$. Then it is a bilinear form on $U_q^-(\mathfrak{g})\phi'_\lambda$ which satisfies

$$\begin{aligned} &(\!(\phi'_\lambda, \phi'_\lambda)\!) = 1,\ (\!(F_i u, u')\!) = (\!(u, E_i u')\!),\ (\!(E_i u, u')\!) = (\!(u, F_i u')\!),\ \text{and} \\ &(\!(T_i u, u')\!) = (\!(u, T_i u')\!). \end{aligned} \tag{2.6}$$

It is easy to see that a bilinear form which satisfies (2.6) is unique. Since $(\!(u',u)\!)$ also satisfies (2.6), $(\!(u,u')\!)$ is a symmetric bilinear form on $U_q^-(\mathfrak{g})\phi'_\lambda$. Since $\psi(u')=0$ implies $(\!(u,u')\!)=0$, $(\!(u,u')\!)$ induces a symmetric bilinear form on $V_\theta(\lambda)$. Since $(\,\bullet\,,\,\bullet\,)$ is non-degenerate on $U_q^-(\mathfrak{g})$, $(\!(\,\bullet\,,\,\bullet\,)\!)$ is a non-degenerate symmetric bilinear form on $V_\theta(\lambda)$.

□

Lemma 2.15. *There exists a unique endomorphism* $-$ *of* $V_\theta(\lambda)$ *such that* $\overline{\phi_\lambda}=\phi_\lambda$ *and* $\overline{av}=\bar{a}\bar{v}$, $\overline{F_iv}=F_i\bar{v}$ *for any* $a\in\mathbf{K}$ *and* $v\in V_\theta(\lambda)$.

Proof. The uniqueness is obvious.

Let ξ be an anti-involution of $U_q^-(\mathfrak{g})$ such that $\xi(q)=q^{-1}$ and $\xi(f_i)=f_{\theta(i)}$. Let $\tilde\rho$ be an element of $\mathbb{Q}\otimes P$ such that $(\tilde\rho,\alpha_i)=(\alpha_i,\alpha_{\theta(i)})/2$. Define $c(\mu)=\big((\mu+\tilde\rho,\theta(\mu+\tilde\rho))-(\tilde\rho,\theta(\tilde\rho))\big)/2+(\lambda,\mu)$ for $\mu\in P$. Then it satisfies

$$c(\mu)-c(\mu-\alpha_i)=(\lambda+\mu,\alpha_{\theta(i)}).$$

Hence c takes integral values on $Q:=\sum_i\mathbb{Z}\alpha_i$.

We define the endomorphism Φ of $U_q^-(\mathfrak{g})\phi''_\lambda$ by $\Phi(a\phi''_\lambda)=q^{-c(\mu)}\xi(a)\phi''_\lambda$ for $a\in U_q^-(\mathfrak{g})_\mu$. Let us show that

$$\Phi(F_i(a\phi''_\lambda))=F_i\Phi(a\phi''_\lambda)\quad\text{for any } a\in U_q^-(\mathfrak{g}). \tag{2.7}$$

For $a\in U_q^-(\mathfrak{g})_\mu$, we have

$$\begin{aligned}\Phi(F_i(a\phi''_\lambda))&=\Phi\big(f_ia+q^{(\alpha_i,\lambda+\mu)}af_{\theta(i)}\big)\phi''_\lambda\\&=\big(q^{-c(\mu-\alpha_i)}\xi(a)f_{\theta(i)}+q^{-(\alpha_i,\lambda+\mu)-c(\mu-\alpha_{\theta(i)})}f_i\xi(a)\big)\phi''_\lambda.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}F_i\Phi(a\phi''_\lambda)&=F_i\big(q^{-c(\mu)}\xi(a)\phi''_\lambda\big)\\&=q^{-c(\mu)}\big(f_i\xi(a)+q^{(\alpha_i,\lambda+\theta(\mu))}\xi(a)f_{\theta(i)}\big)\phi''_\lambda.\end{aligned}$$

Therefore we obtain (2.7).

Hence Φ induces the desired endomorphism of $V_\theta(\lambda)\subset U_q^-(\mathfrak{g})\phi''_\lambda$. □

Hereafter we assume further that

$$\text{there is no } i\in I \text{ such that } \theta(i)=i.$$

We conjecture that $V_\theta(\lambda)$ has a crystal basis under this assumption. This means the following. Since E_i and F_i satisfy the q-boson relation, any $u\in V_\theta(\lambda)$ can be uniquely written as $u = \sum_{n \geqslant 0} F_i^{(n)} u_n$ with $E_i u_n = 0$. We define the modified root operators $\widetilde{E}_i$ and $\widetilde{F}_i$ by:

$$\widetilde{E}_i(u) = \sum_{n \geqslant 1} F_i^{(n-1)} u_n \text{ and } \widetilde{F}_i(u) = \sum_{n \geqslant 0} F_i^{(n+1)} u_n.$$

Let $L_\theta(\lambda)$ be the $\mathbf{A}_0$-submodule of $V_\theta(\lambda)$ generated by $\widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell} \phi_\lambda$ ($\ell \geqslant 0$ and $i_1, \dots, i_\ell \in I$), and let $\mathrm{B}_\theta(\lambda)$ be the subset

$$\left\{ \widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell} \phi_\lambda \bmod qL_\theta(\lambda) \mid \ell \geqslant 0,\, i_1, \dots, i_\ell \in I \right\}$$

of $L_\theta(\lambda)/qL_\theta(\lambda)$.

Conjecture 2.16. For a dominant integral weight λ such that $\theta(\lambda) = \lambda$, we have

(1) $\widetilde{F}_i L_\theta(\lambda) \subset L_\theta(\lambda)$ and $\widetilde{E}_i L_\theta(\lambda) \subset L_\theta(\lambda)$,

(2) $\mathrm{B}_\theta(\lambda)$ is a basis of $L_\theta(\lambda)/qL_\theta(\lambda)$,

(3) $\widetilde{F}_i \mathrm{B}_\theta(\lambda) \subset \mathrm{B}_\theta(\lambda)$, and $\widetilde{E}_i \mathrm{B}_\theta(\lambda) \subset \mathrm{B}_\theta(\lambda) \sqcup \{0\}$,

(4) $\widetilde{F}_i \widetilde{E}_i(b) = b$ for any $b \in \mathrm{B}_\theta(\lambda)$ such that $\widetilde{E}_i b \neq 0$, and $\widetilde{E}_i \widetilde{F}_i(b) = b$ for any $b \in \mathrm{B}_\theta(\lambda)$.

As in [K1], we have

Lemma 2.17. *Assume Conjecture* 2.16*. Then we have*

(i) $L_\theta(\lambda) = \{v \in V_\theta(\lambda) \mid (L_\theta(\lambda), v) \subset \mathbf{A}_0\}$,

(ii) *Let* $(\,\bullet\,,\,\bullet\,)_0$ *be the* $\mathbb{Q}$*-valued symmetric bilinear form on* $L_\theta(\lambda)/qL_\theta(\lambda)$ *induced by* $(\,\bullet\,,\,\bullet\,)$*. Then* $\mathrm{B}_\theta(\lambda)$ *is an orthonormal basis with respect to* $(\,\bullet\,,\,\bullet\,)_0$.

Moreover we conjecture that $V_\theta(\lambda)$ has a global crystal basis. Namely we have

Conjecture 2.18. The triplet $(L_\theta(\lambda),\, L_\theta(\lambda)^-,\, V_\theta(\lambda)^{\mathrm{low}}_{\mathbf{A}})$ is balanced. Here $V_\theta(\lambda)^{\mathrm{low}}_{\mathbf{A}} := U_q^-(\mathfrak{g})_{\mathbf{A}} \phi_\lambda$.

Its dual version is as follows.

Let us denote by $V_\theta(\lambda)^{\mathrm{up}}_{\mathbf{A}}$ the dual space $\left\{ v \in V_\theta(\lambda) \mid (V_\theta(\lambda)^{\mathrm{low}}_{\mathbf{A}}, v) \subset \mathbf{A} \right\}$. Then Conjecture 2.18 is equivalent to the following conjecture.

Conjecture 2.19. $(L_\theta(\lambda), c(L_\theta(\lambda)), V_\theta(\lambda)^{\mathrm{up}}_{\mathbf{A}})$ is balanced.

Here c is a unique endomorphism of $V_\theta(\lambda)$ such that $c(\phi_\lambda) = \phi_\lambda$ and $c(av) = \bar{a}c(v)$, $c(E_i v) = E_i c(v)$ for any $a \in \mathbf{K}$ and $v \in V_\theta(\lambda)$. We have $(c(v'), v) = \overline{(v', \bar{v})}$ for any $v, v' \in V_\theta(\lambda)$.

Note that $V_\theta(\lambda)^{\mathrm{up}}_{\mathbf{A}}$ is the largest $\mathbf{A}$-submodule M of $V_\theta(\lambda)$ such that M is invariant by the $E_i^{(n)}$'s and $M \cap \mathbf{K}\phi_\lambda = \mathbf{A}\phi_\lambda$.

By Conjecture 2.19, $L_\theta(\lambda) \cap c(L_\theta(\lambda)) \cap V_\theta(\lambda)^{\mathrm{up}}_{\mathbf{A}} \to L_\theta(\lambda)/qL_\theta(\lambda)$ is an isomorphism. Let $\mathrm{G}^{\mathrm{up}}_\theta$ be its inverse. Then $\{\mathrm{G}^{\mathrm{up}}_\theta(b)\}_{b \in \mathrm{B}_\theta(\lambda)}$ is a basis of $V_\theta(\lambda)$, which we call the *upper global basis* of $V_\theta(\lambda)$. Note that $\{\mathrm{G}^{\mathrm{up}}_\theta(b)\}_{b \in \mathrm{B}_\theta(\lambda)}$ is the dual basis to $\{\mathrm{G}^{\mathrm{low}}_\theta(b)\}_{b \in \mathrm{B}_\theta(\lambda)}$ with respect to the inner product of $V_\theta(\lambda)$.

We shall prove these conjectures in the case $\mathfrak{g} = \mathfrak{gl}_\infty$ and $\lambda = 0$.

§3. PBW Basis of $V_\theta(0)$ for $\mathfrak{g} = \mathfrak{gl}_\infty$

§3.1. Review on the PBW basis

In the sequel, we set $I = \mathbb{Z}_{\mathrm{odd}}$ and

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{for } i = j, \\ -1 & \text{for } j = i \pm 2, \\ 0 & \text{otherwise,} \end{cases}$$

and we consider the corresponding quantum group $U_q(\mathfrak{gl}_\infty)$. In this case, we have $q_i = q$. We write $[n]$ and $[n]!$ for $[n]_i$ and $[n]_i!$ for short.

We can parametrize the crystal basis $\mathrm{B}(\infty)$ by the multisegments. We shall recall this parametrization and the PBW basis.

Definition 3.1. For $i, j \in I$ such that $i \leqslant j$, we define a segment $\langle i, j \rangle$ as the interval $[i, j] \subset I := \mathbb{Z}_{\mathrm{odd}}$. A multisegment is a formal finite sum of segments:

$$\mathfrak{m} = \sum_{i \leqslant j} m_{ij} \langle i, j \rangle$$

with $m_{i,j} \in \mathbb{Z}_{\geqslant 0}$. We call m_{ij} the multiplicity of a segment $\langle i, j \rangle$. If $m_{i,j} > 0$, we sometimes say that $\langle i, j \rangle$ appears in $\mathfrak{m}$. We sometimes write $m_{i,j}(\mathfrak{m})$ for $m_{i,j}$. We sometimes write $\langle i \rangle$ for $\langle i, i \rangle$. We denote by $\mathcal{M}$ the set of multisegments. We denote by $\emptyset$ the zero element (or the empty multisegment) of $\mathcal{M}$.

Definition 3.2. For two segments $\langle i_1, j_1 \rangle$ and $\langle i_2, j_2 \rangle$, we define the ordering $\geqslant_{\mathrm{PBW}}$ by the following:

$$\langle i_1, j_1 \rangle \geqslant_{\mathrm{PBW}} \langle i_2, j_2 \rangle \iff \begin{cases} j_1 > j_2 \\ \text{or} \\ j_1 = j_2 \text{ and } i_1 \geqslant i_2. \end{cases}$$

We call this ordering the *PBW-ordering*.

Definition 3.3. For a multisegment $\mathfrak{m}$, we define the element $P(\mathfrak{m}) \in U_q^-(\mathfrak{gl}_\infty)$ as follows.

(1) For a segment $\langle i,j\rangle$, we define the element $\langle i,j\rangle \in U_q^-(\mathfrak{gl}_\infty)$ inductively by

$$\begin{aligned}&\langle i,i\rangle = f_i,\\ &\langle i,j\rangle = \langle i,j-2\rangle\langle j,j\rangle - q\langle j,j\rangle\langle i,j-2\rangle \quad \text{for } i<j.\end{aligned}$$

(2) For a multisegment $\mathfrak{m} = \sum_{i\leqslant j} m_{ij}\langle i,j\rangle$, we define

$$P(\mathfrak{m}) = \overrightarrow{\prod} \langle i,j\rangle^{(m_{ij})}.$$

Here the product $\overrightarrow{\prod}$ is taken over segments appearing in $\mathfrak{m}$ from large to small with respect to the PBW-ordering. The element $\langle i,j\rangle^{(m_{ij})}$ is the divided power of $\langle i,j\rangle$ i.e.

$$\langle i,j\rangle^{(n)} = \begin{cases} \dfrac{1}{[n]!}\langle i,j\rangle^n & \text{for } n>0,\\ 1 & \text{for } n=0,\\ 0 & \text{for } n<0.\end{cases}$$

Hence the weight of $P(\mathfrak{m})$ is equal to $\mathrm{wt}(\mathfrak{m}) := -\sum_{i\leqslant k\leqslant j} m_{i,j}\alpha_k$: $t_iP(\mathfrak{m})t_i^{-1} = q^{(\alpha_i,\mathrm{wt}(\mathfrak{m}))}P(\mathfrak{m})$.

Theorem 3.4 ([L])**.** *The set of elements $\{P(\mathfrak{m}) \mid \mathfrak{m}\in\mathcal{M}\}$ is a $\mathbf{K}$-basis of $U_q^-(\mathfrak{gl}_\infty)$. Moreover this is an $\mathbf{A}$-basis of $U_q^-(\mathfrak{gl}_\infty)_{\mathbf{A}}$. We call this basis the* PBW basis *of $U_q^-(\mathfrak{gl}_\infty)$.*

Definition 3.5. For two segments $\langle i_1,j_1\rangle$ and $\langle i_2,j_2\rangle$, we define the ordering $\geqslant_{\mathrm{cry}}$ by the following:

$$\langle i_1,j_1\rangle \geqslant_{\mathrm{cry}} \langle i_2,j_2\rangle \Longleftrightarrow \begin{cases} j_1>j_2\\ \text{or}\\ j_1=j_2 \text{ and } i_1\leqslant i_2.\end{cases}$$

We call this ordering the *crystal ordering*.

Example 3.6. The crystal ordering is different from the PBW-ordering. For example, we have $\langle -1,1\rangle >_{\mathrm{cry}} \langle 1,1\rangle >_{\mathrm{cry}} \langle -1\rangle$, while we have $\langle 1,1\rangle >_{\mathrm{PBW}} \langle -1,1\rangle >_{\mathrm{PBW}} \langle -1\rangle$.

Definition 3.7. We define the crystal structure on $\mathcal{M}$ as follows: for $\mathfrak{m} = \sum m_{i,j}\langle i,j\rangle \in \mathcal{M}$ and $i \in I$, set $A_k^{(i)}(\mathfrak{m}) = \sum_{k' \geqslant k}(m_{i,k'} - m_{i+2,k'+2})$ for $k \geqslant i$. Define $\varepsilon_i(\mathfrak{m})$ as $\max\left\{A_k^{(i)}(\mathfrak{m}) \mid k \geqslant i\right\} \geqslant 0$.

(i) If $\varepsilon_i(\mathfrak{m}) = 0$, then define $\tilde{e}_i(\mathfrak{m}) = 0$. If $\varepsilon_i(\mathfrak{m}) > 0$, let k_e be the largest $k \geqslant i$ such that $\varepsilon_i(\mathfrak{m}) = A_k^{(i)}(\mathfrak{m})$ and define $\tilde{e}_i(\mathfrak{m}) = \mathfrak{m} - \langle i, k_e\rangle + \delta_{k_e \neq i}\langle i+2, k_e\rangle$.

(ii) Let k_f be the smallest $k \geqslant i$ such that $\varepsilon_i(\mathfrak{m}) = A_k^{(i)}(\mathfrak{m})$ and define $\tilde{f}_i(\mathfrak{m}) = \mathfrak{m} - \delta_{k_f \neq i}\langle i+2, k_f\rangle + \langle i, k_f\rangle$.

Remark 3.8. For $i \in I$, the actions of the operators $\widetilde{e}_i$ and $\widetilde{f}_i$ on $\mathfrak{m} \in \mathcal{M}$ are also described by the following algorithm:

Step 1. Arrange the segments in $\mathfrak{m}$ in the crystal ordering.

Step 2. For each segment $\langle i,j\rangle$, write $-$, and for each segment $\langle i+2,j\rangle$, write $+$.

Step 3. In the resulting sequence of $+$ and $-$, delete a subsequence of the form $+-$ and keep on deleting until no such subsequence remains.

Then we obtain a sequence of the form $--\cdots-++\cdots+$.

(1) $\varepsilon_i(\mathfrak{m})$ is the total number of $-$ in the resulting sequence.

(2) $\widetilde{f}_i(\mathfrak{m})$ is given as follows:

(a) if the leftmost $+$ corresponds to a segment $\langle i+2,j\rangle$, then replace it with $\langle i,j\rangle$,

(b) if no $+$ exists, add a segment $\langle i,i\rangle$ to $\mathfrak{m}$.

(3) $\widetilde{e}_i(\mathfrak{m})$ is given as follows:

(a) if the rightmost $-$ corresponds to a segment $\langle i,j\rangle$ with $i < j$, then replace it with $\langle i+2,j\rangle$,

(b) if the rightmost $-$ corresponds to a segment $\langle i,i\rangle$, then remove it,

(c) if no $-$ exists, then $\widetilde{e}_i(\mathfrak{m}) = 0$.

Let us introduce a linear ordering on the set $\mathcal{M}$ of multisegments, lexicographic with respect to the crystal ordering on the set of segments.

Definition 3.9. For $\mathfrak{m} = \sum_{i \leqslant j} m_{i,j}\langle i,j\rangle \in \mathcal{M}$ and $\mathfrak{m}' = \sum_{i \leqslant j} m'_{i,j}\langle i,j\rangle \in \mathcal{M}$, we define $\mathfrak{m}' \underset{\mathrm{cry}}{<} \mathfrak{m}$ if there exist $i_0 \leqslant j_0$ such that $m'_{i_0,j_0} < m_{i_0,j_0}$, $m'_{i,j_0} = m_{i,j_0}$ for $i < i_0$, and $m'_{i,j} = m_{i,j}$ for $j > j_0$ and $i \leqslant j$.

Theorem 3.10.

(i) $L(\infty) = \bigoplus_{\mathfrak{m} \in \mathcal{M}} \mathbf{A}_0 P(\mathfrak{m})$.

(ii) $\mathrm{B}(\infty) = \{P(\mathfrak{m}) \bmod qL(\infty) \mid \mathfrak{m} \in \mathcal{M}\}$.

(iii) *We have*

$$\begin{aligned} \widetilde{e}_i P(\mathfrak{m}) &\equiv P(\widetilde{e}_i(\mathfrak{m})) \quad \bmod qL(\infty), \\ \widetilde{f}_i P(\mathfrak{m}) &\equiv P(\widetilde{f}_i(\mathfrak{m})) \quad \bmod qL(\infty). \end{aligned}$$

Note that $\widetilde{e}_i$ and $\widetilde{f}_i$ in the left-hand-side is the modified root operators.

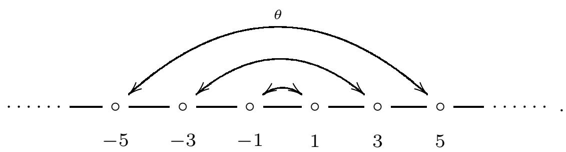
(iv) *We have*

$$\overline{P(\mathfrak{m})} \in P(\mathfrak{m}) + \sum_{\mathfrak{m}' \underset{\mathrm{cry}}{<} \mathfrak{m}} \mathbf{A} P(\mathfrak{m}').$$

Therefore we can index the crystal basis by multisegments. By this theorem we can easily see by a standard argument that $(L(\infty), L(\infty)^-, U_q^-(\mathfrak{g})_{\mathbf{A}})$ is balanced, and there exists a unique $\mathrm{G}^{\mathrm{low}}(\mathfrak{m}) \in L(\infty) \cap U_q^-(\mathfrak{g})_{\mathbf{A}}$ such that $\mathrm{G}^{\mathrm{low}}(\mathfrak{m})^- = \mathrm{G}^{\mathrm{low}}(\mathfrak{m})$ and $\mathrm{G}^{\mathrm{low}}(\mathfrak{m}) \equiv P(\mathfrak{m}) \bmod qL(\infty)$. Then $\{\mathrm{G}^{\mathrm{low}}(\mathfrak{m})\}_{\mathfrak{m} \in \mathcal{M}}$ is a lower global basis.

§3.2. θ-restricted multisegments

We consider the Dynkin diagram involution θ of $I := \mathbb{Z}_{\mathrm{odd}}$ defined by $\theta(i) = -i$ for $i \in I$.



We shall prove in this case Conjectures 2.16 and 2.18 for $\lambda = 0$ (Theorems 4.15 and 5.5).

We set

$$\begin{aligned} \widetilde{V}_\theta(0) &:= \mathcal{B}_\theta(\mathfrak{gl}_\infty) / \sum_{i \in I} \big(\mathcal{B}_\theta(\mathfrak{gl}_\infty) E_i + \mathcal{B}_\theta(\mathfrak{gl}_\infty)(T_i - 1) + \mathcal{B}_\theta(\mathfrak{gl}_\infty)(F_i - F_{\theta(i)}) \big) \\ &\simeq U_q^-(\mathfrak{gl}_\infty) / \sum_i U_q^-(\mathfrak{gl}_\infty)(f_i - f_{\theta(i)}). \end{aligned}$$

Let $\widetilde{\phi}$ be the generator of $\widetilde{V}_\theta(0)$ corresponding to $1 \in \mathcal{B}_\theta(\mathfrak{gl}_\infty)$. Since $F_i\phi_0'' = (f_i + f_{\theta(i)})\phi_0'' = F_{\theta(i)}\phi_0''$, we have an epimorphism of $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$-modules

$$\widetilde{V}_\theta(0) \twoheadrightarrow V_\theta(0). \tag{3.1}$$

We shall see later that it is in fact an isomorphism (see Theorem 4.15).

Definition 3.11. If a multisegment $\mathfrak{m}$ has the form

$$\mathfrak{m} = \sum_{-j \leqslant i \leqslant j} m_{ij}\langle i, j\rangle,$$

we call $\mathfrak{m}$ a θ*-restricted* multisegment. We denote by $\mathcal{M}_\theta$ the set of θ-restricted multisegments.

Definition 3.12. For a θ-restricted segment $\langle i, j\rangle$, we define its modified divided power by

$$\langle i, j\rangle^{[m]} = \begin{cases} \langle i, j\rangle^{(m)} = \dfrac{1}{[m]!}\langle i, j\rangle^m & (i \neq -j), \\ \dfrac{1}{\prod_{\nu=1}^m [2\nu]}\langle -j, j\rangle^m & (i = -j). \end{cases}$$

We understand that $\langle i, j\rangle^{[m]}$ is equal to 1 for $m = 0$ and vanishes for $m < 0$.

Definition 3.13. For $\mathfrak{m} \in \mathcal{M}_\theta$, we define $P_\theta(\mathfrak{m}) \in U_q^-(\mathfrak{gl}_\infty) \subset \mathcal{B}_\theta(\mathfrak{gl}_\infty)$ by

$$P_\theta(\mathfrak{m}) = \overrightarrow{\prod_{\langle i,j\rangle \in \mathfrak{m}}} \langle i, j\rangle^{[m_{ij}]}.$$

Here the product $\overrightarrow{\prod}$ is taken over the segments appearing in $\mathfrak{m}$ from large to small with respect to the PBW-ordering.

If an element $\mathfrak{m}$ of the free abelian group generated by $\langle i, j\rangle$ does not belong to $\mathcal{M}_\theta$, we understand $P_\theta(\mathfrak{m}) = 0$.

We will prove later that $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m} \in \mathcal{M}_\theta}$ is a basis of $V_\theta(0)$ (see Theorem 4.15). Here and hereafter, we write ϕ instead of $\phi_0 \in V_\theta(0)$.

§3.3. Commutation relations of $\langle i, j\rangle$

In the sequel, we regard $U_q^-(\mathfrak{gl}_\infty)$ as a subalgebra of $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$ by $f_i \mapsto F_i$. We shall give formulas to express products of segments by a PBW basis.

Proposition 3.14. *For $i, j, k, l \in I$, we have*

(1) $\langle i,j\rangle\langle k,\ell\rangle = \langle k,\ell\rangle\langle i,j\rangle$ *for* $i \leqslant j,\ k \leqslant \ell$ *and* $j < k-2$,

(2) $\langle i,j\rangle\langle j+2,k\rangle = \langle i,k\rangle + q\langle j+2,k\rangle\langle i,j\rangle$ *for* $i \leqslant j < k$,

(3) $\langle j,k\rangle\langle i,\ell\rangle = \langle i,\ell\rangle\langle j,k\rangle$ *for* $i < j \leqslant k < \ell$,

(4) $\langle i,k\rangle\langle j,k\rangle = q^{-1}\langle j,k\rangle\langle i,k\rangle$ *for* $i < j \leqslant k$,

(5) $\langle i,j\rangle\langle i,k\rangle = q^{-1}\langle i,k\rangle\langle i,j\rangle$ *for* $i \leqslant j < k$,

(6) $\langle i,k\rangle\langle j,\ell\rangle = \langle j,\ell\rangle\langle i,k\rangle + (q^{-1}-q)\langle i,\ell\rangle\langle j,k\rangle$ *for* $i < j \leqslant k < \ell$.

Proof. (1) is obvious. We prove (2) by the induction on $k-j$. If $k-j=2$, it is trivial by the definition. If $j < k-2$, then $\langle k\rangle$ and $\langle i,j\rangle$ commute. Thus, we have

$$\begin{aligned}\langle i,j\rangle\langle j+2,k\rangle &= \langle i,j\rangle\big(\langle j+2,k-2\rangle\langle k\rangle - q\langle k\rangle\langle j+2,k-2\rangle\big)\\ &= \big(\langle i,k-2\rangle + q\langle j+2,k-2\rangle\langle i,j\rangle\big)\langle k\rangle - q\langle k\rangle\langle i,j\rangle\langle j+2,k-2\rangle\\ &= \langle i,k-2\rangle\langle k\rangle + q\langle j+2,k-2\rangle\langle k\rangle\langle i,j\rangle\\ &\qquad\qquad -q\langle k\rangle\big(\langle i,k-2\rangle + q\langle j+2,k-2\rangle\langle i,j\rangle\big)\\ &= \langle i,k\rangle + \langle j+2,k\rangle\langle i,j\rangle.\end{aligned}$$

In order to prove the other relations, we first show the following special cases.

Lemma 3.15. *We have for any* $j \in I$

(a) $\langle j-2,j\rangle\langle j\rangle = q^{-1}\langle j\rangle\langle j-2,j\rangle$ *and* $\langle j\rangle\langle j,j+2\rangle = q^{-1}\langle j,j+2\rangle\langle j\rangle$,

(b) $\langle j\rangle\langle j-2,j+2\rangle = \langle j-2,j+2\rangle\langle j\rangle$,

(c) $\langle j-2,j\rangle\langle j,j+2\rangle = \langle j,j+2\rangle\langle j-2,j\rangle + (q^{-1}-q)\langle j-2,j+2\rangle\langle j\rangle$.

Proof. The first equality in (a) follows from

$$\begin{aligned}&\langle j-2,j\rangle\langle j\rangle - q^{-1}\langle j\rangle\langle j-2,j\rangle\\ &\qquad = \big(f_{j-2}f_j - qf_jf_{j-2}\big)f_j - q^{-1}f_j\big(f_{j-2}f_j - qf_jf_{j-2}\big)\\ &\qquad = f_{j-2}f_j^2 - (q+q^{-1})f_jf_{j-2}f_j + f_j^2f_{j-2} = 0.\end{aligned}$$

We can similarly prove the second.

Let us show (b) and (c). We have, by (a)

$$\begin{aligned}
&\langle j-2,j\rangle\langle j,j+2\rangle = \langle j-2,j\rangle\big(\langle j\rangle\langle j+2\rangle - q\langle j+2\rangle\langle j\rangle\big)\\
&\quad = q^{-1}\langle j\rangle\langle j-2,j\rangle\langle j+2\rangle - q\big(\langle j-2,j+2\rangle + q\langle j+2\rangle\langle j-2,j\rangle\big)\langle j\rangle\\
&\quad = q^{-1}\langle j\rangle\big(\langle j-2,j+2\rangle + q\langle j+2\rangle\langle j-2,j\rangle\big)\\
&\qquad\qquad -q\langle j-2,j+2\rangle\langle j\rangle - q\langle j+2\rangle\langle j\rangle\langle j-2,j\rangle\\
&\quad = \big(\langle j\rangle\langle j+2\rangle - q\langle j+2\rangle\langle j\rangle\big)\langle j-2,j\rangle\\
&\qquad\qquad +q^{-1}\langle j\rangle\langle j-2,j+2\rangle - q\langle j-2,j+2\rangle\langle j\rangle\\
&\quad = \langle j,j+2\rangle\langle j-2,j\rangle + q^{-1}\langle j\rangle\langle j-2,j+2\rangle - q\langle j-2,j+2\rangle\langle j\rangle.
\end{aligned}\tag{3.2}$$

Similarly, we have

$$\begin{aligned}
&\langle j-2,j\rangle\langle j,j+2\rangle = \big(\langle j-2\rangle\langle j\rangle - q\langle j\rangle\langle j-2\rangle\big)\langle j,j+2\rangle\\
&\quad = q^{-1}\langle j-2\rangle\langle j,j+2\rangle\langle j\rangle - q\langle j\rangle\big(\langle j-2,j+2\rangle + q\langle j,j+2\rangle\langle j-2\rangle\big)\\
&\quad = q^{-1}\big(\langle j-2,j+2\rangle + q\langle j,j+2\rangle\langle j-2\rangle\big)\langle j\rangle\\
&\qquad -q\langle j\rangle\langle j-2,j+2\rangle - q\langle j,j+2\rangle\langle j\rangle\langle j-2\rangle\\
&\quad = \langle j,j+2\rangle\big(\langle j-2\rangle\langle j\rangle - q\langle j\rangle\langle j-2\rangle\big)\\
&\qquad\qquad +q^{-1}\langle j-2,j+2\rangle\langle j\rangle - q\langle j\rangle\langle j-2,j+2\rangle\\
&\quad = \langle j,j+2\rangle\langle j-2,j\rangle + q^{-1}\langle j-2,j+2\rangle\langle j\rangle - q\langle j\rangle\langle j-2,j+2\rangle.
\end{aligned}\tag{3.3}$$

Then, (3.2) and (3.3) imply (b) and (c). □

We shall resume the proof of Proposition 3.14. By Lemma 3.15 (b), $\langle i,k\rangle$ commutes with $\langle j\rangle$ for $i<j<k$. Thus we obtain (3).

We shall show (4) by the induction on $k-j$. Suppose $k-j=0$. The case $i=k-2$ is nothing but Lemma 3.15 (a).

If $i<k-2$, then

$$\begin{aligned}
\langle i,k\rangle\langle k\rangle &= \langle i,k-4\rangle\langle k-2,k\rangle\langle k\rangle - q\langle k-2,k\rangle\langle i,k-4\rangle\langle k\rangle\\
&= q^{-1}\langle k\rangle\langle i,k-4\rangle\langle k-2,k\rangle - \langle k\rangle\langle k-2,k\rangle\langle i,k-4\rangle = q^{-1}\langle k\rangle\langle i,k\rangle.
\end{aligned}$$

Suppose $k-j>0$. By using the induction hypothesis and (3), we have

$$\begin{aligned}
\langle i,k\rangle\langle j,k\rangle &= \langle i,k\rangle\langle j\rangle\langle j+2,k\rangle - q\langle i,k\rangle\langle j+2,k\rangle\langle j\rangle\\
&= \langle j\rangle\langle i,k\rangle\langle j+2,k\rangle - \langle j+2,k\rangle\langle i,k\rangle\langle j\rangle\\
&= q^{-1}\langle j\rangle\langle j+2,k\rangle\langle i,k\rangle - \langle j+2,k\rangle\langle j\rangle\langle i,k\rangle = q^{-1}\langle j,k\rangle\langle i,k\rangle.
\end{aligned}$$

Similarly we can prove (5).

Let us prove (6). We have

$$\begin{aligned}\langle i,k\rangle\langle j,\ell\rangle &= \big(\langle i,j-2\rangle\langle j,k\rangle - q\langle j,k\rangle\langle i,j-2\rangle\big)\langle j,\ell\rangle\\ &= q^{-1}\langle i,j-2\rangle\langle j,\ell\rangle\langle j,k\rangle - q\langle j,k\rangle\big(\langle i,\ell\rangle + q\langle j,\ell\rangle\langle i,j-2\rangle\big)\\ &= q^{-1}\big(\langle i,\ell\rangle + q\langle j,\ell\rangle\langle i,j-2\rangle\big)\langle j,k\rangle\\ &\qquad\qquad -q\langle i,\ell\rangle\langle j,k\rangle - q\langle j,\ell\rangle\langle j,k\rangle\langle i,j-2\rangle\\ &= \langle j,\ell\rangle\langle i,k\rangle + (q^{-1}-q)\langle i,\ell\rangle\langle j,k\rangle.\end{aligned}$$

□

Lemma 3.16.

(i) *For* $1 \leqslant i \leqslant j$, *we have* $\langle -j,-i\rangle\widetilde{\phi} = \langle i,j\rangle\widetilde{\phi}$.

(ii) *For* $1 \leqslant i < j$, *we have* $\langle -j,i\rangle\widetilde{\phi} = q^{-1}\langle -i,j\rangle\widetilde{\phi}$.

Proof. (i) If $i=j$, it is obvious. By the induction on $j-i$, we have

$$\begin{aligned}\langle -j,-i\rangle\widetilde{\phi} &= (\langle -j,-i-2\rangle\langle -i\rangle - q\langle -i\rangle\langle -j,-i-2\rangle)\widetilde{\phi}\\ &= (\langle -j,-i-2\rangle\langle i\rangle - q\langle -i\rangle\langle i+2,j\rangle)\widetilde{\phi}\\ &= (\langle i\rangle\langle -j,-i-2\rangle - q\langle i+2,j\rangle\langle -i\rangle)\widetilde{\phi}\\ &= (\langle i\rangle\langle i+2,j\rangle - q\langle i+2,j\rangle\langle i\rangle)\widetilde{\phi} = \langle i,j\rangle\widetilde{\phi}.\end{aligned}$$

(ii) By (i), we have

$$\begin{aligned}\langle -j,i\rangle\widetilde{\phi} &= (\langle -j,-1\rangle\langle 1,i\rangle - q\langle 1,i\rangle\langle -j,-1\rangle)\widetilde{\phi}\\ &= (\langle -j,-1\rangle\langle -i,-1\rangle - q\langle 1,i\rangle\langle 1,j\rangle)\widetilde{\phi}\\ &= (q^{-1}\langle -i,-1\rangle\langle -j,-1\rangle - \langle 1,j\rangle\langle 1,i\rangle)\widetilde{\phi}\\ &= (q^{-1}\langle -i,-1\rangle\langle 1,j\rangle - \langle 1,j\rangle\langle -i,-1\rangle)\widetilde{\phi} = q^{-1}\langle -i,j\rangle\widetilde{\phi}.\end{aligned}$$

□

Proposition 3.17.

(i) *For a multisegment* $\mathfrak{m} = \sum_{i\leqslant j} m_{i,j}\langle i,j\rangle$, *we have*

$$\mathrm{Ad}(t_k)P(\mathfrak{m}) = q^{\sum_i(m_{i,k-2}-m_{i,k})+\sum_j(m_{k+2,j}-m_{k,j})}P(\mathfrak{m}).$$

(ii)

$$e'_k\langle i,j\rangle^{(n)} = \begin{cases} q^{1-n}\langle i\rangle^{(n-1)} & \textit{if } k=i=j,\\ (1-q^2)q^{1-n}\langle i+2,j\rangle\langle i,j\rangle^{(n-1)} & \textit{if } k=i<j,\\ 0 & \textit{otherwise,}\end{cases}$$

$$e^*_k\langle i,j\rangle^{(n)} = \begin{cases} q^{1-n}\langle i\rangle^{(n-1)} & \textit{if } i=j=k,\\ (1-q^2)q^{1-n}\langle i,j\rangle^{(n-1)}\langle i,j-2\rangle & \textit{if } i<j=k,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. (i) is obvious. Let us show (ii). It is obvious that $e'_k\langle i,j\rangle^{(n)}=0$ unless $i \leqslant k \leqslant j$. It is known ([K1]) that we have $e'_k\langle k\rangle^{(n)} = q^{1-n}\langle k\rangle^{(n-1)}$. We shall prove $e'_k\langle k,j\rangle^{(n)} = (1-q^2)q^{1-n}\langle k+2,j\rangle\langle k,j\rangle^{(n-1)}$ for $k<j$ by the induction on n. By (2.1), we have

$$\begin{aligned} e'_k\langle k,j\rangle &= e'_k(\langle k\rangle\langle k+2,j\rangle - q\langle k+2,j\rangle\langle k\rangle)\\ &= \langle k+2,j\rangle - q^2\langle k+2,j\rangle = (1-q^2)\langle k+2,j\rangle. \end{aligned}$$

For $n \geqslant 1$, by the induction hypothesis and Proposition 3.14 (4), we get

$$\begin{aligned} &[n]e'_k\langle k,j\rangle^{(n)} = e'_k\langle k,j\rangle\langle k,j\rangle^{(n-1)}\\ &\quad= (1-q^2)\langle k+2,j\rangle\langle k,j\ \rangle^{(n-1)} + q^{-1}\langle k,j\rangle\cdot (1-q^2)q^{2-n}\langle k+2,j\rangle\langle k,j\rangle^{(n-2)}\\ &\quad= (1-q^2)\left\{\langle k+2,j\rangle\langle k,j\rangle^{(n-1)} + q^{1-n}\langle k,j\rangle\langle k+2,j\rangle\langle k,j\rangle^{(n-2)}\right\}\\ &\quad= (1-q^2)(1+q^{-n}[n-1])\langle k+2,j\rangle\langle k,j\rangle^{(n-1)}\\ &\quad= (1-q^2)q^{1-n}[n]\langle k+2,j\rangle\langle k,j\rangle^{(n-1)}. \end{aligned}$$

Finally we show $e'_k\langle i,j\rangle = 0$ if $k \neq i$. We may assume $i<k\leqslant j$. If $i<k<j$, we have

$$\begin{aligned} e'_k\langle i,j\rangle &= e'_k(\langle i,k-2\rangle\langle k,j\rangle - q\langle k,j\rangle\langle i,k-2\rangle)\\ &= q\langle i,k-2\rangle e'_k\langle k,j\rangle - q(e'_k\langle k,j\rangle)\langle i,k-2\rangle\\ &= q(1-q^2)\langle i,k-2\rangle\langle k+2,j\rangle - q(1-q^2)\langle k+2,j\rangle\langle i,k-2\rangle\\ &= 0. \end{aligned}$$

The case $k=j$ is similarly proved.

The proof for e^*_k is similar. □

§3.4. Actions of divided powers

Lemma 3.18. *Let a, b be non-negative integers, and let $k \in I_{>0} := \{k \in I \mid k > 0\}$.*

(1) *For $\ell > k$, we have*

$$\langle -k\rangle\langle -k+2,\ell\rangle^{(a)}\langle -k,\ell\rangle^{(b)} = [b+1]\langle -k+2,\ell\rangle^{(a-1)}\langle -k,\ell\rangle^{(b+1)} + q^{a-b}\langle -k+2,\ell\rangle^{(a)}\langle -k,\ell\rangle^{(b)}\langle -k\rangle.$$

(2) *We have*

$$\langle -k\rangle\langle -k+2,k\rangle^{(a)}\langle -k,k\rangle^{[b]} = [2b+2]\langle -k+2,k\rangle^{(a-1)}\langle -k,k\rangle^{[b+1]} + q^{a-b}\langle -k+2,k\rangle^{(a)}\langle -k,k\rangle^{[b]}\langle -k\rangle.$$

(3) *For $k > 1$, we have*

$$\langle -k\rangle\langle -k+2,k-2\rangle^{[a]} = (q^a+q^{-a})^{-1}\langle -k+2,k-2\rangle^{[a-1]}\langle -k,k-2\rangle + q^a\langle -k+2,k-2\rangle^{[a]}\langle -k\rangle.$$

(4) *If $\ell \leqslant k-2$, we have*

$$\langle \ell,k-2\rangle^{(a)}\langle k\rangle = \langle \ell,k\rangle\langle \ell,k-2\rangle^{(a-1)} + q^a\langle k\rangle\langle \ell,k-2\rangle^{(a)}.$$

(5) *For $k > 1$, we have*

$$\langle -k+2,k-2\rangle^{[a]}\langle k\rangle = (q^a+q^{-a})^{-1}\langle -k+2,k\rangle\langle -k+2,k-2\rangle^{[a-1]} + q^a\langle k\rangle\langle -k+2,k-2\rangle^{[a]}.$$

Proof. We show (1) by the induction on a. If $a = 0$, it is trivial. For $a > 0$, we have

$$\begin{aligned}
&[a]\langle -k\rangle\langle -k+2,\ell\rangle^{(a)}\langle -k,\ell\rangle^{(b)} \\
&\quad = \big(\langle -k,\ell\rangle + q\langle -k+2,\ell\rangle\langle -k\rangle\big)\langle -k+2,\ell\rangle^{(a-1)}\langle -k,\ell\rangle^{(b)} \\
&\quad = [b+1]q^{1-a}\langle -k+2,\ell\rangle^{(a-1)}\langle -k,\ell\rangle^{(b+1)} \\
&\qquad\quad + q\langle -k+2,\ell\rangle\big\{[b+1]\langle -k+2,\ell\rangle^{(a-2)}\langle -k,\ell\rangle^{(b+1)} \\
&\qquad\qquad\quad + q^{a-b-1}\langle -k+2,\ell\rangle^{(a-1)}\langle -k,\ell\rangle^{(b)}\langle -k\rangle\big\} \\
&\quad = [b+1](q^{1-a}+q[a-1])\langle -k+2,\ell\rangle^{(a-1)}\langle -k,\ell\rangle^{(b+1)} \\
&\qquad\qquad + q^{a-b}[a]\langle -k+2,\ell\rangle^{(a)}\langle -k,\ell\rangle^{(b)}\langle -k\rangle.
\end{aligned}$$

Since $q^{1-a}+q[a-1]=[a]$, the induction proceeds.

The proof of (2) is similar by using $\langle -k,k\rangle^{[b]}=[2b]\langle -k,k\rangle^{[b-1]}\langle -k,k\rangle$.

We prove (3) by the induction on a. The case $a=0$ is trivial. For $a>0$, we have

$$\begin{aligned}
&[2a]\langle -k\rangle\langle -k+2,k-2\rangle^{[a]}\\
&\quad=\big(\langle -k,k-2\rangle+q\langle -k+2,k-2\rangle\langle -k\rangle\big)\langle -k+2,k-2\rangle^{[a-1]}\\
&\quad=q^{1-a}\langle -k+2,k-2\rangle^{[a-1]}\langle -k,k-2\rangle\\
&\qquad+q\langle -k+2,k-2\rangle\big\{(q^{a-1}+q^{1-a})^{-1}\langle -k+2,k-2\rangle^{[a-2]}\langle -k,k-2\rangle\\
&\qquad\qquad+q^{a-1}\langle -k+2,k-2\rangle^{[a-1]}\langle -k\rangle\big\}\\
&\quad=\big(q^{1-a}+\frac{q[2a-2]}{q^{a-1}+q^{1-a}}\big)\langle -k+2,k-2\rangle^{[a-1]}\langle -k,k-2\rangle\\
&\qquad\qquad+q^{a}[2a]\langle -k+2,k-2\rangle^{[a]}\langle -k\rangle\\
&\quad=(q^{a}+q^{-a})^{-1}[2a]\langle -k+2,k-2\rangle^{[a-1]}\langle -k,k-2\rangle\\
&\qquad\qquad+q^{a}[2a]\langle -k+2,k-2\rangle^{[a]}\langle -k\rangle.
\end{aligned}$$

Similarly, we can prove (4) and (5) by the induction on a. □

Lemma 3.19. *For $k>1$ and $a,b,c,d\geqslant 0$, set*

$$(a,b,c,d)=\langle k\rangle^{(a)}\langle -k+2,k\rangle^{(b)}\langle -k,k\rangle^{[c]}\langle -k+2,k-2\rangle^{[d]}\widetilde{\phi}.$$

Then, we have

$$\begin{aligned}
\langle -k\rangle(a,b,c,d)=&[2c+2](a,b-1,c+1,d)\\
&+[b+1]q^{b-2c}(a,b+1,c,d-1)\\
&+[a+1]q^{2d-2c}(a+1,b,c,d).
\end{aligned}\tag{3.4}$$

Proof. We shall show first

$$\begin{aligned}
&\langle -k\rangle\langle -k+2,k-2\rangle^{[d]}\widetilde{\phi}\\
&\quad=\big(\langle -k+2,k\rangle\langle -k+2,k-2\rangle^{[d-1]}+q^{2d}\langle k\rangle\langle -k+2,k-2\rangle^{[d]}\big)\widetilde{\phi}.
\end{aligned}\tag{3.5}$$

By Lemma 3.18 (3), we have

$$\begin{aligned}
&\langle -k\rangle\langle -k+2,k-2\rangle^{[d]}\widetilde{\phi}\\
&\quad=\big((q^{d}+q^{-d})^{-1}\langle -k+2,k-2\rangle^{[d-1]}\langle -k,k-2\rangle\\
&\qquad\qquad+q^{d}\langle -k+2,k-2\rangle^{[d]}\langle -k\rangle\big)\widetilde{\phi}.
\end{aligned}$$

By Lemma 3.16 and Lemma 3.18 (5), it is equal to

$$\begin{aligned}
&\big((q^d+q^{-d})^{-1}q^{-1}\langle -k+2,k-2\rangle^{[d-1]}\langle -k+2,k\rangle+q^d\langle -k+2,k-2\rangle^{[d]}\langle k\rangle\big)\widetilde{\phi}\\
&\quad=\Big((q^d+q^{-d})^{-1}q^{-1}q^{1-d}\langle -k+2,k\rangle\langle -k+2,k-2\rangle^{[d-1]}\\
&\qquad\qquad+q^d\big((q^d+q^{-d})^{-1}\langle -k+2,k\rangle\langle -k+2,k-2\rangle^{[d-1]}\\
&\qquad\qquad\qquad+q^d\langle k\rangle\langle -k+2,k-2\rangle^{[d]}\big)\Big)\widetilde{\phi}.
\end{aligned}$$

Thus we obtain (3.5). Applying Lemma 3.18 (2), we have

$$\begin{aligned}
\langle -k\rangle(a,b,c,d)&=\langle k\rangle^{(a)}\Big([2c+2]\langle -k+2,k\rangle^{(b-1)}\langle -k,k\rangle^{[c+1]}\\
&\qquad+q^{b-c}\langle -k+2,k\rangle^{(b)}\langle -k,k\rangle^{[c]}\langle -k\rangle\Big)\langle -k+2,k-2\rangle^{[d]}\widetilde{\phi}\\
&=[2c+2](a,b-1,c+1,d)+q^{b-c}\langle k\rangle^{(a)}\langle -k+2,k\rangle^{(b)}\langle -k,k\rangle^{[c]}\\
&\qquad\times\big(\langle -k+2,k\rangle\langle -k+2,k-2\rangle^{[d-1]}+q^{2d}\langle k\rangle\langle -k+2,k-2\rangle^{[d]}\big)\widetilde{\phi}\\
&=[2c+2](a,b-1,c+1,d)+q^{b-2c}[b+1](a,b+1,c,d-1)\\
&\qquad+q^{(b-c)+2d-c-b}[a+1](a+1,b,c,d).
\end{aligned}$$

Hence we have (3.4). □

Proposition 3.20.

(1) *We have*

$$\langle -1\rangle^{(a)}\langle -1,1\rangle^{[m]}\widetilde{\phi}=\sum_{s=0}^{\lfloor a/2\rfloor}\left(\prod_{\nu=1}^{s}\frac{[2m+2\nu]}{[2\nu]}\right)q^{-2(a-s)m+\frac{(a-2s)(a-2s-1)}{2}}\times\langle 1\rangle^{(a-2s)}\langle -1,1\rangle^{[m+s]}\widetilde{\phi}.$$

(2) *For $k>1$, we have*

$$\begin{aligned}
&\langle -k\rangle^{(n)}\langle -k+2,k-2\rangle^{[a]}\widetilde{\phi}\\
&\quad=\sum_{i+j+2t=n,\,j+t=u}q^{2ai+\frac{j(j-1)}{2}-i(t+u)}\\
&\qquad\qquad\times\langle k\rangle^{(i)}\langle -k+2,k\rangle^{(j)}\langle -k,k\rangle^{[t]}\langle -k+2,k-2\rangle^{[a-u]}\widetilde{\phi}.
\end{aligned}$$

(3) *If $\ell>k$, we have*

$$\langle k\rangle^{(n)}\langle k+2,\ell\rangle^{(a)}=\sum_{s=0}^{n}q^{(n-s)(a-s)}\langle k+2,\ell\rangle^{(a-s)}\langle k,\ell\rangle^{(s)}\langle k\rangle^{(n-s)}.$$

Proof. We prove (1) by the induction on a. The case $a = 0$ is trivial. Assume $a > 0$. Then, Lemma 3.18 (2) implies

$$\begin{aligned}
&\langle -1\rangle\langle 1\rangle^{(n)}\langle -1,1\rangle^{[m]}\widetilde{\phi}\\
&\quad= \big([2m+2]\langle 1\rangle^{(n-1)}\langle -1,1\rangle^{[m+1]} + q^{n-m}\langle 1\rangle^{(n)}\langle -1,1\rangle^{[m]}\langle -1\rangle\big)\widetilde{\phi}\\
&\quad= \big([2m+2]\langle 1\rangle^{(n-1)}\langle -1,1\rangle^{[m+1]} + q^{n-m}\langle 1\rangle^{(n)}\langle -1,1\rangle^{[m]}\langle 1\rangle\big)\widetilde{\phi}\\
&\quad= \big([2m+2]\langle 1\rangle^{(n-1)}\langle -1,1\rangle^{[m+1]} + q^{n-2m}[n+1]\langle 1\rangle^{(n+1)}\langle -1,1\rangle^{[m]}\big)\widetilde{\phi}.
\end{aligned}$$

Put

$$c_s = \left(\prod_{\nu=1}^{s}\frac{[2m+2\nu]}{[2\nu]}\right) q^{-2(a-s)m+\frac{(a-2s)(a-2s-1)}{2}}.$$

Then we have

$$\begin{aligned}
[a+1]\langle -1\rangle^{(a+1)}\langle -1,1\rangle^{[m]}\widetilde{\phi} &= \langle -1\rangle\langle -1\rangle^{(a)}\langle -1,1\rangle^{[m]}\widetilde{\phi}\\
&= \langle -1\rangle\sum_{s=0}^{\lfloor a/2\rfloor} c_s\langle 1\rangle^{(a-2s)}\langle -1,1\rangle^{[m+s]}\widetilde{\phi}\\
&= \sum_{s=0}^{\lfloor a/2\rfloor} c_s\big\{[2(m+s+1)]\langle 1\rangle^{(a-2s-1)}\langle -1,1\rangle^{[m+s+1]}\\
&\qquad + q^{a-2s-2(m+s)}[a-2s+1]\langle 1\rangle^{(a-2s+1)}\langle -1,1\rangle^{[m+s]}\big\}\widetilde{\phi}.
\end{aligned}$$

In the right-hand-side, the coefficients of $\langle 1\rangle^{a+1-2r}\langle -1,1\rangle^{[m+r]}\widetilde{\phi}$ are

$$\begin{aligned}
&[2(m+r)]c_{r-1} + q^{a-2m-4r}[a-2r+1]c_r\\
&= \prod_{\nu=1}^{r}\frac{[2m+2\nu]}{[2\nu]} q^{-2(a-r+1)m+\frac{(a-2r)(a-2r+1)}{2}}\Big([2r]q^{a-2r+1} + [a-2r+1]q^{-2r}\Big)\\
&= [a+1]\prod_{\nu=1}^{r}\frac{[2m+2\nu]}{[2\nu]} q^{-2(a-r+1)m+\frac{(a-2r)((a-2r+1)}{2}}.
\end{aligned}$$

Hence we obtain (1).

We prove (2) by the induction on n. We use the following notation for short:

$$(i,j,t,a) := \langle k\rangle^{(i)}\langle -k+2,k\rangle^{(j)}\langle -k,k\rangle^{[t]}\langle -k+2,k-2\rangle^{[a]}\widetilde{\phi}.$$

Then Lemma 3.19 implies that

$$\begin{aligned}
\langle -k\rangle(i,j,t,a) = {}&[2t+2](i,j-1,t+1,a)\\
&+[j+1]q^{j-2t}(i,j+1,t,a-1)\\
&+[i+1]q^{2a-2t}(i+1,j,t,a).
\end{aligned}$$

Hence, by assuming (2) for n, we have

$$[n+1]\langle -k\rangle^{(n+1)}\langle -k+2,k-2\rangle^{[a]}\widetilde{\phi} = \langle -k\rangle\langle -k\rangle^{(n)}\langle -k+2,k-2\rangle^{[a]}\widetilde{\phi}$$
$$= \sum_{i+j+2t=n, j+t=u} \left\{ \begin{array}{l} [2t+2]q^{2ai+\frac{j(j-1)}{2}-i(t+u)}(i,j-1,t+1,a-u) \\ +[j+1]q^{2ai+\frac{j(j-1)}{2}-i(t+u)+j-2t}(i,j+1,t,a-u-1) \\ +[i+1]q^{2ai+\frac{j(j-1)}{2}-i(t+u)+2a-2u-2t}(i+1,j,t,a-u) \end{array} \right\}.$$

Then in the right hand side, the coefficients of $(i',j',t',a-u')$ satisfying $i'+j'+2t'=n+1, j'+t'=u'$ are

$$\begin{aligned} &[2t']q^{2ai'+\frac{(j'+1)j'}{2}-i'(t'-1+u')} + [j']q^{2ai'+\frac{(j'-1)(j'-2)}{2}-i'(t'+u'-1)+j'-1-2t'} \\ &\qquad +[i']q^{2a(i'-1)+\frac{j'(j'-1)}{2}-(i'-1)(t'+u')+2a-2u'-2t'} \\ &= q^{2ai'+\frac{j'(j'-1)}{2}-i'(t'+u')}\Big([2t']q^{j'+i'} + [j']q^{i'-2t'} + [i']q^{-(t'+u')}\Big) \\ &= q^{2ai'+\frac{j'(j'-1)}{2}-i'(t'+u')}[n+1]. \end{aligned}$$

We can prove (3) similarly as above. □

§3.5. Actions of E_k, F_k on the PBW basis

For a θ-restricted multisegment $\mathfrak{m}$, we set

$$\widetilde{P}_\theta(\mathfrak{m}) = P_\theta(\mathfrak{m})\widetilde{\phi}.$$

We understand $\widetilde{P}_\theta(\mathfrak{m}) = 0$ if $\mathfrak{m}$ is not a multisegment.

Theorem 3.21. *For $k \in I_{>0}$ and a θ-restricted multisegment $\mathfrak{m} = \sum_{-j\leqslant i\leqslant j} m_{i,j}\langle i,j\rangle$, we have*

$$\begin{aligned} &F_{-k}\widetilde{P}_\theta(\mathfrak{m}) \\ &= \sum_{\ell>k}[m_{-k,\ell}+1]q^{\sum_{\ell'>\ell}(m_{-k+2,\ell'}-m_{-k,\ell'})}\widetilde{P}_\theta(\mathfrak{m}-\langle -k+2,\ell\rangle+\langle -k,\ell\rangle) \\ &\quad + q^{\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})}[2m_{-k,k}+2]\widetilde{P}_\theta(\mathfrak{m}-\langle -k+2,k\rangle+\langle -k,k\rangle) \\ &\quad + q^{\sum_{\ell>k}(m_{-k+2,k}-m_{-k,k})+m_{-k+2,k}-2m_{-k,k}} \\ &\qquad\qquad \times[m_{-k+2,k}+1]\widetilde{P}_\theta(\mathfrak{m}-\delta_{k\neq 1}\langle -k+2,k-2\rangle+\langle -k+2,k\rangle) \\ &\quad + \sum_{-k+2<i\leqslant k} q^{\sum_{\ell>k}(m_{-k+2,k}-m_{-k,k})+2m_{-k+2,k-2}-2m_{-k,k}+\sum_{-k+2<j<i}(m_{j,k-2}-m_{j,k})} \\ &\qquad\qquad \times[m_{i,k}+1]\widetilde{P}_\theta(\mathfrak{m}-\delta_{i<k}\langle i,k-2\rangle+\langle i,k\rangle). \end{aligned}$$

Proof. We divide $\mathfrak{m}$ into four parts

$$\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 + \delta_{k\neq 1} m_{-k+2,k-2}\langle -k+2, k-2\rangle,$$

where $\mathfrak{m}_1 = \sum_{j>k} m_{i,j}\langle i,j\rangle$, $\mathfrak{m}_2 = \sum_{j=k} m_{i,j}\langle i,j\rangle$, $\mathfrak{m}_3 = \sum_{-k+2<i\leqslant j\leqslant k-2} m_{i,j}\langle i,j\rangle$. Then Proposition 3.14 implies

$$\widetilde{P}_\theta(\mathfrak{m}) = P_\theta(\mathfrak{m}_1)P_\theta(\mathfrak{m}_2)P_\theta(\mathfrak{m}_3)\langle -k+2, k-2\rangle^{[m_{-k+2,k-2}]}\widetilde{\phi}.$$

If $k=1$, we understand $\langle -k+2, k-2\rangle^{[n]} = 1$. By Lemma 3.18 (1), we have

$$\begin{aligned}
&\langle -k\rangle P_\theta(\mathfrak{m}_1) \\
&\quad = \sum_{\ell>k} q^{\sum_{\ell'>\ell}(m_{-k+2,\ell'}-m_{-k,\ell'})}[m_{-k,\ell}+1]P_\theta(\mathfrak{m}_1 - \langle -k+2,\ell\rangle + \langle -k,\ell\rangle) \\
&\qquad\qquad + q^{\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})}P_\theta(\mathfrak{m}_1)\langle -k\rangle,
\end{aligned}$$

and Lemma 3.18 (2) implies

$$\begin{aligned}
\langle -k\rangle P_\theta(\mathfrak{m}_2) &= [2m_{-k,k}+2]P_\theta(\mathfrak{m}_2 - \langle -k+2,k\rangle + \langle -k,k\rangle) \\
&\qquad + q^{m_{-k+2,k}-m_{-k,k}}P_\theta(\mathfrak{m}_2)\langle -k\rangle.
\end{aligned}$$

Since we have $\langle -k\rangle P_\theta(\mathfrak{m}_3) = P_\theta(\mathfrak{m}_3)\langle -k\rangle$, we obtain

$$\begin{aligned}
(3.6)\quad \langle -k\rangle \widetilde{P}_\theta(\mathfrak{m}) = &\textstyle\sum_{\ell>k} q^{\sum_{\ell'>\ell}(m_{-k+2,\ell'}-m_{-k,\ell'})}[m_{-k,\ell}+1] \\
&\qquad \times \widetilde{P}_\theta(\mathfrak{m} - \langle -k+2,\ell\rangle + \langle -k,\ell\rangle) \\
&+ q^{\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})}[2m_{-k,k}+2] \\
&\qquad \times \widetilde{P}_\theta(\mathfrak{m} - \langle -k+2,k\rangle + \langle -k,k\rangle) \\
&+ q^{\sum_{\ell\geqslant k}(m_{-k+2,\ell}-m_{-k,\ell})}P_\theta(\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{m}_3) \\
&\qquad \times \langle -k\rangle\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}]}\widetilde{\phi}.
\end{aligned}$$

By (3.5), we have

$$\begin{aligned}
&\langle -k\rangle\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}]}\widetilde{\phi} \\
&\quad = \langle -k+2,k\rangle\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}-1]}\widetilde{\phi} \\
&\qquad + \delta_{k\neq 1} q^{2m_{-k+2,k-2}}\langle k\rangle\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}]}\widetilde{\phi}.
\end{aligned}$$

Hence the last term in (3.6) is equal to

$$\begin{aligned}
&q^{\sum_{\ell\geqslant k}(m_{-k+2,\ell}-m_{-k,\ell})-m_{-k,k}} \\
&\qquad \times [m_{-k+2,k}+1]\widetilde{P}_\theta(\mathfrak{m} - \delta_{k\neq 1}\langle -k+2,k-2\rangle + \langle -k+2,k\rangle) \\
&+ \delta_{k\neq 1} q^{\sum_{\ell\geqslant k}(m_{-k+2,\ell}-m_{-k,\ell})+2m_{-k+2,k-2}} \\
&\qquad \times P_\theta(\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{m}_3)\langle k\rangle\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}]}\widetilde{\phi}.
\end{aligned}$$

For $k \neq 1$, Lemma 3.18 (4) implies

$$P_\theta(\mathfrak{m}_3)\langle k\rangle = \sum_{-k+2<i\leqslant k} q^{\sum_{-k+2<j<i} m_{j,k-2}}\langle i,k\rangle P_\theta(\mathfrak{m}_3 - \delta_{i<k}\langle i,k-2\rangle),$$

and Proposition 3.14 implies

$$P_\theta(\mathfrak{m}_2)\langle i,k\rangle = q^{-\sum_{j<i} m_{j,k}}[m_{i,k}+1]P_\theta(\mathfrak{m}_2 + \langle i,k\rangle).$$

Hence we obtain

$$\begin{aligned} &P_\theta(\mathfrak{m}_1)P_\theta(\mathfrak{m}_2)P_\theta(\mathfrak{m}_3)\langle k\rangle\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}]}\widetilde{\phi} \\ &\quad= \sum_{-k+2<i\leqslant k} q^{\sum_{-k+2<j<i} m_{j,k-2} - \sum_{-k\leqslant j<i} m_{j,k}} \\ &\qquad\qquad \times[m_{i,k}+1]\widetilde{P}_\theta(\mathfrak{m} - \delta_{i<k}\langle i,k-2\rangle + \langle i,k\rangle). \end{aligned}$$

Thus we obtain the desired result. □

Theorem 3.22. *For $k \in I_{>0}$ and a θ-restricted multisegment $\mathfrak{m} = \sum_{-j\leqslant i\leqslant j} m_{i,j}\langle i,j\rangle$, we have*

$$\begin{aligned} &E_{-k}\widetilde{P}_\theta(\mathfrak{m}) \\ &= (1-q^2)\sum_{\ell>k} q^{1+\sum_{\ell'\geqslant\ell}(m_{-k+2,\ell'}-m_{-k,\ell'})} \\ &\qquad\qquad\times[m_{-k+2,\ell}+1]\widetilde{P}_\theta(\mathfrak{m} - \langle -k,\ell\rangle + \langle -k+2,\ell\rangle) \\ &+(1-q^2)q^{1+\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})+m_{-k+2,k}-2m_{-k,k}} \\ &\qquad\qquad\times[m_{-k+2,k}+1]\widetilde{P}_\theta(\mathfrak{m} - \langle -k,k\rangle + \langle -k+2,k\rangle) \\ &+(1-q^2)\sum_{-k+2<i\leqslant k-2} q^{1+\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})+2m_{-k+2,k-2}-2m_{-k,k}+\sum_{-k+2<i'\leqslant i}(m_{i,k-2}-m_{i'k})} \\ &\qquad\qquad\times[m_{i,k-2}+1]\widetilde{P}_\theta(\mathfrak{m} - \langle i,k\rangle + \langle i,k-2\rangle) \\ &+\delta_{k\neq1}(1-q^2)q^{1+\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})+2m_{-k+2,k-2}-2m_{-k,k}} \\ &\qquad\times[2(m_{-k+2,k-2}+1)]\widetilde{P}_\theta(\mathfrak{m} - \langle -k+2,k\rangle + \langle -k+2,k-2\rangle) \\ &+q^{\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})-2m_{-k,k}+\delta_{k\neq1}\left(1-m_{k,k}+2m_{-k+2,k-2}+\sum_{-k+2<i\leqslant k-2}(m_{i,k-2}-m_{i,k})\right)} \\ &\qquad\qquad\qquad\qquad\times\widetilde{P}_\theta(\mathfrak{m} - \langle k\rangle). \end{aligned}$$

Proof. We shall divide $\mathfrak{m}$ into

$$\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3$$

where $\mathfrak{m}_1 = \sum_{i\leqslant j, j>k} m_{i,j}\langle i,j\rangle$ and $\mathfrak{m}_2 = \sum_{i\leqslant k} m_{i,k}\langle i,k\rangle$ and $\mathfrak{m}_3 = \sum_{i\leqslant j<k} m_{i,j}\langle i,j\rangle$. By (2.3) and Proposition 3.17, we have

$$
\begin{aligned}
E_{-k}\widetilde{P}_\theta(\mathfrak{m}) = \Big(& \big(e'_{-k}P_\theta(\mathfrak{m}_1)\big)P_\theta(\mathfrak{m}_2+\mathfrak{m}_3) \\
& +(\mathrm{Ad}(t_{-k})P_\theta(\mathfrak{m}_1))(e'_{-k}P_\theta(\mathfrak{m}_2+\mathfrak{m}_3)) \\
& + \mathrm{Ad}(t_{-k})\left\{P_\theta(\mathfrak{m}_1)\big(e^*_kP_\theta(\mathfrak{m}_2)\big)\,\mathrm{Ad}(t_k)P_\theta(\mathfrak{m}_3)\right\}\Big)\widetilde{\phi}.
\end{aligned} \tag{3.7}
$$

By Proposition 3.17, the first term is

$$
\begin{aligned}
&\big(e'_{-k}P_\theta(\mathfrak{m}_1)\big)P_\theta(\mathfrak{m}_2+\mathfrak{m}_3) \\
&\quad = (1-q^2)\textstyle\sum_{\ell>k} q^{1+\sum_{\ell'\geqslant\ell}(m_{-k+2,\ell'}-m_{-k,\ell'})} \\
&\qquad\qquad \times[m_{-k+2,\ell}+1]P_\theta(\mathfrak{m}-\langle -k,\ell\rangle+\langle -k+2,\ell\rangle).
\end{aligned} \tag{3.8}
$$

The second term is

$$
\begin{aligned}
&(\mathrm{Ad}(t_{-k})P_\theta(\mathfrak{m}_1))(e'_{-k}P_\theta(\mathfrak{m}_2+\mathfrak{m}_3)) \\
&\quad = q^{\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})}\frac{[m_{-k,k}][m_{-k+2,k}+1]}{[2m_{-k,k}]} \\
&\qquad \times(1-q^2)q^{1-m_{-k,k}+m_{-k+2,k}}P_\theta(\mathfrak{m}-\langle -k,k\rangle+\langle -k+2,k\rangle).
\end{aligned} \tag{3.9}
$$

Let us calculate the last part of (3.7). We have

$$
\begin{aligned}
&\mathrm{Ad}(t_{-k})\Big(P_\theta(\mathfrak{m}_1)\big(e^*_kP_\theta(\mathfrak{m}_2)\big)\,\mathrm{Ad}(t_k)P_\theta(\mathfrak{m}_3)\Big) \\
&\quad = q^{\sum_{\ell}(m_{-k+2,\ell}-m_{-k,\ell})+\sum_{i\leqslant k-2}m_{i,k-2}-\delta_{k=1}}P_\theta(\mathfrak{m}_1)\big(e^*_kP_\theta(\mathfrak{m}_2)\big)P_\theta(\mathfrak{m}_3).
\end{aligned}
$$

We have

$$
\begin{aligned}
e^*_kP_\theta(\mathfrak{m}_2) = {} & q^{1-m_k-\sum_{i<k}m_{i,k}}P_\theta(\mathfrak{m}_2-\langle k\rangle) \\
& +(1-q^2)\sum_{-k<i<k} q^{1-m_{i,k}-\sum_{i'<i}m_{i',k}}P_\theta(\mathfrak{m}_2-\langle i,k\rangle)\langle i,k-2\rangle \\
& +\frac{[m_{-k,k}]}{[2m_{-k,k}]}(1-q^2)q^{1-m_{-k,k}}P(\mathfrak{m}_2-\langle -k,k\rangle)\langle -k,k-2\rangle.
\end{aligned}
$$

For $-k<i<k$, we have

$$
\begin{aligned}
&\langle i,k-2\rangle P_\theta(\mathfrak{m}_3) \\
&\quad = q^{-\sum_{i'>i}m_{i',k-2}}[(1+\delta_{i=-k+2})(m_{i,k-2}+1)]P_\theta(\mathfrak{m}_3+\langle i,k-2\rangle).
\end{aligned}
$$

By Lemma 3.16, we have

$$\begin{aligned}
&\langle -k, k-2\rangle P_\theta(\mathfrak{m}_3)\widetilde{\phi} \\
&\quad = q^{-\sum_{-k+2\leqslant k\leqslant k-2} m_{i,k-2}} P_\theta(\mathfrak{m}_3)\langle -k, k-2\rangle\widetilde{\phi} \\
&\quad = q^{-\sum_{-k+2\leqslant k\leqslant k-2} m_{i,k-2}-\delta_{k\neq 1}} P_\theta(\mathfrak{m}_3)\langle -k+2, k\rangle\widetilde{\phi} \\
&\quad = q^{-m_{-k+2,k-2}-\sum_{-k+2\leqslant i\leqslant k-2} m_{i,k-2}-\delta_{k\neq 1}} \langle -k+2, k\rangle P_\theta(\mathfrak{m}_3)\widetilde{\phi}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&P_\theta(\mathfrak{m}_1)\big(e_k^* P_\theta(\mathfrak{m}_2)\big)P_\theta(\mathfrak{m}_3)\widetilde{\phi} \\
&= q^{1-\sum_{i\leqslant k} m_{i,k}} \widetilde{P}_\theta(\mathfrak{m}-\langle k\rangle) \\
&+(1-q^2)\sum_{-k+2<i\leqslant k-2} q^{1-\sum_{i'\leqslant i} m_{i',k}-\sum_{i'>i} m_{i',k-2}} \\
&\qquad\qquad \times[m_{i,k-2}+1]\widetilde{P}_\theta(\mathfrak{m}-\langle i,k\rangle+\langle i,k-2\rangle) \\
&+(1-q^2)\delta_{k\neq 1} q^{1-m_{-k,k}-m_{-k+2,k}-\sum_{-k+2<i} m_{i,k-2}} \\
&\qquad \times[2(m_{-k+2,k-2}+1)]\widetilde{P}_\theta(\mathfrak{m}-\langle -k+2,k\rangle+\langle -k+2,k-2\rangle) \\
&+(1-q^2)q^{2(1-m_{-k,k})-m_{-k+2,k-2}-\sum_{-k+2\leqslant i\leqslant k-2} m_{i,k-2}-\delta_{k\neq 1}} \\
&\qquad\qquad \times\frac{[m_{-k+2,k}+1][m_{-k,k}]}{[2m_{-k,k}]} P(\mathfrak{m}-\langle -k,k\rangle+\langle -k+2,k\rangle).
\end{aligned}$$

Hence the coefficient of $\widetilde{P}_\theta(\mathfrak{m}-\langle k\rangle)$ in $E_{-k}\widetilde{P}_\theta(\mathfrak{m})$ is

$$\begin{aligned}
&q^{\sum_\ell (m_{-k+2,\ell}-m_{-k,\ell})+\sum_{i\leqslant k-2} m_{i,k-2}-\delta_{k=1}+1-\sum_{i\leqslant k} m_{i,k}} \\
&= q^{\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})-2m_{-k,k}+\delta_{k\neq 1}\left(1-m_{k,k}+2m_{-k+2,k-2}+\sum_{-k+2<i\leqslant k-2}(m_{i,k-2}-m_{i,k})\right)}.
\end{aligned}$$

The coefficient of $\widetilde{P}_\theta(\mathfrak{m}-\langle -k,k\rangle+\langle -k+2,k\rangle)$ in $E_{-k}\widetilde{P}_\theta(\mathfrak{m})$ is

$$\begin{aligned}
&(1-q^2)q^{1+\sum_{\ell\geqslant k}(m_{-k+2,\ell}-m_{-k,\ell})}\frac{[m_{-k,k}][m_{-k+2,k}+1]}{[2m_{-k,k}]} \\
&+q^{\sum_\ell (m_{-k+2,\ell}-m_{-k,\ell})+\sum_{i\leqslant k-2} m_{i,k-2}-\delta_{k=1}+2(1-m_{-k,k})-m_{-k+2,k-2}-\sum_{-k+2\leqslant i\leqslant k-2} m_{i,k-2}-\delta_{k\neq 1}} \\
&\qquad\qquad \times(1-q^2)\frac{[m_{-k+2,k}+1][m_{-k,k}]}{[2m_{-k,k}]} \\
&= (1-q^2)q^{1+\sum_{\ell\geqslant k}(m_{-k+2,\ell}-m_{-k,\ell})}\frac{[m_{-k,k}][m_{-k+2,k}+1]}{[2m_{-k,k}]}(1+q^{-2m_{-k,k}}) \\
&= (1-q^2)q^{1-m_{-k,k}+\sum_{\ell\geqslant k}(m_{-k+2,\ell}-m_{-k,\ell})}[m_{-k+2,k}+1] \\
&= (1-q^2)q^{1+m_{-k+2,k}-2m_{-k,k}+\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})}[m_{-k+2,k}+1].
\end{aligned}$$

For $-k+2 < i \leqslant k-2$, the coefficient of $\widetilde{P}_\theta(\mathfrak{m} - \langle i,k\rangle + \langle i,k-2\rangle)$ in $E_{-k}\widetilde{P}_\theta(\mathfrak{m})$ is

$$\begin{aligned}&(1-q^2)q^{\sum_\ell (m_{-k+2,\ell}-m_{-k,\ell})+\sum_{i'\leqslant k-2} m_{i',k-2}-\delta_{k=1}+1-\sum_{i'\leqslant i} m_{i',k}-\sum_{i'>i} m_{i',k-2}}[m_{i,k-2}+1]\\ &= (1-q^2)\\ &\quad\times q^{1+\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})+2m_{-k+2,k-2}-2m_{-k,k}+\sum_{-k+2<i'\leqslant i}(m_{i,k-2}-m_{i',k})}[m_{i,k-2}+1].\end{aligned}$$

Finally, for $k \neq 1$, the coefficient of $\widetilde{P}_\theta(\mathfrak{m} - \langle -k+2,k\rangle + \langle -k+2,k-2\rangle)$ in $E_{-k}\widetilde{P}_\theta(\mathfrak{m})$ is

$$\begin{aligned}&(1-q^2)q^{\sum_\ell (m_{-k+2,\ell}-m_{-k,\ell})+\sum_{i\leqslant k-2} m_{i,k-2}-\delta_{k=1}+1-m_{-k,k}-m_{-k+2,k}-\sum_{-k+2<i} m_{i,k-2}}\\ &\qquad\times[2(m_{-k+2,k-2}+1)]\\ &= (1-q^2)q^{1+\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})+2m_{-k+2,k-2}-2m_{-k,k}}[2(m_{-k+2,k-2}+1)].\end{aligned}$$

□

Theorem 3.23. *For $k > 0$ and $\mathfrak{m} \in \mathcal{M}_\theta$, we have*

$$\begin{aligned}E_k\widetilde{P}_\theta(\mathfrak{m}) &= \textstyle\sum_{\ell>k}(1-q^2)q^{1+\sum_{\ell'\geqslant\ell}(m_{k+2,\ell'}-m_{k,\ell'})}\\ &\qquad\times[m_{k+2,\ell}+1]\widetilde{P}_\theta(\mathfrak{m} - \langle k,\ell\rangle + \langle k+2,\ell\rangle)\\ &\quad+q^{1+\sum_{\ell>k}(m_{k+2,\ell}-m_{k,\ell})-m_{k,k}}\widetilde{P}_\theta(\mathfrak{m} - \langle k\rangle),\\ F_k\widetilde{P}_\theta(\mathfrak{m}) &= \sum_{\ell\geqslant k} q^{\sum_{\ell'>\ell}(m_{k+2,\ell'}-m_{k,\ell'})}[m_{k,\ell}+1]\widetilde{P}_\theta(\mathfrak{m} - \delta_{\ell\neq k}\langle k+2,\ell\rangle + \langle k,\ell\rangle).\end{aligned}$$

Proof. The first follows from $e^*_{-k}P_\theta(\mathfrak{m}) = 0$ and Proposition 3.17, and the second follows from Proposition 3.20. □

§4. Crystal Basis of $V_\theta(0)$

§4.1. A criterion for crystals

We shall give a criterion for a basis to be a crystal basis. Although we treat the case for modules over $\mathcal{B}(\mathfrak{g})$ in this paper, similar results hold also for $U_q(\mathfrak{g})$.

Let $\mathbf{K}[e,f]$ be the ring generated by e and f with the defining relation $ef = q^{-2}fe + 1$. We define the divided power by $f^{(n)} = f^n/[n]!$.

Let P be a free $\mathbb{Z}$-module, and let α be a non-zero element of P.

Let M be a $\mathbf{K}[e,f]$-module. Assume that M has a weight decomposition $M = \oplus_{\xi \in P} M_\xi$, and $eM_\lambda \subset M_{\lambda+\alpha}$ and $fM_\lambda \subset M_{\lambda-\alpha}$.

Assume the following finiteness conditions:

(4.1) for any $\lambda \in P$, $\dim M_\lambda < \infty$ and $M_{\lambda+n\alpha} = 0$ for $n \gg 0$.

Hence for any $u \in M$, we can write $u = \sum_{n \geqslant 0} f^{(n)} u_n$ with $eu_n = 0$. We define endomorphisms $\tilde{e}$ and $\tilde{f}$ of M by

$$\tilde{e}u = \sum_{n \geqslant 1} f^{(n-1)} u_n,$$
$$\tilde{f}u = \sum_{n \geqslant 0} f^{(n+1)} u_n.$$

Let B be a crystal with weight decomposition by P. In this paper, we consider only the following type of crystals. We have $\mathrm{wt}\colon B \to P$, $\tilde{f}\colon B \to B$, $\tilde{e}\colon B \to B \sqcup \{0\}$, $\varepsilon\colon B \to \mathbb{Z}_{\geqslant 0}$ satisfying the following properties, where $B_\lambda := \mathrm{wt}^{-1}(\lambda)$:

(i) $\tilde{f} B_\lambda \subset B_{\lambda-\alpha}$ and $\tilde{e} B_\lambda \subset B_{\lambda+\alpha} \sqcup \{0\}$ for any $\lambda \in P$,

(ii) $\tilde{f}\tilde{e}(b) = b$ if $\tilde{e}b \neq 0$, and $\tilde{e} \circ \tilde{f} = \mathrm{id}_B$,

(iii) for any $\lambda \in P$, B_λ is a finite set and $B_{\lambda+n\alpha} = \emptyset$ for $n \gg 0$,

(iv) $\varepsilon(b) = \max\{n \geqslant 0 \mid \tilde{e}^n b \neq 0\}$ for any $b \in B$.

Set $\mathrm{ord}(a) = \sup\{n \in \mathbb{Z} \mid a \in q^n \mathbf{A}_0\}$ for $a \in \mathbf{K}$. We understand $\mathrm{ord}(0) = \infty$.

Let $\{C(b)\}_{b \in B}$ be a system of generators of M with $C(b) \in M_{\mathrm{wt}(b)}$: $M = \sum_{b \in B} \mathbf{K} C(b)$.

Let ξ be a map from B to an ordered set. Let $c\colon \mathbb{Z} \to \mathbb{R}$, $f\colon \mathbb{Z} \to \mathbb{R}$ and $e\colon \mathbb{Z} \to \mathbb{R}$. Assume that a decomposition $B = B' \cup B''$ is given.

Assume that we have expressions:

$$eC(b) = \sum_{b' \in B} E_{b,b'} C(b'), \tag{4.2}$$

$$fC(b) = \sum_{b' \in B} F_{b,b'} C(b'). \tag{4.3}$$

Now consider the following conditions for these data, where $\ell = \varepsilon(b)$ and $\ell' = \varepsilon(b')$:

(4.4) $c(0) = 0$, and $c(n) > 0$ for $n \neq 0$, (4.5) $c(n) \leqslant n + c(m+n) + e(m)$ for $n \geqslant 0$,

(4.6) $c(n) \leqslant c(m+n) + f(m)$ for $n \leqslant 0$,

(4.7) $c(n) + f(n) > 0$ for $n > 0$,

(4.8) $c(n) + e(n) > 0$ for $n > 0$,

(4.9) $\operatorname{ord}(F_{b,b'}) \geqslant -\ell + f(\ell + 1 - \ell')$,

(4.10) $\operatorname{ord}(E_{b,b'}) \geqslant 1 - \ell + e(\ell - 1 - \ell')$,

(4.11) $F_{b,\tilde{f}b} \in q^{-\ell}(1 + q\mathbf{A}_0)$,

(4.12) $E_{b,\tilde{e}b} \in q^{1-\ell}(1 + q\mathbf{A}_0)$ if $\ell > 0$,

(4.13) $\operatorname{ord}(F_{b,b'}) > -\ell + f(\ell + 1 - \ell')$ if $b' \neq \tilde{f}b$, $\xi(\tilde{f}b) \not> \xi(b')$,

(4.14) $\operatorname{ord}(F_{b,b'}) > -\ell + f(\ell + 1 - \ell')$ if $\tilde{f}b \in B'$, $b' \neq \tilde{f}b$ and $\ell \leqslant \ell' - 1$,

(4.15) $\operatorname{ord}(E_{b,b'}) > 1 - \ell + e(\ell - 1 - \ell')$ if $b \in B''$, $b' \neq \tilde{e}b$ and $\ell \leqslant \ell' + 1$.

Theorem 4.1. *Assume the conditions* (4.4)–(4.15)*. Let L be the $\mathbf{A}_0$-submodule $\sum_{b\in B} \mathbf{A}_0 C(b)$ of M. Then we have $\tilde{e}L \subset L$ and $\tilde{f}L \subset L$. Moreover we have*

$$\tilde{e}C(b) \equiv C(\tilde{e}b) \bmod qL \quad \textit{and} \quad \tilde{f}C(b) \equiv C(\tilde{f}b) \bmod qL \quad \textit{for any } b \in B.$$

Here we understand $C(0) = 0$.

We shall divide the proof into several steps.

Write

$$C(b) = \sum_{n\geqslant 0} f^{(n)} C_n(b) \quad \text{with } eC_n(b) = 0.$$

Set

$$L_0 = \sum_{b\in B,\ n\geqslant 0} \mathbf{A}_0 f^{(n)} C_0(b).$$

Set for $u \in M$, $\operatorname{ord}(u) = \sup\{n \in \mathbb{Z} \mid u \in q^n L_0\}$. If $u = 0$ we set $\operatorname{ord}(u) = \infty$, and if $u \notin \cup_{n\in\mathbb{Z}} q^n L_0$, then $\operatorname{ord}(u) = -\infty$.

We shall use the following two recursion formulas (4.16) and (4.17).

We have

$$\begin{aligned} eC(b) &= \sum_{n\geqslant 1} q^{1-n} f^{(n-1)} C_n(b) \\ &= \sum_{n\geqslant 0} E_{b,b'} f^{(n)} C_n(b'). \end{aligned}$$

Hence we have

$$C_n(b) = \sum_{b' \in B_{\lambda+\alpha}} q^{n-1} E_{b,b'} C_{n-1}(b') \quad \text{for } n > 0 \text{ and } b \in B_\lambda. \tag{4.16}$$

If $\ell := \varepsilon(b) > 0$, then we have

$$\begin{aligned} fC(\tilde{e}b) &= \sum_{b' \in B,\, n \geqslant 0} F_{\tilde{e}b,b'} f^{(n)} C_n(b') \\ &= \sum_{n \geqslant 0} [n+1] f^{(n+1)} C_n(\tilde{e}b). \end{aligned}$$

Hence, we have by (4.11)

$$\begin{aligned} \delta_{n \neq 0}[n] C_{n-1}(\tilde{e}b) &= \textstyle\sum_{b'} F_{\tilde{e}b,b'} C_n(b') \\ &\in q^{1-\ell}(1 + q\mathbf{A}_0) C_n(b) + \textstyle\sum_{b' \neq b} F_{\tilde{e}b,b'} C_n(b'). \end{aligned}$$

Therefore we obtain

$$C_n(b) \in \delta_{n \neq 0}(1 + q\mathbf{A}_0) q^{\ell-n} C_{n-1}(\tilde{e}b) + \sum_{b' \neq b} q^{\ell-1} \mathbf{A}_0 F_{\tilde{e}b,b'} C_n(b') \quad \text{if } \ell > 0. \tag{4.17}$$

Lemma 4.2. *$\mathrm{ord}(C_n(b)) \geqslant c(n-\ell)$ for any $n \in \mathbb{Z}_{\geqslant 0}$ and $b \in B$, where $\ell := \varepsilon(b)$.*

Proof. For $\lambda \in P$, we shall show the assertion for $b \in B_\lambda$ by the induction on $\sup\{n \in \mathbb{Z} \mid M_{\lambda+n\alpha} \neq 0\}$. Hence we may assume

$$\mathrm{ord}(C_n(b)) \geqslant c(n-\ell) \text{ for any } n \in \mathbb{Z}_{\geqslant 0} \text{ and } b \in B_{\lambda+\alpha}. \tag{4.18}$$

(i) Let us first show $C_n(b) \in \mathbf{K}L_0$.

Since it is trivial for $n = 0$, assume that $n > 0$. Since $C_{n-1}(b') \in \mathbf{K}L_0$ for $b' \in B_{\lambda+\alpha}$ by the induction assumption (4.18), we have $C_n(b) \in \mathbf{K}L_0$ by (4.16).

(ii) Let us show that $\mathrm{ord}(C_n(b)) \geqslant c(n-\ell)$ for $n \geqslant \ell$.

If $n = 0$, then $\ell = 0$ and the assertion is trivial by (4.4). Hence we may assume that $n > 0$.

We shall use (4.16). For $b' \in B_{\lambda+\alpha}$, we have

$$\mathrm{ord}(C_{n-1}(b')) \geqslant c(n-1-\ell') \quad \text{where } \ell' = \varepsilon(b')$$

by the induction hypothesis (4.18). On the other hand, $\mathrm{ord}(E_{b,b'}) \geqslant 1 - \ell + e(\ell - 1 - \ell')$ by (4.10). Hence,

$$\begin{aligned} \mathrm{ord}(q^{n-1} E_{b,b'} C_{n-1}(b')) &\geqslant (n-1) + \big(1 - \ell + e(\ell-1-\ell')\big) + c(n-1-\ell') \\ &= (n-\ell) + e(\ell-1-\ell') + c((n-\ell) + (\ell-1-\ell')) \\ &\geqslant c(n-\ell) \end{aligned}$$

by (4.5).

(iii) In the general case, let us set

$$r = \min\{\mathrm{ord}(C_n(b)) - c(n - \varepsilon(b)) \mid b \in B_\lambda,\ n \geqslant 0\} \in \mathbb{R} \cup \{\infty\}.$$

Assuming $r < 0$, we shall prove

$$\mathrm{ord}(C_n(b)) > c(n-\ell) + r \quad \text{for any } b \in B_\lambda,$$

which leads a contradiction.

By the induction on $\xi(b)$, we may assume that

(4.19) if $\xi(b') < \xi(b)$, then $\mathrm{ord}(C_n(b')) > c(n - \ell') + r$ where $\ell' := \varepsilon(b')$.

By (ii), we may assume that $n < \ell$. Hence $\tilde{e}b \in B$. By the induction hypothesis (4.18), we have $\mathrm{ord}(q^{\ell-n} C_{n-1}(\tilde{e}b)) \geqslant \ell - n + c((n-1) - (\ell-1)) \geqslant c(n-\ell) > c(n-\ell) + r$. By (4.17), it is enough to show

$$\mathrm{ord}(q^{\ell-1} F_{\tilde{e}b,b'} C_n(b')) > c(n-\ell) + r \quad \text{for } b' \neq b.$$

We shall divide its proof into two cases.

(a) $\xi(b') < \xi(b)$.

In this case, (4.19) implies $\mathrm{ord}(C_n(b')) > c(n-\ell') + r$. Hence

$$\begin{aligned}\mathrm{ord}(q^{\ell-1} F_{\tilde{e}b,b'} C_n(b')) &> (\ell - 1) + (1 - \ell + f(\ell - \ell')) + c(n - \ell') + r \\ &= f(\ell-\ell') + c((n-\ell) + (\ell-\ell')) + r \geqslant c(n-\ell) + r\end{aligned}$$

by (4.9) and (4.6).

(b) Case $\xi(b') \not< \xi(b)$.

In this case, $\mathrm{ord}(F_{\tilde{e}b,b'}) > 1 - \ell + f(\ell - \ell')$ by (4.13), and $\mathrm{ord}(C_n(b')) \geqslant c(n-\ell') + r$. Hence,

$$\begin{aligned}\mathrm{ord}(q^{\ell-1} F_{\tilde{e}b,b'} C_n(b')) &> (\ell - 1) + (1 - \ell + f(\ell - \ell')) + c(n - \ell') + r \\ &= f(\ell-\ell') + c((n-\ell) + (\ell-\ell')) + r \geqslant c(n-\ell) + r.\end{aligned}$$

□

Lemma 4.3. $\mathrm{ord}(C_\ell(b) - C_{\ell-1}(\tilde{e}b)) > 0$ *for* $\ell := \varepsilon(b) > 0$.

Proof.
We divide the proof into two cases: $b \in B'$ and $b \in B''$.

(i) $b \in B'$.

By (4.17), it is enough to show

$$\operatorname{ord}(q^{\ell-1} F_{\tilde{e}b,b'} C_\ell(b')) > 0 \quad \text{for } b' \neq b.$$

(a) Case $\ell > \ell' := \varepsilon(b')$.

We have

$$\operatorname{ord}(q^{\ell-1} F_{\tilde{e}b,b'} C_\ell(b')) \geqslant (\ell-1) + (1-\ell+f(\ell-\ell')) + c(\ell-\ell') > 0$$

by (4.7).

(b) Case $\ell \leqslant \ell'$.

We have $\operatorname{ord}(F_{\tilde{e}b,b'}) > 1-\ell+f(\ell-\ell')$ by (4.14). Hence

$$\operatorname{ord}(q^{\ell-1} F_{\tilde{e}b,b'} C_\ell(b')) > (\ell-1) + (1-\ell+f(\ell-\ell')) + c(\ell-\ell') \geqslant 0$$

by (4.6) with $n = 0$.

(ii) Case $b \in B''$.

We use (4.16). By (4.12), it is enough to show that

$$\operatorname{ord}(q^{\ell-1} E_{b,b'} C_{\ell-1}(b')) > 0 \text{ for } b' \neq \tilde{e}b.$$

(a) Case $\ell - 1 > \ell'$.

$\operatorname{ord}(q^{\ell-1} E_{b,b'} C_{\ell-1}(b')) \geqslant e(\ell-1-\ell') + c(\ell-1-\ell') > 0$ by (4.10) and (4.8).

(b) Case $\ell - 1 \leqslant \ell'$.

$\operatorname{ord}(E_{b,b'}) > 1-\ell+e(\ell-1-\ell')$ by (4.15), and $\operatorname{ord}(q^{\ell-1} E_{b,b'} C_{\ell-1}(b')) > e(\ell-1-\ell') + c(\ell-1-\ell') \geqslant 0$ by (4.5) with $n = 0$.

□

Hence we have

$$\begin{aligned}
&C_n(b) \equiv 0 \bmod qL_0 \quad \text{for } n \neq \ell := \varepsilon(b),\\
&C_\ell(b) \equiv C_0(\tilde{e}^\ell b) \bmod qL_0,\\
&C(b) \equiv f^{(\ell)} C_\ell(b) \bmod qL_0,\\
&\tilde{f}C(b) \equiv C(\tilde{f}b) \bmod qL_0,\\
&\tilde{e}C(b) \equiv C(\tilde{e}b) \bmod qL_0,\\
&L_0 := \sum_{b \in B,\ n \geqslant 0} \mathbf{A}_0 f^{(n)} C_0(b) = \sum_{b \in B} \mathbf{A}_0 C(b).
\end{aligned}$$

Indeed, the last equality follows from the fact that $\{C(b)\}_{b\in B}$ generates L_0/qL_0.

Thus we have completed the proof of Theorem 4.1.

The following is the special case where $B' = B'' = B$ and $\xi(b) = \varepsilon(b)$.

Corollary 4.4. *Assume* (4.4)–(4.12) *and*

(4.20) $\operatorname{ord}(F_{b,b'}) > -\ell + f(1+\ell-\ell')$ *if* $\ell < \ell'$ *and* $b' \neq \tilde{f}b$,

(4.21) $\operatorname{ord}(E_{b,b'}) > 1 - \ell + e(\ell - 1 - \ell')$ *if* $\ell \leqslant \ell' + 1$ *and* $b' \neq \tilde{e}b$.

Then the assertions of Theorem 4.1 *hold.*

§4.2. Crystal structure on $\mathcal{M}_\theta$

We shall define the crystal structure on $\mathcal{M}_\theta$.

Definition 4.5. Suppose $k > 0$. For a θ-restricted multisegment $\mathfrak{m} = \sum_{-j\leqslant i\leqslant j} m_{i,j}\langle i,j\rangle$, we set

$$\varepsilon_{-k}(\mathfrak{m}) = \max\left\{A_j^{(-k)}(\mathfrak{m}) \mid j \geqslant -k+2\right\},$$

where

$$A_j^{(-k)}(\mathfrak{m}) = \sum_{\ell\geqslant j}(m_{-k,\ell} - m_{-k+2,\ell+2}) \quad \text{for } j > k,$$

$$A_k^{(-k)}(\mathfrak{m}) = \sum_{\ell>k}(m_{-k,\ell} - m_{-k+2,\ell}) + 2m_{-k,k} + \delta(m_{-k+2,k} \text{ is odd}),$$

$$A_j^{(-k)}(\mathfrak{m}) = \sum_{\ell>k}(m_{-k,\ell} - m_{-k+2,\ell}) + 2m_{-k,k} - 2m_{-k+2,k-2} + \sum_{-k+2<i\leqslant j+2} m_{i,k} - \sum_{-k+2<i\leqslant j} m_{i,k-2} \quad \text{for } -k+2 \leqslant j \leqslant k-2.$$

(i) Let n_f be the smallest $\ell \geqslant -k+2$, with respect to the ordering $\cdots > k+2 > k > -k+2 > \cdots > k-2$, such that $\varepsilon_{-k}(\mathfrak{m}) = A_\ell^{(-k)}(\mathfrak{m})$. We define

$$\widetilde{F}_{-k}(\mathfrak{m}) = \begin{cases} \mathfrak{m} - \langle -k+2, n_f\rangle + \langle -k, n_f\rangle & \text{if } n_f > k,\\ \mathfrak{m} - \langle -k+2, k\rangle + \langle -k, k\rangle & \text{if } n_f = k \text{ and } m_{-k+2,k} \text{ is odd},\\ \mathfrak{m} - \delta_{k\neq 1}\langle -k+2, k-2\rangle + \langle -k+2, k\rangle & \text{if } n_f = k \text{ and } m_{-k+2,k} \text{ is even},\\ \mathfrak{m} - \delta_{n_f\neq k-2}\langle n_f+2, k-2\rangle + \langle n_f+2, k\rangle & \text{if } -k+2 \leqslant n_f \leqslant k-2. \end{cases}$$

(ii) If $\varepsilon_{-k}(\mathfrak{m}) = 0$, then $\widetilde{E}_{-k}(\mathfrak{m}) = 0$. If $\varepsilon_{-k}(\mathfrak{m}) > 0$, then let n_e be the largest $\ell \geqslant -k+2$, with respect to the above ordering, such that $\varepsilon_{-k}(\mathfrak{m}) = A_\ell^{(-k)}(\mathfrak{m})$. We define

$$\widetilde{E}_{-k}(\mathfrak{m}) = \begin{cases} \mathfrak{m} - \langle -k, n_e\rangle + \langle -k+2, n_e\rangle & \text{if } n_e > k, \\ \mathfrak{m} - \langle -k, k\rangle + \langle -k+2, k\rangle & \text{if } n_e = k \text{ and } m_{-k+2,k} \text{ is even,} \\ \mathfrak{m} - \langle -k+2, k\rangle + \delta_{k\neq 1}\langle -k+2, k-2\rangle & \text{if } n_e = k \text{ and } m_{-k+2,k} \text{ is odd,} \\ \mathfrak{m} - \langle n_e+2, k\rangle + \delta_{n_e\neq k-2}\langle n_e+2, k-2\rangle & \text{if } -k+2 \leqslant n_e \leqslant k-2. \end{cases}$$

Remark 4.6. For $0 < k \in I$, the actions of $\widetilde{E}_{-k}$ and $\widetilde{F}_{-k}$ on $\mathfrak{m} \in \mathcal{M}_\theta$ are described by the following algorithm.

Step 1. Arrange segments in $\mathfrak{m}$ of the form $\langle -k, j\rangle$ $(j > k)$, $\langle -k+2, j\rangle$ $(j > k)$, $\langle i, k\rangle$ $(-k \leqslant i \leqslant \mathrm{k})$, $\langle i, k-2\rangle$ $(-k+2 \leqslant i \leqslant k-2)$ in the order

$$\cdots, \langle -k, k+2\rangle, \langle -k+2, k+2\rangle, \langle -k, k\rangle, \langle -k+2, k\rangle, \langle -k+2, k-2\rangle, \\ \langle -k+4, k\rangle, \langle -k+4, k-2\rangle, \cdots, \langle k-2, k\rangle, \langle k-2, k-2\rangle, \langle k\rangle.$$

Step 2. Write signatures for each segment contained in $\mathfrak{m}$ by the following rules.

(i) If a segment is not $\langle -k+2, k\rangle$, then

- For $\langle -k, k\rangle$, write $--$,
- For $\langle -k, j\rangle$ with $j > k$, write $-$,
- For $\langle -k+2, k-2\rangle$ with $k > 1$, write $++$,
- For $\langle -k+2, j\rangle$ with $j > k$, write $+$,
- For $\langle j, k\rangle$ with $-k+2 < j \leqslant k$, write $-$,
- For $\langle j, k-2\rangle$ with $-k+2 < j \leqslant k-2$, write $+$,
- Otherwise, write no signature.

(ii) For segments $m_{-k+2,k}\langle -k+2, k\rangle$, if $m_{-k+2,k}$ is even, then write no signature, and if $m_{-k+2,k}$ is odd, then write $-+$.

Step 3. In the resulting sequence of $+$ and $-$, delete a subsequence of the form $+-$ and keep on deleting until no such subsequence remains.

Then we obtain a sequence of the form $- - \cdots - + + \cdots +$.

(1) $\varepsilon_{-k}(\mathfrak{m})$ is the total number of $-$ in the resulting sequence.

(2) $\widetilde{F}_{-k}(\mathfrak{m})$ is given as follows:

(i) if the leftmost $+$ corresponds to a segment $\langle -k+2, j\rangle$ for $j > k$, then replace it with $\langle -k, j\rangle$,

(ii) if the leftmost $+$ corresponds to a segment $\langle j, k-2\rangle$ for $-k+2 \leqslant j \leqslant k-2$, then replace it with $\langle j, k\rangle$,

(iii) if the leftmost $+$ corresponds to segment $m_{-k+2,k}\langle -k+2, k\rangle$, then replace one of the segments with $\langle -k, k\rangle$,

(iv) if no $+$ exists, add a segment $\langle k, k\rangle$ to $\mathfrak{m}$.

(3) $\widetilde{E}_{-k}(\mathfrak{m})$ is given as follows:

(i) if the rightmost $-$ corresponds to a segment $\langle -k, j\rangle$ for $j \geqslant k$, then replace it with $\langle -k+2, j\rangle$,

(ii) if the rightmost $-$ corresponds to a segment $\langle j, k\rangle$ for $-k+2 < j < k$, then replace it with $\langle j, k-2\rangle$,

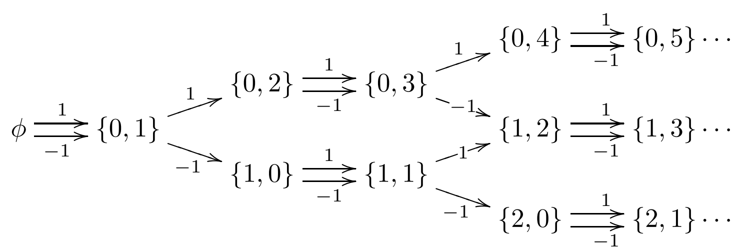
(iii) if the rightmost $-$ corresponds to segments $m_{-k+2,k}\langle -k+2, k\rangle$, then replace one of the segment with $\langle -k+2, k-2\rangle$,

(iv) if the rightmost $-$ corresponds to a segment $\langle k, k\rangle$ for $k > 1$, then delete it,

(v) if no $-$ exists, then $\widetilde{E}_{-k}(\mathfrak{m}) = 0$.

Example 4.7.

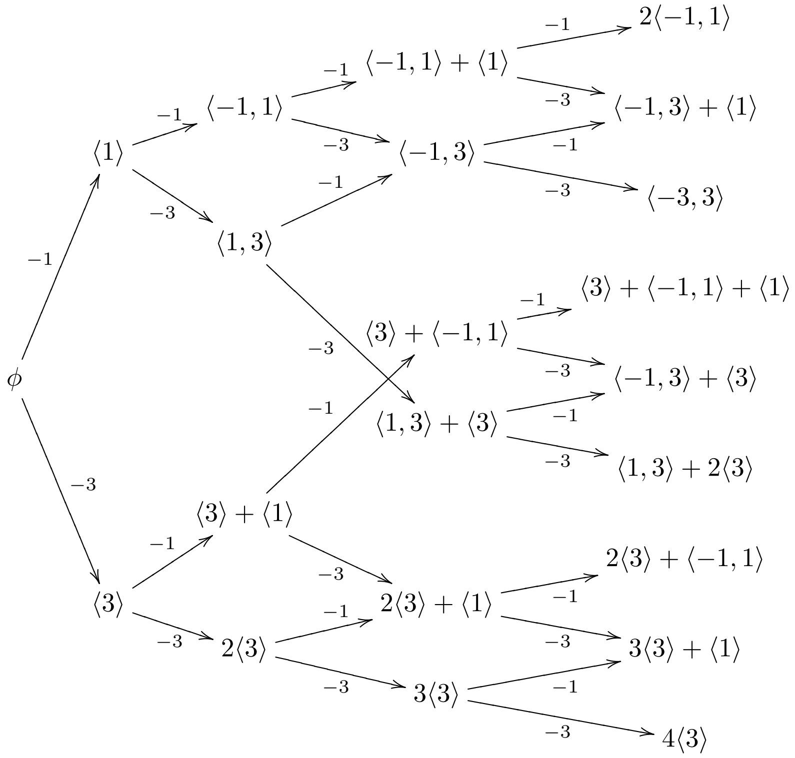
(1) We shall write $\{a, b\}$ for $a\langle -1, 1\rangle + b\langle 1\rangle$. The following diagram is the part of the crystal graph of $\mathrm{B}_\theta(0)$ that concerns only the 1-arrows and the (-1)-arrows.



Especially the part of (-1)-arrows is the following diagram.

$$\{0,2n\} \xrightarrow{-1} \{0,2n+1\} \xrightarrow{-1} \{1,2n\} \xrightarrow{-1} \{1,2n+1\} \xrightarrow{-1} \{2,2n\}\cdots$$

(2) The following diagram is the part of the crystal graph of $\mathrm{B}_\theta(0)$ that concerns only the (-1)-arrows and the (-3)-arrows. This diagram is, as a graph, isomorphic to the crystal graph of A_2.



(3) Here is the part of the crystal graph of $\mathrm{B}_\theta(0)$ that concerns only the n-arrows and the $(-n)$-arrows for an odd integer $n \geqslant 3$:

$$\phi \underset{-n}{\overset{n}{\rightrightarrows}} \langle n\rangle \underset{-n}{\overset{n}{\rightrightarrows}} 2\langle n\rangle \underset{-n}{\overset{n}{\rightrightarrows}} 3\langle n\rangle \underset{-n}{\overset{n}{\rightrightarrows}} \cdots\cdots$$

Lemma 4.8. *For $k \in I_{>0}$, the data $\widetilde{E}_{-k}$, $\widetilde{F}_{-k}$, ε_{-k} define a crystal structure on $\mathcal{M}_\theta$, namely we have*

(i) $\widetilde{F}_{-k}\mathcal{M}_\theta \subset \mathcal{M}_\theta$ *and* $\widetilde{E}_{-k}\mathcal{M}_\theta \subset \mathcal{M}_\theta \sqcup \{0\}$,

(ii) $\widetilde{F}_{-k}\widetilde{E}_{-k}(\mathfrak{m}) = \mathfrak{m}$ *if* $\widetilde{E}_{-k}(\mathfrak{m}) \neq 0$, *and* $\widetilde{E}_{-k} \circ \widetilde{F}_{-k} = \mathrm{id}$,

(iii) $\varepsilon_{-k}(\mathfrak{m}) = \max\left\{n \geqslant 0 \mid \widetilde{E}^n_{-k}(\mathfrak{m}) \neq 0\right\}$ *for any* $\mathfrak{m} \in \mathcal{M}_\theta$.

Proof. We shall first show that, for $\mathfrak{m} = \sum_{-j\leqslant i\leqslant j} m_{i,j}\langle i,j\rangle \in \mathcal{M}_\theta$, $\widetilde{F}_{-k}(\mathfrak{m})$ is θ-restricted, $\widetilde{E}_{-k}\widetilde{F}_{-k}(\mathfrak{m}) = \mathfrak{m}$ and $\varepsilon_{-k}(\widetilde{F}_{-k}\mathfrak{m}) = \varepsilon_{-k}(\mathfrak{m}) + 1$. Let $A_j := A_j^{(-k)}(\mathfrak{m})$ $(j \geqslant -k+2)$ and let n_f be as in Definition 4.5. Set $\mathfrak{m}' = \widetilde{F}_{-k}\mathfrak{m}$. Let $A'_j = A_j^{(-k)}(\mathfrak{m}')$ and let n'_e be n_e for $\mathfrak{m}'$.

(i) Assume $n_f > k$. Since $A_{n_f} > A_{n_f-2} = A_{n_f} + m_{-k,n_f-2} - m_{-k+2,n_f}$, we have $m_{-k,n_f-2} < m_{-k+2,n_f}$. Hence $\mathfrak{m}' = \mathfrak{m} - \langle -k+2, n_f\rangle + \langle -k, n_f\rangle$ is θ-restricted. Then we have

$$A'_j = \begin{cases} A_j & \text{if } j > n_f, \\ A_j + 1 & \text{if } j = n_f, \\ A_j + 2 & \text{if } j < n_f. \end{cases}$$

Hence $\varepsilon_{-k}(\mathfrak{m}') = A_{n_f} + 1 = \varepsilon_{-k}(\mathfrak{m}) + 1$ and $n'_e = n_f$, which implies $\mathfrak{m} = \widetilde{E}_{-k}(\mathfrak{m}')$.

(ii) Assume $n_f = k$.

(a) If $m_{-k+2,k}$ is odd, then $\mathfrak{m}' = \mathfrak{m} - \langle -k+2, k\rangle + \langle -k, k\rangle$ is θ-restricted. We have

$$A'_j = \begin{cases} A_j & \text{if } j > k, \\ A_j + 1 & \text{if } j = k, \\ A_j + 2 & \text{if } j < k, \end{cases}$$

Hence $\varepsilon_{-k}(\mathfrak{m}') = \varepsilon_{-k}(\mathfrak{m}) + 1$ and $n'_e = k$, which implies $\mathfrak{m} = \widetilde{E}_{-k}(\mathfrak{m}')$.

(b) Assume that $m_{-k+2,k}$ is even. If $k \neq 1$, then $A_k > A_{-k+2} = A_k - 2m_{-k+2,k-2}$, and hence $m_{-k+2,k-2} > 0$. Therefore $\mathfrak{m}' = \mathfrak{m} - \delta_{k\neq 1}\langle -k+2, k-2\rangle + \langle -k+2, k\rangle$ is θ-restricted. We have

$$A'_j = \begin{cases} A_j & \text{if } j > k, \\ A_j + 1 & \text{if } j = k, \\ A_j + 2 & \text{if } j < k. \end{cases}$$

Hence $\varepsilon_{-k}(\mathfrak{m}') = \varepsilon_{-k}(\mathfrak{m}) + 1$ and $n'_e = k$, which implies $\mathfrak{m} = \widetilde{E}_{-k}(\mathfrak{m}')$.

(iii) Assume $-k+2 \leqslant n_f < k-2$. Since $A_{n_f} > A_{n_f+2} = A_{n_f} + m_{n_f+4,k} - m_{n_f+2,k-2}$, we have $m_{n_f+2,k-2} > m_{n_f+4,k}$. Hence $\mathfrak{m}' = \mathfrak{m} - \langle n_f+2, k-2\rangle + \langle n_f+2, k\rangle$ is θ-restricted. Then we have

$$A'_j = \begin{cases} A_j & \text{if } j > n_f, \\ A_j + 1 & \text{if } j = n_f, \\ A_j + 2 & \text{if } j < n_f. \end{cases}$$

(Here the ordering is as in Definition 4.5 (i).) Hence $\varepsilon_{-k}(\mathfrak{m}') = \varepsilon_{-k}(\mathfrak{m})+1$ and $n'_e = n_f$, which implies $\mathfrak{m} = \widetilde{E}_{-k}\mathfrak{m}'$.

(iv) Assume $n_f = k-2$. It is obvious that $\mathfrak{m}' = \mathfrak{m} + \langle k\rangle$ is θ-restricted. We have

$$A'_j = \begin{cases} A_j & \text{if } j \neq n_f, \\ A_j + 1 & \text{if } j = n_f. \end{cases}$$

Hence $\varepsilon_{-k}(\mathfrak{m}') = \varepsilon_{-k}(\mathfrak{m}) + 1$ and $n'_e = n_f$, which implies $\mathfrak{m} = \widetilde{E}_{-k}(\mathfrak{m}')$.

Similarly, we can prove that if $\varepsilon_{-k}(\mathfrak{m}) > 0$, then $\widetilde{E}_{-k}(\mathfrak{m})$ is θ-restricted and $\widetilde{F}_{-k}\widetilde{E}_{-k}(\mathfrak{m}) = \mathfrak{m}$. Hence we obtain the desired results. □

Definition 4.9. For $k \in I_{>0}$, we define $\widetilde{F}_k$, $\widetilde{E}_k$ and ε_k by the same rule as in Definition 3.7 for $\tilde{f}_k$, $\tilde{e}_k$ and ε_k.

Since it is well-known that it gives a crystal structure on $\mathcal{M}$, we obtain the following result.

Theorem 4.10. *By $\widetilde{F}_k$, $\widetilde{E}_k$, ε_k $(k \in I)$, $\mathcal{M}_\theta$ is a crystal, namely, we have*

(i) *$\widetilde{F}_k\mathcal{M}_\theta \subset \mathcal{M}_\theta$ and $\widetilde{E}_k\mathcal{M}_\theta \subset \mathcal{M}_\theta \sqcup \{0\}$,*

(ii) *$\widetilde{F}_k\widetilde{E}_k(\mathfrak{m}) = \mathfrak{m}$ if $\widetilde{E}_k(\mathfrak{m}) \neq 0$, and $\widetilde{E}_k \circ \widetilde{F}_k = \mathrm{id}$,*

(iii) *$\varepsilon_k(\mathfrak{m}) = \max\left\{n \geqslant 0 \mid \widetilde{E}_k^n(\mathfrak{m}) \neq 0\right\}$ for any $\mathfrak{m} \in \mathcal{M}_\theta$.*

The crystal $\mathcal{M}_\theta$ has a unique highest weight vector.

Lemma 4.11. *If $\mathfrak{m} \in \mathcal{M}_\theta$ satisfies that $\varepsilon_k(\mathfrak{m}) = 0$ for any $k \in I$, then $\mathfrak{m} = \emptyset$. Here $\emptyset$ is the empty multisegment. In particular, for any $\mathfrak{m} \in \mathcal{M}_\theta$, there exist $\ell \geqslant 0$ and $i_1, \ldots, i_\ell \in I$ such that $\mathfrak{m} = \widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell}\emptyset$.*

Proof. Assume $\mathfrak{m} \neq \emptyset$. Let k be the largest k such that $m_{k,j} \neq 0$ for some j. Then take the largest j such that $m_{k,j} \neq 0$. Then $j \geqslant |k|$. Moreover, we have $m_{k+2,\ell} = 0$ for any ℓ, and $m_{k,\ell} = 0$ for any $\ell > j$. Hence we have

$$A_j^{(k)}(\mathfrak{m}) = \begin{cases} 2m_{k,j} & \text{if } k = -j, \\ m_{k,j} & \text{otherwise.} \end{cases}$$

Hence $\varepsilon_k(\mathfrak{m}) \geqslant A_j^{(k)}(\mathfrak{m}) > 0$. □

§4.3. Estimates of the order of coefficients

By applying Theorem 4.1, we shall show that $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ is a crystal basis of $V_\theta(0)$ and its crystal structure coincides with the one given in § 4.2.

Let k be a positive odd integer. We define $c, f, e \colon \mathbb{Z} \to \mathbb{Q}$ by $c(n) = \lfloor n/2 \rfloor$ and $f(n) = e(n) = n/2$. Then the conditions (4.4)–(4.8) are obvious. Set $\xi(\mathfrak{m}) = (-1)^{m_{-k+2,k}} m_{-k,k}$ and

$$\begin{aligned} B'' &= \{\mathfrak{m} \in \mathcal{M}_\theta \mid -k+2 \leqslant n_e(\mathfrak{m}) < k\} \cup \{\mathfrak{m} \in \mathcal{M}_\theta \mid m_{-k+2,k}(\mathfrak{m}) \text{ is odd}\}, \\ B' &= \mathcal{M}_\theta \setminus B''. \end{aligned}$$

Here $n_e(\mathfrak{m})$ is n_e given in Definition 4.5 (ii). If $\varepsilon_{-k}(\mathfrak{m}) = 0$, then we understand $n_e(\mathfrak{m}) = \infty$.

We define $F^{-k}_{\mathfrak{m},\mathfrak{m}'}$ and $E^{-k}_{\mathfrak{m},\mathfrak{m}'}$ by the coefficients of the following expansion:

$$\begin{aligned} F_{-k} P_\theta(\mathfrak{m})\widetilde{\phi} &= \sum_{\mathfrak{m}'} F^{-k}_{\mathfrak{m},\mathfrak{m}'} P_\theta(\mathfrak{m}')\widetilde{\phi}, \\ E_{-k} P_\theta(\mathfrak{m})\widetilde{\phi} &= \sum_{\mathfrak{m}'} E^{-k}_{\mathfrak{m},\mathfrak{m}'} P_\theta(\mathfrak{m}')\widetilde{\phi}, \end{aligned}$$

as given in Theorems 3.21 and 3.22. Put $\ell = \varepsilon_{-k}(\mathfrak{m})$ and $\ell' = \varepsilon_{-k}(\mathfrak{m}')$.

Proposition 4.12. *The conditions* (4.9), (4.11), (4.13) *and* (4.14) *are satisfied for* $\widetilde{E}_{-k}$, $\widetilde{F}_{-k}$, ε_{-k}, *namely, we have*

(a) *if* $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, *then* $F^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-\ell}(1 + q\mathbf{A}_0)$,

(b) *if* $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$, *then* $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) \geqslant -\ell + f(\ell + 1 - \ell') = -(\ell + \ell' - 1)/2$,

(c) *if* $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$ *and* $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell + \ell' - 1)/2$, *then the following two conditions hold:*

   (1) $\xi(\widetilde{F}_{-k}(\mathfrak{m})) > \xi(\mathfrak{m}')$,

(2) $\ell \geqslant \ell'$ *or* $\widetilde{F}_{-k}(\mathfrak{m}) \in B''$.

Proof. We shall write A_j for $A_j^{-k}(\mathfrak{m})$. Let n_f be as in Definition 4.5 (i). Note that $F^{-k}_{\mathfrak{m},\widetilde{F}_{-k}(\mathfrak{m})} \neq 0$.

If $F^{-k}_{\mathfrak{m},\mathfrak{m}'} \neq 0$, we have the following four cases. We shall use $[n] \in q^{1-n}(1+q\mathbf{A}_0)$ for $n > 0$.

Case 1. $\mathfrak{m}' = \mathfrak{m} - \langle -k+2, n\rangle + \langle -k, n\rangle$ for $n > k$.

In this case, we have

$$F^{-k}_{\mathfrak{m},\mathfrak{m}'} = [m_{-k,n}+1]q^{\sum_{j>n}(m_{-k+2,j}-m_{-k,j})} \in q^{-A_n}(1+q\mathbf{A}_0)$$

and

$$\begin{aligned}\ell &= \max\{A_j(j \geqslant -k+2)\},\\ \ell' &= \max\{A_j\ (j > n), A_n+1, A_j+2\ (j < n)\}.\end{aligned}$$

If $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, then $\ell = A_n$ and we obtain (a). Assume $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$. Since $A_n \leqslant \ell, \ell'-1$, we have $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_n \geqslant -(\ell+\ell'-1)/2$. Hence we obtain (b). If $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then we have $A_n = \ell = \ell'-1$. Since $A_j + 2 \leqslant \ell' = A_n + 1$ for $j < n$, we have $n_f = n$ and $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, which is a contradiction.

Case 2. $\mathfrak{m}' = \mathfrak{m} - \langle -k+2, k\rangle + \langle -k, k\rangle$.

In this case we have

$$F^{-k}_{\mathfrak{m},\mathfrak{m}'} = [2m_{-k,k}+2]q^{\sum_{j>k}(m_{-k+2,j}-m_{-k,j})} \in q^{-A_k-\delta(\mathfrak{m}_{-k+2,k} \text{ is even})}(1+q\mathbf{A}_0).$$

(i) Assume that $m_{-k+2,k}$ is odd. We have $F^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-A_k}(1+q\mathbf{A}_0)$ and

$$\ell' = \max\{A_j\ (j > k), A_k+1, A_j+2\ (j < k)\}.$$

If $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, then $\ell = A_k$ and (a) holds. Assume that $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$. We have $A_k \leqslant \ell, \ell'-1$ and hence $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_k \geqslant -(\ell+\ell'-1)/2$. If $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k = \ell = \ell'-1$, and we have $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, which is a contradiction.

(ii) Assume that $m_{-k+2,k}$ is even. Then $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$, $F^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-A_k-1}(1+q\mathbf{A}_0)$ and

$$\ell' = \max\{A_j\ (j > k), A_k+3, A_j+2\ (j < k)\}.$$

We have $A_k \leqslant \ell, \ell'-3$ and hence $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_k - 1 \geqslant -(\ell+\ell'-1)/2$. Hence (b) holds. Let us show (c). Assume $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$, and

$\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$. Then we have $A_k = \ell = \ell' - 3$. Hence $n_f \leqslant k$ and we have either $\widetilde{F}_{-k}(\mathfrak{m}) = \mathfrak{m} - \delta_{i\neq k}\langle i, k-2\rangle + \langle i,k\rangle$ with $-k+2 < i \leqslant k$ or $\widetilde{F}_{-k}(\mathfrak{m}) = \mathfrak{m} - \delta_{k\neq 1}\langle -k+2, k-2\rangle + \langle -k+2, k\rangle$. Hence we have $\xi(\widetilde{F}_{-k}(\mathfrak{m})) = \pm m_{-k,k} > -m_{-k,k} - 1 = \xi(\mathfrak{m}')$. Hence we obtain (c) (1).

(1) Assume $\widetilde{F}_{-k}(\mathfrak{m}) = \mathfrak{m} - \delta_{i\neq k}\langle i, k-2\rangle + \langle i,k\rangle$ with $-k+2 < i \leqslant k$. Then $k \neq 1$ and $\widetilde{E}_{-k}(\widetilde{F}_{-k}(\mathfrak{m})) = \widetilde{F}_{-k}(\mathfrak{m}) - \langle i,k\rangle + \delta_{i\neq k}\langle i, k-2\rangle$. Hence $n_e(\widetilde{F}_{-k}(\mathfrak{m})) = i - 2 < k$. Hence $\widetilde{F}_{-k}(\mathfrak{m}) \in B''$. Therefore we obtain (c) (2).

(2) Assume $\widetilde{F}_{-k}(\mathfrak{m}) = \mathfrak{m} - \delta_{k\neq 1}\langle -k+2, k-2\rangle + \langle -k+2, k\rangle$. Then $m_{-k+2,k}(\widetilde{F}_{-k}(\mathfrak{m})) = m_{-k+2,k} + 1$ is odd. Hence $\widetilde{F}_{-k}(\mathfrak{m}) \in B''$.

Case 3. $\mathfrak{m}' = \mathfrak{m} - \delta_{k\neq 1}\langle -k+2, k-2\rangle + \langle -k+2, k\rangle$. In this case, we have

$$
\begin{aligned}
F^{-k}_{\mathfrak{m},\mathfrak{m}'} &= [m_{-k+2,k}+1] q^{\sum_{j>k}(m_{-k+2,j} - m_{-k,j}) + m_{-k+2,k} - 2m_{-k,k}} \\
&\in q^{-A_k + \delta(m_{-k+2,k} \text{ is odd})}(1 + q\mathbf{A}_0).
\end{aligned}
$$

(i) If $m_{-k+2,k}$ is odd, then $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$, $F^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-A_k+1}(1+q\mathbf{A}_0)$, and

$$
\ell' = \max\{A_j \ (j > k), A_k - 1, A_j + 2 \ (j < k)\}.
$$

We have $A_k \leqslant \ell, \ell' + 1$ and hence $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_k + 1 \geqslant -(\ell+\ell'-1)/2$. If $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k = \ell = \ell' + 1$, and $n_f = k$. Hence we obtain (c) (2), and $\widetilde{F}_{-k}(\mathfrak{m}) = \mathfrak{m} - \langle -k+2, k\rangle + \langle -k, k\rangle$. Hence $\xi(\widetilde{F}_{-k}(\mathfrak{m})) = m_{-k,k} + 1 > m_{-k,k} = \xi(\mathfrak{m}')$. Hence we obtain (c) (1).

(ii) If $m_{-k+2,k}$ is even, then $F^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-A_k}(1+q\mathbf{A}_0)$ and

$$
\ell' = \max\{A_j \ (j > k), A_k + 1, A_j + 2 \ (j < k)\}.
$$

If $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, then $\ell = A_k$ and (a) is satisfied. Assume $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$. We have $A_k \leqslant \ell, \ell' - 1$ and hence $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_k \geqslant -(\ell+\ell'-1)/2$. If $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k = \ell = \ell' - 1$, and hence $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, which is a contradiction.

Case 4. $\mathfrak{m}' = \mathfrak{m} - \delta_{i\neq k}\langle i, k-2\rangle + \langle i,k\rangle$ for $-k+2 < i \leqslant k$. We have

$$
\begin{aligned}
F^{-k}_{\mathfrak{m},\mathfrak{m}'} &= [m_{i,k}+1] \\
&\quad \times q^{\sum_{j>k}(m_{-k+2,j} - m_{-k,j}) + 2m_{-k+2,k-2} - 2m_{-k,k} + \sum_{-k+2<j<i}(m_{j,k-2} - m_{j,k})} \\
&\in q^{-A_{i-2}}(1 + q\mathbf{A}_0),
\end{aligned}
$$

and

$$\ell' = \max\{A_j\ (j \geqslant k), A_j\ (j < i-2), A_{i-2}+1, A_j+2\ (i-2 < j \leqslant k-2)\}.$$

If $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, then $\ell = A_{i-2}$ and (a) holds. Assume $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$. Since $A_{i-2} \leqslant \ell, \ell'-1$, we have $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_{i-2} \geqslant -(\ell+\ell'-1)/2$. Hence we obtain (b). If $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then we have $A_{i-2} = \ell = \ell'-1$. Hence $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, which is a contradiction.

□

Proposition 4.13. *Suppose $k > 0$. The conditions* (4.10), (4.12), *and* (4.15) *hold, namely, we have*

(a) *if $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, then $E^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{1-\ell}(1+q\mathbf{A}_0)$,*

(b) *if $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$, then $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) \geqslant 1-\ell+e(\ell-1-\ell') = -(\ell+\ell'-1)/2$,*

(c) *if $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$, $\ell \leqslant \ell'+1$ and $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $b \notin B''$.*

Proof. The proof is similar to the one of the above proposition.

We shall write A_j for $A^{-k}_j(\mathfrak{m})$. Let n_e be as in Definition 4.5 (ii).

Note that $E^{-k}_{\mathfrak{m},\widetilde{E}_{-k}(\mathfrak{m})} \neq 0$ if $\widetilde{E}_{-k}(\mathfrak{m}) \neq 0$. If $E^{-k}_{\mathfrak{m},\mathfrak{m}'} \neq 0$, we have the following five cases.

Case 1. $\mathfrak{m}' = \mathfrak{m} - \langle -k, n\rangle + \langle -k+2, n\rangle$ for $n > k$.

In this case, we have

$$E^{-k}_{\mathfrak{m},\mathfrak{m}'} = (1-q^2)[m_{-k+2,n}+1]q^{1+\sum_{j\geqslant n}(m_{-k+2,j}-m_{-k,j})} \in q^{1-A_n}(1+q\mathbf{A}_0)$$

and

$$\ell = \max\{A_j (j \geqslant -k+2)\},$$
$$\ell' = \max\{A_j\ (j > n), A_n-1, A_j-2\ (j < n)\}.$$

If $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, then $\ell = A_n$ and we obtain (a). Assume $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$. Since $A_n \leqslant \ell, \ell'+1$, we have $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = 1-A_n \geqslant -(\ell+\ell'-1)/2$. Hence we obtain (b). If $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then we have $A_n = \ell = \ell'+1$. Since $A_j \leqslant \ell' = A_n - 1$ for $j > n$, we have $n_e = n$ and $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, which is a contradiction.

Case 2. $\mathfrak{m}' = \mathfrak{m} - \langle -k, k\rangle + \langle -k+2, k\rangle$.

In this case we have

$$\begin{aligned}E^{-k}_{\mathfrak{m},\mathfrak{m}'} &= (1-q^2)[m_{-k+2,k}+1]q^{1+\sum_{j>k}(m_{-k+2,j}-m_{-k,j})+m_{-k+2,k}-2m_{-k,k}} \\ &\in q^{1-A_k+\delta(\mathfrak{m}_{-k+2,k}\text{ is odd})}(1+q\mathbf{A}_0).\end{aligned}$$

(i) Assume that $m_{-k+2,k}$ is odd. Then $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$, $E^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{2-A_k}(1+q\mathbf{A}_0)$ and

$$\ell' = \max\{A_j\ (j>k), A_k-3, A_j-2\ (j<k)\}.$$

We have $A_k \leqslant \ell, \ell'+3$ and hence $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = 2-A_k \geqslant -(\ell+\ell'-1)/2$. Hence (b) holds. If $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k=\ell=\ell'+3$. Hence $\ell > \ell'+1$ and (c) holds.

(ii) Assume that $m_{-k+2,k}$ is even. Then $E^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{1-A_k}(1+q\mathbf{A}_0)$ and

$$\ell' = \max\{A_j\ (j>k), A_k-1, A_j-2\ (j<k)\}.$$

If $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, then $\ell = A_k$, and we obtain (a). Assume $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$. We have $A_k \leqslant \ell, \ell'+1$ and hence $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = 1-A_k \geqslant -(\ell+\ell'-1)/2$. If $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k=\ell=\ell'+1$ and $n_e=k$. Hence $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, which is a contradiction.

Case 3. $\mathfrak{m}' = \mathfrak{m} - \langle -k+2, k\rangle + \delta_{k\neq 1}\langle -k+2, k-2\rangle$. If $k \neq 1$, we have

$$\begin{aligned}E^{-k}_{\mathfrak{m},\mathfrak{m}'} &= (1-q^2)[2(m_{-k+2,k-2}+1)]q^{1+\sum_{j>k}(m_{-k+2,j}-m_{-k,j})+2m_{-k+2,k-2}-2m_{-k,k}} \\ &\in q^{-A_k+\delta(m_{-k+2,k}\text{ is odd})}(1+q\mathbf{A}_0).\end{aligned}$$

If $k=1$, we have

$$E^{-k}_{\mathfrak{m},\mathfrak{m}'} = q^{\sum_{j>k}(m_{-k+2,j}-m_{-k,j})-2m_{-k,k}} = q^{-A_k+\delta(m_{-k+2,k}\text{ is odd})}.$$

In the both cases, we have

$$E^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-A_k+\delta(m_{-k+2,k}\text{ is odd})}(1+q\mathbf{A}_0).$$

(i) If $m_{-k+2,k}$ is odd, then $E^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{1-A_k}(1+q\mathbf{A}_0)$ and

$$\ell' = \max\{A_j\ (j>k), A_k-1, A_j-2\ (j<k)\}.$$

If $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, then $\ell = A_k$ and (a) is satisfied. We have $A_k \leqslant \ell, \ell'+1$ and hence $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = 1-A_k \geqslant -(\ell+\ell'-1)/2$. Assume $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$. If $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k=\ell=\ell'+1$, and $n_e=k$. Hence $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, which is a contradiction.

(ii) If $m_{-k+2,k}$ is even, then $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$, $E^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-A_k}(1+q\mathbf{A}_0)$, and

$$\ell' = \max\{A_j\ (j>k), A_k+1, A_j-2\ (j<k)\}.$$

We have $A_k \leqslant \ell, \ell'-1$ and hence $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_k \geqslant -(\ell+\ell'-1)/2$. Hence we obtain (b). If $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k=\ell=\ell'-1$. Hence $n_e(\mathfrak{m}) \geqslant k$ and $m_{-k+2,k}(\mathfrak{m})$ is even. Hence $\mathfrak{m} \notin B''$.

Case 4. $\mathfrak{m}' = \mathfrak{m} - \langle i,k\rangle + \langle i,k-2\rangle$ for $-k+2 < i \leqslant k-2$.

We have

$$\begin{aligned} E^{-k}_{\mathfrak{m},\mathfrak{m}'} &= (1-q^2)[m_{i,k-2}+1] \\ &\quad \times q^{1+\sum_{j>k}(m_{-k+2,j}-m_{-k,j})+2m_{-k+2,k-2}-2m_{-k,k}+\sum_{-k+2<j\leqslant i}(m_{j,k-2}-m_{j,k})} \\ &\in q^{1-A_{i-2}}(1+q\mathbf{A}_0), \end{aligned}$$

and

$$\ell' = \max\{A_j\ (j \geqslant k),\ A_j\ (j < i-2),\ A_{i-2}-1,\ A_j - 2\ (i \leqslant j \leqslant k-2)\}.$$

If $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, then $\ell = A_{i-2}$ and (a) holds. Assume $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$. Since $A_{i-2} \leqslant \ell, \ell'+1$, we have $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = 1 - A_{i-2} \geqslant -(\ell+\ell'-1)/2$. Hence we obtain (b). If $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then we have $A_{i-2} = \ell = \ell'+1$. Hence $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, which is a contradiction.

Case 5. $k \neq 1$ and $\mathfrak{m}' = \mathfrak{m} - \langle k\rangle$. In this case,

$$\begin{aligned} E^{-k}_{\mathfrak{m},\mathfrak{m}'} &= q^{\sum\limits_{j>k}(m_{-k+2,j}-m_{-k,j})-2m_{-k,k}+1-m_{k,k}+2m_{-k+2,k-2}+\sum\limits_{-k+2<i\leqslant k-2}(m_{i,k-2}-m_{i,k})} \\ &\in q^{1-A_{k-2}}(1+q\mathbf{A}_0), \end{aligned}$$

and

$$\ell' = \max\{A_j\ (j \neq k-2), A_{k-2}-1\}.$$

If $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, then $\ell = A_{k-2}$ and (a) holds. Assume $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$. Since $A_{k-2} \leqslant \ell, \ell'+1$, we have $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = 1 - A_{k-2} \geqslant -(\ell+\ell'-1)/2$. Hence we obtain (b). If $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then we have $A_{k-2} = \ell = \ell'+1$. Hence $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, which is a contradiction. □

Proposition 4.14. *Let $k \in I_{>0}$. Then the conditions in Corollary* 4.4 *holds for $\widetilde{E}_k$, $\widetilde{F}_k$ and ε_k, with the same functions c, e, f.*

Since the proof is similar to and simpler than the one of the preceding two propositions, we omit the proof.

As a corollary we have the following result. We write ϕ for the generator ϕ_0 of $V_\theta(0)$ for short.

Theorem 4.15.

(i) *The morphism*

$$\widetilde{V}_\theta(0) := U_q^-(\mathfrak{g}) / \sum_{k\in I} U_q^-(\mathfrak{g})(f_k - f_{-k}) \to V_\theta(0)$$

is an isomorphism.

(ii) $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ *is a basis of the* $\mathbf{K}$*-vector space* $V_\theta(0)$.

(iii) *Set*

$$L_\theta(0) := \sum_{\ell\geqslant 0,\ i_1,\dots,i_\ell\in I} \mathbf{A}_0 \widetilde{F}_{i_1}\cdots\widetilde{F}_{i_\ell}\phi \subset V_\theta(0),$$
$$\mathrm{B}_\theta(0) = \left\{\widetilde{F}_{i_1}\cdots\widetilde{F}_{i_\ell}\phi \bmod qL_\theta(0) \mid \ell\geqslant 0, i_1,\dots,i_\ell\in I\right\}.$$

Then, $\mathrm{B}_\theta(0)$ *is a basis of* $L_\theta(0)/qL_\theta(0)$ *and* $(L_\theta(0),\mathrm{B}_\theta(0))$ *is a crystal basis of* $V_\theta(0)$*, and the crystal structure coincides with the one of* $\mathcal{M}_\theta$.

(iv) *More precisely, we have*

(a) $L_\theta(0) = \bigoplus_{\mathfrak{m}\in\mathcal{M}_\theta} \mathbf{A}_0 P_\theta(\mathfrak{m})\phi$,

(b) $\mathrm{B}_\theta(0) = \{P_\theta(\mathfrak{m})\phi \bmod qL_\theta(0) \mid \mathfrak{m}\in\mathcal{M}_\theta\}$,

(c) *for any* $k\in I$ *and* $\mathfrak{m}\in\mathcal{M}_\theta$*, we have*

(1) $\widetilde{F}_k P_\theta(\mathfrak{m})\phi \equiv P_\theta(\widetilde{F}_k(\mathfrak{m}))\phi \bmod qL_\theta(0)$,

(2) $\widetilde{E}_k P_\theta(\mathfrak{m})\phi \equiv P_\theta(\widetilde{E}_k(\mathfrak{m}))\phi \bmod qL_\theta(0)$, *where we understand* $P_\theta(0)=0$,

(3) $\widetilde{E}_k^n P_\theta(\mathfrak{m})\phi \in qL_\theta(0)$ *if and only if* $n>\varepsilon_k(\mathfrak{m})$.

Proof. Let us recall that $P_\theta(\mathfrak{m})\phi \in V_\theta(0)$ is the image of $\widetilde{P}_\theta(\mathfrak{m}) \in \widetilde{V}_\theta(0)$. By Theorem 3.21, $\{\widetilde{P}_\theta(\mathfrak{m})\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ generates $\widetilde{V}_\theta(0)$. Let us set $\widetilde{L} = \sum_{\mathfrak{m}\in\mathcal{M}_\theta} \mathbf{A}_0\widetilde{P}_\theta(\mathfrak{m}) \subset \widetilde{V}_\theta(0)$. Then Theorem 4.1 implies that

$$\widetilde{F}_k\widetilde{P}_\theta(\mathfrak{m}) \equiv \widetilde{P}_\theta(\widetilde{F}_k(\mathfrak{m})) \bmod q\widetilde{L} \text{ and } \widetilde{E}_k\widetilde{P}_\theta(\mathfrak{m}) \equiv \widetilde{P}_\theta(\widetilde{E}_k(\mathfrak{m})) \bmod q\widetilde{L}.$$

Hence the similar results hold for $L_0 := \sum_{\mathfrak{m}\in\mathcal{M}_\theta} \mathbf{A}_0 P_\theta(\mathfrak{m})\phi \subset V_\theta(0)$ and $P_\theta(\mathfrak{m})\phi$.

Let us show that

(A) $\{P_\theta(\mathfrak{m})\phi \bmod qL_0\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ is linearly independent in L_0/qL_0,

by the induction of the θ-weight (see Remark 2.12). Assume that we have a linear relation $\sum_{\mathfrak{m}\in S} a_{\mathfrak{m}} P_\theta(\mathfrak{m})\phi \equiv 0 \bmod qL_0$ for a finite subset S and $a_{\mathfrak{m}} \in \mathbb{Q}\setminus\{0\}$. We may assume that all $\mathfrak{m}$ in S have the same θ-weight. Take $\mathfrak{m}_0\in S$. If $\mathfrak{m}_0$ is the empty multisegment $\emptyset$, then $S=\{\emptyset\}$ and $P_\theta(\mathfrak{m}_0)\phi=\phi$ is non-zero, which is a contradiction. Otherwise, there exists k such that $\varepsilon_k(\mathfrak{m}_0)>0$ by Lemma 4.11. Applying $\widetilde{E}_k$, we have $\sum_{\mathfrak{m}\in S} a_{\mathfrak{m}}\widetilde{E}_k P_\theta(\mathfrak{m})\phi \equiv \sum_{\mathfrak{m}\in S,\ \widetilde{E}_k(\mathfrak{m})\neq 0} a_{\mathfrak{m}} P_\theta(\widetilde{E}_k(\mathfrak{m}))\phi \equiv 0 \bmod qL_0$. Since $\widetilde{E}_k(\mathfrak{m})$ ($\widetilde{E}_k(\mathfrak{m})\neq 0$) are mutually distinct, we have $a_{\mathfrak{m}_0}=0$ by the induction hypothesis. It is a contradiction.

Thus we have proved (A). Hence $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ is a basis of $V_\theta(0)$, which implies that $\{\widetilde{P}_\theta(\mathfrak{m})\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ is a basis of $\widetilde{V}_\theta(0)$. Thus we obtain (i) and (ii).

Let us show (iv) (a). Since $\widetilde{F}_{i_1}\cdots\widetilde{F}_{i\ell}\phi \equiv P_\theta(\widetilde{F}_{i_1}\cdots\widetilde{F}_{i\ell}\emptyset)\phi \bmod qL_0$, we have $L_\theta(0)\subset L_0$ and $L_0\subset L_\theta(0)+qL_0$. Hence Nakayama's lemma implies $L_0 = L_\theta(0)$. The other statements are now obvious. ∎

§5. Global Basis of $V_\theta(0)$

§5.1. Integral form of $V_\theta(0)$

In this section, we shall prove that $V_\theta(0)$ has a lower global basis. In order to see this, we shall first prove that $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ is a basis of the $\mathbf{A}$-module $V_\theta(0)_{\mathbf{A}}$. Recall that $\mathbf{A} = \mathbb{Q}[q,q^{-1}]$, and $V_\theta(0)_{\mathbf{A}} = U_q^-(\mathfrak{gl}_\infty)_{\mathbf{A}}\phi$.

Lemma 5.1. $V_\theta(0)_{\mathbf{A}} = \bigoplus_{\mathfrak{m}\in\mathcal{M}_\theta} \mathbf{A}P_\theta(\mathfrak{m})\phi.$

Proof. It is clear that $\bigoplus_{\mathfrak{m}\in\mathcal{M}_\theta}\mathbf{A}P_\theta(\mathfrak{m})\phi$ is stable by the actions of $F_k^{(n)}$ by Proposition 3.20. Hence we obtain $V_\theta(0)_{\mathbf{A}}\subset \bigoplus_{\mathfrak{m}\in\mathcal{M}_\theta}\mathbf{A}P_\theta(\mathfrak{m})\phi$.

We shall prove $P_\theta(\mathfrak{m})\phi\in U_q^-(\mathfrak{gl}_\infty)_{\mathbf{A}}\phi$. It is well-known that $\langle i,j\rangle^{(m)}$ is contained in $U_q^-(\mathfrak{gl}_\infty)_{\mathbf{A}}$, which is also seen by Proposition 3.20 (3). We divide $\mathfrak{m}$ as $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2$, where $\mathfrak{m}_1=\sum_{-j<i\leqslant j} m_{ij}\langle i,j\rangle$ and $\mathfrak{m}_2=\sum_{k>0} m_k\langle -k,k\rangle$. Then $P_\theta(\mathfrak{m}) = P(\mathfrak{m}_1)P_\theta(\mathfrak{m}_2)$ and $P(\mathfrak{m}_1)\in U_q^-(\mathfrak{gl}_\infty)_{\mathbf{A}}$. Hence we may assume from the beginning that $\mathfrak{m}=\sum_{0<k\leqslant a} m_k\langle -k,k\rangle$. We shall show that $P_\theta(\mathfrak{m})\phi\in V_\theta(0)_{\mathbf{A}}$ by the induction on a.

Assume $a>1$. Set $\mathfrak{m}'=\sum_{0<k\leqslant a-4} m_k\langle -k,k\rangle$ and $v=P_\theta(\mathfrak{m}')\phi$. Then $\langle -a+2,a-2\rangle^{[m]}v\in V_\theta(0)_{\mathbf{A}}$ for any m by the induction hypothesis.

We shall show that $\langle -a,a\rangle^{[n]}\langle -a+2,a-2\rangle^{[m]}v$ is contained in $V_\theta(0)_{\mathbf{A}}$ by the induction on n. Since $P_\theta(\mathfrak{m}')$ commutes with $\langle a\rangle$, $\langle -a\rangle$, $\langle -a+2,a-2\rangle$, $\langle -a+2,a\rangle$ and $\langle -a,a\rangle$, Proposition 3.20 (2) implies

$$\begin{aligned}&\langle -a\rangle^{(2n)}\langle -a+2,a-2\rangle^{[n+m]}v\\ &=\sum_{i+j+2t=2n,\; j+t=u} q^{2(n+m)i+j(j-1)/2-i(t+u)}\\ &\qquad\times\langle a\rangle^{(i)}\langle -a+2,a\rangle^{(j)}\langle -a,a\rangle^{[t]}\langle -a+2,-2\rangle^{[n+m-u]}v,\end{aligned}$$

which is contained in $V_\theta(0)_{\mathbf{A}}$. Since we have

$$\langle a\rangle^{(i)}\langle -a+2,a\rangle^{(j)}\langle -a,a\rangle^{[t]}\langle -a+2,a-2\rangle^{[n+m-u]}v\in V_\theta(0)_{\mathbf{A}}$$

if $(i,j,t,u)\neq(0,0,n,n)$ by the induction hypothesis on n, $\langle -a,a\rangle^{[n]}\langle -a+2,a-2\rangle^{[m]}v$ is contained in $V_\theta(0)_{\mathbf{A}}$.

If $a = 1$, we similarly prove $P_\theta(\mathfrak{m})\phi \in V_\theta(0)_{\mathbf{A}}$ using Proposition 3.20 (1) instead of (2). □

§5.2. Conjugate of the PBW basis

We will prove that the bar involution is upper triangular with respect to the PBW basis $\{P_\theta(\mathfrak{m})\}_{\mathfrak{m}\in\mathcal{M}_\theta}$.

First we shall prove Theorem 3.10 (4).

For $a, b \in \mathcal{M}$ such that $a \leqslant b$, we denote by $\mathcal{M}_{[a,b]}$ (resp. $\mathcal{M}_{\leqslant b}$) the set of $\mathfrak{m} \in \mathcal{M}$ of the form $\mathfrak{m} = \sum_{a\leqslant i\leqslant j\leqslant b} m_{i,j}\langle i,j\rangle$ (resp. $\mathfrak{m} = \sum_{i\leqslant j\leqslant b} m_{i,j}\langle i,j\rangle$). Similarly we define $(\mathcal{M}_\theta)_{\leqslant b}$. For a multisegment $\mathfrak{m} \in \mathcal{M}_{\leqslant b}$, we divide $\mathfrak{m}$ into $\mathfrak{m} = \mathfrak{m}_b + \mathfrak{m}_{<b}$, where $\mathfrak{m}_b = \sum_{i\leqslant b} m_{i,j}\langle i,b\rangle$ and $\mathfrak{m}_{<b} = \sum_{i\leqslant j<b} m_{i,j}\langle i,j\rangle$.

Lemma 5.2. *For $n \geqslant 0$ and $a, b \in I$ such that $a \leqslant b$, we have*

$$\overline{\langle a,b\rangle^{(n)}} \in \langle a,b\rangle^{(n)} + \sum_{\mathfrak{m} \underset{\mathrm{cry}}{<} n\langle a,b\rangle} \mathbf{K}P(\mathfrak{m}).$$

Proof. We shall first show

$$\overline{\langle a,b\rangle} \in \langle a,b\rangle + \sum_{a+2\leqslant k\leqslant b} \langle k,b\rangle U_q^-(\mathfrak{g}) \tag{5.1}$$

by the induction on $b-a$. If $a = b$, it is trivial. If $a < b$, we have

$$\begin{aligned}\overline{\langle a,b\rangle} &= \langle a\rangle\overline{\langle a+2,b\rangle} - q^{-1}\overline{\langle a+2,b\rangle}\langle a\rangle \\ &\in \langle a\rangle\Big(\langle a+2,b\rangle + \sum_{a+2<k\leqslant b}\langle k,b\rangle U_q^-(\mathfrak{g})\Big) \\ &\qquad\qquad -q^{-1}\Big(\langle a+2,b\rangle + \sum_{a+2<k\leqslant b}\langle k,b\rangle U_q^-(\mathfrak{g})\Big)\langle a\rangle \\ &\subset \langle a,b\rangle + (q-q^{-1})\langle a+2,b\rangle\langle a\rangle + \sum_{a+2<k\leqslant b}(\langle k,b\rangle\langle a\rangle U_q^-(\mathfrak{g}) + \langle k,b\rangle U_q^-(\mathfrak{g})).\end{aligned}$$

Hence we obtain (5.1). We shall show the lemma by the induction on n. We may assume $n > 0$ and

$$\overline{\langle a,b\rangle^{n-1}} \in \langle a,b\rangle^{n-1} + \sum_{\mathfrak{m} \underset{\mathrm{cry}}{<} (n-1)\langle a,b\rangle} \mathbf{K}P(\mathfrak{m}).$$

Hence we have

$$\overline{\langle a,b\rangle^n} = \overline{\langle a,b\rangle}\ \overline{\langle a,b\rangle^{n-1}} \in \langle a,b\rangle^n + \sum_{a<k\leqslant b}\langle k,b\rangle U_q^-(\mathfrak{g}) + \sum_{\mathfrak{m} \underset{\mathrm{cry}}{<} (n-1)\langle a,b\rangle} \mathbf{K}\langle a,b\rangle P(\mathfrak{m}).$$

For $a < k \leqslant b$ and $\mathfrak{m} \in \mathcal{M}$ such that $\mathrm{wt}(\mathfrak{m}) = \mathrm{wt}(n\langle a,b\rangle) - \mathrm{wt}(\langle k,b\rangle)$, we have $\mathfrak{m} \in \mathcal{M}_{[a,b]}$ and $\mathfrak{m}_b = \sum_{a\leqslant i\leqslant b} m_{i,b}\langle i,b\rangle$ with $\sum_i m_{i,b} = n-1$. In particular, $m_{a,b} \leqslant n-1$. Hence $\langle k,b\rangle P(\mathfrak{m}) \in \mathbf{K}P(\mathfrak{m}+\langle k,b\rangle)$ and $\mathfrak{m}+\langle k,b\rangle \underset{\text{cry}}{<} n\langle a,b\rangle$.

If $\mathfrak{m} \underset{\text{cry}}{<} (n-1)\langle a,b\rangle$, then $\langle a,b\rangle P(\mathfrak{m}) \in \mathbf{K}P(\langle a,b\rangle+\mathfrak{m})$ and $\langle a,b\rangle+\mathfrak{m} \underset{\text{cry}}{<} n\langle a,b\rangle$. □

Proposition 5.3. *For* $\mathfrak{m} \in \mathcal{M}$,

$$\overline{P(\mathfrak{m})} \in P(\mathfrak{m}) + \sum_{\mathfrak{n} \underset{\text{cry}}{<} \mathfrak{m}} \mathbf{K}P(\mathfrak{n}).$$

Proof. Put $\mathfrak{m} = \sum_{i\leqslant j\leqslant b} m_{i,j}\langle i,j\rangle$ and divide $\mathfrak{m} = \mathfrak{m}_b + \mathfrak{m}_{<b}$. We prove the claim by the induction on b and the number of segments in $\mathfrak{m}_b$. Suppose $\mathfrak{m}_b = m\langle a,b\rangle + \mathfrak{m}_1$ with $m = m_{a,b} > 0$, where $\mathfrak{m}_1 = \sum_{a<i\leqslant b} m_{i,b}\langle i,b\rangle$.
(i) Let us first show that

$$\overline{P(\mathfrak{m}_b)} \in P(\mathfrak{m}_b) + \sum_{\mathfrak{m}' \underset{\text{cry}}{<} \mathfrak{m}_b} \mathbf{K}P(\mathfrak{m}'). \tag{5.2}$$

We have $\overline{P(\mathfrak{m}_b)} = \overline{P(\mathfrak{m}_1)} \cdot \overline{\langle a,b\rangle^{(m)}}$. Since $\overline{P(\mathfrak{m}_1)} \in P(\mathfrak{m}_1) + \sum_{\mathfrak{m}'_1 \underset{\text{cry}}{<} \mathfrak{m}_1} \mathbf{K}P(\mathfrak{m}'_1)$ by the induction hypothesis, and $\overline{\langle a,b\rangle^{(m)}} \in \langle a,b\rangle^{(m)} + \sum_{\mathfrak{m}'' \underset{\text{cry}}{<} m\langle a,b\rangle} \mathbf{K}P(\mathfrak{m}'')$, we have

$$\overline{P(\mathfrak{m}_b)} \in P(\mathfrak{m}_b) + \sum_{\mathfrak{m}'_1 \underset{\text{cry}}{<} \mathfrak{m}_1,\ \mathfrak{m}'_1 \in \mathcal{M}_{[a+2,b]}} \mathbf{K}P(\mathfrak{m}'_1)\langle a,b\rangle^{(m)} + \sum_{\mathfrak{m}'_1 \underset{\text{cry}}{\leqslant} \mathfrak{m}_1,\ \mathfrak{m}'' \underset{\text{cry}}{<} m\langle a,b\rangle} \mathbf{K}P(\mathfrak{m}'_1)P(\mathfrak{m}'').$$

If $\mathfrak{m}'_1 \underset{\text{cry}}{<} \mathfrak{m}_1$ and $\mathfrak{m}'_1 \in \mathcal{M}_{[a+2,b]}$, then $P((\mathfrak{m}'_1)_{<b})$ and $\langle a,b\rangle^{(m)}$ commute. Hence $P(\mathfrak{m}'_1)\langle a,b\rangle^{(m)} = P(\mathfrak{m}'_1 + m\langle a,b\rangle)$ and $\mathfrak{m}'_1 + m\langle a,b\rangle \underset{\text{cry}}{<} \mathfrak{m}_b$.

If $\mathfrak{m}'_1 \underset{\text{cry}}{\leqslant} \mathfrak{m}_1$, $\mathfrak{m}'_1 \in \mathcal{M}_{[a+2,b]}$ and $\mathfrak{m}'' \underset{\text{cry}}{<} m\langle a,b\rangle$, then we can write $\mathfrak{m}''_b = j\langle a,b\rangle + \mathfrak{m}_2$ with $j < m$ and $\mathfrak{m}_2 \in \mathcal{M}_{[a+2,b]}$. Hence we have

$$P(\mathfrak{m}'_1)P(\mathfrak{m}'') \in \mathbf{K}P((\mathfrak{m}'_1)_b)P(j\langle a,b\rangle)P((\mathfrak{m}'_1)_{<b})P(\mathfrak{m}_2)P(\mathfrak{m}''_{<b}).$$

Since $(\mathfrak{m}'_1)_{<b}$, $\mathfrak{m}_2 \in \mathcal{M}_{[a+2,b]}$ we have $P((\mathfrak{m}'_1)_{<b})P(\mathfrak{m}_2)P(\mathfrak{m}''_{<b}) \in \sum_{\mathfrak{n}_b \in \mathcal{M}_{[a+2,b]}} \mathbf{K}P(\mathfrak{n})$. Hence we have $P(\mathfrak{m}'_1)P(\mathfrak{m}'') \in \sum_{\mathfrak{n}_b \in \mathcal{M}_{[a+2,b]}} \mathbf{K}P((\mathfrak{m}'_1)_b + j\langle a,b\rangle + \mathfrak{n})$ and $(\mathfrak{m}'_1)_b + j\langle a,b\rangle + \mathfrak{n} \underset{\text{cry}}{<} \mathfrak{m}_b$. Hence we obtain (5.2).

(ii) By the induction hypothesis, $\overline{P(\mathfrak{m}_{<b})} \in P(\mathfrak{m}_{<b}) + \sum_{\mathfrak{m}'' \underset{\text{cry}}{<} \mathfrak{m}_{<b}} \mathbf{K}P(\mathfrak{m}'')$. Since $\overline{P(\mathfrak{m})} = \overline{P(\mathfrak{m}_b)}\ \overline{P(\mathfrak{m}_{<b})}$, (5.2) implies that

$$\overline{P(\mathfrak{m})} \in P(\mathfrak{m}) + \sum_{\mathfrak{m}' \underset{\text{cry}}{<} \mathfrak{m}_b, \mathfrak{m}'' \in \mathcal{M}_{<b}} \mathbf{K}P(\mathfrak{m}')P(\mathfrak{m}'') + \sum_{\mathfrak{m}'' \underset{\text{cry}}{<} \mathfrak{m}_{<b}} \mathbf{K}P(\mathfrak{m}_b)P(\mathfrak{m}'').$$

For $\mathfrak{m}' \underset{\text{cry}}{<} \mathfrak{m}_b$ and $\mathfrak{m}'' \in \mathcal{M}_{<b}$, we have

$$P(\mathfrak{m}')P(\mathfrak{m}'') = P(\mathfrak{m}'_b)P(\mathfrak{m}'_{<b})P(\mathfrak{m}'') \in \sum_{\mathfrak{n} \in \mathcal{M}_{\leqslant b},\, \mathfrak{n}_b = \mathfrak{m}'_b} \mathbf{K}P(\mathfrak{n}) \subset \sum_{\mathfrak{n} \underset{\text{cry}}{<} \mathfrak{m}} \mathbf{K}P(\mathfrak{n}).$$

For $\mathfrak{m}'' \underset{\text{cry}}{<} \mathfrak{m}_{<b}$, we have $P(\mathfrak{m}_b)P(\mathfrak{m}'') = P(\mathfrak{m}_b + \mathfrak{m}'')$ and $\mathfrak{m}_b + \mathfrak{m}'' \underset{\text{cry}}{<} \mathfrak{m}$. Thus we obtain the desired result. □

Proposition 5.4. *For $\mathfrak{m} \in \mathcal{M}_\theta$, we have*

$$\overline{P_\theta(\mathfrak{m})}\phi \in P_\theta(\mathfrak{m})\phi + \sum_{\mathfrak{m}' \in \mathcal{M}_\theta, \mathfrak{m}' \underset{\text{cry}}{<} \mathfrak{m}} \mathbf{K}P_\theta(\mathfrak{m}')\phi.$$

Proof. First note that

$$P(\mathfrak{m})\phi \in \sum_{\mathfrak{n} \in (\mathcal{M}_\theta)_{\leqslant b}} \mathbf{K}P_\theta(\mathfrak{n})\phi \quad \text{for any } b \in I_{>0} \text{ and } \mathfrak{m} \in \mathcal{M}_{[-b,b]}, \tag{5.3}$$

by the weight consideration.

For $\mathfrak{m} \in \mathcal{M}_\theta$, $P_\theta(\mathfrak{m})$ and $P(\mathfrak{m})$ are equal up to a multiple of bar-invariant scalar. Thus we have

$$\overline{P_\theta(\mathfrak{m})} \in P_\theta(\mathfrak{m}) + \sum_{\mathfrak{m}' \in \mathcal{M},\ \mathfrak{m}' \underset{\text{cry}}{<} \mathfrak{m}} \mathbf{K}P(\mathfrak{m}')$$

by Proposition 5.3. Hence it is enough to show that

$$P(\mathfrak{m}')\phi \in \sum_{\mathfrak{n} \in \mathcal{M}_\theta,\ \mathfrak{n} \underset{\text{cry}}{<} \mathfrak{m}} \mathbf{K}P_\theta(\mathfrak{n})\phi \tag{5.4}$$

for $\mathfrak{m}' \in \mathcal{M}$ such that $\mathfrak{m}' \underset{\text{cry}}{<} \mathfrak{m}$ and $\mathrm{wt}(\mathfrak{m}') = \mathrm{wt}(\mathfrak{m})$. Put $\mathfrak{m} = \sum_{i \leqslant j \leqslant b} m_{i,j}\langle i, j\rangle$ and write $\mathfrak{m} = \mathfrak{m}_b + \mathfrak{m}_{<b}$. We prove (5.4) by the induction on b. By the assumption on $\mathfrak{m}'$, we have $\mathfrak{m}' \in \mathcal{M}_{[-b,b]}$ and $\mathfrak{m}'_b \underset{\text{cry}}{\leqslant} \mathfrak{m}_b$. Thus $\mathfrak{m}'_b \in \mathcal{M}_\theta$. Hence $\mathbf{K}P(\mathfrak{m}')\phi = \mathbf{K}P_\theta(\mathfrak{m}'_b)P(\mathfrak{m}'_{<b})\phi$.

If $\mathfrak{m}'_b = \mathfrak{m}_b$, then $\mathfrak{m}'_{<b} <_{\mathrm{cry}} \mathfrak{m}_{<b}$, and the induction hypothesis implies $P(\mathfrak{m}'_{<b})\phi \in \sum_{\mathfrak{n}\in\mathcal{M}_\theta,\ \mathfrak{n} \underset{\mathrm{cry}}{<} \mathfrak{m}_{<b}} \mathbf{K}P_\theta(\mathfrak{n})\phi$. Since $P_\theta(\mathfrak{m}'_b)P_\theta(\mathfrak{n}) = P_\theta(\mathfrak{m}'_b + \mathfrak{n})$ and $\mathfrak{m}'_b + \mathfrak{n} \underset{\mathrm{cry}}{<} \mathfrak{m}$, we obtain (5.4).

If $\mathfrak{m}'_b \underset{\mathrm{cry}}{<} \mathfrak{m}_b$, write $\mathfrak{m}' = \sum_{-b\leqslant i\leqslant j\leqslant b} m'_{i,j}\langle i,j\rangle$. Set $s = m_{-b.b} - m'_{-b,b} \geqslant 0$. Since $\mathrm{wt}(\mathfrak{m}') = \mathrm{wt}(\mathfrak{m})$, we have $\sum_{j<b} m'_{-b,j} = s$. If $s = 0$, then $\mathfrak{m}'_{<b} \in \mathcal{M}_{[-b+2,b-2]}$, and $P(\mathfrak{m}'_{<b})\phi \in \sum_{\mathfrak{n}\in(\mathcal{M}_\theta)_{<b}} \mathbf{K}P_\theta(\mathfrak{n})\phi$ by (5.3). Then (5.4) follows from $\mathfrak{m}'_b + \mathfrak{n} \underset{\mathrm{cry}}{<} \mathfrak{m}$.

Assume $s > 0$. Since $\mathfrak{m}'_{<b} \in \mathcal{M}_{[-b,b]}$, we have $P(\mathfrak{m}'_{<b})\phi \in \sum\limits_{\mathfrak{n}\in(\mathcal{M}_\theta)_{\leqslant b}} \mathbf{K}P_\theta(\mathfrak{n})\phi$ by (5.3). We may assume $(1+\theta)\,\mathrm{wt}(\mathfrak{m}'_{<b}) = (1+\theta)\,\mathrm{wt}(\mathfrak{n})$ (see Remark 2.12). Hence, we have $s = 2m_{-b,b}(\mathfrak{n}) + \sum\limits_{-b<i\leqslant b} m_{i,b}(\mathfrak{n})$. In particular, $m_{-b,b}(\mathfrak{n}) \leqslant s/2$. We have $\mathfrak{m}'_b+\mathfrak{n} \in \mathcal{M}_\theta$ and $P_\theta(\mathfrak{m}'_b)P_\theta(\mathfrak{n})\phi = P_\theta(\mathfrak{m}'_b+\mathfrak{n})\phi$. Since $m_{-b,b}(\mathfrak{m}'_b+\mathfrak{n}) \leqslant (m_{-b,b} - s) + s/2 < m_{-b,b}$, we have $\mathfrak{m}'_b + \mathfrak{n} \underset{\mathrm{cry}}{<} \mathfrak{m}$. Hence we obtain (5.4). □

§5.3. Existence of a global basis

As a consequence of the preceding subsections, we obtain the following theorem.

Theorem 5.5.

(i) $(L_\theta(0), L_\theta(0)^-, V_\theta(0)_{\mathbf{A}})$ *is balanced.*

(ii) *For any* $\mathfrak{m} \in \mathcal{M}_\theta$, *there exists a unique* $\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m}) \in L_\theta(0) \cap V_\theta(0)_{\mathbf{A}}$ *such that* $\overline{\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m})} = \mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m})$ *and* $\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m}) \equiv P_\theta(\mathfrak{m})\phi \mod qL_\theta(0)$.

(iii) $\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m}) \in P_\theta(\mathfrak{m})\phi + \sum_{\mathfrak{n} \underset{\mathrm{cry}}{<} \mathfrak{m}} q\mathbb{Q}[q]P_\theta(\mathfrak{n})\phi$ *for any* $\mathfrak{m} \in \mathcal{M}_\theta$.

(iv) $\{\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m})\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ *is a basis of the* $\mathbf{A}$*-module* $V_\theta(0)_{\mathbf{A}}$, *the* $\mathbf{A}_0$*-module* $L_\theta(0)$ *and the* $\mathbf{K}$*-vector space* $V_\theta(0)$.

Proof. We have already seen that $\overline{P_\theta(\mathfrak{m})\phi} = \sum_{\mathfrak{m}' \underset{\mathrm{cry}}{\leqslant} \mathfrak{m}} c_{\mathfrak{m},\mathfrak{m}'}P_\theta(\mathfrak{m}')\phi$ for $c_{\mathfrak{m},\mathfrak{m}'} \in \mathbf{A}$ with $c_{\mathfrak{m},\mathfrak{m}} = 1$. Let us denote by C the matrix $(c_{\mathfrak{m},\mathfrak{m}'})_{\mathfrak{m},\mathfrak{m}'\in\mathcal{M}_\theta}$. Then $\overline{C}C = \mathrm{id}$ and it is well-known that there is a matrix $A = (a_{\mathfrak{m},\mathfrak{m}'})_{\mathfrak{m},\mathfrak{m}'\in\mathcal{M}_\theta}$ such that $\overline{A}C = A$, $a_{\mathfrak{m},\mathfrak{m}'} = 0$ unless $\mathfrak{m}' \underset{\mathrm{cry}}{\leqslant} \mathfrak{m}$, $a_{\mathfrak{m},\mathfrak{m}} = 1$ and $a_{\mathfrak{m},\mathfrak{m}'} \in q\mathbb{Q}[q]$ for $\mathfrak{m}' \underset{\mathrm{cry}}{<} \mathfrak{m}$. Set $\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m}) = \sum_{\mathfrak{m}' \underset{\mathrm{cry}}{\leqslant} \mathfrak{m}} a_{\mathfrak{m},\mathfrak{m}'}P_\theta(\mathfrak{m}')\phi$. Then we have $\overline{\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m})} = \mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m})$ and $\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m}) \equiv P_\theta(\mathfrak{m})\phi \mod qL_\theta(0)$. Since $\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m})$ is a basis of $V_\theta(0)_{\mathbf{A}}$, we obtain the desired results. □

Errata to "Symmetric crystals and affine Hecke algebras of type B, Proc. Japan Acad., 82, no. 8, 2006, 131–136" :

(i) In Conjecture 3.8, $\lambda = \Lambda_{p_0} + \Lambda_{p_0^{-1}}$ should be read as $\lambda = \sum_{a \in A} \Lambda_a$, where $A = I \cap \{p_0, p_0^{-1}, -p_0, -p_0^{-1}\}$. We thank S. Ariki who informed us that the original conjecture is false.

(ii) In the two diagrams of $B_\theta(\lambda)$ at the end of § 2, λ should be 0.

(iii) Throughout the paper, $A^{(1)}_\ell$ should be read as $A^{(1)}_{\ell-1}$.

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