Publ. RIMS, Kyoto Univ.
44 (2008), 955–971

On $\mathbb{Q}$-conic Bundles, II

By

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Abstract

A $\mathbb{Q}$-conic bundle germ is a proper morphism from a threefold with only terminal singularities to the germ $(Z \ni o)$ of a normal surface such that fibers are connected and the anti-canonical divisor is relatively ample. We obtain the complete classification of $\mathbb{Q}$-conic bundle germs when the base surface germ is singular. This is a generalization of [MP08], which further assumed that the fiber over o is irreducible.

§1. Introduction

This note is a continuation of our previous work [MP08] where we studied the local structure of $\mathbb{Q}$-conic bundles.

(1.1) Definition. A $\mathbb{Q}$-*conic bundle* is a projective morphism $f: X \to Z$ from a threefold with only terminal singularities to a surface such that

(i) $f_*\mathcal{O}_X = \mathcal{O}_Z$ and all fibers are one-dimensional,

(ii) $-K_X$ is f-ample.

For $f: X \to Z$ as above and for a point $o \in Z$, we call the analytic germ $(X, f^{-1}(o)_{\mathrm{red}})$ a $\mathbb{Q}$-*conic bundle germ.*

In [MP08] we completely classified $\mathbb{Q}$-conic bundle germs over a singular base and such that the central fiber is irreducible. For convenience of quotations

Communicated by S. Mukai. Received October 4, 2007.
2000 Mathematics Subject Classification(s): 14J30, 14E35, 14E30.
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we reproduce briefly the classification. For more detailed explanations we refer to the original paper [MP08].

(1.2) Theorem. *Let $f\colon (X,C) \to (Z,o)$ be a $\mathbb{Q}$-conic bundle germ, where C is irreducible and (Z,o) is singular. Then we are in one of the following cases*:

Type	No.	singularities	(Z,o)
toroidal	(1.2.1)	$\frac{1}{n}(1,a,-a)$ and $\frac{1}{n}(-1,a,-a)$, $\gcd(n,a)=1$	A_{n-1}
(IA)+(IA)	(1.2.2)	$\frac{1}{n}(a,-1,1)$ and $\frac{1}{n}(a+1,1,-1)$, $n = 2a+1$	A_{n-1}
(IE$^\vee$)	(1.2.3)	$\frac{1}{8}(5,1,3)$	A_3
(ID$^\vee$)	(1.2.4)	$cA/2$ or $cAx/2$	A_1
(IA$^\vee$)	(1.2.5)	$\frac{1}{4}(1,1,3)$ (+(III))	A_1
(II$^\vee$)	(1.2.6)	$cAx/4$ (+(III))	A_1

In this paper we consider the case where the base surface is singular and the central fiber is reducible. Our main result is the following.

(1.3) Theorem. *Let $f\colon (X,C) \to (Z,o)$ be a $\mathbb{Q}$-conic bundle germ. Assume that C is reducible and the base surface (Z,o) is singular. Then (Z,o) is Du Val of type A_1 and (X,C) is the $\boldsymbol{\mu}_2$-quotient of the index-two $\mathbb{Q}$-conic bundle $f'\colon (X',C') \to (Z',o')$ over a smooth base, where $\boldsymbol{\mu}_2$ acts on (Z',o') freely in codimension one. Moreover, C' has four irreducible components, $\boldsymbol{\mu}_2$ does not fix any of them and X has a unique non-Gorenstein point P. Furthermore, X' is given by the following two equations in $\mathbb{P}(1,1,1,2)_{y_1,\dots,y_4} \times \mathbb{C}^2_{u,v}$*

$$\begin{cases} y_1^2 - y_3^2 = \psi_1(y_1,\dots,y_4;u,v), \\ y_2^2 - y_3^2 = \psi_2(y_1,\dots,y_4;u,v), \end{cases}$$

where $\boldsymbol{\mu}_2$ acts as follows:

$$(y_1,y_2,y_3,y_4;u,v) \longmapsto (-y_1,-y_2,y_3,-y_4;-u,-v).$$

Here $\psi_i = \psi_i(y_1,\dots,y_4;u,v)$ are weighted quadratic in $y_1,\dots,y_4$ with respect to $\mathrm{wt}(y_1,\dots,y_4) = (1,1,1,2)$ and $\psi_i(y_1,\dots,y_4;0,0) = 0$. The following are the only possibilities:

(1.3.1) *(X,P) is a cyclic quotient singularity of type $\frac{1}{4}(1,1,-1)$ and for any component $C_i \subset C$ germ (X,C_i) is of type* (IA$^\vee$),

(1.3.2) *(X, P) is a singularity of type $cAx/4$ and for any component $C_i \subset C$ germ (X, C_i) is of type* $(\mathrm{II}^\vee)$.

Conversely, if the quotient $(X, C) = (X', C')/\boldsymbol{\mu}_2$, where (X', C') and the action of $\boldsymbol{\mu}_2$ are as above, has only terminal singularities, then (X, C) is a conic bundle germ over $\mathbb{C}^2_{u,v}/\boldsymbol{\mu}_2$ with reducible central fiber C.

(1.4) Corollary (Reid's general elephant conjecture). *Let $f\colon (X, C) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ. Assume that (Z, o) is singular. Then a general member $F \in |-K_X|$ has only Du Val singularities. Moreover, in cases* (1.3.1) *and* (1.3.2) *F does not contain any component of C and is of type A_3 and D respectively.*

Below are a series of explicit examples of $\mathbb{Q}$-conic bundles as in (1.3).

(1.5) Example. Consider the subvariety $X' \subset \mathbb{P}(1, 1, 1, 2) \times \mathbb{C}^2$ defined by the following two equations:

$$\begin{cases} y_1^2 - y_3^2 + u^{2k+1} y_4 + v^2 y_2^2 = 0, \\ y_2^2 - y_3^2 + v y_4 \qquad\qquad\quad = 0. \end{cases}$$

The projection $f'\colon X' \to \mathbb{C}^2$ is a $\mathbb{Q}$-conic bundle of index 2 (cf. [MP08, 12.1.3]). Define the action of $\boldsymbol{\mu}_2$ on X' as follows

$$(y_1,\ y_2,\ y_3,\ y_4;\ u,\ v) \longmapsto (-y_1,\ -y_2,\ y_3,\ -y_4;\ -u,\ -v).$$

Then $X'/\boldsymbol{\mu}_2 \to \mathbb{C}^2/\boldsymbol{\mu}_2$ is a $\mathbb{Q}$-conic bundle with a unique non-Gorenstein point P. The point P is of type (1.3.1) if $k = 0$ and of type (1.3.2) if $k \geq 1$.

The basic idea of the proof is to reduce the problem of classifying $\mathbb{Q}$-conic bundles (X, C) as in Theorem (1.3) to the case where the central fiber is irreducible by applying the MMP to a $\mathbb{Q}$-factorialization $(X^{\mathrm{q}}, C^{\mathrm{q}})$. Then the resulting $\mathbb{Q}$-conic bundle $(\bar{X}, \bar{C})$ belongs to the list (1.2). We trace back from $(\bar{X}, \bar{C})$ to (X, C). It turns out that in many cases the steps of the MMP do not affect the singularities of $(\bar{X}, \bar{C})$. Here we use some results about divisorial contractions and flips (see §2) based on [KM92] and [Kaw96]. Then the base change trick allows us to show that (X, C) is a $\boldsymbol{\mu}_2$-quotient of an index-two conic bundle, see §3.

§2. Preliminary Results on Extremal Contractions

(2.1) Let $(E^\sharp, P^\sharp)$ be a Du Val singularity. (We assume that $(E^\sharp, P^\sharp)$ is *singular*). Assume that $\boldsymbol{\mu}_m$ acts on $E^\sharp$ freely outside $P^\sharp$ and the quotient

$(E, P) = (E^\sharp, P^\sharp)/\boldsymbol{\mu}_m$ is also Du Val. Then there is a $\boldsymbol{\mu}_m$-equivariant embedding $(E^\sharp, P^\sharp) \subset (\mathbb{C}^3_{x,y,z}, 0)$ such that x, y, z and the equation of $E^\sharp$ are semi-invariant. Let $F^\sharp \subset \mathbb{C}^3$ be the locus of points at which the action of $\boldsymbol{\mu}_m$ is not free. By our assumption $F^\sharp$ is a curve. Define the invariant $\varsigma(E^\sharp, P^\sharp, \boldsymbol{\mu}_m)$ as the local intersection number $(E^\sharp \cdot F^\sharp)_0$. According to [Rei87, 4.10] we have only the following cases:

(2.1.1)

m	$(E^\sharp, P^\sharp) \to (E, P)$	$\varsigma(E^\sharp, P^\sharp, \boldsymbol{\mu}_m)$
any	$A_{r-1} \to A_{mr-1}$	r
4	$A_{2r-2} \to D_{2r+1}$	$2r-1$
2	$A_{2r-1} \to D_{r+2}$	2
3	$D_4 \to E_6$	2
2	$D_{r+1} \to D_{2r}$	r
2	$E_6 \to E_7$	3

(2.1.2) Let (W, P) be a three-dimensional terminal singularity of index $m > 1$ and let $E \in |-K_W|$ be a divisor having a Du Val singularity at P. Assume that (W, P) is not a cyclic quotient. Let $\pi\colon (W^\sharp, P^\sharp) \to (W, P)$ be the index-one $\boldsymbol{\mu}_m$-cover and let $(W^\sharp, P^\sharp) = \{\phi = 0\} \subset \mathbb{C}^4_{x_1,x_2,x_3,x_4}$ be a $\boldsymbol{\mu}_m$-equivariant embedding. Let $E^\sharp := \pi^{-1}(E)$ and $F^\sharp \subset \mathbb{C}^4$ be the locus of points at which the action of $\boldsymbol{\mu}_m$ is not free. Since π is free in codimension two, $F^\sharp$ is a curve. Recall that the local intersection number $(W^\sharp \cdot F^\sharp)_0$ is called the *axial multiplicity* of (W, P) [Mor88, 1a.5]. We denote it by $\mathrm{am}(W, P)$. By the classification of terminal singularities we may assume that $F^\sharp$ is the x_4-axis, and either $\mathrm{wt}(x_1, x_2, x_3, x_4, \phi) \equiv (1, -1, a, 0, 0) \bmod m$, or $m = 4$ and $\mathrm{wt}(x_1, x_2, x_3, x_4, \phi) \equiv (1, -1, a, 2, 2) \bmod 4$, where $\gcd(a, m) = 1$. Since $(E^\sharp, P^\sharp)$ is a Du Val singularity, its Zariski tangent space at the origin is three-dimensional. Hence there is a $\boldsymbol{\mu}_m$-stable hypersurface $H^\sharp \subset \mathbb{C}^4$ such that $E^\sharp = H^\sharp \cap W^\sharp$ and $H^\sharp$ is smooth.

(2.1.3) Claim. $F^\sharp \subset H^\sharp$.

Proof. Let ψ be the $\boldsymbol{\mu}_m$-semi-invariant equation of $H^\sharp$. Then $\mathrm{wt}\,\psi \equiv a$. Hence ψ does not contain terms x_4^k and so it vanishes on $F^\sharp$. $\square$

(2.1.4) We define the invariant $\varsigma(W, E, P)$ as the local intersection number $(E^\sharp \cdot F^\sharp)_0$ inside $H^\sharp$. Clearly it coincides with $\varsigma(E^\sharp, P^\sharp, \boldsymbol{\mu}_m)$ defined above.

(2.1.5) Lemma. *Assume that* (W, P) *is not a cyclic quotient singularity. The invariant* $\varsigma(W, E, P)$ *does not depend on the choice of* E *and* $\varsigma(W, E, P) = \operatorname{am}(W, P)$.

Proof. Both sides of the equality coincide with the order of vanishing of $\phi|_{F^\sharp}$. □

(2.1.6) Corollary. *Let* (W, P) *is a three-dimensional terminal singularity of index* $m > 1$ *which is not a cyclic quotient and let* $E \in |-K_{(W,P)}|$ *be a member having a Du Val singularity of A-type at* P. *Then* E *is isomorphic to a general member* $E_{\mathrm{gen}} \in |-K_{(W,P)}|$.

Proof. By the above lemma we have $\varsigma(E^\sharp, P^\sharp, \boldsymbol{\mu}_m) = \varsigma(E^\sharp_{\mathrm{gen}}, P^\sharp, \boldsymbol{\mu}_m) = \operatorname{am}(W, P)$. Then the statement follows by the first line in (2.1.1). □

(2.2) Proposition. *Let* $\varphi\colon (V, \Gamma) \to (W, o)$ *be the analytic germ of a divisorial extremal contraction of threefolds with terminal singularities (in particular,* W *is* $\mathbb{Q}$*-Gorenstein) such that the central fiber* $\Gamma := \varphi^{-1}(o)_{\mathrm{red}}$ *is one-dimensional and irreducible.*

(i) *The point* (W, o) *cannot be of type* $cAx/4$.

(ii) *If* (W, o) *is of type* $cAx/2$, *then* (V, Γ) *has a unique non-Gorenstein point which is of type* $(\mathrm{II}^\vee)$.

(iii) *If* (W, o) *is analytically isomorphic to*

$$\{x_1x_2 + x_3^2 + x_4^{2k} = 0\}/\boldsymbol{\mu}_2(1, 1, 0, 1), \tag{2.2.1}$$

then (V, Γ) *has a unique non-Gorenstein point* P *which is locally imprimitive of index* 4 *and splitting degree* 2. *Moreover,* $P \in (V, \Gamma)$ *is either of type* $(\mathrm{II}^\vee)$ *or* $(\mathrm{IA}^\vee)$ *and in the second case* (X, P) *is a cyclic quotient singularity.*

Proof. For the proof we assume that (W, o) is of type $cAx/4$, $cAx/2$, or as in (2.2.1). We will use the classification [KM92, Th. 2.2]. Let m be the index of (W, o). Then the canonical class K_W is an m-torsion element in $\mathrm{Cl}^{\mathrm{sc}}(W, o)$. Its pull-back φ^*K_W is a well-defined Cartier divisor on $V \setminus \Gamma$ such that $m(\varphi^*K_W) \sim 0$. Hence $\mathrm{Cl}^{\mathrm{sc}}(V, \Gamma)$ contains an m-torsion element, say ξ. By the classification [KM92, Th. 2.2] and by [Mor88, (1.10)] the group $\mathrm{Cl}^{\mathrm{sc}}(V, \Gamma)$ can contain a torsion only when (V, Γ) is of type (k1A) (with a point of type $(\mathrm{IA}^\vee)$), $(\mathrm{II}^\vee)$, or (k2A).

Assume that (V, Γ) is of type (k2A). Then by [KM92, Th. 2.2] a general member $D \in |-K_V|$ and its image $\varphi(D) \in |-K_W|$ have only Du Val singularities. Moreover, $(\varphi(D), o)$ is a singularity of type A_* and so (W, o) is of type $cA/*$. Clearly, the contraction $\varphi|_D : D \to \varphi(D)$ is crepant. By our assumptions (W, o) is a singularity given by (2.2.1). So, $\mathrm{am}(W, o) = 2$. By Corollary (2.1.6) the singularity $(\varphi(D), o)$ is of type A_3. Since $\varphi_D : D \to \varphi(D)$ is crepant and V has two singular points, the only possibility is that D has two singularities of type A_1. But in this case V is of index two and then by [KM92, Th. 4.7] V has a unique non-Gorenstein point, a contradiction.

In the remaining cases ($\mathrm{II}^\vee$) and (k1A), V has a unique non-Gorenstein point P. Then (V, Γ) is locally imprimitive at P and the splitting degree equals m. In particular, the index of P is $> m$ [Mor88, Cor. 1.16]. Thus if (V, Γ) is of type ($\mathrm{II}^\vee$), then we are in the case (ii) or (iii).

Assume that (V, Γ) is of type (k1A). Then by [KM92, Th. 2.2] a general member $D \in |-K_V|$ does not contain Γ, has only Du Val singularity at $P := \{D \cap \Gamma\}$, and $\varphi|_D : D \to \varphi(D)$ is an isomorphism. Hence $\varphi(D) \in |-K_W|$ has a Du Val singularity of type A at o. In this case, (W, o) cannot be of type $cAx/*$. Thus (W, o) is given by (2.2.1). By Corollary (2.1.6) $D \simeq \varphi(D)$ is of type A_3. Since the index of (V, P) is > 2, (V, P) must be a cyclic quotient singularity $\frac{1}{4}(1, 1, -1)$. So we are in the case (iii). This proves the proposition. $\square$

(2.3) Proposition. *Let $\chi : (V, \Gamma) \dashrightarrow (V^+, \Gamma^+)$ be a flip of threefolds with terminal singularities with irreducible flipping curve Γ. Then (V^+, Γ^+) contains none of the following configurations of singularities*:

(i) *two cyclic quotient singularities P_1^+ and P_2^+ of indices m_1 and m_2 with* $\gcd(m_1, m_2) > 1$ *such that (V^+, Γ^+) is locally primitive at P_1^+ and P_2^+*;

(ii) *an imprimitive point P^+ of splitting degree $s > 1$.*

Proof. By [KM92, Cor. 13.4] Γ^+ is irreducible. Assume that one of the cases (i)-(ii) holds. As in [Mor88, Cor. 1.12] there is a d-torsion element $\xi^+ \in \mathrm{Cl}^{\mathrm{sc}}\, V^+$ for some $d > 1$. Its proper transform ξ on V is a d-torsion element in $\mathrm{Cl}^{\mathrm{sc}}\, V$. In [KM92] flips are classified into 6 types (k1A), (k2A), (cD/3), (IIA), (IC), (kAD) according to a general member of the anti-canonical linear system $|-K_V|$ [KM92, Th. 2.2]. The group $\mathrm{Cl}^{\mathrm{sc}}\, V$ can contain a torsion only in cases (k1A) and (k2A) (in all other cases the flipping variety is locally primitive and indices of non-Gorenstein points are coprime, cf. [Mor88, (1.10)]). The torsion elements ξ and ξ^+ induce the following cyclic $\boldsymbol{\mu}_d$-coverings:

(2.3.1)
$$\begin{tikzcd}
(V',\Gamma') \arrow[r, dashed, "\chi'"] \arrow[d, "\pi"] & (V^{+\prime},\Gamma^{+\prime}) \arrow[d, "\pi^+"] \\
(V,\Gamma) \arrow[r, dashed, "\chi"] & (V^+,\Gamma^+)
\end{tikzcd}$$

Consider the flipping diagram

$$\begin{tikzcd}
(V,\Gamma) \arrow[rr, dashed, "\chi"] \arrow[dr, "\varphi"'] & & (V^+,\Gamma^+) \arrow[dl, "\varphi^+"] \\
& (W,o) &
\end{tikzcd}$$

By [Mor88, Th. 7.3, 9.10] and [KM92, Th. 2.2], a general member $D \in |-K_V|$ has only Du Val singularities. Since the restriction $\varphi_D\colon D \to \varphi(D)$ is crepant, the same holds for $\varphi(D) \in |-K_W|$. Further, if we put $D^+ = \chi(D)$, then $D^+ \in |-K_{V^+}|$ and D^+ also has only Du Val singularities. Since $K_{V^+} \cdot \Gamma^+ > 0$, $D^+ \supset \Gamma^+$.

(2.3.2) First we consider the case where our flip is of type (k1A). Then V has a unique non-Gorenstein point P and P is of type $cA/*$. In this case $D \cap \Gamma = \{P\}$ and $(\varphi(D), o) \simeq (D, P)$ is of type A_*. Since $\mathrm{Cl}^{\mathrm{sc}}\, V$ has a torsion, (V,Γ) is locally imprimitive at P.

(2.3.3) Assume that we are in the case (i). We claim that V^+ has at least one analytically non-$\mathbb{Q}$-factorial singular point ($\neq P_1^+$, P_2^+). Indeed, since the germ (V,Γ) has only one non-Gorenstein point, it is locally imprimitive and in the diagram (2.3.1) π is the splitting cover [Mor88, Cor. 1.12]. Here Γ' has exactly d components and $V^{+\prime}$ is the relative canonical model of V'. Since (V^+,Γ^+) is locally primitive at P_1^+ and P_2^+, the curve $\Gamma^{+\prime}$ is irreducible. Now the map χ' can be decomposed as follows

$$\chi'\colon V' = V_0' \dashrightarrow V_1' \dashrightarrow \cdots \dashrightarrow V_n' \to V^{+\prime},$$

where every $V_i' \dashrightarrow V_{i+1}'$ is a flip along an irreducible curve and $V_n' \to V^{+\prime}$ is a crepant small contraction (cf. [KM92, Proof of 13.5]). Every step $V_i' \dashrightarrow V_{i+1}'$ preserves the number of components of the central fiber. Hence the crepant contraction $V_n' \to V^{+\prime}$ is nontrivial and gives us an analytically non-$\mathbb{Q}$-factorial point $Q \in \Gamma^+ \subset V^+$ (because $V^{+\prime} \to V^+$ is étale outside of P_1^+ and P_2^+). This proves our claim. Thus the divisor D^+ has at least three singular points: P_1^+,

P_2^+, and Q. But then $\varphi_D^+: D^+ \to \varphi(D)$ contracts Γ^+ to a Du Val singularity of type D_* or E_*, a contradiction.

(2.3.4) Now we assume that we are in the case (ii). We claim that the log divisor $K_{D^+} + \Gamma^+$ is not plt at P^+. Indeed, in the diagram (2.3.1) π^+ is the splitting cover (see [Mor88, Cor. 1.12.1]). In particular, π^+ is étale outside P^+, $\pi^{+-1}(P^+)$ is one point, and $\Gamma^{+\prime}$ has $s > 1$ irreducible components, all of them pass through $\pi^{+-1}(P^+)$. Let $D^{+\prime} := \pi^{+-1}(D^+)$. Since $\Gamma^{+\prime}$ is singular at $\pi^{+-1}(P^+)$, the log divisor $K_{D^{+\prime}} + \Gamma^{+\prime}$ is not plt at this point. This proves our claim because the restriction $\pi_D^+: D^{+\prime} \to D^+$ is étale in codimension one (see, e.g., [Kol92, Cor. 20.4]). Now since the contraction $D^+ \to \varphi(D)$ is crepant, D^+ is dominated by the minimal resolution $D^{\min}$ of $\varphi(D)$: $D^{\min} \to D^+ \to \varphi(D)$. Since $K_{D^+} + \Gamma^+$ is not plt, the exceptional divisor of $D^{\min} \to \varphi(D)$ is not a chain of smooth rational curves. Hence $(\varphi(D), o)$ is not a singularity of type A_*, a contradiction.

(2.3.5) Finally, we consider the case where our flip is of type (k2A). These flips are described in [Mor02]. We will use notation of [Mor02]. By [Mor02, Th. 4.7] (V^+, Γ^+) is locally primitive. Hence we have the case (i). Moreover, V^+ has exactly two singular points and they are analytically isomorphic to germs of the following cA/m_i singularities:

$$\{\xi_i \eta_i = G_{k-i}(\zeta_i^{m_i}, u^{e(k+2-i)})\}/\boldsymbol{\mu}_{m_i} \subset \mathbb{C}^4_{\xi_i,\eta_i,\zeta_i,u}/\boldsymbol{\mu}_{m_i}(1, -1, a_i, 0),$$

where k, a_i are some positive numbers and $e(j)$ is some function. Hence these points coincide with P_1^+ and P_2^+. Since $P_i^+ \in \Gamma^+ \subset V^+$ are cyclic quotient singularities, we have $e(k) = e(k+1) = 1$ (u needs to be eliminated). If we put $\delta := a_1 m_2 + a_2 m_1 - m_1 m_2$, then $\delta \geq d$ and by definition [Mor02, Def. 3.2] we have $e(3) = 0$, $e(4) = \delta\alpha_1 \geq d > 1$, $e(5) = (\delta^2\rho_2 - 1)\alpha_1 + \delta\alpha_2 \geq d > 1$ (see [Mor02, Rem. 3.6]). Thus, $k \geq 6$. On the other hand, by [Mor02, Lemma 3.5, Cor. 3.7] we have $k \leq 5$, a contradiction.

□

(2.4) Proposition. *Let $\varphi: (V, \Gamma) \to (W, o)$ be the germ of a birational crepant contraction of threefolds with terminal singularities, where Γ is irreducible.*

(i) *(V, Γ) contains at most two non-Gorenstein points.*

(ii) *If (V, Γ) is imprimitive at some point P, then (W, o) cannot be a singularity of type $cA/*$.*

Proof. For the proof we assume that V is not Gorenstein. Since φ is crepant, the point (W, o) is not Gorenstein. Let m be its index. Let $D \in |-K_{(W,o)}|$ be a general member and let $S := \varphi^{-1}(D)$. Then $S \in |-K_{(V,\Gamma)}|$ and both S and D have only Du Val singularities. Moreover, the restriction map $\varphi_S \colon S \to D$ is crepant. Hence S is dominated by the minimal resolution $D^{\min}$ of D and obtained from $D^{\min}$ by contracting all but one exceptional curves.

First assume that (V, Γ) has at least three non-Gorenstein points, say P, Q, and R. By the classification of Du Val singularities (D, o) is a singularity of type D_* or E_* and S is obtained from D by blowing up the exceptional curve corresponding to the central vertex in the Dynkin diagram. In this case exceptional curves on $D^{\min}$ over (S, P), (S, Q) and (S, R) form strings and the proper transform of Γ is adjacent to the ends of them. This means that the log divisor $K_S + \Gamma$ is plt. The latter implies that the germ (V, Γ) is locally primitive (cf. (2.3.4)). Now consider the index-one cover $\pi \colon (W^\sharp, o^\sharp) \to (W, o)$. It induces the following diagram

(2.4.1)
$$\begin{tikzcd} (V^\sharp, \Gamma^\sharp) \arrow[r, "\upsilon"] \arrow[d, "\varphi^\sharp"] & (V, \Gamma) \arrow[d, "\varphi"] \\ (W^\sharp, o^\sharp) \arrow[r, "\pi"] & (W, o) \end{tikzcd}$$

Since (V, Γ) is locally primitive, $\Gamma^\sharp = \varphi^{\sharp -1}(o^\sharp)$ is irreducible. The group $\boldsymbol{\mu}_m$ naturally acts on $\Gamma^\sharp \simeq \mathbb{P}^1$ and has exactly two fixed points. Thus we may assume that $\upsilon^{-1}(R)$ contains no fixed points. But then $\upsilon^{-1}(R)$ consists of $m > 1$ non-Gorenstein points of the same index. By [Mor88, Cor. 1.12] there is a torsion element in $\mathrm{Cl}^{\mathrm{sc}}(V^\sharp, \Gamma^\sharp) \simeq \mathrm{Cl}^{\mathrm{sc}}(W^\sharp, o^\sharp)$. This contradicts the fact that $W^\sharp \setminus \{o^\sharp\}$ is simply connected. Thus (i) is proved.

Now assume that (V, Γ) contains an imprimitive point P. By the proof of (i) S has at most two singular points and the log divisor $K_S + \Gamma$ is not plt at P. On the other hand, assume that (D, o) is a point of type A_*. Then the exceptional curves of the minimal resolution $D^{\min} \to S$ and Γ form a chain. Hence $K_S + \Gamma$ is not plt, a contradiction. $\square$

(2.5) Proposition (cf. [Mor88, 1.14]). *Let $f \colon (X, C) \to (Z, o)$ be the germ of a contraction from a threefold with only terminal singularities to a surface such that*

(i) $-K_X$ *is nef and big,*

(ii) $C := f^{-1}(o)_{\mathrm{red}}$ *is a curve having at least three components,*

(iii) *each K_X-trivial component $C_j \subset C$ contains a non-Gorenstein point.*

Then X has index > 1 at all singular points of C.

Proof. By the Kawamata-Viehweg vanishing theorem we have $R^1 f_* \mathcal{O}_X = 0$. Hence C is a union of $\mathbb{P}^1$'s whose configuration is a tree. Let $P \in C$ be a singular point and let $C_i \subset C$ be a component passing through P. We have $\operatorname{gr}^0_{C_i} \omega \simeq \mathcal{O}(-1)$. Indeed, take a positive integer m such that mK_X is Cartier. Then there is a natural embedding $(\operatorname{gr}^0_{C_i} \omega)^{\otimes m} \hookrightarrow \mathcal{O}_{C_i}(mK_X)$. Since $K_X \cdot C_i \le 0$ we have $\deg \operatorname{gr}^0_{C_i} \omega \le 0$. Moreover, if $K_X \cdot C_i < 0$, then $\deg \operatorname{gr}^0_{C_i} \omega < 0$. Assume that $K_X \cdot C_i = 0$ Since C_i contains a non-Gorenstein point, the above embedding is not an isomorphism and so again $\deg \operatorname{gr}^0_{C_i} \omega < 0$. On the other hand, C_i is contractible over Z. Hence, by the Grauert-Riemenshneider vanishing theorem we have $H^1(\operatorname{gr}^0_{C_i} \omega) = 0$. This shows $\operatorname{gr}^0_{C_i} \omega \simeq \mathcal{O}(-1)$.

Now let C_j be another component of C passing through P. As above, $\operatorname{gr}^0_{C_j} \omega \simeq \mathcal{O}(-1)$. Consider the following exact sequence

$$0 \longrightarrow \operatorname{gr}^0_{C_i \cup C_j} \omega \longrightarrow \operatorname{gr}^0_{C_i} \omega \oplus \operatorname{gr}^0_{C_j} \omega \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\operatorname{Supp} \mathcal{F} = P$. Since $C_i \cup C_j \neq C$, $C_i \cup C_j$ is contractible over Z and again by the Grauert-Riemenshneider vanishing $H^1(\operatorname{gr}^0_{C_i \cup C_j} \omega) = 0$. This implies $\operatorname{gr}^0_{C_i \cup C_j} \omega \simeq \operatorname{gr}^0_{C_i} \omega \oplus \operatorname{gr}^0_{C_j} \omega$. So $\operatorname{gr}^0_{C_i \cup C_j} \omega$ is not locally free at P and this point cannot be Gorenstein. □

§3. The Proof of the Main Theorem

In this section we prove Theorem (1.3).

(3.1) Notation. Let $f\colon (X, C) \to (Z, o)$ be a $\mathbb{Q}$-conic bundle germ with reducible central fiber C. Then $\rho(X/Z) > 1$. Recall that according to [MP08, Th. 1.2.7] (Z, o) is either smooth or Du Val of type A (see also the construction (3.1.2) below). We assume that (Z, o) is singular of type A_{n-1}, $n \ge 2$.

(3.1.1) Lemma. *Notation as above.*

(i) *If (X, C) has a point P such that either*

   (a) *P is of type $cAx/4$, or*

   (b) *for each component $C_i \subset C$ passing through P the germ (X, C_i) is locally imprimitive at P.*

Then P is the only non-Gorenstein point on X.

(ii) *Conversely, if P is a unique non-Gorenstein point on X, then all the components $C_i \subset C$ pass through P and the germ (X, C_i) is locally imprimitive at P. If furthermore (X, P) is of index 4, then (X, C) is a quotient of an index two $\mathbb{Q}$-conic bundle germ (X', C') over a smooth base by $\boldsymbol{\mu}_2$, where the action is free in codimension one, C' has four irreducible components and $\boldsymbol{\mu}_2$ does not fix any of them.*

Proof. Let $P \in X$ be a point as in (i). For each component $C_i \subset C$ passing through P the germ (X, C_i) is an extremal neighborhood and by [KM92, Th. 2.2] (X, C_i) has no non-Gorenstein point other than P. Since each singular point of C is not Gorenstein [Kol99, Prop. 4.2], [MP08, 4.4.2] and C is connected, P is the only non-Gorenstein point on the whole X.

Now assume that P is the only non-Gorenstein point. Consider the base change [MP08, 2.4]: $(X', C') \to (X, C)$. Here (X', C') is a conic bundle germ over a smooth base and $X' \to X$ is a $\boldsymbol{\mu}_n$-cover étale outside P. Thus $(X, C) = (X', C')/\boldsymbol{\mu}_n$. If $\boldsymbol{\mu}_n$ fixes a component $C'_i \subset C$, then there are two $\boldsymbol{\mu}_n$-fixed points on C_i and they give us two non-Gorenstein points on X, a contradiction. So the first assertion of (ii) is proved.

Finally assume that (X, P) is of index 4. Since the index of (X, P) is divisible by n, $n = 4$ or 2. If $n = 4$, then X' is Gorenstein. In this case, by [Pro97, Th. 2.4] C is irreducible, a contradiction. Thus $n = 2$ and (X', C') is of index 2. By the above, $\boldsymbol{\mu}_2$ does not fix any component of C'. On the other hand, C' has at most four components [MP08, Th. 12.1]. Hence C' has exactly four components. This proves the lemma. □

(3.1.2) Let $q: X^{\mathrm{q}} \to X$ be a $\mathbb{Q}$-factorialization. (It is possible that q is the identity map.) Run the MMP over Z: $X^{\mathrm{q}} = X_0 \dashrightarrow X_{N+1} = \bar{X}$. Since X/Z is a rational curve fibration, X_{N+1} is not a minimal model over Z. Therefore, at the end we get an extremal contraction $\bar{f}: \bar{X} \to \bar{Z}$ of Fano type over Z. Since the composition $f^{\mathrm{q}}: X^{\mathrm{q}} \to Z$ has only one-dimensional fibers, $Z = \bar{Z}$ and $X^{\mathrm{q}} \dashrightarrow \bar{X}$ is a sequence of flips and extremal divisorial contractions that contract a divisor to a curve which is not contained in the fiber over $o \in Z$. Thus we have the following diagram:

$$\begin{tikzcd}
(X^{\mathrm{q}}, C^{\mathrm{q}}) \arrow[r, dashed, "g_0"] \arrow[d, "q"'] \arrow[ddrr, bend left, "f^{\mathrm{q}}"] & (X_1, C_1) \arrow[r, dashed] & \cdots \arrow[r, dashed] & (X_N, C_N) \arrow[r, "g_N"] & (\bar{X}, \bar{C}) \arrow[ddll, bend left, "\bar{f}"] \\
(X, C) \arrow[drr, bend right, "f"] & & & & \\
& & (Z, o) & &
\end{tikzcd}$$

Here each X_k has a morphism $f_k\colon X_k \to Z$ with connected one-dimensional fibers and $C_k := f_k^{-1}(o)$ is the central fiber (with reduced structure). Since $\rho(\bar{X}/Z) = 1$, $\bar{f}\colon \bar{X} \to \bar{Z}$ is a $\mathbb{Q}$-conic bundle with irreducible central fiber $\bar{C}$. Since the base (Z, o) is singular, $\bar{X}$ is not Gorenstein. So $\bar{f}$ is classified in [MP08], see also (1.2).

(3.1.3) Note that each component of the central fiber C_k is contractible and the resulting variety is again projective over Z (because it has one-dimensional fibers over Z). Hence each component of C_k generates an extremal ray (not necessarily K-negative). This implies that all our flipping curves are irreducible and all the divisorial contractions have irreducible fibers. Note also that all the varieties X_k are analytically $\mathbb{Q}$-factorial at each point on C_k (again because $X_k \to Z$ has one-dimensional fibers, cf. [Mor88, Proof of 1.7]).

The following is the key argument in the proof.

(3.2) Proposition. *In the above notation one of the following holds.*

(3.2.1) *There is a component $C_0^{\mathrm{q}} \subset C^{\mathrm{q}}$ containing two cyclic quotient singularities P^{q} and Q^{q} of index n. No other components of C^{q} pass through P^{q} and Q^{q}.*

(3.2.2) *There is a point $P^{\mathrm{q}} \in (X^{\mathrm{q}}, C^{\mathrm{q}})$ of index $m > 1$ which is contained in only one component $C_0^{\mathrm{q}} \subset C^{\mathrm{q}}$ and such that $(X^{\mathrm{q}}, C_0^{\mathrm{q}})$ is locally imprimitive at P^{q}. The following are the possibilities for (n, m): $(4, 8)$, $(2, 4)$, and $(2, 2)$.*

(3.2.3) *There is a point $P^{\mathrm{q}} \in (X^{\mathrm{q}}, C^{\mathrm{q}})$ which is contained in exactly two components $C_0^{\mathrm{q}}, C_1^{\mathrm{q}} \subset C^{\mathrm{q}}$ and such that both germs $(X^{\mathrm{q}}, C_i^{\mathrm{q}})$ are locally imprimitive at P^{q}. The point $(X^{\mathrm{q}}, P^{\mathrm{q}})$ is of type $cAx/4$ or $\frac{1}{4}(1, 1, -1)$. Here $n = 2$.*

Moreover, there is an n-torsion element $\xi^{\mathrm{q}} \in \mathrm{Cl}^{\mathrm{sc}}(X^{\mathrm{q}}, C^{\mathrm{q}})$ which is not Cartier at P^{q} (and at Q^{q} is the case (3.2.1)).

Proof. Since (Z,o) is of type A_{n-1}, there is an n-torsion element $\eta \in \mathrm{Cl}(Z,o)$. Put $\bar{\xi} := \bar{f}^*\eta$, $\xi_l := f_l^*\eta$, and $\xi^{\mathrm{q}} := f^{\mathrm{q}\,*}\eta$.

Assume that $(\bar{X},\bar{C})$ is either toroidal of type (IA)+(IA). Let $\bar{P}$, $\bar{Q}$ be the singular points of $\bar{X}$. Then $\bar{\xi}$ is not Cartier at $\bar{P}$ and $\bar{Q}$. We claim that the map $\psi\colon \bar{X} \dashrightarrow X^{\mathrm{q}}$ is an isomorphism near $\bar{P}$ and $\bar{Q}$. Indeed, by induction, since $\bar{P}$, $\bar{Q}$ are cyclic quotient singularities of index n, there is no divisorial contractions over these points by [Kaw96] and by Proposition (2.3) on each step the proper transform of $\bar{C}$ cannot be a flipped curve. So if we put $P^{\mathrm{q}} := \psi(\bar{P})$, $Q^{\mathrm{q}} := \psi(\bar{Q})$, and $C_0^{\mathrm{q}} := \psi(\bar{C})$, we get the case (3.2.1).

Now assume that $(\bar{X},\bar{C})$ is of type (IE$^\vee$), (IA$^\vee$), or (II$^\vee$). Let $\bar{P}$ be a (unique) non-Gorenstein point. Then $(\bar{X},\bar{P})$ is either a cyclic quotient singularity or of type $cAx/4$ and again $\bar{\xi}$ is not Cartier at $\bar{P}$. Moreover, $(\bar{X},\bar{C})$ is locally imprimitive at $\bar{P}$. As above, there is no divisorial contractions over $\bar{P}$ by [Kaw96] and Proposition (2.2) and the proper transform of $\bar{C}$ cannot be a flipped curve by Proposition (2.3). Put $P^{\mathrm{q}} := \psi(\bar{P})$ and $C_0^{\mathrm{q}} := \psi(\bar{C})$. We get the case (3.2.2).

Finally consider the case where $(\bar{X},\bar{C})$ is of type (ID$^\vee$). Then $n = 2$, i.e., (Z,o) is of type A_1. Let $\bar{P}$ be a (unique) non-Gorenstein point. Then $(\bar{X},\bar{C})$ is locally imprimitive at $\bar{P}$ and $(\bar{X},\bar{P})$ is of type $cA/2$ or $cAx/2$. Moreover, in the first case, $(\bar{X},\bar{P})$ is analytically isomorphic to a singularity given by (2.2.1). If there is no divisorial contractions over $\bar{P}$, we can argue as above and get the case (3.2.2). Otherwise on some step, the map $\psi_{k+1}\colon \bar{X} \dashrightarrow X_{k+1}$ is an isomorphism near $\bar{P}$ and there is a divisorial contraction $g_k\colon X_k \to X_{k+1}$ which blows up a curve passsing through $P_{k+1} := \psi_{k+1}(\bar{P})$. Let $C_{k,0} := g_k^{-1}(P_{k+1})$ and let $C_{k,1}$ be the proper transform of $\bar{C}$ on X_k. By Proposition (2.2) X_k has exactly one non-Gorenstein point P_k on $C_{k,0}$. Moreover, P_k is either a cyclic quotient singularity $\frac{1}{4}(1,1,-1)$ or of type $cAx/4$ and $(X_k, C_{k,0})$ is locally imprimitive at P_k of splitting degree 2. Note that $\xi_k = g_k^*\xi_{k+1}$ is not Cartier at some point of $C_{k,0}$. Since P_k is the only non-Gorenstein point on $C_{k,0}$, ξ_k is not Cartier at P_k. Now if $C_{k,1}$ does not pass through P_k, then as above we get the case (3.2.2). Assume that $C_{k,0} \cap C_{k,1} = \{P_k\}$.

We claim that $(X_k, C_{k,1})$ is locally imprimitive at P_k. Indeed, ξ_k defines the double cover $\pi_k\colon (X_k', C_k') \to (X_k, C_k)$ which is étale outside $\operatorname{Sing} X_k$. Since ξ_k is not Cartier at P_k, π_k does not split over P_k. Hence, $C_{k,1}' := \pi_k^{-1}(C_{k,1})$ is connected. On the other hand, since $(\bar{X},\bar{C})$ is locally imprimitive at $\bar{P}$, the curve $C_{k,1}'$ is reducible. This means that $C_{k,1}$ is locally imprimitive at P_k. Finally as above the map $X_k \dashrightarrow X^{\mathrm{q}}$ is an isomorphism near P_k. We get case (3.2.3). $\square$

(3.3) Proposition. *Notation as in* (3.1)*. Then* (X, C) *contains only one non-Gorenstein point* P*. This point is either a cyclic quotient* $\frac{1}{4}(1,1,-1)$ *or of type* $cAx/4$*. Moreover, for each component* $C_i \subset C$ *the germ* (X, C_i) *is imprimitive at* P *and* (Z, o) *is of type* A_1*.*

Proof. By Proposition (3.2) there is a component $C_0^{\mathrm{q}} \subsetneq C^{\mathrm{q}}$ as in (3.2.1), (3.2.2), or (3.2.3). First assume that C_0^{q} is not contracted by $q: X^{\mathrm{q}} \to X$. Put $C_0 := q(C_0^{\mathrm{q}})$. Then (X, C_0) is an extremal neighborhood. In the case (3.2.1) it has two cyclic quotient singularities at $q(P^{\mathrm{q}})$ and $q(Q^{\mathrm{q}})$ and no other components of C pass through $q(P^{\mathrm{q}})$ and $q(Q^{\mathrm{q}})$. On the other hand, $C \neq C_0$ and intersection points $C_0 \cap (C - C_0)$ are non-Gorenstein [Kol99, Prop. 4.2], [MP08, 4.4.2]. Thus the extremal neighborhood (X, C_0) has at least three non-Gorenstein points. This contradicts [Mor88, Th. 6.2]. Similarly, in the case (3.2.2), (X, C_0) is locally imprimitive at $q(P^{\mathrm{q}})$ and no other components of C pass through $q(P^{\mathrm{q}})$. We get a contradiction by Lemma (3.1.1). Consider the case (3.2.3). If C_1^{q} is not contracted by q, then we are done by Lemma (3.1.1). If C_1^{q} is contracted by q, then $q(C_1)$ is a point of type $cAx/4$ by Proposition (2.4) and because P^{q} is of index 4. Then again the assertion follows by Lemma (3.1.1).

From now on we assume that q contracts C_0^{q}, i.e., $K_{X^{\mathrm{q}}} \cdot C_0^{\mathrm{q}} = 0$. In the case (3.2.3) by symmetry and by the above arguments we may assume that q contracts C_1^{q}. Consider the decomposition

$$q: X^{\mathrm{q}} \xrightarrow{\varphi} X^{\delta} \xrightarrow{\delta} X,$$

where φ contracts all the $K_{X^{\mathrm{q}}}$-trivial components of C^{q} except for C_0^{q}. Put $C^{\delta} := \varphi(C^{\mathrm{q}})$ and $C_0^{\delta} := \varphi(C_0^{\mathrm{q}})$. Thus $-K_{X^{\delta}}$ is nef and big over Z and C_0^{δ} is the only $K_{X^{\delta}}$-trivial curve on X^{δ}/Z. Let $C^{\delta\delta} := C^{\delta} - C_0^{\delta}$. Then $C^{\delta\delta}$ has at least two components. Let $P := \delta(C_0^{\delta})$ and $R^{\delta} = C^{\delta\delta} \cap C_0^{\delta}$. By Proposition (2.5) R^{δ} is not Gorenstein.

In the case (3.2.1), C_0^{δ} contains at least three non-Gorenstein points: R^{δ}, $P^{\delta} := \varphi(P^{\mathrm{q}})$, and $Q^{\delta} := \varphi(Q^{\mathrm{q}})$. This contradicts Proposition (2.4).

In the case (3.2.2), $P^{\delta} := \varphi(P^{\mathrm{q}})$ is a locally imprimitive point of $(X^{\delta}, C_0^{\delta})$. By Proposition (2.4) the singularity $(X, P = \delta(C_0^{\delta}))$ is not of type $cA/*$. If the index of (X, P) is ≥ 4, then (X, P) is of type $cAx/4$ and we can apply Lemma (3.1.1). Thus we assume that (X, P) is of index 2 and $n = 2$. Let $C_i \subset C$ be a component passing through P. By [Mor88, Cor. 1.16] (X, C_i) is primitive at P. Further, $\xi := f^*\eta = q_*\xi^{\mathrm{q}}$ is an 2-torsion element of $\mathrm{Cl}^{\mathrm{sc}}(X, C)$ and is not Cartier at P. This defines a double étale in codimension one cover $(X', C_i') \to (X, C_i)$ which does not splits over P. Hence there is a point $Q \in (X, C_i)$ of even index.

This contradicts the classification [KM92, Th. 2.2] (cf. [Mor07]).

Consider the case (3.2.3). Then $P^{\delta} := \varphi(P^{\mathrm{q}})$ is a point of index ≥ 4 (because φ is a crepant contraction). Recall that φ contracts C_1^{q} by our assumption. Then by Proposition (2.4) (X^{δ}, P^{δ}) is a point of type $cAx/4$. As in the proof of Proposition (2.4), let $D \in |-K_{(X,\delta(P^{\delta}))}|$ be a general element and let $S := \delta^{-1}(D)$. Then both D and S have only Du Val singularities and the contraction $\delta_S \colon S \to D$ is crepant. Since (S, P^{δ}) is not of type A_*, the germ (D, P) also cannot be of type A_*. Hence, (X, P) is not of type $cA/*$ and so it is of type $cAx/4$ (because its index is ≥ 4). Then the assertion follows by Lemma (3.1.1). $\square$

(3.4) Explicit forms. By Proposition (3.3) and Lemma (3.1.1) $f\colon (X, C) \to (Z, o)$ is a quotient of an index-two $\mathbb{Q}$-conic bundle $f'\colon (X', C') \to (Z', o')$ over a smooth base by $\boldsymbol{\mu}_2$, where $\boldsymbol{\mu}_2$ acts on X' and Z' freely in codimension one. By [MP08, Prop. 12.1.10] there is a $\boldsymbol{\mu}_2$-equivariant diagram

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}(1,1,1,2) \times \mathbb{C}^2 \arrow[d, "p"] \\ & \mathbb{C}^2 \end{tikzcd}$$

where the actions of $\boldsymbol{\mu}_2$ on $(\mathbb{C}^2, 0) \simeq (Z', o')$ and $\mathbb{P}(1,1,1,2)$ are linear. Further, we can make coordinates y_1, y_2, y_3, u, v in $\mathbb{P}(1,1,1,2)$ and $\mathbb{C}^2$ to be semi-invariant. By [MP08, Th. 12.1] X' is given by two semi-invariant equations

$$\begin{cases} q_1(y_1, y_2, y_3) - \psi_1(y_1, \dots, y_4; u, v) = 0, \\ q_2(y_1, y_2, y_3) - \psi_2(y_1, \dots, y_4; u, v) = 0, \end{cases}$$

where ψ_i and q_i are weighted quadratic in $y_1, \dots, y_4$ with respect to $\mathrm{wt}(y_1, \dots, y_4) = (1, 1, 1, 2)$ and $\psi_i(y_1, \dots, y_4; 0, 0) = 0$. Since the action of $\boldsymbol{\mu}_2$ on $Z \simeq \mathbb{C}^2$ is free outside 0, this action is given by $u \mapsto -u$, $v \mapsto -v$. Modulo multiplication on ± 1 and permutations of y_1, y_2, y_3, we may assume also that $y_1 \mapsto -y_1$, $y_2 \mapsto -y_2$, $y_3 \mapsto y_3$. Otherwise all the points of $\{y_4 = 0\} \cap C'$ are fixed by $\boldsymbol{\mu}_2$, while P is the only non-Gorenstein on X.

The central fiber C' is defined by $q_1 = q_2 = 0$. By Lemma (3.1.1) C' has exactly four components and $\boldsymbol{\mu}_2$ does not fix any of them. Thus we may assume that $C' = \cup C_i'$, $i = 1, 2, 3, 4$ and $\boldsymbol{\mu}_2$ interchanges C_1' and C_2' (resp. C_3' and C_4'). For any two components $C_i' \neq C_j'$ of C', there is a linear form $l_{i,j}(y_1, \dots, y_3)$ that vanishes along $C_i' \cup C_j'$. Then quadratic forms $l_{1,2}l_{3,4}$, $l_{1,3}l_{2,4}$, $l_{1,4}l_{2,3}$ vanish along C'. Hence they belong to the pencil $\lambda_1 q_1 + \lambda_2 q_2$ and semi-invariant. This implies that the action of $\boldsymbol{\mu}_2$ on the pencil is trivial. Moreover, we can put $q_1 = l_{1,3}l_{2,4}$ and $q_2 = l_{1,4}l_{2,3}$. In view of the $\boldsymbol{\mu}_2$-action we may assume that $l_{1,3} = y_1 + y_3$, $l_{2,4} = y_1 - y_3$, $l_{1,4} = y_2 + y_3$, $l_{2,3} = y_2 - y_3$ after some linear coordinate change of y_1, y_2, y_3.

We claim that $y_4 \mapsto -y_4$. The arguments below are similar to ones in the proof of [MP08, Lemma 12.1.12]. Assume to the contrary that $y_4 \mapsto y_4$. Let $U \subset \mathbb{P}(1,1,1,2)$ be the chart $y_4 \neq 0$. Then $U \simeq \mathbb{C}^3_{z_1,z_2,z_3}/\boldsymbol{\mu}_2(1,1,1)$. Let $X^\sharp$ be the pull-back of $X \cap (U \times \mathbb{C}^2_{u,v})$ on $\mathbb{C}^3_{z_1,z_2,z_3} \times \mathbb{C}^2_{u,v}$ and let $P^\sharp \in X^\sharp$ be the preimage of P. Since the induced map $X^\sharp \to X$ is étale in codimension one, $(X^\sharp, P^\sharp) \to (X, P)$ is the index-one cover. Hence $(X^\sharp, P^\sharp) \to (X, P)/\boldsymbol{\mu}_2$ is also the index-one cover of the terminal point $(X, P)/\boldsymbol{\mu}_2$ of index 4 (the last is true because the action of $\boldsymbol{\mu}_2$ is free in codimension one). Hence the morphism is a $\boldsymbol{\mu}_4$-covering by the structure of terminal singularities. However $(X, P)/\boldsymbol{\mu}_2$ is the quotient of $(X^\sharp, P^\sharp)$ by commuting $\boldsymbol{\mu}_2$-actions:

$$(z_1, z_2, z_3, u, v) \mapsto (-z_1, -z_2, -z_3, u, v), (-z_1, -z_2, z_3, -u, -v)$$

This is a contradiction, and we have $y_4 \mapsto -y_4$ as claimed. This finishes the proof of Theorem (1.3).

Proof of Corollary (1.4). If C is irreducible, the assertion follows by [MP08, Proposition (1.3.7)], so we have to check only cases (1.3.1) and (1.3.2). Thus we assume that X has a unique non-Gorenstein point, say P, and C is reducible. For each component $C_i \subset C$, the germ (X, C_i) is an extremal neighborhood with a unique non-Gorenstein point. Let $F \in |-K_{(X,P)}|$ be a general member of the anti-canonical linear system of the germ (X, P). The point (F, P) is Du Val by [Rei87, (6.4), (B)]. Further, F is also a member of $|-K_{(X,C_i)}|$ for each C_i, see [Mor88, Theorem (7.3)]. Hence $F \in |-K_X|$. $\square$

Acknowledgments

The work was carried out at Research Institute for Mathematical Sciences (RIMS), Kyoto University. The second author would like to thank RIMS for invitations to work there in February 2007, for hospitality and wonderful conditions of work. Both of the authors are grateful to the referee for his/her careful reading and helpful suggestions.

The research of the first author was supported in part by JSPS Grant-in-Aid for Scientific Research (B)(2), No. 16340004. The second author was partially supported by grants CRDF-RUM, No. 1-2692-MO-05 and RFBR, No. 08-01-00395-a, 06-01-72017.

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